Solvability and spectral properties of integral equations on the real line: I. Weighted spaces of continuous functions

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Abstract

We consider in this paper the solvability of linear integral equations on the real line, in operator form \((\lambda - K)\phi = \psi\), where \(\lambda \in \mathbb{C}\) and \(K\) is an integral operator. We impose conditions on the kernel, \(k\), of \(K\) which ensure that \(K\) is bounded as an operator on \(X := BC(\mathbb{R})\). Let \(X_a\) denote the weighted space \(X_a := \{\chi \in X : \chi(s) = O(|s|^{-a})\) as \(|s| \to \infty\}\). Our first result is that if, additionally, \(|k(s, t)| \leq \kappa(s - t)\), with \(\kappa \in L^1(\mathbb{R})\) and \(\kappa(s) = O(|s|^{-b})\) as \(|s| \to \infty\), for some \(b > 1\), then the spectrum of \(K\) is the same on \(X_a\) as on \(X\), for \(0 < a \leq b\). Using this result we then establish conditions on families of operators, \(\{K_k : k \in W\}\), which ensure that, if \(\lambda \neq 0\) and \(\lambda \phi = K_k \phi\) has only the trivial solution in \(X\), for all \(k \in W\), then, for \(0 \leq a \leq b\), \((\lambda - K)\phi = \psi\) has exactly one solution \(\phi \in X_a\) for every \(k \in W\) and \(\psi \in X_a\). These conditions ensure further that \((\lambda - K)^{-1} : X_a \to X_a\) is bounded uniformly in \(k \in W\), for \(0 \leq a \leq b\). As a particular application we consider the case when the kernel takes the form \(k(s, t) = \kappa(s - t)z(t)\), with \(\kappa \in L^1(\mathbb{R})\), \(z \in L^\infty(\mathbb{R})\), and \(\kappa(s) = O(|s|^{-b})\) as \(|s| \to \infty\), for some \(b > 1\). As an example where kernels of this latter form occur we discuss a boundary integral equation formulation of an impedance

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1. Introduction

We consider in this paper integral equations of the form
\[
\lambda \phi(s) - \int_{-\infty}^{+\infty} k(s, t) \phi(t) \, dt = \psi(s), \quad s \in \mathbb{R},
\]
where \( \lambda \in \mathbb{C} \), the functions \( k : \mathbb{R}^2 \to \mathbb{C} \) and \( \psi \) are assumed known, and \( \phi \) is the solution to be determined. Define the integral operator \( K \) by
\[
K \psi(s) = \int_{-\infty}^{+\infty} k(s, t) \psi(t) \, dt, \quad s \in \mathbb{R}.
\]
Then (1.1) can be abbreviated in operator notation as
\[
(\lambda - K) \phi = \psi.
\]

We assume throughout that \( k \) is (Lebesgue) measurable and that the following assumptions on \( k \) hold:

(A) \( \sup_{s \in \mathbb{R}} \int_{-\infty}^{+\infty} |k(s, t)| \, dt < \infty. \)

(B) For all \( s \in \mathbb{R} \),
\[
\int_{-\infty}^{+\infty} |k(s, t) - k(s', t)| \, dt \to 0,
\]
as \( s' \to s \).

Assumption (A) ensures that \( K \) is a bounded operator on \( L^\infty(\mathbb{R}) \), with norm
\[
\|K\| = \sup_{s \in \mathbb{R}} \int_{-\infty}^{+\infty} |k(s, t)| \, dt < \infty.
\]
Assumptions (A) and (B) together ensure that \( K : L^\infty(\mathbb{R}) \to BC(\mathbb{R}) \) and is bounded, where \( BC(\mathbb{R}) \subset L^\infty(\mathbb{R}) \) is the subspace of functions bounded and continuous on \( \mathbb{R} \).
For most of the paper we will make the following assumption which is stronger than (A):

\[(A') \text{ For some } \kappa \in L^1(\mathbb{R}),\]

\[|k(s, t)| \leq |\kappa(s - t)|, \quad s, t \in \mathbb{R}.\]

Assumption \((A')\) implies that, by Young’s inequality, \(K : L^p(\mathbb{R}) \to L^p(\mathbb{R})\) and is bounded, for \(1 \leq p \leq \infty\), with norm \(\|K\| \leq \|\kappa\|_1\).

