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Scattering Theory for the Klein–Gordon Equation

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We develop the scattering theory for the Klein–Gordon equation. We follow the usual procedure of considering an equivalent equation, which is first order in time, in the Hilbert space of vector valued functions which have a finite energy norm. We prove existence and completeness of the wave operators, the intertwining relations, and the invariance principle as well. This is done for a large class of potentials. In particular, the magnetic potential may even be divergent at infinity. Electric and scalar potentials that behave at infinity as $|x|^{-\epsilon-1}$, $\epsilon > 0$ are contained in our class.

INTRODUCTION

We develop the scattering theory for the Klein–Gordon equation [1]:

$$\left(i\frac{\partial}{\partial t} - b_0\right)^2 \psi(x, t) = \left[\sum_{i=1}^{n} (D_i - b_i)^2 + m^2 + q_s(x)\right] \psi(x, t),$$

$$x \in \mathbb{R}^n, \qquad t \in \mathbb{R}, \qquad D_j = -i\frac{\partial}{\partial x_j},$$

$b_i(x)$ and $q_s(x)$ are real valued functions and m is a positive constant. The Klein–Gordon equation describes a relativistic spin zero particle of mass m in the presence of an electric potential $b_0(x)$, a magnetic potential $b_i(x)$, $1 \leqslant i \leqslant n$ and $q_s(x)$ may be interpreted as a scalar potential.

We follow the usual procedure of considering an equivalent equation, which is first order in time, in the Hilbert space of vector valued functions which have finite energy norm.

In our main theorem (Theorem 5) we prove existence and completeness of the wave operators, the intertwining relations, and the invariance principle as well. This is done for a large class of potentials. In particular the magnetic potential, $b_i(x)$, $1 \leqslant i \leqslant n$ may even be divergent at infinity. Electric and scalar potentials that behave at infinity as $|x|^{-1-\epsilon}$, $\epsilon > 0$ are contained in our class.

The essential point in the proof of our main Theorems 1, 2, 3, and 5 is that

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we show that the unperturbed and perturbed Hamiltonians are unitary equivalent to pseudodifferential operators defined in $\mathscr{L}_2^2 = \mathscr{L}^2 \oplus \mathscr{L}^2$. That allows us to use the methods developed for pseudodifferential operators in [5, 6].

The Klein–Gordon equation has been studied by many authors. We will just mention the more recent results [2, 7, 8, 9], where a list of references is given.

Lundberg [2] considers the case $n = 3$, $b_i(x) = 0$, $1 \leqslant i \leqslant n$ and:

(i) q_0 and q_s real valued, locally Hölder continuous except at a finite number of singularities;

(ii) $b_0^2(x)$, $q_s(x)$ square integrable;

(iii) $b_0(x)$ and $q_s(x)$ behave as $O(|x|^{-3-\epsilon})$, $\epsilon > 0$ for $|x| \to \infty$;

(iv) $\int dx(-b_0^2 + q_s) \mid f(x)|^2 \geqslant -\alpha \int dx(|\nabla f|^2 + m^2 |f|^2)$, with $0 < \alpha < 1$ and $f(x) \in C_0^\infty$.

Eckhardt [8] considers the case $n \geqslant 3$, $b_i(x) = 0$, $1 \leqslant i \leqslant n$. He assumes [2, IV] and conditions of Stummel type on $q_s(x)$ and $b_0^2(x)$; and some other conditions [7]. The method of the proof of [2, 8] is different from our method; they consider eigenfunction expansions, as in [10]. Kako [9] considers the case $n = 3$, and $b_i(x)$ $0 \leqslant i \leqslant n$ and $q_s(x)$ bounded, measurable, and satisfying:

(1) $|b_i(x)| \leqslant C |x|^{-2-\epsilon}$, $0 \leqslant i \leqslant 3$;

(2) $b_i(x)$, $1 \leqslant i \leqslant 3$ are differentiable and $|(\partial/\partial x_i)\, b_i| \leqslant C |x|^{-2-\epsilon}$;

(3) $|q_s| \leqslant C |x|^{-2-\epsilon}$.

All the functions we consider are assumed to be measurable. Since we are interested in scattering theory we did not obtain here the stronger possible local singularities in the potentials. See [15] concerning that question.

1

The Klein–Gordon equation [1] is the partial differential equation:

$$\left(i \frac{\partial}{\partial t} - b_0\right)^2 \psi(x, t) = \left[\sum_{i=1}^{n} (D_i - b_i)^2 + m^2 + q_s(x)\right] \psi(x, t),$$

$$x \in \mathbb{R}^m, \qquad t \in \mathbb{R}, \qquad D_j = -i \frac{\partial}{\partial x_j}. \tag{1.1}$$

$b_i(x)$, $0 \leqslant i \leqslant n$, and $q_s(x)$ are real valued functions, and m is a positive constant.

The Klein–Gordon equation describes a relativistic spin zero particle of mass m in the presence of an electric potential $b_0(x)$, a magnetic potential $b_i(x)$, $1 \leqslant i \leqslant n$, and $q_s(x)$ may be interpreted as a scalar potential.

As is well known, Eq. (1.1) together with the initial conditions

$$\left\{\psi(x,0),\, i\frac{\partial}{\partial t}\psi(x,0)\right\} = \{f_1(x), f_2(x)\}$$

define a well-posed initial value problem. Moreover, the energy integral

$$E(\psi) = \int d^n x \left\{\sum_{i=1}^{n} |(D_i - b_i)\psi|^2 + (m^2 + q)\,|\psi|^2 + \left|\frac{\partial}{\partial t}\psi\right|^2\right\}, \tag{1.2}$$

where $q(x) = q_s(x) - b_0^2(x)$, is constant in time. We will follow the usual procedure of reducing (1.1) to an equivalent equation which is first order in time. Let $f_1(x) = \psi(x, t)$, $f_2 = i(\partial/\partial t)\,\psi(x, t)$, and $f = \binom{f_1}{f_2}$. Then (1.1) is equivalent to the equation

$$i(\partial/\partial t)\, f = hf, \tag{1.3}$$

where

$$h = \begin{bmatrix} 0 & 1 \\ L & Q \end{bmatrix}, \qquad D(h) = C_0^{\infty,2},$$

$$L = \sum_{i=1}^{n} (D_i - b_i)^2 + m^2 + q(x), \qquad q(x) = q_s - b_0^2, \tag{1.4}$$

$$Q = 2b_0(x).$$

C_0^∞ is the space of infinitely differentiable functions of compact support on $\mathbb{R}^n$, and $C_0^{\infty,2} = C_0^\infty \oplus C_0^\infty$.

