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Further results on nearly Kirkman triple systems with subsystems

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Abstract

In this paper we further discuss the embedding problem for nearly Kirkman triple systems and get the result that: (1) For $u \equiv v \equiv 0 \pmod 6$, $v \geqslant 78$, and $u \geqslant 3.5v$, there exists an NKTS(u) containing a sub-NKTS(v). (2) For $v = 18, 24, 30, 36, 42, 48, 54, 60, 66$ or 72, there exists an NKTS(u) containing a sub-NKTS(v) if and only if $u \equiv 0 \pmod 6$ and $u \geqslant 3v$.

1. Introduction

Let v be a positive integer, and K and M be two sets of positive integers. A group divisible design GD($K, M; v$) is an ordered triple ($\mathbf{X}$,$\mathbf{G}$,$\mathbf{B}$) where $\mathbf{X}$ is a set with cardinality v, $\mathbf{G}$ is a set of subsets (called groups) of $\mathbf{X}$ such that $\mathbf{G}$ partitions $\mathbf{X}$ and $|G| \in M$ for each $G \in \mathbf{G}$, and $\mathbf{B}$ is a set of subsets (called blocks) of $\mathbf{X}$ such that $|B| \in K$ for each $B \in \mathbf{B}$, with the property that each pair of distinct elements of $\mathbf{X}$ is contained either in a unique group or in a unique block, but not both. The number v is called the order of the group divisible design.

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If $K=\{k\}$ and $M=\{m\}$, then a GD($\{k\},\{m\};v$) is called uniform and is denoted GD($k,m;v$).

Let (**X**,**G**,**B**) be a GD(K,M,v); it is sometimes called a K-GDD of group type T where $T=\{|G|:G\in\mathbf{G}\}$ is a multiset. We also write $T=\prod_{i=1}^{s} m_i^{u_i}$ if **G** contains exactly u_i groups of size m_i, $1\leqslant i\leqslant s$.

Now, we define the idea of a GDD with a hole. Informally, an incomplete GDD (IGDD) is a GDD from which a sub-GDD is missing (thus creating a "hole"). We give a formal definition. An IGDD is a quadruple (**X**,**Y**,**G**,**B**) which satisfies the following properties:

(1) **X** is a set of points, and $\mathbf{Y}\subset\mathbf{X}$ (**Y** is called the hole);
(2) **G** is a partition of **X** into groups;
(3) **B** is a set of subsets of **X** (blocks), each of which intersects each group in at most one point;
(4) no block contains two members of **Y**;
(5) every pair of points $\{x,y\}$ from distinct groups, such that at least one of x,y is in $\mathbf{X}\backslash\mathbf{Y}$, occurs in a unique block of **B**.

We say that an IGDD(**X**,**Y**,**G**,**B**) is a K-IGDD if $|B|\in K$ for every block $B\in\mathbf{B}$. The type of the IGDD is defined to be the multiset of ordered pairs $\{(|G|,|G\cap\mathbf{Y}|:G\in\mathbf{G}\}$. Note that if $\mathbf{Y}=\emptyset$, then the IGDD is a GDD.

Let (**X**,**G**,**B**) be a GD($K,M;v$). A subset P of **B** is called a parallel class if P forms a partition of **X**. A GD(K,M,v) is called resolvable and is denoted RGD($K,M;v$) if its block set can be partitioned into parallel classes. When $K=\{k\}$ and $M=\{m\}$ we may also denote an RGD($\{k\},\{m\};v$) as a $\{k\}$-RGDD of type $m^{v/m}$. As two important special cases, an RGD($3,1;v$) (or equivalently, an RGD($3,3;v$)) is usually known as a Kirkman triple system of order v and is denoted KTS(v), and an RGD($3,2;v$) is usually called a nearly Kirkman triple system of order v and is denoted NKTS(v). For the existence of Kirkman triple systems and nearly Kirkman triple systems, we have the following results:

Theorem 1.1 (Ray-Chaudhuri and Wilson [7]). *There exists an KTS*(v) *if and only if* $v\equiv 3 \pmod 6$.

Theorem 1.2 (Baker and Wilson [1], Brouwer [3], Kotzig and Rosa [7] and Rees and Stinson [13]). *There exists an* NKTS(v) *if and only if* $v\equiv 0 \pmod 6$, $v\geqslant 18$.

Now let ($\mathbf{X},\mathbf{G}_1,\mathbf{B}_1$) be an RGD($K,M;v$) and ($\mathbf{Y},\mathbf{G}_2,\mathbf{B}_2$) be an RGD($K,M;u$). If $\mathbf{X}\subset\mathbf{Y}$, $\mathbf{G}_1\subset\mathbf{G}_2$ and each parallel class of $\mathbf{B}_1$ is a part of some parallel class of $\mathbf{B}_2$, then ($\mathbf{X},\mathbf{G}_1,\mathbf{B}_1$) is said to be embedded in ($\mathbf{Y},\mathbf{G}_2,\mathbf{B}_2$). If we allow a subdesign of an RGDD to be missing (i.e., creating a hole), we have an incomplete RGDD. Note that the subdesign need not exist. We will be exclusively concerned with constructing incomplete RGD($3,2;v$)s, which we henceforth denote as INKTS(v,h), h being the number of points in the hole.

The problem of constructing Kirkman triple systems containing subsystems was studied by Rees and Stinson [9,10,12]. The obvious necessary conditions for the existence of a KTS(u) containing a KTS(v) as a subsystem are $u \geqslant 3v$, $u \equiv v \equiv 3 \pmod 6$. In [9], it is shown that these necessary conditions are sufficient.

In this paper, we are interested in NKTS(u) which contain NKTS(v) as a subsystem. Here the obvious necessary conditions are that $u \geqslant 3v$, $u \equiv v \equiv 0 \pmod 6$ and $v \geqslant 18$. This problem has been studied in a couple of recent papers, and the following results have been proved:

Theorem 1.3 (Tang and Shen [13]). *For any* $v \equiv 0 \pmod 6$ *with* $v \geqslant 18$ *and any* $k \geqslant 3$, *there exists an* INKTS(kv, v).

Theorem 1.4 (Deng et al. [4]). *For each* $h \in \{6, 12\}$ *there exists an* INKTS(v, h) *if and only if* $v \equiv 0 \pmod 6$ *and* $v \geqslant 3h$.

Theorem 1.5 (Deng et al. [4]). (i) *For* $u \equiv v \equiv 0 \pmod 6$, $v \geqslant 48$ *and* $u \geqslant 4v - 18$, *there exists an* NKTS(u) *containing a* sub-NKTS(v).

(ii) *For* $v \in \{18, 24, 30, 36, 42\}$ *and* $u \equiv 0 \pmod 6$, $u \geqslant 3v$, *there exists an* NKTS(u) *containing a* sub-NKTS(v).

In this paper, we further discuss the embedding problem for nearly Kirkman triple systems and get the following result:

Theorem 1.6. (1) *For* $u \equiv v \equiv 0 \pmod 6$, $v \geqslant 78$, *and* $u \geqslant 3.5v$, *there exists an* NKTS(u) *containing a* sub-NKTS(v).

(2) *For* $v = 18, 24, 30, 36, 42, 48, 54, 60, 66$ *or* 72, *there exists an* NKTS(u) *containing a* sub-NKTS(v) *if and only if* $u \equiv 0 \pmod 6$ *and* $u \geqslant 3v$.

As in [4] the direct constructions for the designs appearing in this paper were obtained by appropriate adaptations of the standard backtracking algorithm, beginning with a feasible tactical configuration (subscript pattern) with respect to a particular automorphism group.

2. Constructions for nearly Kirkman triple systems containing subsystems

First we introduce the idea of Kirkman frames, which play a very important role in solving the embedding problem for nearly Kirkman triple systems. Here we give the definition [12].

Let $(\mathbf{X}, \mathbf{G}, \mathbf{B})$ be a GD($K, M; v$) and let P be a subset of $\mathbf{B}$. If P forms a partition of $\mathbf{X} \backslash G$ for some group $G \in \mathbf{G}$, then P is called a holey parallel class with hole G. A GD($K, M; v$) is called a Kirkman K-frame if the block set $\mathbf{B}$ can be partitioned into holey parallel classes. For $K = \{3\}$, a Kirkman $\{3\}$-frame is called a Kirkman frame.

The following theorem gives a powerful construction for Kirkman frames from group divisible designs [12].

Theorem 2.1. *Let* $(\mathbf{X},\mathbf{G},\mathbf{B})$ *be a group divisible design. Let* $w:\mathbf{X}\to Z^+\cup\{0\}$ *be a weight function on* $\mathbf{X}$. *Suppose that for each block* $B\in\mathbf{B}$, *there exists a Kirkman frame of type* $\{w(x):x\in B\}$. *Then there is a Kirkman frame of type* $\{\sum_{x\in G}w(x):G\in\mathbf{G}\}$.

The spectrum of uniform Kirkman frames has been completely determined [12].

Theorem 2.2. *There exists a Kirkman frame of type* t^u *if and only if* $t\equiv 0\pmod 2$, $u\geqslant 4$ *and* $t(u-1)\equiv 0\pmod 3$.

The following "filling in holes" construction provides a powerful tool for the embedding problem for nearly Kirkman triple systems [13].

Construction 2.3. *Suppose there is a Kirkman frame of type* T *on* v *points. If, for some* $a>0$, *there exists an* INKTS$(t+a,a)$ *for all* $t\in T$, *then there is an* INKTS$(v+a,a)$, *and for every* $t\in T$, *an* INKTS$(v+a,t+a)$.

It will be necessary to build families of GDDs. Our basic construction for GDDs is a recursive one. It is usually referred to as the "Fundamental GDD construction" (see [3]).

Construction 2.4. *Let* $(\mathbf{X},\mathbf{G},\mathbf{B})$ *be a group divisible design. Let* $w:\mathbf{X}\to Z^+\cup\{0\}$ *be a weight function on* $\mathbf{X}$. *Suppose that for each block* $B\in\mathbf{B}$, *there exists a* K-GDD *of type* $\{w(x):x\in B\}$. *Then there is a* K-GDD *of type* $\{\sum_{x\in G}w(x):G\in\mathbf{G}\}$.

3. Applications of the constructions

We use the above constructions to discuss the embedding problem for nearly Kirkman triple systems. First, we present a specific construction using GDDs with block-size four.

Lemma 3.1. *Suppose there is a* TD$(6,m)$, $m\geqslant 5$ *and* $m\leqslant w\leqslant 2m$. *Let* $a=6$ *or* 12. *Then there is an* NKTS$(36m+6w+a)$ *containing a* sub-NKTS$(12m+a)$.