Given a Banach space \(Y\) and \(A \in B(Y)\), the set of bounded linear operators on \(Y\), let \(\|A\|_Y\) denote the norm of \(A\), set \(\mathcal{R}_Y(A) := \{\lambda \in \mathbb{C} : (\lambda - A)^{-1} \in B(Y)\}\), and let \(\Sigma_Y(A) := \mathbb{C} \setminus \mathcal{R}_Y(A)\) denote the spectrum of \(A \in B(Y)\). Let \(\Sigma_Y^p(A) \subset \Sigma_Y(A)\) denote the point spectrum, the set of eigenvalues of \(A\), in other words, the set of \(\lambda\) for which \((\lambda - A) : Y \to Y\) is not injective. Let \(\Phi(Y) \subset B(Y)\) denote the set of Fredholm operators on \(Y\). (We remind the reader that \(A \in B(Y)\) is said to be Fredholm if \(A(Y)\) is closed and both the kernel of \(A\), \(\ker A := \{\phi \in Y : A\phi = 0\}\), and the cokernel, \(\coker A := Y/A(Y)\), have finite dimension—see, e.g., [18]. If \(A\) is Fredholm the index of \(A\) is the integer \(\text{ind } A := \dim \ker A - \dim \coker A\).)

Let \(\Sigma_Y^e(A) \subset \Sigma_Y(A)\) denote the essential spectrum of \(A\), those to of \(\lambda \in \mathbb{C}\) for which \((\lambda - A)/w \in \Phi(Y)\). In terms of these definitions and supposing that \(K \in B(Y)\), it follows from the Banach theorem that (1.3) has exactly one solution \(\phi \in Y\) for every \(\psi \in Y\) iff \(\lambda \notin \Sigma_Y(K)\).

In the remainder of the paper, let us abbreviate \(L^p(\mathbb{R})\) by \(L^p\) and \(BC(\mathbb{R})\) by \(X\). For \(a \geq 0\) let \(X_a\) denote the weighted space of continuous functions defined by

\[X_a := \{\phi \in X : \phi(s) = O(|s|^{-a}), \quad |s| \to \infty\}.\]  

(1.4)

\(X_a\) is a Banach space under the norm \(\|\cdot\|_{X_a}\), defined by \(\|\phi\|_{X_a} = \|\phi w_a\|_{\infty}\), where \(w_a(s) = (1 + |s|)^a\) and it is shown in [6] that, if \(k\) satisfies \((A')\) and \((B)\) and

\[\kappa(s) = O(|s|^{-b}), \quad |s| \to \infty,\]  

(1.5)

for some \(b > 1\), then \(K \in B(X_a)\) for \(0 \leq a \leq b\) and \(\Sigma_{X_a}(K) \subset \{0\} \cup \Sigma_X(K)\), for \(0 \leq a < b\).

The main objective of Section 2 is to establish the deeper and sharper result that these same assumptions ensure that

\[\Sigma_{X_a}(K) = \Sigma_X(K), \quad 0 \leq a \leq b.\]  

(1.6)

For the special case \(k(s, t) = \kappa(s - t), \quad s, t \in \mathbb{R},\) with \(\kappa \in L^1\), and for the related Wiener–Hopf integral equation,

\[\lambda \phi(s) - \int_{0}^{\infty} \kappa(s - t)\phi(t)dt = \psi(s), \quad s \in \mathbb{R},\]  

(1.7)
and certain other variants, this relationship is already known via quite different and more specific arguments which characterise the spectrum explicitly [20]. But, even for these specific cases, the arguments in [20] require the (in most applications more stringent) assumption that

\[ \int_{-\infty}^{+\infty} |\kappa(t)| \left(1 + |t|\right)^b \, dt < \infty \] (1.8)

to obtain (1.6). In particular, in the case that, for some constants \(c\) and \(q\), \(|\kappa(t)| \sim c|t|^{-q}\) as \(|t| \to \infty\), our arguments require \(q \geq b + 1\) while \((1.8)\) requires \(q > b + 1\). (If \(b > 1\), our condition \(q \geq b\) for (1.6) to hold is optimal, since \(K \in B(X_b)\) only if \(q \geq b\) in the case that \(|\kappa(t)| \sim c|t|^{-q}\) as \(|t| \to \infty\).)

In Section 3 we start by considering the relationship between \(\Sigma_X(K)\) and \(\Sigma^p_X(K)\). It is shown already in [13] that if, for some \(r \in \mathbb{R}\) \(\{0\}\),

\[ k(s + r, t + r) = k(s, t), \quad s, t \in \mathbb{R}, \] (1.9)

and (A) and (B) hold, then

\[ \Sigma_X(K) = \Sigma^p_X(K) \cup \{0\}. \] (1.10)

It is easy to see that (1.9) is equivalent to the operator equation

\[ T_r K = K T_r, \] (1.11)

where \(T_r\) is the translation operator defined by

\[ T_r \psi(s) = \psi(s - r), \quad s \in \mathbb{R}. \] (1.12)

Obviously (1.9) and (1.11) hold for all \(r \in \mathbb{R}\) in the case when \(k(s, t) = \kappa(s - t)\).

