Tensor Product Representations

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1. Introduction

In this note we interpret the table $T$ of tensor products of irreducible representations $\lambda$ of a finite group $G$ as yielding a representation $\{\lambda\}$ of the algebra of such representations. If $G$ is Abelian, the resulting structure is particularly simple [7]. We treat the general case, writing

$$\lambda \times \mu = \lambda \mu = \mu \lambda = \sum_v g^{\nu}_{\lambda \mu} \nu,$$  \hspace{1cm} (1.1)

where

$$g^{\nu}_{\lambda \mu} = \frac{1}{g} \sum_i g_{i \lambda_i} x_i \mu_i \chi_i \chi_i \nu,$$  \hspace{1cm} (1.2)

so that

$$T = \sum_v (g^{\nu}_{\lambda \mu})_v$$

In Section 2, it is proved that the matrices

$$\{\lambda_i\} = (\lambda_{\mu}) = (g^{\nu}_{\lambda \mu})$$

have the desired properties.

We illustrate the construction, in particular, in relation to the inducing and restricting processes, as applied to the symmetric and alternating groups, since Young diagrams are convenient in this context, but the results are quite general.

It should be said in conclusion that the author's interest in these ideas was stimulated by their importance in Theoretical physics [3, 6] which suggested a geometrical model described elsewhere [5].

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2. Even though the reduction (1.1) is completely determined by the character relations

\[ \sum_{i} g_i \chi_i^p \chi_i^q = \begin{cases} g, & \sigma = p \\ 0, & \sigma \neq p \end{cases}, \quad g_i \sum_{i} \chi_i^p \chi_i^q = \begin{cases} g, & j = i \\ 0, & j \neq i \end{cases}, \]

the subject has been studied for \( S_n \) in some detail by Murnagahan and others (cf. [3, Ch. 7]) and by the author [4, p. 20], using the relation

\[ (\rho H \uparrow G) \times \lambda G = ((\lambda G \downarrow H) \times \rho H) \uparrow G. \]

Here \( H \) is any subgroup of \( G \) and \( \uparrow, \downarrow \) indicate the inducing and restricting processes. If \([\alpha], [\beta], \ldots\) are irreducible representations of \( S_n \), we have

\[ [\alpha] \times [\beta] = \prod (1 - R_{ik})([\beta] \downarrow H) \uparrow S_n, \]

where \( R_{ik} \) is Young’s raising operator and

\[ H = S_{a_1} \times S_{a_2} \times \cdots \]

corresponding to the rows of the diagram \([\alpha]\). While 2.3 is explicit, it involves cancellation, so that the reduction is still obscure.

In order to establish the proposed representation of the \( \lambda \)-algebra of \( G \), we prove the following

**Theorem.**

\[ \{\lambda, \mu\} = \left( \frac{S_{\beta \gamma}}{S_{\alpha \delta}} \right) \left( \frac{S_{\delta \alpha}}{S_{\beta \gamma}} \right) = \left( \frac{S_{\beta \gamma}}{S_{\alpha \delta}} \right) \]

\[ = \sum_{\nu} g_{\lambda \mu}^{\nu} \{\nu\}. \]

**Proof.** Since

\[ \chi_i^\lambda \chi_i^\mu = \sum_{\nu} g_{\lambda \mu}^{\nu} \chi_i^\nu, \]

we may write the characters as diagonal matrices \( D^\lambda \) and, taking conjugates,

\[ D^\lambda D^{\mu'} = \sum g_{\lambda \mu}^{\nu} D^{\nu}. \]

If we denote the character table of \( G \) by \( \chi \), then

\[ \chi D^\lambda \chi^{-1} \cdot \chi D^{\mu'} \chi^{-1} = \sum g_{\lambda \mu}^{\nu} \chi D^{\nu} \chi^{-1}. \]

Then setting

\[ \chi D^\lambda \chi^{-1} = \{\lambda\} = (\lambda, \gamma) = (g_{\beta \gamma}^\lambda), \]

we have
we verify that (1.2) is satisfied. Clearly,
\[
\begin{align*}
\begin{cases}
1, & \beta = \gamma \\
0, & \beta \neq \gamma
\end{cases}
\end{align*}
\]
and
\[
\sum_{\gamma} g_{\alpha \gamma} g_{\gamma \gamma} = \sum_{\gamma} g_{\alpha \gamma} g_{\gamma \gamma} ;
\]
so we have the following.

**Corollary.** If for some \( \gamma \) we have \( g_{\alpha \gamma} g_{\gamma \gamma} \neq 0 \), then for some \( \nu \) we have \( g_{\beta \nu} g_{\nu \nu} \neq 0 \).