Proof. Give points in four groups of the TD weight 3, give the points in the fifth group weight 3 or 6, and give the points in the sixth group weight 6. Apply Construction 2.4, filling in $\{4\}$-GDDs of type 3^46^2 or 3^56^1 [11], to get a $\{4\}$-GDD of type $(3m)^4(6m)^1(3w)^1$. Give the points of the resultant GDD weight 2, applying Theorems 2.1 and 2.2, to get a Kirkman frame of type $(6m)^4(12m)^1(6w)^1$. Adjoin a ideal points and apply Construction 2.3 and Theorem 1.4 to yield an INKTS$(36m+6w+a,12m+a)$. Now construct an NKTS$(12m+a)$ on the hole, giving rise to an NKTS$(36m+6w+a)$ containing a sub-NKTS$(12m+a)$. □

Now we use the following corollaries to Lemma 3.1.

Lemma 3.2. *Suppose* $v \equiv 6 \pmod{12}$, $v \geqslant 66$, $v \neq 78, 126, 174, 222, 270$, $u \equiv 0 \pmod{6}$, *and* $3.5v - 15 \leqslant u \leqslant 4v - 18$. *Then there is an* INKTS(u, v).

Proof. Apply Lemma 3.1 with $m = (v - 6)/12$, $w = (u - 36m - 6)/6$ and $a = 6$. Then a TD$(6, m)$ exists, and $m \leqslant w \leqslant 2m$. This builds an NKTS$(36m + 6w + 6)$ containing a sub-NKTS$(12m + 6)$. □

Lemma 3.3. *Suppose* $v \equiv 0 \pmod{12}$, $v \geqslant 72$, $v \neq 84, 132, 180, 228, 276$, $u \equiv 0 \pmod{6}$, *and* $3.5v - 30 \leqslant u \leqslant 4v - 36$. *Then there is an* INKTS(u, v).

Proof. Apply Lemma 3.1 with $m = (v - 12)/12$, $w = (u - 36m - 12)/6$ and $a = 12$. Then a TD$(6, m)$ exists, and $m \leqslant w \leqslant 2m$. This builds an NKTS$(36m + 6w + 12)$ containing a sub-NKTS$(12m + 12)$. □

Lemma 3.4. *Suppose* $v \in \{78, 126, 174, 222, 270\}$, $u \equiv 0 \pmod{6}$, *and* $3.5v < u \leqslant 4v + 18$. *Then there is an* INKTS(u, v).

Proof. Let $m = (v - 18)/12 + 2$; then $m \in \{7, 11, 15, 19, 23\}$. Take a TD$(6, m)$ and give all points on four of the groups weight 6. On the fifth group give 2 of the points weight 6 and all remaining points weight 12. Assign weight 6 or 12 to each point on the sixth group. Use Kirkman frames of type 6^6, $6^5 12^1$ and $6^4 12^2$, and adjoin 6 ideal points. This gives INKTS(u, v) where $v = 12m - 6$ and $12m - 6 + 30m \leqslant u \leqslant 12m - 6 + 36m$, i.e. $3.5v + 15 \leqslant u \leqslant 4v + 18$.

To get $u = 3.5v + 3$ and $3.5v + 9$ proceed as follows. Suppose first that $v \neq 78$. Let $m = (v - 18)/12$, then $m \in \{9, 13, 17, 21\}$. Proceed as above, taking a TD$(6, m)$ and giving all points on four of the groups weight 6 and giving all points on the fifth group weight 12. On the sixth group give either 8 or 9 of the points weight 12 and all remaining points weight 6. Now adjoin 18 ideal points and apply Construction 2.3 with $a = 18$ (see Theorem 1.5(ii)) to obtain an INKTS(u, v) where $v = 12m + 18$ and $u = 12m + 18 + 30m + 48$ or $12m + 18 + 30m + 54$, i.e. $u = 3.5v + 3$ or $3.5v + 9$. Now let $v = 78$. For $u = 276$ adjoin 12 ideal points to a Kirkman frame of type 66^4 and fill in INKTS$(78, 12)$s and an NKTS(78), while for $u = 282$ take a $\{4\}$-GDD of type $18^4 30^1 36^1$ (see the appendix) and apply Theorem 2.1 and Construction 2.3, using weight 2 with $a = 6$ ideal points. □

Lemma 3.5. *Suppose* $v \in \{60, 84, 132, 180, 228, 276\}$, $u \equiv 0 \pmod{6}$, *and* $3.5v \leqslant u \leqslant 4v$. *Then there is an* INKTS(u, v).

Proof. Let $m = (v - 12)/12 + 1$, then $m \in \{5, 7, 11, 15, 19, 23\}$. Take a TD$(6, m)$ and proceed as in the first part of Lemma 3.4, giving just one point on the fifth group weight 6 and adjoining 6 ideal points. This gives INKTS(u, v) where $v = 12m$, and $12m + 30m \leqslant u \leqslant 12m + 36m$, i.e. $3.5v \leqslant u \leqslant 4v$. □

Lemma 3.6. *Suppose* $v \equiv 0 \pmod{12}$, $v \geqslant 48$, *and* $u = 4v - 30$ *or* $4v - 24$. *Then there is an* INKTS(u, v).

Proof. Here we just proceed as in Lemma 3.5, unless $v \in \{48, 72, 120, 168, 216, 264\}$. For $v \in \{120, 168, 216, 264\}$, take $m = (v - 24)/12 + 3$, then $m \in \{11, 15, 19, 23\}$. Take a TD$(6, m)$ and proceed as above, giving three points on the fifth group weight 6 and either 11 or 10 points on the sixth group weight 6, adjoining 6 ideal points.

There remain INKTS(162,48), INKTS(168,48), INKTS(258,72) and INKTS(264,72).

For INKTS(168, 48), take a $\{4\}$-GDD of type $15^4 21^1$ (see [5]); apply Theorems 2.1 and Construction 2.3, using weight 2 with 6 ideal points.

For INKTS(264, 72), take a $\{4\}$-GDD of type $24^4 33^1$ (see [5]); apply weight 2 and adjoin 6 ideal points.

For INKTS(162, 48) and INKTS(258, 72), we present the following direct constructions:

INKTS(162, 48). Point set: $(Z(38) \times Z(3)) \cup \{x_1, x_2, \ldots, x_{48}\}$. Groups: $\{0_0, 19_0\}$, $\{0_1, 19_1\}$, $\{0_2, 19_2\}$ mod$(38, -)$. Hole: $\{x_1, x_2, \ldots, x_{48}\}$.

Parallel classes of triples: Thirty-eight of them are obtained by developing the following triples mod$(38, -)$:

$\{31_0, 35_1, 0_2\}$, $\{22_0, 31_1, 37_2\}$, $\{25_0, 37_1, 7_2\}$, $\{0_0, 0_1, 17_1\}$, $\{1_1, 1_2, 16_2\}$, $\{2_2, 2_0, 15_0\}$, $\{1_0, 23_1, x_1\}$, $\{3_0, 26_1, x_2\}$, $\{4_0, 28_1, x_3\}$, $\{5_0, 30_1, x_4\}$, $\{6_0, 32_1, x_5\}$, $\{7_0, 34_1, x_6\}$, $\{8_0, 36_1, x_7\}$, $\{11_0, 2_1, x_8\}$, $\{12_0, 4_1, x_9\}$, $\{10_0, 3_1, x_{10}\}$, $\{13_0, 7_1, x_{11}\}$, $\{14_0, 9_1, x_{12}\}$, $\{9_0, 5_1, x_{13}\}$, $\{16_0, 13_1, x_{14}\}$, $\{17_0, 15_1, x_{15}\}$, $\{19_0, 18_1, x_{16}\}$, $\{6_1, 19_2, x_{17}\}$, $\{8_1, 22_2, x_{18}\}$, $\{10_1, 27_2, x_{19}\}$, $\{12_1, 28_2, x_{20}\}$, $\{11_1, 29_2, x_{21}\}$, $\{14_1, 33_2, x_{22}\}$, $\{16_1, 36_2, x_{23}\}$, $\{20_1, 3_2, x_{24}\}$, $\{21_1, 5_2, x_{25}\}$, $\{19_1, 4_2, x_{26}\}$, $\{22_1, 8_2, x_{27}\}$, $\{24_1, 11_2, x_{28}\}$, $\{25_1, 13_2, x_{29}\}$, $\{29_1, 18_2, x_{30}\}$, $\{27_1, 17_2, x_{31}\}$, $\{33_1, 24_2, x_{32}\}$, $\{20_2, 21_0, x_{33}\}$, $\{25_2, 27_0, x_{34}\}$, $\{30_2, 33_0, x_{35}\}$, $\{32_2, 36_0, x_{36}\}$, $\{31_2, 37_0, x_{37}\}$, $\{23_2, 30_0, x_{38}\}$, $\{26_2, 35_0, x_{39}\}$, $\{14_2, 24_0, x_{40}\}$, $\{21_2, 32_0, x_{41}\}$, $\{6_2, 23_0, x_{42}\}$, $\{15_2, 29_0, x_{43}\}$, $\{10_2, 26_0, x_{44}\}$, $\{9_2, 28_0, x_{45}\}$, $\{35_2, 18_0, x_{46}\}$, $\{12_2, 34_0, x_{47}\}$, $\{34_2, 20_0, x_{48}\}$.

Nineteen of them are obtained by adding $0, 2, 4, \ldots, 36$ to the following triples mod$(38, -)$:

$\{16_0, 17_1, 18_2\}$, $\{35_0, 36_1, 37_2\}$, $\{17_0, 31_1, 2_2\}$, $\{36_0, 12_1, 21_2\}$, $\{14_0, 33_1, 6_2\}$, $\{33_0, 14_1, 25_2\}$, $\{0_0, 1_0, x_{17}\}$, $\{19_0, 20_0, x_{18}\}$, $\{2_0, 5_0, x_{19}\}$, $\{21_0, 24_0, x_{20}\}$, $\{3_0, 8_0, x_{21}\}$, $\{22_0, 27_0, x_{22}\}$, $\{4_0, 11_0, x_{23}\}$, $\{23_0, 30_0, x_{24}\}$, $\{6_0, 15_0, x_{25}\}$, $\{25_0, 34_0, x_{26}\}$, $\{7_0, 18_0, x_{27}\}$, $\{26_0, 37_0, x_{28}\}$, $\{12_0, 29_0, x_{29}\}$, $\{31_0, 10_0, x_{30}\}$, $\{13_0, 28_0, x_{31}\}$, $\{32_0, 9_0, x_{32}\}$, $\{0_1, 1_1, x_{33}\}$, $\{19_1, 20_1, x_{34}\}$, $\{2_1, 5_1, x_{35}\}$, $\{21_1, 24_1, x_{36}\}$, $\{3_1, 8_1, x_{37}\}$, $\{22_1, 27_1, x_{38}\}$, $\{4_1, 11_1, x_{39}\}$, $\{23_1, 30_1, x_{40}\}$, $\{6_1, 15_1, x_{41}\}$, $\{25_1, 34_1, x_{42}\}$, $\{7_1, 18_1, x_{43}\}$, $\{26_1, 37_1, x_{44}\}$, $\{16_1, 29_1, x_{45}\}$, $\{35_1, 10_1, x_{46}\}$, $\{13_1, 28_1, x_{47}\}$, $\{32_1, 9_1, x_{48}\}$, $\{0_2, 1_2, x_1\}$, $\{19_2, 20_2, x_2\}$, $\{4_2, 7_2, x_3\}$, $\{23_2, 26_2, x_4\}$, $\{8_2, 13_2, x_5\}$, $\{27_2, 32_2, x_6\}$, $\{9_2, 16_2, x_7\}$, $\{28_2, 35_2, x_8\}$, $\{15_2, 24_2, x_9\}$, $\{34_2, 5_2, x_{10}\}$, $\{3_2, 14_2, x_{11}\}$, $\{22_2, 33_2, x_{12}\}$, $\{17_2, 30_2, x_{13}\}$, $\{36_2, 11_2, x_{14}\}$, $\{12_2, 29_2, x_{15}\}$, $\{31_2, 10_2, x_{16}\}$.