Let us make it clear at this point, however, that (1.10) certainly does not hold as it stands for all kernels satisfying (A) and (B). As an example, (1.10) does not hold in general for the Wiener–Hopf case (1.7), in which

\[ k(s, t) = \begin{cases} \kappa(s - t), & s \in \mathbb{R}, \ t \geq 0, \\ 0, & s \in \mathbb{R}, \ t < 0, \end{cases} \] (1.13)

with \(\kappa \in L^1\) [18,20]. However, the following result, which may be viewed as a generalisation of (1.10), is shown in [15]. Denote the integral operator \(K\) by \(K_k\) to indicate its dependence on its kernel \(k\), and let \(W\) denote a family of kernels which satisfy (A) and (B) uniformly and have the translation invariance property (cf. (1.11)) that

\[ \{T_r K_k : k \in W\} = \{K_k T_r : k \in W\}, \] (1.14)

for some \(r \in \mathbb{R} \setminus \{0\}\). Then, provided \(W\) also has certain compactness properties (explained in Section 3—see Theorem 3.1 below) with respect to the \(\sigma\)-topology proposed in [15], it holds that
\[ \bigcup_{k \in W} \Sigma_X(K_k) = \{0\} \cup \bigcup_{k \in W} \Sigma_X^p(K_k) =: \Sigma_X^p(W). \]

Moreover, for \( \lambda \not\in \Sigma_X^p(W) \), it holds that
\[ \sup_{k \in W} \| (\lambda - K_k)^{-1} \|_X < \infty. \]

The main conclusion of Section 3, extending the results of Section 2 and [15], is to show that, if \( W \) has the translation invariance and compactness properties alluded to above and, in addition, every \( k \in W \) satisfies \( (A') \) with the same function \( \kappa \), and (1.5) holds for some \( b > 1 \), then
\[ \bigcup_{k \in W} \Sigma_{X_a}(K_k) = \Sigma_X^p(W), \quad 0 \leq a \leq b. \]

Moreover, for \( \lambda \not\in \Sigma_X^p(W) \), we obtain that
\[ \sup_{k \in W} \| (\lambda - K_k)^{-1} \|_{X_a} < \infty, \quad 0 \leq a \leq b. \]

In Section 4, as an application of the results of Section 3, we consider kernels of the form
\[ k(s, t) = \kappa(s - t)z(t), \]
for some \( \kappa \in L^1 \), \( z \in L^\infty \), so that \( K = \mathcal{K}M_z \), where \( \mathcal{K} \) is the convolution integral operator with kernel \( \kappa(s - t) \) and \( M_z \) is the operation of multiplication by \( z \). In the case that \( \kappa(s) = O(|s|^{-b}) \) as \( |s| \to \infty \), for some \( b > 1 \), and \( V \subset L^\infty \) is weak\(^*\) sequentially compact and has certain translation invariance properties, specified in Theorem 4.4 below, we show that
\[ \bigcup_{z \in V} \Sigma_{X_a}(\mathcal{K}M_z) = \{0\} \cup \bigcup_{z \in V} \Sigma_X^p(\mathcal{K}M_z), \]
for \( 0 \leq a \leq b \). As an interesting example, (1.17) holds for the case when \( V = \{z \in L^\infty : \text{ess range } z \subset Q\} \) with \( Q \subset \mathbb{C} \) compact and convex. We describe a problem of time harmonic acoustic scattering in the half-plane, with inhomogeneous impedance boundary condition, in which this case arises when boundary integral equation methods are employed.

In part II of this study [4] we utilise the denseness of \( X_a \) in \( L^1 \) for \( a > 1 \), and combine the results of Section 3 with standard properties of adjoint operators to study the solvability of (1.1) in \( L^p \) spaces. In particular, analogously to (1.15), we show that, under similar conditions on the set \( W \),
\[ \bigcup_{k \in W} \Sigma_{L^p}(K_k) \subseteq \Sigma_X^p(W), \]
for \( 1 \leq p \leq \infty \), with equality for \( p = 1, \infty \). We also, in [4], consider further applications of the results of Section 3 to boundary integral equations arising in problems of scattering by unbounded surfaces.
There are many possibilities for extension of the results discussed in this paper. In particular, we point out that there is no problem, in principle, in extending the results of Sections 2–4 to multidimensional integral equations and to systems of integral equations. For kernels of the form (1.16) extensions of Theorem 3.1 to the multidimensional case and to systems of equations are carried out in [13,22]. A partial extension of the results of Section 2 to the multidimensional case is given in [11], and of the results of Sections 2 and 3 to systems of equations in the multidimensional case in [16,17].