We associate with the energy integral (1.2) a sesquilinear form, the energy sesquilinear form, defined in $C_0^{\infty,2}$, namely,

$$(f, g)_{\mathrm{E}} = \sum_{i=1}^{n} ((D_i - b_i)f_1\,,(D_i - b_i)g_1) + ((m^2 + q)f_1\,, g_1) + (f_2\,, g_2), \tag{1.5}$$

$f, g \in C_0^{\infty,2}$ $(\cdot, \cdot)$ denotes the usual $\mathscr{L}^2$ scalar product, and $q(x) = q_s(x) - b_0^2(x)$. It is easy to verify that h is symmetric on the energy sesquilinear form, i.e.,

$$(hf, g)_{\mathrm{E}} = (f, hg)_{\mathrm{E}}\,. \tag{1.6}$$

We consider first the free case, i.e., the situation where $b_i(x) \equiv 0$, $q_s(x) \equiv 0$. In this case the energy norm is given by

$$(f, g)_0 = \sum_{i=1}^{n} (D_i f_1\,, D_i g_1) + m^2(f_1\,, g_1) + (f_2\,, g_2). \tag{1.7}$$

Let $\mathscr{H}_0$ be the completion of $C_0^{\infty,2}$ with this norm. Clearly $(\cdot, \cdot)_0$ is equivalent with the norm of $H_1 \otimes \mathscr{L}^2$, where H_s, $s \in \mathbb{R}$ is the Sobolev space of order s. Equation (1.3) reduces to

$$i \frac{\partial}{\partial t} f = H_0 f, \qquad H_0 = \begin{bmatrix} 0 & 1 \\ -\Delta + m^2 & 0 \end{bmatrix}. \tag{1.8}$$

We prove in Section 2 that

THEOREM 1. *H_0 is self-adjoint on $\mathscr{H}_0$ with domain $D(H_0) = H_2 \otimes H_1$, and is essentially self-adjoint on $C_0^{\infty,2}$. It is absolutely continuous and $\sigma(H_0) = \sigma_e(H_0) = (-\infty, -m] \cup [m, \infty)$. $\sigma_e(H_0)$ denotes the essential spectrum of H_0.*

In the interacting case, i.e., the case where $b_i(x)$ and $q_s(x)$ are not identically zero, we introduce an assumption assuring that the energy sesquilinear form is strictly positive, i.e., it defines a norm.

(A_0) There is a constant $\epsilon > 0$ such that

$$\int q^-(x) \, | f(x)|^2 \, d^n x \leqslant \sum_{i=1}^{n} \| D_i f \|^2 + (m^2 - \epsilon) \| f \|^2, \qquad f \in C_0^\infty.$$

$q^\pm(x)$ denotes the positive and negative parts of $q(x)$.

We give in the Appendix a necessary and sufficient condition for A_0 to be satisfied. Note that A_0 is weaker than [2, Condition iv]. Then $(\cdot, \cdot)_E$ is a norm (see Lemma 2.1 of Section 2), and we denote by $\mathscr{H}_E$ the completion of $C_0^{\infty,2}$ with the energy norm.

We give our conditions in terms of the following quantities [3]:

$$N_{\alpha,\delta}(q) = \sup_x \int_{|x-y|<\delta} | q(y)|^2 \, \omega_\alpha(x-y) \, dy,$$

where

$$\begin{aligned} \omega_\alpha(x) &= | x |^{\alpha-n} && \text{for } \alpha < n, \\ &= 1 - \lg | x | && \text{for } \alpha = n, \\ &= 1 && \text{for } \alpha > n. \end{aligned}$$

We denote $N_\alpha(q) = N_{\alpha,1}(q)$, and we say that $q \in N_\alpha$ if $N_\alpha(q) < \infty$.

Our next assumptions are:

(A_1) For $1 \leqslant i \leqslant n$, $b_i(x) \in N_2$ and if $n \geqslant 2$ $N_{2,s}(b_i) \to_{s\to 0} 0$.

(A_2) $| q(x)|^{1/2} \in N_2$ and if $n \geqslant 2$ $N_{2,s}(q) \underset{s\to 0}{\to} 0$.

We prove in Lemma 2.4 that $A_0 - A_2$ imply that the norm of $\mathscr{H}_E$ and $\mathscr{H}_0$ are equivalent, then they coincide as a set. Let J be the identification operator from $\mathscr{H}_0$ onto $\mathscr{H}_E$. J, is a bijection.

We introduce three more assumptions:

(A_3) $C(x) = \sum_{i=1}^{n} ((\partial/\partial x_i)\, b_i(x)) \in N_4$ and if $n \geqslant 4$ $N_{4s}(C) \underset{s\to 0}{\to} 0$.

(A_4) $q(x) \in N_4$ and if $n \geqslant 4$ $N_{4,s}(q) \underset{s\to 0}{\to} 0$, $b^2 = \sum_{i=1}^{n} b_i^2(x) \in N_4$, and if $n \geqslant 4$ $N_{4,s}(b^2) \underset{s\to 0}{\to} 0$.

(A_5) $b_0(x) \in N_2$ and if $n \geqslant 2$ $N_{2,s}(b_0) \underset{s\to 0}{\to} 0$.

We prove in Section 2 that

THEOREM 2. *If A_0–A_5 are satisfied h has a self-adjoint extension, H, in $\mathscr{H}_E$ with domain*

$$D(H) = H_2 \otimes H_1 .$$

The wave operators are defined, when they exist in the following way:

$$\omega_{\pm} = \underset{t\to\pm\infty}{\text{s-lim}}\, e^{iHt} J e^{-iH_0 t},$$

where s-lim *means strong limit. Note that H_0 is absolutely continuous, i.e., $P_{\mathrm{ac}}(H_0) = \mathbb{1}$.*

$\omega_{\pm}$ are said to be complete if their range coincides with the absolutely continuous subspace of H. In that case the scattering matrix, S, is defined as

$$S = \omega_+{}^* \omega_- \qquad \text{and is unitary.}$$

We say that the intertwining relations hold is $\psi(H)\,\omega_{\pm} = \omega_{\pm}\psi(H_0)$ for each Borel function ψ. The invariance principle holds for a class of functions φ if

$$\omega_{\pm} f = \lim_{t\to\pm\infty} e^{it\varphi(H)} J e^{-it\varphi(H_0)} f$$

for φ in the class.