Holey parallel classes of triples: Nineteen of them are obtained by adding $0, 2, 4, \ldots,$ 36 to the following triples mod$(38, -)$:

$\{0_0, 6_0, 24_0\}$, $\{19_0, 25_0, 5_0\}$, $\{4_0, 8_0, 20_0\}$, $\{23_0, 27_0, 1_0\}$, $\{7_0, 9_0, 17_0\}$, $\{26_0, 28_0, 36_0\}$, $\{0_1, 6_1, 24_1\}$, $\{19_1, 25_1, 5_1\}$, $\{4_1, 8_1, 20_1\}$, $\{23_1, 27_1, 1_1\}$, $\{11_1, 13_1, 21_1\}$, $\{30_1, 32_1, 2_1\}$, $\{12_2, 18_2, 36_2\}$, $\{31_2, 37_2, 17_2\}$, $\{4_2, 8_2, 20_2\}$, $\{23_2, 27_2, 1_2\}$, $\{9_2, 11_2, 19_2\}$, $\{28_2, 30_2, 0_2\}$, $\{14_0, 16_1, 15_2\}$, $\{33_0, 35_1, 34_2\}$, $\{11_0, 14_1, 16_2\}$, $\{30_0, 33_1, 35_2\}$, $\{10_0, 15_1, 13_2\}$, $\{29_0, 34_1, 32_2\}$, $\{12_0, 18_1, 22_2\}$, $\{31_0, 37_1, 3_2\}$, $\{3_0, 10_1, 7_2\}$, $\{22_0, 29_1, 26_2\}$, $\{18_0, 28_1, 24_2\}$, $\{37_0, 9_1, 5_2\}$, $\{13_0, 26_1, 21_2\}$, $\{32_0, 7_1, 2_2\}$, $\{16_0, 31_1, 25_2\}$, $\{35_0, 12_1, 6_2\}$, $\{2_0, 22_1, 14_2\}$, $\{21_0, 3_1, 33_2\}$, $\{15_0, 36_1, 10_2\}$, $\{34_0, 17_1, 29_2\}$.

Four of them are obtained by developing each of the following triples mod$(38, -)$:

$\{0_0, 8_1, 13_2\}$, $\{0_0, 11_1, 18_2\}$, $\{0_0, 16_1, 26_2\}$, $\{0_0, 18_1, 11_2\}$.

INKTS(258, 72). Point set: $(Z(62) \times Z(3)) \cup \{x_1, x_2, \ldots, x_{72}\}$. Groups: $\{0_0, 31_0\}$, $\{0_1, 31_1\}$, $\{0_2, 31_2\}$ mod$(62, -)$. Hole: $\{x_1, x_2, \ldots, x_{72}\}$.

Parallel classes of triples: Sixty-two of them are obtained by developing the following triples mod$(62, -)$:

$\{36_0, 52_0, 54_0\}$, $\{43_1, 53_1, 57_1\}$, $\{46_2, 50_2, 60_2\}$, $\{43_0, 55_1, 49_2\}$, $\{32_0, 51_1, 0_2\}$, $\{26_0, 33_1, 36_2\}$, $\{57_0, 1_1, 3_2\}$, $\{24_0, 27_1, 25_2\}$, $\{61_0, 3_1, 2_2\}$, $\{25_0, 30_1, 27_2\}$, $\{59_0, 5_1, 1_2\}$, $\{27_0, 36_1, 40_2\}$, $\{51_0, 61_1, 56_2\}$, $\{38_0, 49_1, 55_2\}$, $\{7_1, 26_2, x_1\}$, $\{59_1, 17_2, x_2\}$, $\{9_1, 30_2, x_3\}$, $\{11_1, 33_2, x_4\}$, $\{15_1, 38_2, x_5\}$, $\{13_1, 39_2, x_6\}$, $\{17_1, 44_2, x_7\}$, $\{19_1, 48_2, x_8\}$, $\{21_1, 51_2, x_9\}$, $\{23_1, 54_2, x_{10}\}$, $\{25_1, 57_2, x_{11}\}$, $\{26_1, 59_2, x_{12}\}$, $\{24_1, 58_2, x_{13}\}$, $\{31_1, 4_2, x_{14}\}$, $\{32_1, 6_2, x_{15}\}$, $\{34_1, 9_2, x_{16}\}$, $\{29_1, 5_2, x_{17}\}$, $\{35_1, 12_2, x_{18}\}$, $\{37_1, 15_2, x_{19}\}$, $\{28_1, 7_2, x_{20}\}$, $\{39_1, 19_2, x_{21}\}$, $\{41_1, 22_2, x_{22}\}$, $\{47_1, 29_2, x_{23}\}$, $\{45_1, 28_2, x_{24}\}$, $\{31_2, 34_0, x_{25}\}$, $\{24_2, 28_0, x_{26}\}$, $\{32_2, 37_0, x_{27}\}$, $\{34_2, 40_0, x_{28}\}$, $\{37_2, 44_0, x_{29}\}$, $\{41_2, 50_0, x_{30}\}$, $\{47_2, 58_0, x_{31}\}$, $\{43_2, 55_0, x_{32}\}$, $\{42_2, 56_0, x_{33}\}$, $\{45_2, 60_0, x_{34}\}$, $\{14_2, 31_0, x_{35}\}$, $\{35_2, 53_0, x_{36}\}$, $\{18_2, 39_0, x_{37}\}$, $\{10_2, 33_0, x_{38}\}$, $\{21_2, 45_0, x_{39}\}$, $\{23_2, 49_0, x_{40}\}$, $\{20_2, 47_0, x_{41}\}$, $\{16_2, 46_0, x_{42}\}$, $\{11_2, 42_0, x_{43}\}$, $\{8_2, 41_0, x_{44}\}$, $\{13_2, 48_0, x_{45}\}$, $\{61_2, 35_0, x_{46}\}$, $\{53_2, 29_0, x_{47}\}$, $\{52_2, 30_0, x_{48}\}$, $\{0_0, 38_1, x_{49}\}$, $\{1_0, 40_1, x_{50}\}$, $\{2_0, 42_1, x_{51}\}$, $\{3_0, 44_1, x_{52}\}$, $\{4_0, 46_1, x_{53}\}$, $\{5_0, 48_1, x_{54}\}$, $\{6_0, 50_1, x_{55}\}$, $\{7_0, 52_1, x_{56}\}$, $\{8_0, 54_1, x_{57}\}$, $\{9_0, 56_1, x_{58}\}$, $\{10_0, 58_1, x_{59}\}$, $\{11_0, 60_1, x_{60}\}$, $\{12_0, 0_1, x_{61}\}$, $\{13_0, 2_1, x_{62}\}$, $\{14_0, 4_1, x_{63}\}$, $\{15_0, 6_1, x_{64}\}$, $\{16_0, 8_1, x_{65}\}$, $\{17_0, 10_1, x_{66}\}$, $\{18_0, 12_1, x_{67}\}$, $\{19_0, 14_1, x_{68}\}$, $\{20_0, 16_1, x_{69}\}$, $\{21_0, 18_1, x_{70}\}$, $\{22_0, 20_1, x_{71}\}$, $\{23_0, 22_1, x_{72}\}$.

Thirty-one of them are obtained by adding $0, 2, 4, \ldots, 60$ to the following triples mod$(62, -)$:

$\{14_0, 28_1, 21_2\}$, $\{45_0, 59_1, 52_2\}$, $\{17_0, 45_1, 60_2\}$, $\{48_0, 14_1, 29_2\}$, $\{19_0, 50_1, 37_2\}$, $\{50_0, 19_1, 6_2\}$, $\{20_0, 56_1, 12_2\}$, $\{51_0, 25_1, 43_2\}$, $\{12_0, 24_0, 52_0\}$, $\{43_0, 55_0, 21_0\}$, $\{16_1, 46_1, 52_1\}$, $\{47_1, 15_1, 21_1\}$, $\{20_2, 50_2, 56_2\}$, $\{51_2, 19_2, 25_2\}$, $\{30_0, 31_0, x_1\}$, $\{61_0, 0_0, x_2\}$, $\{1_0, 4_0, x_3\}$, $\{32_0, 35_0, x_4\}$, $\{29_0, 34_0, x_5\}$, $\{60_0, 3_0, x_6\}$, $\{2_0, 9_0, x_7\}$,