2. Solvability in weighted spaces of continuous functions

We begin by reviewing properties of integral operators on the real line that we will need for our arguments. We are concerned in this section with properties of the integral operator $K$ as an operator on $X = BC(\mathbb{R})$ and on the weighted spaces $X_a \subset X$ introduced above. Given a sequence $(\phi_n) \subset X$ and $\phi \in X$ we will write $\phi_n \to \phi$ if $\|\phi_n - \phi\|_\infty \to 0$. We shall write $\phi_n \overset{s}{\to} \phi$ and say that $(\phi_n)$ converges strictly to $\phi$ if $(\phi_n)$ is bounded and $\phi_n(s) \to \phi(s)$ uniformly on compact subsets of $\mathbb{R}$. (This is convergence in the strict topology of Buck [5] .) Following [1] we will say that $H \in B(X)$ is $s$-continuous if

$$
\phi_n \overset{s}{\to} \phi \implies H\phi_n \overset{s}{\to} H\phi.
$$

We will say that $H$ is $sn$-continuous if

$$
\phi_n \overset{s}{\to} \phi \implies H\phi_n \to H\phi,
$$

and that $H$ is $s$-sequentially compact if $(H\phi_n)$ has a strictly convergent subsequence whenever $(\phi_n) \subset X$ is bounded. Lemma 2.2 below, expressed in terms of these definitions, is important for our arguments. In the proof of this lemma and throughout the remainder of the paper we use, without further comment, certain basic properties of Fredholm operators summarised in the next lemma. For a proof of this lemma see, e.g., [18].

**Lemma 2.1.** Suppose $Y_1$, $Y_2$, and $Y_3$ are Banach spaces, $B : Y_1 \to Y_2$, $C : Y_2 \to Y_3$, $A : Y_2 \to Y_3$, and $D : Y_1 \to Y_1$ are bounded linear operators, and that $A$ and $B$ are Fredholm and $C$ is compact. Then:

1. $AB$ and $A + C$ are Fredholm and $\text{ind}(AB) = \text{ind} A + \text{ind} B$, $\text{ind}(A + C) = \text{ind} A$.
2. $D$ is Fredholm iff there exists $E \in B(Y_1)$ (called a regulariser of $D$) such that $ED - 1$ and $DE - 1$ are compact operators on $Y_1$.
3. The essential spectrum of $D$, $\Sigma_{Y_1}^e(D)$, is closed and bounded, and $\text{ind}(\lambda - D)$ is constant on connected components of $\mathbb{C} \setminus \Sigma_{Y_1}^e(D)$. 
Lemma 2.2. If $H, L \in B(X)$, $H$ is s-sequentially compact, and $L$ is sn-continuous, then $LH$ is compact. If also $L$ is s-sequentially compact and $\lambda \neq 0$, then $\lambda + L$ is Fredholm of index zero. Moreover,

$$\lambda - H - L \in \Phi(X) \iff \lambda - H \in \Phi(X),$$

and, if $\lambda - H - L$ and $\lambda - H$ are both Fredholm, their indices are the same.

Proof. Let $(\phi_n)$ be bounded. Then $(H\phi_n)$ has a strictly convergent subsequence, $(H\phi_{nm})$. Since $L$ is sn-continuous, $(LH\phi_{nm})$ is convergent. Thus $LH$ is compact. Therefore, if also $L$ is s-sequentially compact, then $L^2$ is compact and so $\lambda^{-1} - \lambda^{-2}L$ is a regulariser for $\lambda + L$. Thus $\lambda + L$ is Fredholm for all $\lambda \neq 0$. But, for $|\lambda| > \|L\|_X$, $(\lambda + L)^{-1} \in B(X)$ and so $\lambda + L$ has index zero. It follows that $\lambda + L$ has index zero for all $\lambda \neq 0$.

If $\lambda - H - L \in \Phi(X)$ then, since also $\lambda + L \in \Phi(X)$, it follows that

$$\lambda^{-1}(\lambda + L)(\lambda - H - L) = \lambda - H - \lambda^{-1}(H + L)$$

is Fredholm. Since $L(H + L)$ is compact it follows further that $\lambda - H \in \Phi(X)$ and

$$\text{ind}(\lambda - H) = \text{ind}(\lambda - H - \lambda^{-1}(H + L)) = \text{ind}(\lambda + L) + \text{ind}(\lambda - H - L) = \text{ind}(\lambda - H - L).$$

An identical argument establishes the reverse implication $\lambda - H \in \Phi(X) \Rightarrow \lambda - H - L \in \Phi(X)$. □

The following simple property of normally solvable operators is another key ingredient in our arguments. For completeness, we sketch the short proof.