We need four more assumptions:

(A_6) Let $\rho(x) = (1 + |x|)$. Then $(\rho^s b_i)$ $0 \leqslant i \leqslant n$ belong to N_2, and $(\rho^s C)$ belongs to N_4, for some $s > \frac{1}{2}$. If $n \geqslant 2$ $N_{2,s}(b_i) \underset{s\to 0}{\to} 0$, $0 \leqslant i \leqslant n$. If $n \geqslant 4$ $N_{4,s}(C) \underset{s\to 0}{\to} 0$.

Note that A_6 implies A_1, A_3, and A_5. We define

$$N_{\alpha,x}(q) = \int_{|y|<1} |q(x-y)|^2\, \omega_{\alpha}(y)\, dy.$$

Then

(A_7) $N_{2,x}(\rho^s b_i) \underset{|x|\to\infty}{\to} 0$; $N_{4,x}(\rho^s C) \underset{|x|\to\infty}{\to} 0$, and $N_{4,x}(q) \underset{|x|\to\infty}{\to} 0$, where $1 \leqslant i \leqslant n$. Moreover, $N_{4,x}(b^2) \underset{|x|\to\infty}{\to} 0$.

The last assumptions are

(A_8) Let d denote any one of $b_i(x)$, $0 \leqslant i \leqslant n$; $C(x)$; then $\rho^{\alpha}(x) \int_{|x-y|<1} |d(y)|^2 \, dy \in \mathscr{L}^p$ for some $p \geqslant 1$, and α satisfying

$$\frac{n-1}{2s-1}\,\frac{n}{1+n} < p \leqslant \infty; \qquad \alpha > 2s + 1 - \frac{2n}{(1+n)p}.$$

(A_9) $\rho^{\beta}(x) \int_{|x-y|<1} |q(y)| \, dy \in \mathscr{L}^l$ for some β, l satisfying $1 \leqslant l \leqslant \infty$, $\beta > 1 - (2n/(1+n)p)$.

Then we prove in Section 2.

THEOREM 3. *Let* A_0, A_2, A_4, *and* A_6–A_9 *be satisfied. Then the wave operators* $\omega_\pm$ *exist and are isometries from* $\mathscr{H}_0$ *onto* $\mathscr{H}_E^{ac}$. *The intertwining relations hold, and the invariance principle holds for all functions* φ *satisfying*

$$\int_0^\infty \left| \int_\Gamma e^{-i\eta s - it\varphi(s)} \, ds \right|^2 d\eta \to 0 \qquad as \quad t \to \infty$$

and

$$\int_\Gamma e^{-it\varphi(s)} \, ds \to 0 \qquad as \quad t \to \infty$$

for any compact Γ *contained in* $(-\infty, -m) \cup (m, \infty)$. *The singular spectrum of* H *has measure zero.*

We denote by $\mathscr{H}_E^{ac}$ the absolutely continuous subspace of H. For the definitions of the absolutely continuous and singular parts of a self-adjoint operator see [13].

Once Theorem 3 is proved the conditions on the magnetic potential $b_i(x)$ $1 \leqslant i \leqslant n$ can be weakened in a considerable amount by the introduction of a Gauge transformation. In particular we can include magnetic potentials which are divergent at infinity. From now on we assume that $b_i(x) \in \mathscr{L}^2_{\mathrm{loc}}$, for $1 \leqslant i \leqslant n$, and, for simplicity, that $n > 2$. We define $(\mathrm{Rot}\, b)_{ij} = \partial_{[i} b_{j]}$ where [ ] means the alternation of the indices i, j. We denote

$$M_{\alpha,1}(q) = \sup_x \int_{|y|<1} |q(x-y)| \, |y|^{\alpha-n} \, dy,$$

$M_{\alpha,1}$ is the set of functions q such that $M_{\alpha,1}(q) < \infty$. We also say that q is locally in $M_{\alpha,1}$ if $\varphi q \in M_{\alpha,1}$ for every $\varphi \in C_0^\infty$.

We introduce the assumption A^T: Let b_i, $1 \leqslant i \leqslant n$ be locally $M_{2,1}$ and suppose that $(\mathrm{Rot}\, b)_{ij}$ is a locally Hölder continuous tensor such that

$$C_{ij}^T(x) = \int |D_i b_j - D_j b_i| \, r^{1-n} \, dy < \infty, \qquad 1 \leqslant i, \quad j \leqslant n,$$

for each x, where $r = |x - y|$.

LEMMA. *If* A^T *is satisfied*

$$b_i(x) = b_i^T(x) + (\partial/\partial x_i)\,\phi(x), \qquad 1 \leqslant i \leqslant n,$$

where

$$b_i^T(x) = K \int (\mathrm{Rot}\, b)_{ji}((\partial/\partial x_j)\, r^{2-n})\, dy,$$

$$\phi(x) = \int_C (b_i - b_i^T)\, ds^i,$$

$$K = -\Gamma(\tfrac{1}{2}n)/2(n-2)\,\pi^{n/2}.$$

C is any curve from a fixed point to x (the integral is independent of the curve), and the summation convention is used.

Proof. See [4, Lemma 2.1]. A gauge transformation is a unitary transformation from a Klein–Gordon equation with magnetic potential $b_i(x)$, $1 \leqslant i \leqslant n$, to a Klein–Gordon equation with magnetic potential $b_i^T(x)$, $1 \leqslant i \leqslant n$. The point is that the $b_i^T(x)$ have a better behavior at infinity than the original $b_i(x)$. For example, the $b_i(x)$ may even be divergent at infinity while the $b_i^T(x)$ go to zero at infinity faster than $K/|x|$.

We introduce the following assumptions.

(A_1^T) $C_{ij}^T \in N_2$ and $N_{2,s}(C_{ij}^T) \underset{s\to 0}{\to} 0$, $1 \leqslant i, j \leqslant n$.