$\{33_0,40_0,x_8\}$, $\{28_0,37_0,x_9\}$, $\{59_0,6_0,x_{10}\}$, $\{5_0,16_0,x_{11}\}$, $\{36_0,47_0,x_{12}\}$, $\{26_0,39_0,x_{13}\}$, $\{57_0,8_0,x_{14}\}$, $\{7_0,22_0,x_{15}\}$, $\{38_0,53_0,x_{16}\}$, $\{27_0,44_0,x_{17}\}$, $\{58_0,13_0,x_{18}\}$, $\{23_0,42_0,x_{19}\}$, $\{54_0,11_0,x_{20}\}$, $\{25_0,46_0,x_{21}\}$, $\{56_0,15_0,x_{22}\}$, $\{18_0,41_0,x_{23}\}$, $\{49_0,10_0,x_{24}\}$, $\{0_1,1_1,x_{25}\}$, $\{31_1,32_1,x_{26}\}$, $\{30_1,33_1,x_{27}\}$, $\{61_1,2_1,x_{28}\}$, $\{3_1,8_1,x_{29}\}$, $\{34_1,39_1,x_{30}\}$, $\{29_1,36_1,x_{31}\}$, $\{60_1,5_1,x_{32}\}$, $\{4_1,13_1,x_{33}\}$, $\{35_1,44_1,x_{34}\}$, $\{6_1,17_1,x_{35}\}$, $\{37_1,48_1,x_{36}\}$, $\{7_1,20_1,x_{37}\}$, $\{38_1,51_1,x_{38}\}$, $\{9_1,24_1,x_{39}\}$, $\{40_1,55_1,x_{40}\}$, $\{10_1,27_1,x_{41}\}$, $\{41_1,58_1,x_{42}\}$, $\{23_1,42_1,x_{43}\}$, $\{54_1,11_1,x_{44}\}$, $\{22_1,43_1,x_{45}\}$, $\{53_1,12_1,x_{46}\}$, $\{26_1,49_1,x_{47}\}$, $\{57_1,18_1,x_{48}\}$, $\{0_2,1_2,x_{49}\}$, $\{31_2,32_2,x_{50}\}$, $\{2_2,5_2,x_{51}\}$, $\{33_2,36_2,x_{52}\}$, $\{3_2,8_2,x_{53}\}$, $\{34_2,39_2,x_{54}\}$, $\{4_2,11_2,x_{55}\}$, $\{35_2,42_2,x_{56}\}$, $\{7_2,16_2,x_{57}\}$, $\{38_2,47_2,x_{58}\}$, $\{13_2,24_2,x_{59}\}$, $\{44_2,55_2,x_{60}\}$, $\{9_2,22_2,x_{61}\}$, $\{40_2,53_2,x_{62}\}$, $\{26_2,41_2,x_{63}\}$, $\{57_2,10_2,x_{64}\}$, $\{28_2,45_2,x_{65}\}$, $\{59_2,14_2,x_{66}\}$, $\{30_2,49_2,x_{67}\}$, $\{61_2,18_2,x_{68}\}$, $\{27_2,48_2,x_{69}\}$, $\{58_2,17_2,x_{70}\}$, $\{23_2,46_2,x_{71}\}$, $\{54_2,15_2,x_{72}\}$.

Holey parallel classes of triples: Thirty-one of them are obtained by adding $0,2,4,\dots,60$ to the following triples mod$(62,-)$:

$\{18_0,43_0,51_0\}$, $\{49_0,12_0,20_0\}$, $\{28_0,38_0,42_0\}$, $\{59_0,7_0,11_0\}$, $\{9_0,39_0,45_0\}$, $\{40_0,8_0,14_0\}$, $\{13_1,38_1,46_1\}$, $\{44_1,7_1,15_1\}$, $\{18_1,34_1,36_1\}$, $\{49_1,3_1,5_1\}$, $\{11_1,39_1,51_1\}$, $\{42_1,8_1,20_1\}$, $\{11_2,19_2,44_2\}$, $\{42_2,50_2,13_2\}$, $\{2_2,4_2,20_2\}$, $\{33_2,35_2,51_2\}$, $\{6_2,18_2,46_2\}$, $\{37_2,49_2,15_2\}$, $\{4_0,41_1,25_2\}$, $\{35_0,10_1,56_2\}$, $\{22_0,57_1,12_2\}$, $\{53_0,26_1,43_2\}$, $\{13_0,47_1,32_2\}$, $\{44_0,16_1,1_2\}$, $\{16_0,48_1,0_2\}$, $\{47_0,17_1,31_2\}$, $\{23_0,53_1,39_2\}$, $\{54_0,22_1,8_2\}$, $\{21_0,45_1,36_2\}$, $\{52_0,14_1,5_2\}$, $\{29_0,52_1,41_2\}$, $\{60_0,21_1,10_2\}$, $\{6_0,28_1,40_2\}$, $\{37_0,59_1,9_2\}$, $\{3_0,24_1,14_2\}$, $\{34_0,55_1,45_2\}$, $\{19_0,37_1,47_2\}$, $\{50_0,6_1,16_2\}$, $\{27_0,43_1,52_2\}$, $\{58_0,12_1,21_2\}$, $\{25_0,40_1,48_2\}$, $\{56_0,9_1,17_2\}$, $\{10_0,23_1,30_2\}$, $\{41_0,54_1,61_2\}$, $\{30_0,30_1,50_1\}$, $\{61_0,61_1,19_1\}$, $\{0_0,1_1,25_1\}$, $\{31_0,32_1,56_1\}$, $\{2_0,4_1,31_1\}$, $\{33_0,35_1,0_1\}$, $\{29_1,29_2,53_2\}$, $\{60_1,60_2,22_2\}$, $\{2_1,7_2,27_2\}$, $\{33_1,38_2,58_2\}$, $\{27_1,28_2,55_2\}$, $\{58_1,59_2,24_2\}$, $\{26_2,26_0,46_0\}$, $\{57_2,57_0,15_0\}$, $\{23_2,24_0,48_0\}$, $\{54_2,55_0,17_0\}$, $\{3_2,5_0,32_0\}$, $\{34_2,36_0,1_0\}$.

Four of them are obtained by developing each of the following triples mod$(62,-)$:

$\{0_0,17_1,9_2\}$, $\{0_0,26_1,14_2\}$, $\{0_0,27_1,40_2\}$, $\{0_0,33_1,49_2\}$.

Thus we complete the proof. □

The foregoing results give us our first part of the main theorem:

Theorem 3.7. *For* $u\equiv v\equiv 0 \pmod 6$, $v\geqslant 78$, *and* $u\geqslant 3.5v$, *there exists an* NKTS(u) *containing a* sub-NKTS(v).

We now consider INKTS(u,v), $v\in\{48,54,60,66,72\}$ in detail.

For $v=48$, we have to consider INKTS$(u,48)$, $u\in\{144,150,156,162,168\}$.

Now $u=144$, 162 and 168 are covered by Theorem 1.3 and Lemma 3.6, respectively.

For $u=156$, take a $\{4\}$-GDD of type 9^4, apply Theorem 2.1 and Construction 2.3, using weight 4 with 12 ideal points. This gives an INKTS(156, 48).

For $u=150$, we have the following lemma.

Lemma 3.8. *There exists an* INKTS(150, 48).

Proof. We present an INKTS(150, 48) as follows:

INKTS(150, 48). Point set: $(Z(34)\times Z(3))\cup\{x_1,x_2,\ldots,x_{48}\}$. Groups: $\{0_0,17_0\}$, $\{0_1,17_1\}$, $\{0_2,17_2\}$ mod$(34,-)$. Hole: $\{x_1,x_2,\ldots,x_{48}\}$.

Parallel classes of triples: Thirty-four of them are obtained by developing the following triples mod$(34,-)$:

$\{23_0,31_1,27_2\}$, $\{24_0,33_1,4_2\}$, $\{0_0,18_1,x_1\}$, $\{1_0,20_1,x_2\}$, $\{2_0,22_1,x_3\}$, $\{3_0,24_1,x_4\}$, $\{4_0,26_1,x_5\}$, $\{5_0,28_1,x_6\}$, $\{6_0,30_1,x_7\}$, $\{7_0,32_1,x_8\}$, $\{8_0,0_1,x_9\}$, $\{9_0,2_1,x_{10}\}$, $\{10_0,4_1,x_{11}\}$, $\{11_0,6_1,x_{12}\}$, $\{12_0,8_1,x_{13}\}$, $\{13_0,10_1,x_{14}\}$, $\{14_0,12_1,x_{15}\}$, $\{15_0,14_1,x_{16}\}$, $\{1_1,10_2,x_{17}\}$, $\{3_1,13_2,x_{18}\}$, $\{5_1,16_2,x_{19}\}$, $\{7_1,20_2,x_{20}\}$, $\{9_1,23_2,x_{21}\}$, $\{11_1,28_2,x_{22}\}$, $\{13_1,29_2,x_{23}\}$, $\{15_1,33_2,x_{24}\}$, $\{16_1,1_2,x_{25}\}$, $\{17_1,3_2,x_{26}\}$, $\{19_1,6_2,x_{27}\}$, $\{21_1,9_2,x_{28}\}$, $\{23_1,12_2,x_{29}\}$, $\{25_1,15_2,x_{30}\}$, $\{27_1,18_2,x_{31}\}$, $\{29_1,21_2,x_{32}\}$, $\{14_2,16_0,x_{33}\}$, $\{30_2,33_0,x_{34}\}$, $\{22_2,26_0,x_{35}\}$, $\{25_2,30_0,x_{36}\}$, $\{26_2,32_0,x_{37}\}$, $\{24_2,31_0,x_{38}\}$, $\{17_2,25_0,x_{39}\}$, $\{19_2,29_0,x_{40}\}$, $\{11_2,22_0,x_{41}\}$, $\{0_2,17_0,x_{42}\}$, $\{5_2,19_0,x_{43}\}$, $\{2_2,18_0,x_{44}\}$, $\{8_2,27_0,x_{45}\}$, $\{7_2,28_0,x_{46}\}$, $\{32_2,20_0,x_{47}\}$, $\{31_2,21_0,x_{48}\}$.

Seventeen of them are obtained by adding $0,2,4,\ldots,32$ to the following triples mod$(34,-)$:

$\{7_0,24_1,32_2\}$, $\{24_0,7_1,15_2\}$, $\{0_0,1_0,x_{17}\}$, $\{17_0,18_0,x_{18}\}$, $\{2_0,5_0,x_{19}\}$, $\{19_0,22_0,x_{20}\}$, $\{3_0,8_0,x_{21}\}$, $\{20_0,25_0,x_{22}\}$, $\{4_0,11_0,x_{23}\}$, $\{21_0,28_0,x_{24}\}$, $\{6_0,15_0,x_{25}\}$, $\{23_0,32_0,x_{26}\}$, $\{16_0,27_0,x_{27}\}$, $\{33_0,10_0,x_{28}\}$, $\{13_0,26_0,x_{29}\}$, $\{30_0,9_0,x_{30}\}$, $\{14_0,29_0,x_{31}\}$, $\{31_0,12_0,x_{32}\}$, $\{0_1,1_1,x_{33}\}$, $\{17_1,18_1,x_{34}\}$, $\{2_1,5_1,x_{35}\}$, $\{19_1,22_1,x_{36}\}$, $\{3_1,8_1,x_{37}\}$, $\{20_1,25_1,x_{38}\}$, $\{4_1,11_1,x_{39}\}$, $\{21_1,28_1,x_{40}\}$, $\{6_1,15_1,x_{41}\}$, $\{23_1,32_1,x_{42}\}$, $\{16_1,27_1,x_{43}\}$, $\{33_1,10_1,x_{44}\}$, $\{13_1,26_1,x_{45}\}$, $\{30_1,9_1,x_{46}\}$, $\{14_1,29_1,x_{47}\}$, $\{31_1,12_1,x_{48}\}$, $\{0_2,1_2,x_1\}$, $\{17_2,18_2,x_2\}$, $\{2_2,5_2,x_3\}$, $\{19_2,22_2,x_4\}$, $\{3_2,8_2,x_5\}$, $\{20_2,25_2,x_6\}$, $\{16_2,23_2,x_7\}$, $\{33_2,6_2,x_8\}$, $\{12_2,21_2,x_9\}$, $\{29_2,4_2,x_{10}\}$, $\{13_2,24_2,x_{11}\}$, $\{30_2,7_2,x_{12}\}$, $\{14_2,27_2,x_{13}\}$, $\{31_2,10_2,x_{14}\}$, $\{11_2,26_2,x_{15}\}$, $\{28_2,9_2,x_{16}\}$.