Lemma 2.3. Suppose $Y_1$ and $Y_2$ are Banach spaces and that $A : Y_1 \to Y_2$ is a bounded linear operator. If $A(Y_1)$ is closed then, for every bounded sequence $(\psi_n) \subseteq A(Y_1)$, there exists a bounded sequence $(\phi_n) \subseteq Y_1$ with $A\phi_n = \psi_n$ for each $n$.

Proof. If $Y_1$ and $Y_2$ are Banach spaces and $A(Y_1)$ is closed then $A(Y_1)$ and the quotient space $Y_1 / \ker A$ are Banach spaces ($Y_1 / \ker A$ with the norm $\|\tilde{\phi}\|_{Y_1 / \ker A} := \inf_{\phi \in \tilde{\phi}} \|\phi\|_{Y_1}$) [21]. Thus the mapping $\tilde{A} : Y_1 / \ker A \to A(Y_1)$, defined by $\tilde{A}\phi = A\phi$, $\phi \in \tilde{\phi} \subseteq Y_1 / \ker A$, is a bijection between Banach spaces, and so is boundedly invertible by the Banach theorem. Thus, there exists $C > 0$ such that, for every bounded sequence $(\psi_n) \subseteq A(Y_1)$ there exists a sequence $(\tilde{\phi}_n) \subseteq Y_1 / \ker A$ with $\|\tilde{\phi}_n\|_{Y_1 / \ker A} \leq C\|\psi_n\|_{Y_2}$ and $\tilde{A}\tilde{\phi}_n = \psi_n$. Clearly we may choose $\phi_n \in \tilde{\phi}_n$ for each $n$ such that $\|\phi_n\|_{Y_1} \leq 2\|\tilde{\phi}_n\|_{Y_1 / \ker A} \leq 2C\|\psi_n\|_{Y_2}$, and it then holds that $A\phi_n = \tilde{A}\phi_n = \psi_n$. □
Conditions (A) and (B) ensure that the integral operator $K \in B(X)$ with
$$\|K\|_X = \sup_{s \in \mathbb{R}} \int_{-\infty}^{+\infty} |k(s, t)| \, dt,$$
and also that $K$ is $s$-continuous and $s$-sequentially compact [2]. Conditions (A) and (B) do not imply that $K$ is compact (for example, if $k(s, t) = \kappa(s - t)$, $s, t \in \mathbb{R}$, with $\kappa \in L^1$, then $K$ has a continuous spectrum [20]). But $K$ is compact if $k$ satisfies (A), (B), and the following assumption [2]:

(C)
$$\int_{-\infty}^{+\infty} |k(s, t)| \, dt \to 0 \quad \text{as } |s| \to \infty.$$

Alternatively, Anselone and Sloan [3] show that $K$ is compact if $k$ is uniformly continuous and satisfies

(D)
$$\sup_{s \in \mathbb{R}} \int_{\mathbb{R} \setminus [-A, A]} |k(s, t)| \, dt \to 0 \quad \text{as } A \to +\infty.$$

Conversely [8], if the integral operator $K$ is compact, then $k$ satisfies (A), (B), and (D).

If $K$ is compact then (since $k$ satisfies (A) and (B) so that it is also $s$-sequentially compact) it is easy to see that $K$ must be $sn$-continuous. More generally (see [8] or Lemma 3.4 below),

(A), (B), (D) $\Rightarrow$ $K$ sn-continuous.

Conditions (A), (B), and (D) are not enough to ensure that $K$ is compact: the counterexample $k(s, t) = e^{ist} \chi(t)$, where $\chi(t) = 1$, $0 < t < 1$, $= 0$, otherwise, is given in [3]. But from the above observations and Lemma 2.2 we deduce that

(A), (B), (D) $\Rightarrow$ $K^2$ compact on $X$.

Moreover, we have the following obvious corollary of Lemma 2.2, in which $L$ denotes the integral operator defined by (1.2) with $K, k$ replaced by $L, l$.

**Corollary 2.4.** If $k$ satisfies (A) and (B) and $l$ satisfies (A), (B), and (D), then $LK$ is a compact operator on $X$. If also $\lambda \neq 0$ then $\lambda + L \in \Phi(X)$ with index zero and

$$\lambda - K - L \in \Phi(X) \iff \lambda - K \in \Phi(X).$$

(2.1)

If the operators in (2.1) are both Fredholm then their indices are the same.
Proof. The assumptions imply that $K$ and $L$ are $s$-sequentially compact and $L$ is $sn$-continuous. The result then follows immediately from Lemma 2.2. 

The case $\lambda = 0$, omitted from the equivalence (2.1), is dealt with in the following result.

Lemma 2.5. If $k$ satisfies (A) and (B) then either $K(X) \neq \overline{K(X)}$ or $X/K(X)$ has infinite dimension. Thus $K \notin \Phi(X)$ and $K(X) \neq X$.