$A_2^T = A_2$: $|q|^{1/2} \in N_2$ and $N_{2,s}(|q|^{1/2}) \underset{s\to 0}{\to} 0$ and we define the energy norm for the $b_i^T(x)$: $f, g \in C_0^{\infty,2}$, $(f, g)_T = \sum_{i=j}^n ((D_i - b_i^T) f_1, (D_i - b_i) g_1) + ((m^2 + q) f_1, g_1) + (f_2, g_2)$. By A_0 $(\cdot,\cdot)_T$ is a norm and let $\mathscr{H}_T$ be the completion of $C_0^{\infty,2}$ with this norm. As before A_1^T and A_2^T imply that the norm $(\cdot,\cdot)_T$ is equivalent to the norm of $H_1 \otimes \mathscr{L}^2$.

The Klein–Gordon equation with $b_i^T(x)$ is

$$i\frac{\partial}{\partial t} f = h_T f, \qquad h_T = \begin{bmatrix} 0 & 1 \\ L_T & Q \end{bmatrix},$$
$$L_T = \sum_{i=1}^n (D_i - b_i^T)^2 + m^2 + q(x). \tag{1.9}$$

Note that $C^T(x) = \sum_{i=1}^n (\partial/\partial x_i)\, b_i^T(x) \equiv 0$. Then we do not need an assumption like A_3.

(A_3^T) $q(x) \in N_4$ and if $n \geqslant 4$ $N_{4,s}(q) \underset{s\to 0}{\to} 0$, $(B^2) = \sum_{ij} (C_{ij})^2 \in N_4$ and if $n \geqslant 4$ $N_{4s}(B^2) \underset{s\to 0}{\to} 0$.

Moreover,

$A_4^T = A_5$: $b_0 \in N_2$ and $N_{2,s}(b_0) \underset{s\to 0}{\to} 0$.

Now Theorem 2 implies (note that $b^{T2} = \sum_{i=1}^n b_i^{T^2} \leqslant K B_T^2$)

THEOREM 4. *Let* A_0, A_T *and* A_1^T–A_4^T *be satisfied. Then* h_T *has a self-adjoint extension,* H_T *in* $\mathscr{H}_T$, *with domain* $D(H) = H_2 \otimes H_1$.

We define $\mathscr{H}_E$ to be the following Hilbert space:

$$\mathscr{H}_E = \{f \in \mathscr{L}_2^2 \text{ such that } f = U^{-1} f^T \text{ for some } f^T \in \mathscr{H}_T\},$$

with the scalar product $(f, g)_E = (f^T, g^T)_T$, where $U^{-1} f^T(x) = e^{-i\phi(x)} f^T(x)$. $\mathscr{H}_E$ is, clearly, the completion of $U^{-1} C_0^{\infty,2}$ with the energy norm (1.5), because

$$(D_i + b_i^T)\, Uf(x) = (D_i + b_i^T)\, e^{i\phi(x)} f(x) = U(D_i + b_i) f(x).$$

U is a unitary operator from $\mathscr{H}_E$ onto $\mathscr{H}_T$ by construction. Moreover, $H = U^{-1} H_T U$ is a self-adjoint extension of h (as defined in 1.4).

Then U is a gauge transformation, i.e., a unitary transformation from Eq. (1.3) in $\mathscr{H}_E$ with magnetic potential $b_i(x)$, $1 \leqslant i \leqslant n$ onto a Klein–Gordon Eq. (1.9) in $\mathscr{H}_T$ with magnetic potential $b_i^T(x)$ $1 \leqslant i \leqslant n$.

The identification operator from $\mathscr{H}_0$ onto $\mathscr{H}_E$ is given by $J = U^{-1} J_T$, where J_T is the identification operator from $\mathscr{H}_0$ onto $\mathscr{H}_T$. The wave operators are given by:

$$\omega_\pm = \operatorname*{s-lim}_{t \to \pm\infty} e^{iHt} J e^{-iH_0 t} = \operatorname*{s-lim}_{t \to \pm\infty} U^{-1} e^{itH_T} e^{-iH_0 t}.$$

Then the existence of the $\omega_\pm^T = s\text{-}\lim_{t\to\pm\infty} e^{itH_T} e^{-itH_0}$ implies the existence of the $\omega_\pm$, and moreover $\omega_\pm = U^{-1} \omega_\pm^T$. Then we assume:

A_5^T, A_6^T, A_7^T, and A_8^T: Assume that A_6, A_7, A_8, and A_9 hold with b_i, $1 \leqslant i \leqslant n$ replaced by C_{ij} $1 \leqslant i, j \leqslant n$. Note that $\sum_{i=1}^n (D_i b_i^T) \equiv 0$, i.e., the assumptions concerning C in A_6–A_9 are excluded. In A_7 we replace b^2 by B_T^2.

THEOREM 5. *Let* A_0, A^T, A_2^T, A_3^T, *and* A_5^T–A_8^T *be satisfied. Then all the conclusions of Theorem* 3 *hold true.*

Proof. By Theorem 3 the $\omega_\pm^T$ exists, then $\omega_\pm = U^{-1} \omega_\pm^T$. Moreover the $\omega_\pm^T$ are isometries from $\mathscr{H}_0$ onto $\mathscr{H}_T^{ac}$. Then the $\omega_\pm$ are isometries from $\mathscr{H}_0$ onto $\mathscr{H}^{ac}$. By the same argument the intertwining relations and the invariance principle hold. Q.E.D.

We note that since $\omega_\pm = U^{-1} \omega_\pm^T$ the S matrix $S = \omega_+^* \omega_- = \omega^{T*} \omega_-^T = S^T$, i.e., the scattering matrix is gauge invariant, as one should expect.

2

In Section 2 we prove Theorems 1, 2, and 3. Let s denote the space of Schwartz. By H_s, the Sobolev space of order, s, $s \in \mathbb{R}$, we denote the completion of C_0^∞ with the norm

$$\|f\|_s = \|(1 + |\eta|^2)^{s/2} Ff\|, \qquad f \in C_0^\infty,$$

where F denotes the Fourier transform, and $\| \ \|$ denotes the $\mathscr{L}^2$ norm.