Holey parallel classes of triples: Seventeen of them are obtained by adding $0,2,4,\ldots,32$ to the following triples mod$(34,-)$:

$\{0_0,2_0,10_0\}$, $\{17_0,19_0,27_0\}$, $\{3_0,11_0,21_0\}$, $\{20_0,28_0,4_0\}$, $\{0_1,2_1,10_1\}$, $\{17_1,19_1,27_1\}$, $\{3_1,11_1,21_1\}$, $\{20_1,28_1,4_1\}$, $\{4_2,6_2,14_2\}$, $\{21_2,23_2,31_2\}$, $\{12_2,20_2,30_2\}$, $\{29_2,3_2,13_2\}$, $\{9_0,14_1,17_2\}$, $\{26_0,31_1,0_2\}$, $\{7_0,13_1,10_2\}$, $\{24_0,30_1,27_2\}$, $\{15_0,22_1,26_2\}$, $\{32_0,5_1,9_2\}$, $\{16_0,26_1,32_2\}$, $\{33_0,9_1,15_2\}$, $\{12_0,25_1,19_2\}$, $\{29_0,8_1,2_2\}$, $\{6_0,6_1,18_1\}$, $\{23_0,23_1,1_1\}$, $\{14_0,15_1,29_1\}$, $\{31_0,32_1,12_1\}$, $\{16_1,16_2,28_2\}$, $\{33_1,33_2,11_2\}$, $\{7_1,8_2,22_2\}$, $\{24_1,25_2,5_2\}$, $\{1_2,1_0,13_0\}$, $\{18_2,18_0,30_0\}$, $\{7_2,8_0,22_0\}$, $\{24_2,25_0,5_0\}$.

Six of them are obtained by developing each of the following triples mod$(34,-)$:

$\{0_0, 2_1, 1_2\}$, $\{0_0, 3_1, 5_2\}$, $\{0_0, 4_1, 2_2\}$, $\{0_0, 11_1, 6_2\}$, $\{0_0, 14_1, 21_2\}$, $\{0_0, 16_1, 9_2\}$. □

We thus obtain:

Theorem 3.9. *For all* $v \equiv 0 \pmod 6$ *with* $v \geqslant 144$, *there exists an* NKTS(v) *containing a* sub-NKTS(48).

For $v = 54$, we have to consider INKTS$(u, 54)$, $u \in \{162, 168, 174, 180, 186, 192\}$.

Now $u = 162$ is covered by Theorem 1.3.

For $u \in \{174, 186, 192\}$ apply Theorem 2.1 and Construction 2.3, using weight 2 with 6 ideal points, to $\{4\}$-GDDs of types $12^5 24^1$ (see [5]), $6^8 18^1 24^1$ or $6^8 21^1 24^1$ (see the appendix), respectively. For $u = 180$ proceed similarly, starting with a TD(4,7) and using weight 6 with 12 ideal points.

For $u = 168$ we have the following lemma.

Lemma 3.10. *There exists an* INKTS(168, 54).

Proof. We present an INKTS(168, 54) as follows:

INKTS(168, 54). Point set: $(Z(38) \times Z(3)) \cup \{x_1, x_2, \ldots, x_{54}\}$. Groups: $\{0_0, 19_0\}$, $\{0_1, 19_1\}$, $\{0_2, 19_2\}$ mod$(38,-)$. Hole: $\{x_1, x_2, \ldots, x_{54}\}$.

Parallel classes of triples: Thirty-eight of them are obtained by developing the following triples mod$(38,-)$:

$\{0_0, 0_1, 0_2\}$, $\{22_0, 34_1, 4_2\}$, $\{1_0, 21_1, x_1\}$, $\{2_0, 23_1, x_2\}$, $\{3_0, 25_1, x_3\}$, $\{4_0, 27_1, x_4\}$,
$\{5_0, 29_1, x_5\}$, $\{6_0, 31_1, x_6\}$, $\{7_0, 33_1, x_7\}$, $\{8_0, 35_1, x_8\}$, $\{9_0, 37_1, x_9\}$, $\{10_0, 1_1, x_{10}\}$,
$\{11_0, 3_1, x_{11}\}$, $\{12_0, 5_1, x_{12}\}$, $\{13_0, 7_1, x_{13}\}$, $\{14_0, 9_1, x_{14}\}$, $\{15_0, 11_1, x_{15}\}$,
$\{16_0, 13_1, x_{16}\}$, $\{17_0, 15_1, x_{17}\}$, $\{18_0, 17_1, x_{18}\}$, $\{2_1, 15_2, x_{19}\}$, $\{4_1, 18_2, x_{20}\}$, $\{6_1, 21_2, x_{21}\}$,
$\{8_1, 24_2, x_{22}\}$, $\{10_1, 27_2, x_{23}\}$, $\{12_1, 30_2, x_{24}\}$, $\{14_1, 33_2, x_{25}\}$, $\{16_1, 36_2, x_{26}\}$,
$\{18_1, 1_2, x_{27}\}$, $\{19_1, 3_2, x_{28}\}$, $\{20_1, 5_2, x_{29}\}$, $\{22_1, 8_2, x_{30}\}$, $\{24_1, 11_2, x_{31}\}$,
$\{26_1, 14_2, x_{32}\}$, $\{28_1, 17_2, x_{33}\}$, $\{30_1, 20_2, x_{34}\}$, $\{32_1, 23_2, x_{35}\}$, $\{36_1, 28_2, x_{36}\}$,
$\{19_2, 20_0, x_{37}\}$, $\{31_2, 33_0, x_{38}\}$, $\{34_2, 37_0, x_{39}\}$, $\{32_2, 36_0, x_{40}\}$,
$\{25_2, 30_0, x_{41}\}$, $\{29_2, 35_0, x_{42}\}$, $\{26_2, 34_0, x_{43}\}$, $\{22_2, 31_0, x_{44}\}$, $\{10_2, 21_0, x_{45}\}$,
$\{16_2, 29_0, x_{46}\}$, $\{9_2, 23_0, x_{47}\}$, $\{12_2, 28_0, x_{48}\}$, $\{7_2, 24_0, x_{49}\}$, $\{13_2, 32_0, x_{50}\}$,
$\{6_2, 27_0, x_{51}\}$, $\{35_2, 19_0, x_{52}\}$, $\{2_2, 26_0, x_{53}\}$, $\{37_2, 25_0, x_{54}\}$.

Nineteen of them are obtained by adding $0, 2, 4, \ldots, 36$ to the following triples mod$(38,-)$:

$\{17_0, 36_1, 10_2\}$, $\{36_0, 17_1, 29_2\}$, $\{0_0, 1_0, x_{19}\}$, $\{19_0, 20_0, x_{20}\}$, $\{2_0, 5_0, x_{21}\}$, $\{21_0, 24_0, x_{22}\}$,
$\{3_0, 8_0, x_{23}\}$, $\{22_0, 27_0, x_{24}\}$, $\{4_0, 11_0, x_{25}\}$, $\{23_0, 30_0, x_{26}\}$, $\{6_0, 15_0, x_{27}\}$, $\{25_0, 34_0, x_{28}\}$,
$\{7_0, 18_0, x_{29}\}$, $\{26_0, 37_0, x_{30}\}$, $\{16_0, 29_0, x_{31}\}$, $\{35_0, 10_0, x_{32}\}$, $\{13_0, 28_0, x_{33}\}$,
$\{32_0, 9_0, x_{34}\}$, $\{14_0, 31_0, x_{35}\}$, $\{33_0, 12_0, x_{36}\}$, $\{0_1, 1_1, x_{37}\}$, $\{19_1, 20_1, x_{38}\}$,
$\{2_1, 5_1, x_{39}\}$, $\{21_1, 24_1, x_{40}\}$, $\{3_1, 8_1, x_{41}\}$, $\{22_1, 27_1, x_{42}\}$, $\{4_1, 11_1, x_{43}\}$,

$\{23_1, 30_1, x_{44}\}$, $\{6_1, 15_1, x_{45}\}$, $\{25_1, 34_1, x_{46}\}$, $\{7_1, 18_1, x_{47}\}$, $\{26_1, 37_1, x_{48}\}$, $\{16_1, 29_1, x_{49}\}$, $\{35_1, 10_1, x_{50}\}$, $\{13_1, 28_1, x_{51}\}$, $\{32_1, 9_1, x_{52}\}$, $\{14_1, 31_1, x_{53}\}$, $\{33_1, 12_1, x_{54}\}$, $\{0_2, 1_2, x_1\}$, $\{19_2, 20_2, x_2\}$, $\{2_2, 5_2, x_3\}$, $\{21_2, 24_2, x_4\}$, $\{3_2, 8_2, x_5\}$, $\{22_2, 27_2, x_6\}$, $\{4_2, 11_2, x_7\}$, $\{23_2, 30_2, x_8\}$, $\{7_2, 16_2, x_9\}$, $\{26_2, 35_2, x_{10}\}$, $\{14_2, 25_2, x_{11}\}$, $\{33_2, 6_2, x_{12}\}$, $\{18_2, 31_2, x_{13}\}$, $\{37_2, 12_2, x_{14}\}$, $\{13_2, 28_2, x_{15}\}$, $\{32_2, 9_2, x_{16}\}$, $\{17_2, 34_2, x_{17}\}$, $\{36_2, 15_2, x_{18}\}$.