Proof. Let $\phi_n(s) = e^{ins}$, $s \in \mathbb{R}$, for $n \in \mathbb{N}$. Suppose that $K(X) = \overline{K(X)}$ and $X/K(X)$ has finite dimension. Then $X = K(X) \oplus M$, with $\dim M = \dim(X/K(X))$. Since $\dim M < \infty$ there exists a projection $P \in B(X)$ with $P(X) = M$. We shall show that this implies that $S := \{\phi_1, \phi_2, \ldots\}$ is equicontinuous at zero which, manifestly, it is not.

Let $\phi_n' := (1 - P)\phi_n \in K(X)$, $\phi_n'' := P\phi_n \in M$, $n \in \mathbb{N}$. Then $(\phi_n')$ and $(\phi_n'')$ are bounded and thus $(\phi_n'')$ is equicontinuous at 0 (as at every point in $\mathbb{R}$) since $M \subset X$ has finite dimension. Further, since $K(X)$ is closed and $(\phi_n'') \subset K(X)$ is bounded, by Lemma 2.3 there exists a bounded sequence $(\psi_n) \subset X$ such that $K\psi_n = \phi_n'$, $n \in \mathbb{N}$. Thus, for $n \in \mathbb{N}$, $s \in \mathbb{R}$,

$$|\phi_n(s) - \phi_n(0)| \leq \sup_n \|\psi_n\|_{\infty} \int_{-\infty}^{+\infty} |k(s, t) - k(0, t)| dt + \sup_n |\phi_n''(s) - \phi_n''(0)|.$$

Since $k$ satisfies (B) and $(\phi_n'')$ is equicontinuous, it follows that $\sup_n |e^{ins} - 1| = \sup_n |\phi_n(s) - \phi_n(0)| \to 0$ as $s \to 0$, a contradiction. 

As a consequence of Corollary 2.4 and Lemma 2.5, we have

Corollary 2.6. If $k$ satisfies (A) and (B) and $l$ satisfies (A), (B), and (D), then

$$0 \in \Sigma_+^e(K) = \Sigma_+^e(K + L).$$

We turn now to a consideration of the solvability of (1.1) in the weighted space $X_a$, defined by (1.4). Let $K^{(a)}$ denote the integral operator defined by

$$K^{(a)} = M_{w_a} K M_{w_a}^{-1},$$

where, for $z \in L^\infty \cup C(\mathbb{R})$, $M_z$ is the operation of multiplication by $z$. $K^{(a)}$ has kernel

$$k^{(a)}(s, t) := \left(w_a(s)/w_a(t)\right)k(s, t).$$
Since $M_{w_a}:X_a \to X$ is an isometric isomorphism with inverse $M_{w_a}^{-1}$, it is easy to see that

$$K^{(a)} \in B(X) \iff K \in B(X_a),$$

that

$$K^{(a)} \in \Phi(X) \iff K \in \Phi(X_a),$$

and that, if they are both Fredholm, their indices are the same. Further,

$$\lambda - K^{(a)} \text{ injective on } X \iff \lambda - K \text{ injective on } X_a$$

$$\iff \lambda - K \text{ injective on } X$$  \hspace{1cm} (2.4)

and

$$((\lambda - K^{(a)})^{-1} \in B(X) \iff (\lambda - K)^{-1} \in B(X_a)).$$  \hspace{1cm} (2.5)

In fact, clearly,

$$((\lambda - K^{(a)})^{-1} = M_{w_a}(\lambda - K)^{-1}M_{w_a}^{-1},$$

so that, if $K, (\lambda - K)^{-1} \in B(X_a), K^{(a)}, (\lambda - K^{(a)})^{-1} \in B(X_a)$, then

$$\|K\|_{X_a} = \|K^{(a)}\|_X, \quad \|(\lambda - K)^{-1}\|_{X_a} = \|(\lambda - K^{(a)})^{-1}\|_X.$$

In [6] it is shown that, if $\kappa \in L^1$ and $\kappa(s) = O(|s|^{-b})$ as $|s| \to \infty$, for some $b > 1$, then

$$\sup_{s \in \mathbb{R}} \int_{-\infty}^{+\infty} |\kappa(s-t)| \frac{w_a(s)}{w_a(t)} \, dt < \infty, \quad 0 \leq a \leq b.$$  \hspace{1cm} (2.6)

As a consequence, the following result is obtained in [6]. We rewrite the short proof to make it clear, for later use in Section 3 where we consider families of integral operators, exactly how the argument depends on $k$.