Proof of Theorem 1. We denote $\mathscr{L}_2^2 = \mathscr{L}^2 \oplus \mathscr{L}^2$. Let U_0 be the operator

$$U_0 = \frac{1}{2^{1/2}} F^{-1} \begin{pmatrix} (\eta^2 + m^2)^{1/2} & 1 \\ (\eta^2 + m^2)^{1/2} & -1 \end{pmatrix} F.$$

U_0 is a unitary operator from $\mathscr{H}_0$ onto $\mathscr{L}_2^2$. Moreover $H_0 = U^{-1}\hat{H}_0 U$, where

$$\hat{H}_0 = F^{-1}(\eta^2 + m^2)^{1/2} FM, \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now by [5, Lemma 1.2 and Theorem 1.7] $\hat{H}_0$ is self-adjoint on $H_1 \otimes H_1$ and essentially self-adjoint on $C_0^{\infty,2}$ and $\sigma(\hat{H}_0) = \sigma_e(\hat{H}_0) = (-\infty, -m] \cup [m, \infty)$. $\hat{H}_0$ is clearly absolutely continuous. The statement of the theorem follows from the unitary equivalence of H_0 and $\hat{H}_0$.

Now we prove that if A_0 is satisfied the energy sesquilinear form is positive.

LEMMA 2.1. *Let* A_0 *be satisfied. Then*

$$(f, f)_E \geqslant \epsilon((f_1, f_1) + (f_2, f_2)), \qquad f \in C_0^{\infty,2}.$$

Proof. It follows from A_0 and [3, Lemma 1.2, p. 168] that

$$(q^- f_1, f_1) \leqslant \sum_{i=1}^{n} \|(D_i + b_i) f_1\|^2 + (m^2 - \epsilon)\|f_1\|^2, \qquad f \in C_0^\infty.$$

Then

$$(f_1, f)_E \geqslant \epsilon(f_1, f_1) + (f_2, f_2) \geqslant \epsilon((f_1, f_1) + (f_2, f_2)).$$

Note that we can always take $\epsilon < 1$. Q.E.D.

The following two lemmas have been proved by Schechter [3].

LEMMA 2.2. *Suppose* $q \in N_{2s}$ *for some* $s > 0$, *and if* $n \geqslant 2s$ *assume that* $N_{2s,\delta}(q) \underset{\delta \to 0}{\to} 0$. *Then for each* $\epsilon > 0$ *there is a constant* K_ϵ *such that*

$$\|qf\| \leqslant \epsilon \|f\|_s + K_\epsilon \|f\|, \qquad f \in C_0^\infty.$$

Proof. See [3, Lemma 7.5, p. 140].

LEMMA 2.3. *Let $q \in N_{2s}$, $s > 0$ and let $N_{2s,x}(q) \underset{x\to 0}{\to} 0$, then $q(x)$ is a compact operator from H_s to $\mathscr{L}^2$.*

Proof. This follows as in [3, p. 145].

LEMMA 2.4. *Let A_0, A_1, A_2 be satisfied. Then there exists C_1, $C_2 > 0$ such that*

$$C_2(\|f_1\|_1^2 + \|f_2\|^2) \leqslant (f, f)_E \leqslant C_1(\|f_1\|_1^2 + \|f_2\|^2).$$

Proof.

$$(f, f)_E = \sum_{i=1}^{n} \|(D_i - b_i)f_1\|^2 + ((m+q)f_1, f_1) + (f_2, f_2)$$

$$\leqslant C(\|f_1\|_1^2 + \|f_2\|)$$

where we applied Lemma 2.2. Moreover

$$(f, f)_E \geqslant \sum_{i=1}^{n} \| D_i f_1\|^2 - 2\sum_{i=1}^{n} \| D_i f_1\| \| b_i f_1\| - \epsilon \|f_1\|_1^2 - K\|f_1\|^2 + \|f_2\|^2$$

$$\geqslant \sum_{i=1}^{n} \| D_i f_1\|^2 - \epsilon' \|f_1\|_1^2 - K' \|f_1\| + \|f_2\|^2$$

by the same argument. Then

$$(1 - \epsilon')(\|f_1\|_1^2 + \|f_2\|^2) \leqslant (f, f)_E + K(f_1, f_1)$$

$$\leqslant C(f, f)_E, \qquad C > 0. \qquad \text{Q.E.D.}$$

Lemma 2.4 implies that the norm of $\mathscr{H}_E$ and the norm of $H_1 \otimes \mathscr{L}^2$ are equivalent, and that they coincide as a set. Then $\mathscr{H}_0$ and $\mathscr{H}_E$ coincide as a set.

LEMMA 2.5. *If A_1, A_3, and A_4 are satisfied L is self-adjoint in $\mathscr{L}^2$ with domain H_2 and essentially self-adjoint on C_0^∞.*

Proof. $Lf = (-\Delta + m^2 + V + b^2 + q)f$, $f \in C_0^\infty$, where $V = 2\sum_{i=1}^n b_i D_i + \sum_{i=1}^n (D_i b_i)$, $b^2 = \sum_{i=1}^n b_i^2$. By Lemma 2.2 we have for any $\epsilon > 0$

$$\left\| \sum_{i=1}^{n} (D_i b_i) f \right\| \leqslant \epsilon \|f\|_2 + K_\epsilon \|f\|,$$

$$\| q(x) f\| \leqslant \epsilon_2 \|f\|_2 + K_\epsilon \|f\|,$$

$$\left\| \sum_{i=1}^{n} b_i D_i f \right\| \leqslant \epsilon \|f\|_2 + K_\epsilon \|f\|,$$

and

$$\| b^2 f \| \leqslant \epsilon \| f \| + K_\epsilon \| b_i f \|.$$

Then $\forall \epsilon > 0$ there exists K_ϵ such that $\|(V + b^2 + q) f\| \leqslant \epsilon \|f\|_2 + K_\epsilon \|f\|$. This implies that $V^2 + b^2 + q$ is $-\Delta + m^2$ bounded with relative bound zero, and the statements of the theorem follow from the Kato–Rellich theorem.

Q.E.D.