Holey parallel classes of triples: Nineteen of them are obtained by adding $0, 2, 4, \ldots, 36$ to the following triples mod$(38, -)$:

$\{0_0, 6_0, 24_0\}$, $\{19_0, 25_0, 5_0\}$, $\{4_0, 8_0, 20_0\}$, $\{23_0, 27_0, 1_0\}$, $\{7_0, 9_0, 17_0\}$, $\{26_0, 28_0, 36_0\}$, $\{0_1, 6_1, 24_1\}$, $\{19_1, 25_1, 5_1\}$, $\{4_1, 8_1, 20_1\}$, $\{23_1, 27_1, 1_1\}$, $\{7_1, 9_1, 17_1\}$, $\{26_1, 28_1, 36_1\}$, $\{4_2, 10_2, 28_2\}$, $\{23_2, 29_2, 9_2\}$, $\{8_2, 12_2, 24_2\}$, $\{27_2, 31_2, 5_2\}$, $\{1_2, 3_2, 11_2\}$, $\{20_2, 22_2, 30_2\}$, $\{11_0, 12_1, 13_2\}$, $\{30_0, 31_1, 32_2\}$, $\{13_0, 15_1, 14_2\}$, $\{32_0, 34_1, 33_2\}$, $\{10_0, 13_1, 15_2\}$, $\{29_0, 32_1, 34_2\}$, $\{14_0, 18_1, 21_2\}$, $\{33_0, 37_1, 2_2\}$, $\{16_0, 21_1, 19_2\}$, $\{35_0, 2_1, 0_2\}$, $\{3_0, 11_1, 16_2\}$, $\{22_0, 30_1, 35_2\}$, $\{12_0, 22_1, 18_2\}$, $\{31_0, 3_1, 37_2\}$, $\{18_0, 29_1, 36_2\}$, $\{37_0, 10_1, 17_2\}$, $\{2_0, 16_1, 25_2\}$, $\{21_0, 35_1, 6_2\}$, $\{15_0, 33_1, 26_2\}$, $\{34_0, 14_1, 7_2\}$.

Seven of them are obtained by developing each of the following triples mod$(38, -)$:

$\{0_0, 6_1, 10_2\}$, $\{0_0, 7_1, 4_2\}$, $\{0_0, 9_1, 15_2\}$, $\{0_0, 13_1, 8_2\}$, $\{0_0, 15_1, 9_2\}$, $\{0_0, 16_1, 26_2\}$, $\{0_0, 17_1, 28_2\}$. □

We thus obtain:

Theorem 3.11. *For all* $v \equiv 0 \pmod 6$ *with* $v \geqslant 162$, *there exists an* NKTS(v) *containing a* sub-NKTS(54).

For $v = 60$, we have to consider INKTS$(u, 60)$, $u \in \{180, 186, 192, 198, 204\}$ (see Lemma 3.5).

Now $u = 180$ is covered by Theorem 1.3.

For $u = 186$, take a TD$(4, 21)$ and apply Theorem 2.1 and Construction 2.3, using weight 2 with $a = 18$ ideal points. For $u = 192$ proceed similarly, starting with a $\{4\}$-GDD of type $6^9 12^1 27^1$ (see the appendix), using weight 2 with 6 ideal points, while for $u = 204$ take a $\{4\}$-GDD of type $6^4 9^1$ (see [4]) and use weight 6 with 6 ideal points.

For $u = 198$ we have the following lemma.

Lemma 3.12. *There exists an* INKTS$(198, 60)$.

Proof. We present an INKTS$(198, 60)$ as follows:

INKTS$(198, 60)$. Point set: $(Z(46) \times Z(3)) \cup \{x_1, x_2, \ldots, x_{60}\}$. Groups: $\{0_0, 23_0\}$, $\{0_1, 23_1\}$, $\{0_2, 23_2\}$ mod$(46, -)$. Hole: $\{x_1, x_2, \ldots, x_{60}\}$.

Parallel classes of triples: Forty-six of them are obtained by developing the following triples mod(46, −):

$\{39_0,42_1,40_2\}$, $\{31_0,36_1,33_2\}$, $\{28_0,34_1,35_2\}$, $\{32_0,40_1,36_2\}$, $\{34_0,44_1,39_2\}$, $\{20_0,38_1,0_2\}$, $\{45_0,25_1,x_1\}$, $\{0_0,27_1,x_2\}$, $\{44_0,26_1,x_3\}$, $\{1_0,30_1,x_4\}$, $\{40_0,24_1,x_5\}$, $\{2_0,33_1,x_6\}$, $\{3_0,35_1,x_7\}$, $\{4_0,37_1,x_8\}$, $\{5_0,39_1,x_9\}$, $\{6_0,41_1,x_{10}\}$, $\{7_0,43_1,x_{11}\}$, $\{8_0,45_1,x_{12}\}$, $\{9_0,1_1,x_{13}\}$, $\{10_0,3_1,x_{14}\}$, $\{11_0,5_1,x_{15}\}$, $\{12_0,7_1,x_{16}\}$, $\{13_0,9_1,x_{17}\}$, $\{14_0,11_1,x_{18}\}$, $\{15_0,13_1,x_{19}\}$, $\{16_0,15_1,x_{20}\}$, $\{0_1,12_2,x_{21}\}$, $\{2_1,15_2,x_{22}\}$, $\{4_1,18_2,x_{23}\}$, $\{6_1,21_2,x_{24}\}$, $\{8_1,24_2,x_{25}\}$, $\{10_1,27_2,x_{26}\}$, $\{12_1,30_2,x_{27}\}$, $\{14_1,34_2,x_{28}\}$, $\{16_1,37_2,x_{29}\}$, $\{17_1,41_2,x_{30}\}$, $\{18_1,43_2,x_{31}\}$, $\{19_1,45_2,x_{32}\}$, $\{20_1,1_2,x_{33}\}$, $\{21_1,3_2,x_{34}\}$, $\{22_1,5_2,x_{35}\}$, $\{23_1,7_2,x_{36}\}$, $\{28_1,13_2,x_{37}\}$, $\{31_1,17_2,x_{38}\}$, $\{29_1,16_2,x_{39}\}$, $\{32_1,22_2,x_{40}\}$, $\{14_2,17_0,x_{41}\}$, $\{19_2,23_0,x_{42}\}$, $\{20_2,25_0,x_{43}\}$, $\{23_2,29_0,x_{44}\}$, $\{11_2,18_0,x_{45}\}$, $\{25_2,33_0,x_{46}\}$, $\{28_2,37_0,x_{47}\}$, $\{31_2,41_0,x_{48}\}$, $\{32_2,43_0,x_{49}\}$, $\{26_2,38_0,x_{50}\}$, $\{29_2,42_0,x_{51}\}$, $\{4_2,19_0,x_{52}\}$, $\{6_2,24_0,x_{53}\}$, $\{2_2,21_0,x_{54}\}$, $\{8_2,30_0,x_{55}\}$, $\{10_2,35_0,x_{56}\}$, $\{9_2,36_0,x_{57}\}$, $\{44_2,26_0,x_{58}\}$, $\{38_2,22_0,x_{59}\}$, $\{42_2,27_0,x_{60}\}$.

Twenty-three of them are obtained by adding $0,2,4,\ldots,44$ to the following triples mod(46, −):

$\{21_0,42_1,7_2\}$, $\{44_0,19_1,30_2\}$, $\{14_0,37_1,26_2\}$, $\{37_0,14_1,3_2\}$, $\{19_0,44_1,32_2\}$, $\{42_0,21_1,9_2\}$, $\{0_0,1_0,x_{21}\}$, $\{23_0,24_0,x_{22}\}$, $\{2_0,5_0,x_{23}\}$, $\{25_0,28_0,x_{24}\}$, $\{3_0,8_0,x_{25}\}$, $\{26_0,31_0,x_{26}\}$, $\{4_0,11_0,x_{27}\}$, $\{27_0,34_0,x_{28}\}$, $\{6_0,15_0,x_{29}\}$, $\{29_0,38_0,x_{30}\}$, $\{7_0,18_0,x_{31}\}$, $\{30_0,41_0,x_{32}\}$, $\{9_0,22_0,x_{33}\}$, $\{32_0,45_0,x_{34}\}$, $\{20_0,35_0,x_{35}\}$, $\{43_0,12_0,x_{36}\}$, $\{16_0,33_0,x_{37}\}$, $\{39_0,10_0,x_{38}\}$, $\{17_0,36_0,x_{39}\}$, $\{40_0,13_0,x_{40}\}$, $\{0_1,1_1,x_{41}\}$, $\{23_1,24_1,x_{42}\}$, $\{2_1,5_1,x_{43}\}$, $\{25_1,28_1,x_{44}\}$, $\{3_1,8_1,x_{45}\}$, $\{26_1,31_1,x_{46}\}$, $\{4_1,11_1,x_{17}\}$, $\{27_1,34_1,x_{48}\}$, $\{6_1,15_1,x_{49}\}$, $\{29_1,38_1,x_{50}\}$, $\{7_1,18_1,x_{51}\}$, $\{30_1,41_1,x_{52}\}$, $\{9_1,22_1,x_{53}\}$, $\{32_1,45_1,x_{54}\}$, $\{20_1,35_1,x_{55}\}$, $\{43_1,12_1,x_{56}\}$, $\{16_1,33_1,x_{57}\}$, $\{39_1,10_1,x_{58}\}$, $\{17_1,36_1,x_{59}\}$, $\{40_1,13_1,x_{60}\}$, $\{0_2,1_2,x_1\}$, $\{23_2,24_2,x_2\}$, $\{2_2,5_2,x_3\}$, $\{25_2,28_2,x_4\}$, $\{6_2,11_2,x_5\}$, $\{29_2,34_2,x_6\}$, $\{10_2,17_2,x_7\}$, $\{33_2,40_2,x_8\}$, $\{12_2,21_2,x_9\}$, $\{35_2,44_2,x_{10}\}$, $\{4_2,15_2,x_{11}\}$, $\{27_2,38_2,x_{12}\}$, $\{18_2,31_2,x_{13}\}$, $\{41_2,8_2,x_{14}\}$, $\{22_2,37_2,x_{15}\}$, $\{45_2,14_2,x_{16}\}$, $\{19_2,36_2,x_{17}\}$, $\{42_2,13_2,x_{18}\}$, $\{20_2,39_2,x_{19}\}$, $\{43_2,16_2,x_{20}\}$.

Holey parallel classes of triples: Twenty-three of them are obtained by adding $0,2,4,\ldots,44$ to the following triples mod(46, −):

$\{0_0,4_0,6_0\}$, $\{23_0,27_0,29_0\}$, $\{1_0,11_0,19_0\}$, $\{24_0,34_0,42_0\}$, $\{21_0,41_0,7_0\}$, $\{44_0,18_0,30_0\}$, $\{0_1,2_1,6_1\}$, $\{23_1,25_1,29_1\}$, $\{1_1,11_1,19_1\}$, $\{24_1,34_1,42_1\}$, $\{8_1,20_1,40_1\}$, $\{31_1,43_1,17_1\}$, $\{0_2,4_2,6_2\}$, $\{23_2,27_2,29_2\}$, $\{1_2,9_2,19_2\}$, $\{24_2,32_2,42_2\}$, $\{3_2,15_2,35_2\}$, $\{26_2,38_2,12_2\}$, $\{17_0,21_1,20_2\}$, $\{40_0,44_1,43_2\}$, $\{5_0,12_1,16_2\}$, $\{28_0,35_1,39_2\}$, $\{20_0,33_1,40_2\}$, $\{43_0,10_1,17_2\}$, $\{3_0,18_1,11_2\}$, $\{26_0,41_1,34_2\}$, $\{13_0,30_1,22_2\}$, $\{36_0,7_1,45_2\}$, $\{16_0,16_1,32_1\}$, $\{39_0,39_1,9_1\}$, $\{14_0,15_1,36_1\}$, $\{37_0,38_1,13_1\}$, $\{2_0,4_1,26_1\}$, $\{25_0,27_1,3_1\}$, $\{22_1,25_2,41_2\}$, $\{45_1,2_2,18_2\}$, $\{14_1,14_2,36_2\}$, $\{37_1,37_2,13_2\}$, $\{5_1,7_2,28_2\}$, $\{28_1,30_2,5_2\}$, $\{21_2,22_0,38_0\}$, $\{44_2,45_0,15_0\}$, $\{10_2,12_0,31_0\}$, $\{33_2,35_0,8_0\}$, $\{8_2,10_0,32_0\}$, $\{31_2,33_0,9_0\}$.