**Theorem 2.7.** If $k$ satisfies (A’) and (B) and (1.5) holds with $b > 1$, then, for $0 \leq a \leq b$, $k^{(a)}$ satisfies (A) and (B) so that $K \in B(X_a)$ and $K^{(a)} \in B(X)$.

**Proof.** Clearly,

$$\sup_{s \in \mathbb{R}} \int_{-\infty}^{+\infty} |k^{(a)}(s,t)| \, dt \leq \sup_{s \in \mathbb{R}} \int_{-\infty}^{+\infty} |\kappa(s-t)| \frac{w_a(s)}{w_a(t)} \, dt < \infty,$$

for $0 \leq a \leq b$, so $k^{(a)}$ satisfies (A). For $s, s' \in \mathbb{R}$,
\[
\int_{-\infty}^{+\infty} |k^{(a)}(s, t) - k^{(a)}(s', t)| \, dt \\
\leq w_a(s) \int_{-\infty}^{+\infty} |k(s, t) - k(s', t)| \, dt + \left| w_a(s') - w_a(s) \right| \|\kappa\|_1 \to 0
\]
as \( s' \to s \), so that \( k^{(a)} \) satisfies (B). \( \square \)

The deeper results we will obtain derive from the following theorem. This is established in [11], but we give here a much shorter proof which, again, makes it clear how the argument depends on \( k \).

**Theorem 2.8.** If \( k \) satisfies (A’) and (1.5) holds with \( b > 1 \), then \( k - k^{(a)} \) satisfies assumption (D) for \( 0 \leq a \leq b \).

**Proof.** For \( a = 0 \) there is nothing to prove. For \( a > 0 \), clearly (A’) implies that

\[
\sup_{s \in \mathbb{R}} \int_{\mathbb{R}\setminus[-A,A]} |k(s, t) - k^{(a)}(s, t)| \, dt \leq \sup_{s \in \mathbb{R}} f_A(s),
\]

where

\[
f_A(s) := \int_{\mathbb{R}\setminus[-A,A]} |\kappa(s - t)| \left| 1 - w_a(s)/w_a(t) \right| \, dt.
\]

We will complete the proof by showing that \( \sup_{s \in \mathbb{R}} f_A(s) \to 0 \) as \( A \to \infty \).

Now

\[
f_A(s) \leq \int_{\mathbb{R}\setminus[-A,A]} |\kappa(s - t)| \, dt, \quad |s| \leq A,
\]

since \( 0 < w_a(s)/w_a(t) \leq 1 \) for \( |s| \leq |t| \). Thus

\[
\sup_{|s| \leq A/2} f_A(s) \leq \int_{\mathbb{R}\setminus[-A/2,A/2]} |\kappa(u)| \, du \to 0
\]
as \( A \to \infty \).

Let

\[
c_a(s) := \sup_{|s-t| \leq |s|^{1/2}} \left| 1 - w_a(s)/w_a(t) \right|.
\]

It is easy to see that \( c_a(s) \to 0 \) as \( |s| \to \infty \). Further,

\[
f_A(s) \leq c_a(s) \|\kappa\|_1 + \int_{S_A,s} |\kappa(s - t)| \left| 1 - w_a(s)/w_a(t) \right| \, dt,
\]

as \( s' \to s \), so that \( k^{(a)} \) satisfies (B). \( \square \)
where \( S_{A,s} := \mathbb{R} \setminus ([-A, A] \cup [s - |s|^{1/2}, s + |s|^{1/2}]) \). Since \( \kappa(s) = O(|s|^{-b}) \) as \( |s| \to \infty \), there exist \( M, C > 0 \) such that
\[
|\kappa(s)| \leq M (1 + |s|)^{-b}, \quad |s| \geq C.
\]

Also
\[
0 < \frac{w_a(s)}{w_a(t)} = \left( 1 + \frac{|s| - |t|}{1 + |t|} \right)^a \leq 2^a \left( 1 + \left( \frac{|s - t|}{1 + |t|} \right)^a \right). \quad (2.7)
\]

Thus, for \( |s|^{1/2} \geq C \), and choosing \( \epsilon > 0 \) with \( \epsilon < \min(b - 1, a) \),
\[
\int_{S_{A,s}} |\kappa(s - t)| \left| 1 - \frac{w_a(s)}{w_a(t)} \right| dt \leq 2^a M (I_1 + I_2),
\]
where
\[
I_1 = \int_{S_{A,s}} (1 + |s - t|)^{-b} dt \leq 2 \int_{|s|^{1/2}} (1 + u)^{-b} du
\]
and
\[
I_2 = \int_{S_{A,s}} (1 + |s - t|)^{a-b} (1 + |t|)^{-a} dt \leq (1 + A)^{-\epsilon} F_{b-a,a-\epsilon},
\]
with
\[
F_{b-a,a-\epsilon} := \sup_{s \in \mathbb{R}} \int_{-\infty}^{+\infty} (1 + |s - t|)^{a-b} (1 + |t|)^{\epsilon-a} dt,
\]
which is finite since \( b - a \geq 0, a - \epsilon \geq 0 \), and \( (b - a) + (a - \epsilon) > 1 \) (see [6, Lemma 3]). So, for all \( A \) sufficiently large,
\[
\sup_{|s| \geq A/2} f_A(s) \leq \sup_{|s| \geq A/2} c_a(s) \| \kappa \|_1 + 2^a M \left( 2 \int_{(A/2)^{1/2}}^{\infty} (1 + u)^{-b} du + (1 + A)^{-\epsilon} F_{b-a,a-\epsilon} \right)
\]
\[
\to 0
\]
as \( A \to \infty \). \( \square \)