3. It follows from the above Corollary that we can now extend to any finite group the geometrical interpretation of the tensor product which is of interest to physicists [3]. To this end, we denote the irreducible representations \( \alpha, \beta, \gamma, \lambda, \mu, \nu \) of \( G \) by points in a finite projective plane (mod 2), where the collinearity of \( \lambda, \beta, \gamma' \) is equivalent to \( g_{\beta \gamma'} \neq 0 \). Denoting conjugate representations by the same point, we obtain the Pasch Axiom

\[
\text{that if a transversal meets two sides of a triangle it must also meet the third side. It follows that Desargues' Theorem is also valid for irreducible representations of } G \text{ considered as points in } PG(n, 2) \text{ for } n \geq 2. \text{ The reader may verify that for } S_5 \text{ we may take } \alpha = [2, 1^5], \beta = [4, 1], \gamma = [3, 1^5], \lambda = [3, 2], \mu = [2^5, 1] \text{ and } \nu = [1^5].
\]

Physicists have largely confined their attention to simply reducible groups whose characters are real; so the distinction between conjugate representations does not arise. However, assuming that conjugate representations are distinct, it follows from the Corollary that the "6j-symbol"

\[
\begin{pmatrix}
\lambda & \mu & \nu \\
\alpha & \beta & \gamma
\end{pmatrix} = g_{\alpha \beta \gamma} g_{\alpha \gamma \nu} g_{\beta \gamma \nu} g_{\alpha \mu \nu} = 0.
\]

If we write \( (\lambda \mu \nu) \) for \( g_{\lambda \mu} \), then, using Burnside's relations [2, p. 291] and interchanging \( \gamma \) and \( \gamma' \),

\[
\begin{pmatrix}
\lambda & \mu & \nu \\
\alpha & \beta & \gamma
\end{pmatrix} = (\beta \gamma \lambda)(\gamma' \alpha \mu)(\beta \gamma \nu)(\lambda \mu \nu) = (\alpha' \beta \gamma')(\gamma \mu \nu')(\lambda' \beta \gamma')(\lambda' \mu \nu'),
\]
which should be compared with (5.1) in the paper by Derome and Sharp [6].
It is not without interest to distribute the irreducible representation of \( S_6 \)
over the Desargues configuration in \( PG(3, 2) \).

4. We conclude these comments on the Theorem and Corollary of
Section 2 by observing that the eigenvalues of \( \{\lambda\} \) are the components of
\( \chi^{\lambda'} \), i.e., the elements of the \( \lambda' \)-row of the matrix \( \chi \). Since the eigenvectors of
\( D^{\lambda'} \) are the \( E_i(0, ..., 0, 1, 0, ..., 0) \) it follows that the eigenvectors of \( \{\lambda\} \) are
the \( \chi E_i \), i.e., the columns of the matrix \( \chi \).

Example 1. For \( S_3 \),

\[
I \quad (1 \ 2) \ (1 \ 2 \ 3)
\]

\[
\chi: [2, 1] \quad 2 \quad 0 \quad -1 \\
[1^3] \quad 1 \quad -1 \quad 1
\]

\[
T = \begin{pmatrix}
\end{pmatrix},
\]

so that

\[
\{3\} = \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad \{2, 1\} = \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}, \quad \{1^3\} = \begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}.
\]

Example 2. For \( A_4 \),

\[
I \quad (1 \ 2)(3 \ 4) \ (1 \ 2 \ 3) \ (1 \ 3 \ 2)
\]

\[
\chi: [3, 1]_A \quad \begin{pmatrix}
1 & 1 & 1 & 1 \\
3 & -1 & 0 & 0
\end{pmatrix}, \\
[2^2]_+ \quad \begin{pmatrix}
1 & 1 & \rho & \rho^2
\end{pmatrix}, \\
[2^2]_- \quad \begin{pmatrix}
1 & 1 & \rho^2 & \rho
\end{pmatrix},
\]

\[
T = \begin{pmatrix}
[4] & [3, 1]_A \quad [2^2]_- \quad [2^2]_+ \\
\end{pmatrix},
\]

with appropriate matrices \( \{\lambda\} \).

Burnside's Theorem is of interest here [1], since the reason for requiring
a faithful representation \( \lambda \) to yield every representation of \( G \) as a component
of some power of \( \lambda \) is abundantly clear. Each \( \{\lambda\} \) is uniquely determined by the
one in its first row or column. If \( H \vartriangleleft G \), then the irreducible representations
of \( G/H \) can be grouped together and the corresponding \( \{\lambda\} \)'s will decompose.
We illustrate these ideas in the case of $S_4$, taking the representations in the order $[4], [2^2], [1^4], [3, 1], [2, 1^2]$ so that

$$\{4\} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \{2^2\} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\{1^4\} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which should be compared with $\{3\}, \{2, 1\}, \{1^3\}$ of $S_3$.

5. The *inducing* and *restricting* processes utilize Frobenius' Reciprocity Theorem written as a matrix $\Delta$. For $A_4 \subseteq S_4$, we find

$$\Delta := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

so that, inducing, we have

$$\Delta \{3, 1\}_A \Delta^t = \{3, 1\} + \{2, 1^2\}$$

corresponding to

$$[3, 1]_A \uparrow [3, 1] + [2, 1^2].$$

Similarly, restricting,

$$\{3, 1\} \Delta = \Delta \{3, 1\}_A,$$

where

$$\{3, 1\} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \{2, 1^2\} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$
Added in proof. The representation theory here described has been dualized to apply to classes in a paper to appear in the October number of the *J. Math. Phys.* 12 (1971). It follows from (3.1c) there that $c_{r,s}^{i} = c_{s}^{i}$ and making this substitution will help the reader in (2.6c).

**References**