Six of them are obtained by developing each of the following triples mod$(46,-)$:

$\{0_0,9_1,14_2\}$, $\{0_0,11_1,17_2\}$, $\{0_0,12_1,6_2\}$, $\{0_0,14_1,23_2\}$, $\{0_0,19_1,10_2\}$, $\{0_0,20_1,30_2\}$. □

We thus obtain:

Theorem 3.13. *For all* $v\equiv 0 \pmod 6$ *with* $v\geqslant 180$, *there exists an* NKTS(v) *containing a* sub-NKTS(60).

For $v=66$, we have to consider INKTS$(u,66)$, $u\in\{198, 204, 210\}$ (see Lemma 3.2).

Now $u=198$ is covered by Theorem 1.3.

For $u=210$, take a $\{4\}$-GDD of type $6^4 9^1$; apply Theorem 2.1 and Construction 2.3, using weight 6 with 12 ideal points. This gives an INKTS(210, 66).

For $u=204$, we have the following lemma.

Lemma 3.14. *There exists an* INKTS(204, 66).

Proof. We present an INKTS(204, 66) as follows:

INKTS(204, 66). Point set: $(Z(46)\times Z(3))\cup\{x_1,x_2,\ldots,x_{66}\}$. Groups: $\{0_0,23_0\}$, $\{0_1,23_1\}$, $\{0_2,23_2\}$ mod$(46,-)$. Hole: $\{x_1,x_2,\ldots,x_{66}\}$.

Parallel classes of triples: Forty-six of them are obtained by developing the following triples mod$(46,-)$:

$\{38_0,42_1,40_2\}$, $\{32_0,44_1,4_2\}$, $\{45_0,23_1,x_1\}$, $\{0_0,25_1,x_2\}$, $\{44_0,24_1,x_3\}$, $\{1_0,28_1,x_4\}$, $\{40_0,22_1,x_5\}$, $\{2_0,31_1,x_6\}$, $\{3_0,33_1,x_7\}$, $\{4_0,35_1,x_8\}$, $\{5_0,37_1,x_9\}$, $\{6_0,39_1,x_{10}\}$, $\{7_0,41_1,x_{11}\}$, $\{8_0,43_1,x_{12}\}$, $\{9_0,45_1,x_{13}\}$, $\{10_0,1_1,x_{14}\}$, $\{11_0,3_1,x_{15}\}$, $\{12_0,5_1,x_{16}\}$, $\{13_0,7_1,x_{17}\}$, $\{14_0,9_1,x_{18}\}$, $\{15_0,11_1,x_{19}\}$, $\{16_0,13_1,x_{20}\}$, $\{17_0,15_1,x_{21}\}$, $\{18_0,17_1,x_{22}\}$, $\{0_1,13_2,x_{23}\}$, $\{2_1,16_2,x_{24}\}$, $\{4_1,19_2,x_{25}\}$, $\{6_1,22_2,x_{26}\}$, $\{8_1,25_2,x_{27}\}$, $\{10_1,28_2,x_{28}\}$, $\{12_1,31_2,x_{29}\}$, $\{14_1,34_2,x_{30}\}$, $\{16_1,37_2,x_{31}\}$, $\{18_1,42_2,x_{32}\}$, $\{19_1,44_2,x_{33}\}$, $\{20_1,0_2,x_{34}\}$, $\{21_1,2_2,x_{35}\}$, $\{26_1,8_2,x_{36}\}$, $\{27_1,10_2,x_{37}\}$, $\{30_1,14_2,x_{38}\}$, $\{32_1,17_2,x_{39}\}$, $\{29_1,15_2,x_{40}\}$, $\{34_1,21_2,x_{41}\}$, $\{36_1,24_2,x_{42}\}$, $\{38_1,27_2,x_{43}\}$, $\{40_1,30_2,x_{44}\}$, $\{18_2,20_0,x_{45}\}$, $\{32_2,35_0,x_{46}\}$, $\{35_2,39_0,x_{47}\}$, $\{38_2,43_0,x_{48}\}$, $\{36_2,42_0,x_{49}\}$, $\{29_2,36_0,x_{50}\}$, $\{33_2,41_0,x_{51}\}$, $\{12_2,21_0,x_{52}\}$, $\{23_2,33_0,x_{53}\}$, $\{26_2,37_0,x_{54}\}$, $\{7_2,19_0,x_{55}\}$, $\{20_2,34_0,x_{56}\}$, $\{5_2,23_0,x_{57}\}$, $\{9_2,28_0,x_{58}\}$, $\{11_2,31_0,x_{59}\}$, $\{1_2,22_0,x_{60}\}$, $\{6_2,30_0,x_{61}\}$, $\{3_2,29_0,x_{62}\}$, $\{45_2,26_0,x_{63}\}$, $\{41_2,24_0,x_{64}\}$, $\{43_2,27_0,x_{65}\}$, $\{39_2,25_0,x_{66}\}$.

Twenty-three of them are obtained by adding $0,2,4,\ldots,44$ to the following triples mod$(46,-)$:

$\{14_0,33_1,43_2\}$, $\{37_0,10_1,20_2\}$, $\{0_0,1_0,x_{23}\}$, $\{23_0,24_0,x_{24}\}$, $\{2_0,5_0,x_{25}\}$, $\{25_0,28_0,x_{26}\}$, $\{3_0,8_0,x_{27}\}$, $\{26_0,31_0,x_{28}\}$, $\{4_0,11_0,x_{29}\}$, $\{27_0,34_0,x_{30}\}$, $\{6_0,15_0,x_{31}\}$, $\{29_0,38_0,x_{32}\}$, $\{7_0,18_0,x_{33}\}$, $\{30_0,41_0,x_{34}\}$, $\{9_0,22_0,x_{35}\}$, $\{32_0,45_0,x_{36}\}$, $\{20_0,35_0,x_{37}\}$, $\{43_0,12_0,x_{38}\}$,

$\{16_0, 33_0, x_{39}\}$, $\{39_0, 10_0, x_{40}\}$, $\{17_0, 36_0, x_{41}\}$, $\{40_0, 13_0, x_{42}\}$, $\{21_0, 42_0, x_{43}\}$, $\{44_0, 19_0, x_{44}\}$, $\{0_1, 1_1, x_{45}\}$, $\{23_1, 24_1, x_{46}\}$, $\{2_1, 5_1, x_{47}\}$, $\{25_1, 28_1, x_{48}\}$, $\{3_1, 8_1, x_{49}\}$, $\{26_1, 31_1, x_{50}\}$, $\{4_1, 11_1, x_{51}\}$, $\{27_1, 34_1, x_{52}\}$, $\{6_1, 15_1, x_{53}\}$, $\{29_1, 38_1, x_{54}\}$, $\{7_1, 18_1, x_{55}\}$, $\{30_1, 41_1, x_{56}\}$, $\{9_1, 22_1, x_{57}\}$, $\{32_1, 45_1, x_{58}\}$, $\{20_1, 35_1, x_{59}\}$, $\{43_1, 12_1, x_{60}\}$, $\{19_1, 36_1, x_{61}\}$, $\{42_1, 13_1, x_{62}\}$, $\{21_1, 40_1, x_{63}\}$, $\{44_1, 17_1, x_{64}\}$, $\{16_1, 37_1, x_{65}\}$, $\{39_1, 14_1, x_{66}\}$, $\{0_2, 1_2, x_1\}$, $\{23_2, 24_2, x_2\}$, $\{2_2, 5_2, x_3\}$, $\{25_2, 28_2, x_4\}$, $\{3_2, 8_2, x_5\}$, $\{26_2, 31_2, x_6\}$, $\{4_2, 11_2, x_7\}$, $\{27_2, 34_2, x_8\}$, $\{6_2, 15_2, x_9\}$, $\{29_2, 38_2, x_{10}\}$, $\{10_2, 21_2, x_{11}\}$, $\{33_2, 44_2, x_{12}\}$, $\{19_2, 32_2, x_{13}\}$, $\{42_2, 9_2, x_{14}\}$, $\{7_2, 22_2, x_{15}\}$, $\{30_2, 45_2, x_{16}\}$, $\{18_2, 35_2, x_{17}\}$, $\{41_2, 12_2, x_{18}\}$, $\{17_2, 36_2, x_{19}\}$, $\{40_2, 13_2, x_{20}\}$, $\{16_2, 37_2, x_{21}\}$, $\{39_2, 14_2, x_{22}\}$.

Holey parallel classes of triples: Twenty-three of them are obtained by adding $0, 2, 4, \ldots, 44$ to the following triples mod$(46, -)$:

$\{4_0, 8_0, 10_0\}$, $\{27_0, 31_0, 33_0\}$, $\{3_0, 13_0, 21_0\}$, $\{26_0, 36_0, 44_0\}$, $\{19_0, 39_0, 5_0\}$, $\{42_0, 16_0, 28_0\}$, $\{6_1, 8_1, 12_1\}$, $\{29_1, 31_1, 35_1\}$, $\{20_1, 30_1, 38_1\}$, $\{43_1, 7_1, 15_1\}$, $\{13_1, 33_1, 45_1\}$, $\{36_1, 10_1, 22_1\}$, $\{11_2, 15_2, 17_2\}$, $\{34_2, 38_2, 40_2\}$, $\{21_2, 31_2, 39_2\}$, $\{44_2, 8_2, 16_2\}$, $\{0_2, 20_2, 32_2\}$, $\{23_2, 43_2, 9_2\}$, $\{12_0, 14_1, 13_2\}$, $\{35_0, 37_1, 36_2\}$, $\{15_0, 18_1, 19_2\}$, $\{38_0, 41_1, 42_2\}$, $\{11_0, 17_1, 14_2\}$, $\{34_0, 40_1, 37_2\}$, $\{20_0, 27_1, 30_2\}$, $\{43_0, 4_1, 7_2\}$, $\{9_0, 19_1, 24_2\}$, $\{32_0, 42_1, 1_2\}$, $\{17_0, 32_1, 41_2\}$, $\{40_0, 9_1, 18_2\}$, $\{14_0, 34_1, 45_2\}$, $\{37_0, 11_1, 22_2\}$, $\{22_0, 44_1, 35_2\}$, $\{45_0, 21_1, 12_2\}$, $\{0_0, 0_1, 16_1\}$, $\{23_0, 23_1, 39_1\}$, $\{1_0, 2_1, 24_1\}$, $\{24_0, 25_1, 1_1\}$, $\{3_1, 10_2, 26_2\}$, $\{26_1, 33_2, 3_2\}$, $\{5_1, 5_2, 27_2\}$, $\{28_1, 28_2, 4_2\}$, $\{2_2, 2_0, 18_0\}$, $\{25_2, 25_0, 41_0\}$, $\{6_2, 7_0, 29_0\}$, $\{29_2, 30_0, 6_0\}$.