Proof of Theorem 2. L is, clearly, the operator associated with the closed form

$$l(f_1, g_1) = \sum_{i=1}^{n} ((D_i + b_i) f_1, (D_i + b_i) g_1) + ((m^2 + q) f_1, g_1),$$

$D(l) = H_1$. The positivity assumption A_0 implies that (see Lemma 2.1): $l(f_1, f_1) \geqslant \epsilon(f_1, f_1)$, $f_1 \in D(l)$. Then $L \geqslant \epsilon > 0$ and $D(L^{1/2}) = H_1$. Moreover $l(f_1, g_1) = (L^{1/2} f_1, L^{1/2} g_1)$, f, $g \in H_1$. It follows that

$$(f, g)_{\mathrm{E}} = (L^{1/2} f_1, L^{1/2} g_1) + (f_2, g_2), \qquad f, g \in \mathscr{H}_{\mathrm{E}}.$$

We define the operator:

$$Uf = \frac{1}{2^{1/2}} \begin{bmatrix} L^{1/2} & 1 \\ L^{1/2} & -1 \end{bmatrix} f, \qquad f \in \mathscr{H}_{\mathrm{E}}.$$

U is a unitary operator from $\mathscr{H}_{\mathrm{E}}$ onto $\mathscr{L}_2^2 = \mathscr{L}^2 \oplus \mathscr{L}^2$.

We define the pseudodifferential operator $\hat{H}$ in $\mathscr{L}_2^2$

$$\hat{H} = \hat{H}_1 + \hat{Q}; \qquad \hat{H}_1 = \begin{pmatrix} L^{1/2} & 0 \\ 0 & -L^{1/2} \end{pmatrix}, \qquad \hat{Q} = 2b_0 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

$\hat{H}_1$ is self-adjoint on $D(\hat{H}_1) = H_1 \otimes H_1$. By A_5 and Lemma 2.2, $\hat{Q}$ is $\hat{H}_1$-bounded with relative bound equal to zero. Then $\hat{H} = \hat{H}_1 + \hat{Q}$ is self-adjoint on $H_1 \otimes H_1$. But

$$H = U^{-1} \hat{H} U = \begin{bmatrix} 0 & 1 \\ L & 0 \end{bmatrix} + Q$$

is a self-adjoint extension of h with domain $D(H) = U^{-1} D(\hat{H}) = (H_2 \otimes H_1)$.

Q.E.D.

Remark 2.7. H can be written in the following way:

$$H = \tilde{H}_0 + \tilde{V},$$

where

$$\tilde{H}_0 = \begin{bmatrix} 0 & 1 \\ -\Delta + m^2 & 0 \end{bmatrix}$$

and

$$\tilde{V} = \begin{bmatrix} 0 & 0 \\ V + b^2 + q & 2b_0 \end{bmatrix}.$$

Clearly $D(\tilde{H}_0) \subset D(\tilde{V})$, then for any $z \in \rho(H) \cap \rho(H_0)$

$$R(z) = \tilde{R}_0(z) - R(z)\,\underset{\sim}{\hat{Q}}\tilde{R}_0(z) = \tilde{R}_0(z) - \tilde{R}_0(z)\,\underset{\sim}{\hat{Q}}R(z),$$

where

$$R(z) = (H - z)^{-1}, \qquad \tilde{R}_0(z) = (\tilde{H}_0 - z)^{-1}.$$

Let J be the identification operator from $\mathscr{H}_0$ onto $\mathscr{H}_E$. Then we have

$$\begin{aligned} R_0(z) - J^{-1}R(z)\,J &= R_0(z)\,\underset{\sim}{\hat{Q}}J^{-1}R(z)\,J \\ &= J^{-1}R(z)\,J\underset{\sim}{\hat{Q}}R_0(z), \end{aligned}$$

where

$$R_0(z) = (H_0 - z)^{-1}.$$

So H_0 and H satisfy a second resolvent equation.

Now we are ready to prove Theorem 3. We will obtain the proof by showing that the conditions of an abstract theorem in scattering by Schechter are satisfied. To be self-contained we will include the statements of the Theorem:

Let $\mathscr{H}_0(\mathscr{H}_1)$ be Hilbert spaces, and let H_0 be a self-adjoint operator on it with spectral family $\{E_0(\lambda)\}$, $(\{E_1(\lambda)\})$. Put $R_0(z) = (z - H_0)^{-1}$ and $R_1(z) = (z - H_1)^{-1}$. Suppose:

(a) There is a linear bijective operator from $\mathscr{H}_0$ to $\mathscr{H}_1$.

(b) There is a Hilbert space K and closed linear operators A, B from $\mathscr{H}_0$ to K such that A is injective and $D(H_0) \subset D(A) \cap D(B)$.

(c) $D(H_1) \subset D(AJ^{-1}) \cap D(BJ^*)$, and

$$\begin{aligned} R_0(z) - J^{-1}R_1(z)\,J &= [R_0(z)\,B^*]\,AJ^{-1}R_1(z)\,J \\ &= [J^{-1}R_1(z)\,JB^*]\,AR_0(z), \end{aligned}$$

where $[W]$ means the closure of the operator W

(d) There is a z_0 in $\rho(H_0)$ such that $BR_0(z_0)\,[R_0(z)\,A^*]$ is a compact operator on K for all nonreal z.

(e) $\underset{\sim}{Q}(z) = [BR_0(z)\,A^*]$ is bounded on K for one z in $\rho(H_0)$, and $\sigma(z) = 1 + \underset{\sim}{Q}(z)$ has a bounded inverse on K for some nonreal z.

(f) There are an open set Λ of real numbers and functions $\underset{\sim}{Q}_\pm(\lambda)$ which are continuous from Λ to $B(K)$ (the set of bounded operators on K) such that $\underset{\sim}{Q}(\lambda \pm ia)$ converges in norm to $\underset{\sim}{Q}_\pm(\lambda)$ as $a \to 0$ for each λ in Λ.

(g) There is a function $M(\lambda)$ from Λ to $B(K)$ which is locally square integrable and such that $i[AR_0(\lambda + ia) - R(\lambda - ia)\, A^*]\, \mu$ converges weakly to $2\pi M(\lambda)\, \mu$ in K as $a \to 0$ for each μ in K and almost every λ in Λ.

(h) There is a closed set e of measure zero such that $[J^*J - J]\, E(\Gamma)$ is a compact operator for each interval $\Gamma \Subset (\Lambda \backslash e)$ ($\Gamma \Subset \Lambda\backslash e$ means that the closure of Γ is compact and contained in $\Lambda\backslash e$).

THEOREM 5 (Schechter). *Under hypotheses* (a)–(h) *the strong limits*

$$\omega_\pm f = \lim_{t\to\pm\infty} e^{itH_1} J e^{-itH_0} E_0^{\mathrm{ac}}(\Lambda) f$$

exist.