Nine of them are obtained by developing each of the following triples mod$(46, -)$:

$\{0_0, 5_1, 7_2\}$, $\{0_0, 8_1, 12_2\}$, $\{0_0, 9_1, 5_2\}$, $\{0_0, 11_1, 6_2\}$, $\{0_0, 13_1, 21_2\}$, $\{0_0, 14_1, 8_2\}$, $\{0_0, 17_1, 9_2\}$, $\{0_0, 18_1, 11_2\}$, $\{0_0, 21_1, 33_2\}$. □

We thus obtain:

Theorem 3.15. *For all* $v \equiv 0 \pmod 6$ *with* $v \geqslant 198$, *there exists an* NKTS(v) *containing a* sub-NKTS(66).

For $v = 72$, no exceptions are left after applying Theorem 1.3 and Lemmas 3.3 and 3.6. Thus we have

Theorem 3.16. *For all* $v \equiv 0 \pmod 6$ *with* $v \geqslant 216$, *there exists an* NKTS(v) *containing a* sub-NKTS(72).

The foregoing results give us the second part of our main theorem:

Theorem 3.17. *For* $v = 18, 24, 30, 36, 42, 48, 54, 60, 66$ *or* 72, *and* $u \equiv 0 \pmod 6$, *there exists an* NKTS(u) *containing a* sub-NKTS(v) *if and only if* $u \geqslant 3v$.

Theorem 1.6 now follows from Theorems 3.7 and 3.17.

Appendix

$\{4\}$-GDD *of type* $6^8 21^1 24^1$: We construct a $\{3,4\}$-GDD of type $6^8 21^1$ with the property that its set of triples can be partitioned into 24 parallel classes. By adding 24 infinite points, we get the desired GDD.

Point set: $(Z(24)\times\{1,2\})\cup\{x_1,x_2,\dots,x_{19}\}\cup(\{a\}\times Z(2))$.

Groups: $\{x_1,x_2,\dots,x_{19},a_0,a_1\}$, with $\{0_1,8_1,16_1,0_2,8_2,16_2\}$ mod$(24,-)$.

Blocks of size four: Develop $\{0_1,12_1,1_2,13_2\}$, $\{0_1,1_1,3_1,7_1\}$, $\{0_2,1_2,3_2,7_2\}$ mod$(24,-)$.

Parallel classes of triples:

$\{12_1,23_1,a_0\}$, $\{13_2,0_2,a_1\}$, $\{17_1,22_1,7_1\}$, $\{17_2,22_2,7_2\}$, $\{0_1,2_2,x_1\}$, $\{1_1,4_2,x_2\}$, $\{2_1,6_2,x_3\}$, $\{3_1,8_2,x_4\}$, $\{4_1,10_2,x_5\}$, $\{5_1,12_2,x_6\}$, $\{6_1,15_2,x_7\}$, $\{9_1,19_2,x_8\}$, $\{10_1,21_2,x_9\}$, $\{11_1,23_2,x_{10}\}$, $\{13_1,3_2,x_{11}\}$, $\{14_1,5_2,x_{12}\}$, $\{8_1,1_2,x_{13}\}$, $\{15_1,9_2,x_{14}\}$, $\{16_1,11_2,x_{15}\}$, $\{18_1,14_2,x_{16}\}$, $\{19_1,16_2,x_{17}\}$, $\{20_1,18_2,x_{18}\}$, $\{21_1,20_2,x_{19}\}$ mod$(24,-)$.

The subscripts on a are to be developed mod 2.

$\{4\}$-GDD *of type* $18^4 30^1 36^1$: We construct a $\{3,4\}$-GDD of type $18^4 30^1$ with the property that its set of triples can be partitioned into 36 parallel classes. By adding 36 infinite points, we get the desired GDD.

Point set: $(Z(36)\times\{1,2\})\cup\{x_1,x_2,\dots,x_{20}\}\cup(\{a,b,c,d,e\}\times Z(2))$.

Groups: $\{x_1,x_2,\dots,x_{20},a_0,a_1,b_0,b_1,c_0,c_1,d_0,d_1,e_0,e_1\}$, with $\{\{0_j,2_j,4_j,\dots,34_j\},\{1_j,3_j,5_j,\dots,35_j\}:\ j=1,2\}$.

Blocks of size four: Develop $\{0_1,15_1,0_2,17_2\}$, $\{0_1,17_1,1_2,16_2\}$ mod$(36,-)$.

Parallel classes of triples:

$\{14_1,17_2,18_2\}$, $\{17_1,22_2,25_2\}$, $\{6_1,7_1,13_2\}$, $\{16_1,19_1,28_2\}$, $\{0_1,10_2,x_1\}$, $\{1_1,12_2,x_2\}$, $\{34_1,11_2,x_3\}$, $\{2_1,16_2,x_4\}$, $\{35_1,14_2,x_5\}$, $\{3_1,21_2,x_6\}$, $\{32_1,\ 15_2,\ x_7\}$, $\{4_1,26_2,x_8\}$, $\{33_1,20_2,x_9\}$, $\{5_1,29_2,x_{10}\}$, $\{9_1,34_2,x_{11}\}$, $\{10_1,0_2,x_{12}\}$, $\{18_1,9_2,x_{13}\}$, $\{11_1,3_2,x_{14}\}$, $\{8_1,1_2,x_{15}\}$, $\{12_1,6_2,x_{16}\}$, $\{29_1,24_2,x_{17}\}$, $\{23_1,19_2,x_{18}\}$, $\{30_1,27_2,x_{19}\}$, $\{25_1,23_2,x_{20}\}$, $\{21_1,26_1,a_0\}$, $\{20_1,27_1,b_0\}$, $\{22_1,31_1,c_0\}$, $\{13_1,24_1,d_0\}$, $\{15_1,28_1,e_0\}$, $\{35_2,4_2,a_1\}$, $\{31_2,2_2,b_1\}$, $\{32_2,5_2,c_1\}$, $\{33_2,8_2,d_1\}$, $\{30_2,7_2,e_1\}$ mod$(36,-)$.

The subscripts on a,b,c,d and e are to be developed mod 2.

$\{4\}$-GDD *of type* $6^8 18^1 24^1$: We construct a $\{3,4\}$-GDD of type $6^8 18^1$ with the property that its set of triples can be partitioned into 24 parallel classes. By adding 24 infinite points, we get the desired GDD.

Point set: $(Z(24)\times\{1,2\})\cup\{x_1,x_2,\dots,x_{15}\}\cup(\{a\}\times Z(3))$.

Groups: $\{x_1,x_2,\dots,x_{15},a_0,a_1,a_2\}$, with $\{0_j,4_j,8_j,12_j,16_j,20_j\}$ mod$(24,-)$ for $j=1,2$.

Blocks of size four: Develop $\{0_1,5_1,7_2,10_2\}$, $\{0_1,6_1,12_2,19_2\}$ mod$(24,-)$.

Parallel classes of triples:

$\{12_1,13_1,a_0\}$, $\{0_1,4_2,a_1\}$, $\{12_2,22_2,a_2\}$, $\{8_1,11_1,18_1\}$, $\{6_1,15_1,17_1\}$, $\{17_2,18_2,23_2\}$, $\{19_2,21_2,6_2\}$, $\{1_1,1_2,x_1\}$, $\{2_1,3_2,x_2\}$, $\{23_1,2_2,x_3\}$, $\{3_1,11_2,x_4\}$, $\{22_1,7_2,x_5\}$,

$\{4_1, 15_2, x_6\}$, $\{20_1, 10_2, x_7\}$, $\{5_1, 20_2, x_8\}$, $\{21_1, 13_2, x_9\}$, $\{7_1, 0_2, x_{10}\}$, $\{14_1, 8_2, x_{11}\}$, $\{9_1, 5_2, x_{12}\}$, $\{19_1, 16_2, x_{13}\}$, $\{16_1, 14_2, x_{14}\}$, $\{10_1, 9_2, x_{15}\}$ mod$(24, -)$.

The subscripts on a are to be developed mod 3.

$\{4\}$-GDD *of type* $6^9 12^1 27^1$: We construct a $\{3,4\}$-GDD of type $6^9 12^1$ with the property that its set of triples can be partitioned into 27 parallel classes. By adding 27 infinite points, we get the desired GDD.

Point set: $(Z(27) \times \{1,2\}) \cup \{x_1, x_2, \ldots, x_{12}\}$.

Groups: $\{x_1, x_2, \ldots, x_{12}\}$, with $\{0_1, 9_1, 18_1, 0_2, 9_2, 18_2\}$ mod$(27, -)$.

Blocks of size four: Develop $\{0_1, 13_1, 8_2, 19_2\}$ mod$(27, -)$.

Parallel classes of triples:

$\{9_1, 15_1, 19_1\}$, $\{11_1, 12_1, 14_1\}$, $\{5_1, 10_1, 17_1\}$, $\{24_2, 1_2, 7_2\}$, $\{23_2, 25_2, 26_2\}$,
$\{6_2, 13_2, 18_2\}$, $\{20_1, 22_2, 3_2\}$, $\{13_1, 16_2, 2_2\}$, $\{8_1, 16_1, 9_2\}$, $\{22_1, 6_1, 0_2\}$,
$\{0_1, 4_2, x_1\}$, $\{1_1, 8_2, x_2\}$, $\{26_1, 10_2, x_3\}$, $\{2_1, 14_2, x_4\}$, $\{25_1, 11_2, x_5\}$, $\{3_1, 17_2, x_6\}$,
$\{24_1, 12_2, x_7\}$, $\{4_1, 21_2, x_8\}$, $\{23_1, 19_2, x_9\}$, $\{18_1, 15_2, x_{10}\}$, $\{7_1, 5_2, x_{11}\}$,
$\{21_1, 20_2, x_{12}\}$ mod$(27, -)$.

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