The operators $\omega_\pm$ *are isometries from* $E_0^{\mathrm{ac}}(\Lambda)\, \mathscr{H}_0$ *onto* $E_1^{\mathrm{ac}}(\Lambda)\, \mathscr{H}_1$. *The intertwining relation holds, i.e.,*

$$\psi(H_1)\, \omega_\pm = \omega_\pm \psi(H_0) \qquad on \quad E_0^{\mathrm{ac}}(\Lambda)\mathscr{H}_0$$

and

$$\omega_\pm f = \lim_{t\to\pm\infty} e^{it\phi(H_1)} J e^{-it\phi(H_0)} E_0^{\mathrm{ac}}(\Lambda) f$$

holds for all function ϕ *satisfying*

$$\int_0^\infty \left| \int_\Gamma e^{-i\eta s - it\varphi(s)}\, ds \right|^2 d\eta \to 0 \qquad as \quad t \to \infty$$

and

$$\int_\Gamma e^{-it\varphi(s)}\, ds \to 0 \qquad as \quad t \to \infty,$$

for any compact Γ *contained in* Λ.

Proof. See [11, Theorem 3.1].

Proof of Theorem 3. Take $K = \mathscr{L}_2^{\,2} \oplus \mathscr{L}_2^{\,2} \oplus \mathscr{L}_2^{\,2} \oplus \mathscr{L}_2^{\,2}$. We define

$$A\colon \mathscr{H}_0 \to K\{Af\} = \{A_1 f, A_2 f, A_3 f, A_4 f\}$$

and

$$B\colon \mathscr{H}_0 \to K\{Bf\} = \{B_1 f, B_2 f, B_3 f, B_4 f\}$$

where

$$A_1 = \begin{bmatrix} \rho^s V & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \rho^{-s}\Lambda, \quad A_3 = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad A_4 = \begin{bmatrix} \operatorname{sig} q \mid q \mid^{1/2} & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & \rho^{-s} \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2\rho^s b_0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad B_4 = \begin{bmatrix} 0 & \mid q \mid^{1/2} \\ 0 & 0 \end{bmatrix}.$$

By A_2 and A_J $D(H_0) \subset D(A) \cap D(B)$. Then (b) is satisfied. Moreover

$$B^*Af = \begin{bmatrix} 0 & 0 \\ V + b^2 + q & 2b_0 \end{bmatrix};$$

then by Remark 2.7 (c) is satisfied.

$BR_0(z)$ is bounded. Moreover

$$R_0(z) = \begin{bmatrix} z & 1 \\ -\Delta + m^2 & z \end{bmatrix} r(z^2)$$

where $r(z^2) = (-\Delta + m^2 - z^2)^{-1}$. $R_0(z)$ is bounded from $H_2 \otimes H_1$ into $H_2 \otimes H_1$ and A_i, $1 \leqslant i \leqslant 4$ is compact from $H_2 \otimes H_1$ into $\mathscr{L}_2^2$ by A_6 and Lemma 2.3. Then $AR_0(z)$ is compact from $\mathscr{H}$ into K, and (d) is satisfied.

Clearly $BR_0A^* = \oplus_{i=1}^4 B_iR_0(z)\, A_i^*$, in an obvious notation. (e) will be proved if we find a $z \in \rho(H_0)$ such that $\| B_iR_0(z)\, A_i^* \| < 1$ as operator from $\mathscr{L}_2^2 \to \mathscr{L}_2^2$, for $1 \leqslant i \leqslant 4$. But

$$B_1R_0A_1^* = \begin{bmatrix} \rho^{-s}r(z^2)(\rho^sV)^* & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_2R_0A_2^* = \begin{bmatrix} 0 & 0 \\ 2\rho^sb_0r(z^2)\rho^{-s} & 2z\rho^sb_0r(z^2)\rho^{-s} \end{bmatrix},$$

$$B_3R_0A_3^* = \begin{bmatrix} br(z^2)b & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$B_4R_0A_4^* = \begin{bmatrix} | q |^{1/2}\, r(z^2) \operatorname{sig} q \,| q |^{1/2} & 0 \\ 0 & 0 \end{bmatrix}.$$

We have

$$\| B_1R_0A_1^* \| \leqslant \| [\rho^sVr(z^2)]^* \| = \| \rho^sVr(z^2) \|.$$

But ρ^sV is compact from H_2 to $\mathscr{L}^2$. Then $\forall\epsilon > 0$ there exists a K_ϵ such that

$$\| \rho^sVr(z^2)f \| \leqslant \epsilon \, \| r(z^2)f \|_2 + K_\epsilon \| r(z^2)f \|.$$

Take $z = i\alpha$, $\alpha > 0$. Then

$$\| \rho^sVr(z^2)f \| \leqslant (\epsilon + (K/\alpha^2)) \| f \|.$$

Then $\forall\epsilon > 0$ there exists a α_0 such that $\| B_1R_0A_1^* \| \leqslant \epsilon$ if $\alpha > \alpha_0$. Moreover

$$\begin{aligned} \| B_2R_0A_2^* \| &\leqslant 2^{3/2} \| z\rho^sb_0r(z^2)\, \rho^{-s} \| \\ &\leqslant 2^{3/2} \| \rho^sb_0(r)^{1/2} \|. \end{aligned}$$

But by A_6 and Lemma 2.2 $\forall\epsilon > 0$ there is a α_0 such that $\| \rho^s b_0(r)^{1/2} \| \leqslant \epsilon/2^{3/2}$, for $\alpha > \alpha_0$, and then $\| B_2 R_0 A_2^* \| \leqslant \epsilon$. By a similar argument

$$\| B_3 R_0 A_3^* \| \leqslant \epsilon, \qquad \| B_4 R_0 A_4^* \| \leqslant \epsilon,$$

and (e) is satisfied.

We take $\Lambda = (-\infty, -m) \cup (m, \infty)$. Since H_0 is absolutely continuous $E_0^{\mathrm{ac}}(\Lambda) = \mathbb{1}$. By [11, Lemmas 3.3 and 3.4] (f) and (g) will be satisfied if we prove that

$$\frac{d}{d\lambda} (E_0(\lambda)\, A^* v, A^* \omega)_0 = (M(\lambda) v, \omega)_K ,$$

and

$$\frac{d}{d\lambda} (E_0(\lambda)\, B^* v, A^* \omega)_0 = (N(\lambda) v, \omega)_K ,$$

where $M(\lambda)$, $N(\lambda)$ are locally Hölder continuous functions from Λ to $B(K)$. We denote by C^* and one of A^* or B^*. Then

$$\frac{d}{d\lambda} (E_0(\lambda)\, C^* v, A^* \omega)_0 = \frac{d}{d\lambda} (U_0 E_0(\lambda)\, C^* v, A^* \omega),$$

where U_0 is the unitary operator from $\mathscr{H}_0$ onto $\mathscr{L}_2^{\,2}$ introduced in the proof of Theorem 1. We consider $\lambda > m$, for $\lambda < -m$ is similar. Then

$$\frac{d}{d\lambda} (E_0(\lambda)\, C^* v, A^* \omega)_0 = \frac{d}{d\lambda} (\hat{E}_0(\lambda)\, M U_0 C^* v, U_0 A^* \omega)$$

where $M = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ (see the proof of Theorem 1), where $\hat{E}_0(\lambda)$ is the spectral family of the operator $F^{-1}(\eta^2 + m^2)^{1/2} F$.

Then

$$\frac{d}{d\lambda} (E_0(\lambda)\, C^* v, A^* \omega)_0 = \sum_{\substack{jk \\ nm}} \frac{d}{d\lambda} (\hat{E}_0(\lambda)(U_0 C^{*n})_{1j}\, v_j^{\,n}, (U_0 A^{*m})_{1k}\, \omega_k^{\,m})$$

where

$$v = \bigoplus_{n=1}^{4} v^n, \qquad \omega = \bigoplus_{m=1}^{4} \omega^m, \qquad v^n, \omega^m \in \mathscr{L}_2^{\,2}.$$

But if A_8 and A_9 are satisfied we prove as in [6] (see also [12]) that

$$\frac{d}{d\lambda} (\hat{E}_0(\lambda)(U_0 C^{*n})_{1j}\, v_j^{\,n}, (U_0 A^{*m})_{1k}\, \omega_k^{\,m}) = (M_{jk,n,m}(\lambda)\, v_j^{\,n}, \omega_k^{\,m}),$$

where $M_{j,k,m,n}^{(\lambda)}$ is a locally Hölder continuous function from (m, ∞) to $B(K)$. This gives the proof of (f) and (g). It remains to prove h, but

$$J^*J - 1 = (V + b^2 + q)\, r(0)$$

$Vr(0)$, $b^2r(0)$, $qr(0)$ are compact by Lemma 2.3. Q.E.D.

Appendix

We give a necessary and sufficient condition for A_0 to be satisfied. We define [14]

$$B_\lambda(q) = \inf_{\psi>0} \sup_x \frac{1}{\psi(x)} \int |q(y)|\, \mathfrak{S}_{2,\lambda}(x-y)\, \psi(y)\, dy,$$

where $\mathfrak{S}_{2,\lambda}(x)$ is the inverse Fourier transform of

$$(2\pi)^{-n/2} (\lambda + |\eta|^2)^{-1}, \qquad \lambda \geqslant 0.$$

Then

LEMMA A.1. *A_0 is satisfied if and only if $B_\lambda(q^-) \leqslant 1$ for some $\lambda < m^2$.*

Proof. A_0 is equivalent to

$$(q^- f, f) \leqslant ((-\Delta + \lambda) f, f), \qquad f \in C_0^\infty$$

and $\lambda = m^2 - \epsilon$; then the lemma follows from [14, Theorem 5.2]. Q.E.D.

We define

$$S_\lambda(q) = \sup_x \int |q(y)|\, \mathfrak{S}_{2,\lambda}(x-y)\, dy.$$

Clearly $B_\lambda(q) \leqslant S_\lambda(q)$; then A_0 is satisfied if $S_\lambda(q^-) \leqslant 1$ for some $\lambda < m^2$. In the case $n = 3$ this takes a particularly simple form, because

$$\mathfrak{S}_{2,\lambda}(x) = \frac{1}{4\pi |x|} \exp[-\lambda^{1/2} |x|], \qquad n = 3.$$

Then A_0 is satisfied if

$$\sup_x \int q^-(y) \frac{1}{|x-y|} (\exp -\lambda^{1/2} |x-y|)\, dy \leqslant 4\pi$$

for some $\lambda < m^2$.

In the case when $q(x)$ is of Coulomb type, i.e., $q(x) = -e/|x|$, $e > 0$, we get

$$e \int \frac{1}{|y|} \frac{1}{|x-y|} (\exp -\lambda^{1/2} |x-y|) \, dy \leqslant 4\pi.$$

By taking polar coordinates and explicit integration we get: $e/\lambda^{1/2} \leqslant 1$; then A_0 is satisfied if $e < m$. Eckardt [8] obtained in this case

$$e < \min \left(\frac{1}{6(3^{1/2})}, \frac{2m^2}{25(3^{1/2})} \right),$$

which is clearly a stronger condition. In the case of a electric potential of Coulomb type, i.e., $q_s \equiv 0$, and $b_0(x) = e/|x|$, one could use the necessary and sufficient condition given in Lemma A.1, but it is easier to use, in this case, Hardy's inequality. A_0 is satisfied if and only if

$$e^2 \int \frac{1}{|x|^2} |f(x)|^2 \, d^3x \leqslant \int (n^2 + \lambda) \, |Ff(n)|^2 \, d^3k,$$

but by Hardy's inequality:

$$\int \frac{1}{|x|^2} |f(x)|^2 \, d^3x \leqslant 4 \int k^2 \, |Ff(n)|^2 \, d^3k.$$

We see that A_0 is satisfied if

$$|e| \leqslant \tfrac{1}{2}.$$

It is known that the constant in Hardy's inequality is the best possible. Returning to the ordinary system of units, $|e| \leqslant \frac{1}{2}$ corresponds to the condition $|z| \leqslant 68.5$, z being the atomic number. These results can be generalized in a trivial way to the case $n \geqslant 3$; we obtain $|e| \leqslant ((n-2)/2)$, $n > 2$.

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Note added in proof. After this paper was finished I learned of the paper [16] where the Klein Gordon equation is studied. The class of potentials studied in [16] is neither contained nor contains the class considered in this paper.

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