**Remark 2.9.** Together, Theorems 2.7 and 2.8 show that, if \( k \) satisfies (A') and (B) and (1.5) holds with \( b > 1 \), then \( k - k^{(a)} \) satisfies assumptions (A), (B), and (D), for \( 0 \leq a \leq b \). These assumptions, as we have seen above, ensure that \((K - K^{(a)})^2\)
is compact. In fact, for $0 \leqslant a < b$ it can be shown [6,11] that $k - k^{(a)}$ also satisfies (C) so that $K - K^{(a)}$ itself is compact. It is not necessarily the case without further assumptions on $k$ that $K - K^{(a)}$ is compact when $a = b$ as the examples and discussion in [8,11] demonstrate.

Combining Corollaries 2.4 and 2.6 with Theorems 2.7 and 2.8 we have the following result.

**Theorem 2.10.** If $k$ satisfies (A′) and (B) and (1.5) holds with $b > 1$, then, for $0 \leqslant a \leqslant b$, $\lambda \in \mathbb{C}$,

$$(\lambda - K) \in \Phi(X) \iff (\lambda - K^{(a)}) \in \Phi(X) \iff (\lambda - K) \in \Phi(X_a)$$

and, if these operators are all Fredholm, their indices are the same. Further,

$$0 \in \Sigma^e_X(K) = \Sigma^e_X(K^{(a)}) = \Sigma^e_{X_a}(K).$$

**Proof.** We have observed already that $(\lambda - K^{(a)}) \in \Phi(X) \iff (\lambda - K) \in \Phi(X_a)$, and that $\lambda - K$ and $\lambda - K^{(a)}$ have the same index if they are both Fredholm. From Corollary 2.4 and Theorems 2.7 and 2.8 it follows that, for $\lambda \neq 0$, $(\lambda - K) \in \Phi(X) \iff (\lambda - K^{(a)}) \in \Phi(X)$ and that their indices are the same if they are both Fredholm. From Corollary 2.6 and Theorem 2.7 it follows that $K, K^{(a)} \notin \Phi(X)$.  □

Our final result of this section is that, under the same assumptions, the spectra of $K$ and $K^{(a)}$ coincide. It depends, in part, upon the following useful result.

**Theorem 2.11.** Suppose $H \in B(X)$ is $s$-continuous and $s$-sequentially compact and $\lambda \neq 0$. If $(\lambda - H)(X)$ is closed and contains all compactly supported continuous functions, then $(\lambda - H)(X) = X$.

**Proof.** Let $\phi \in X$ and let $(\phi_n) \subset X$ be a sequence of compactly supported functions with $\phi_n(s) = \phi(s)$, $|s| \leqslant n$, and $\|\phi_n\|_\infty \leqslant 2\|\phi\|_\infty$. Then $\phi_n \overset{s}{\rightarrow} \phi$ and $(\phi_n) \subset (\lambda - H)(X)$. Since $(\phi_n)$ is bounded and $(\lambda - H)(X)$ is closed, it follows by Lemma 2.3 that there exists a bounded sequence $(\psi_n) \subset X$ such that $\lambda \psi_n - H \psi_n = \phi_n, n \in \mathbb{N}$. Since $H$ is $s$-sequentially compact it follows that $(H \psi_n)$ has a strictly convergent subsequence, denoted again by $(H \psi_n)$, such that, for some $\psi \in X$, $\lambda^{-1}(H \psi_n + \phi_n) \overset{s}{\rightarrow} \psi$. Thus

$$\psi_n = \lambda^{-1}(H \psi_n + \phi_n) \overset{s}{\rightarrow} \psi$$

and, since $H$ is $s$-continuous,

$$\lambda^{-1}(H \psi_n + \phi_n) \overset{s}{\rightarrow} \lambda^{-1}(H \psi + \phi).$$

Thus $\lambda \psi = H \psi + \phi$, i.e., $\phi \in (\lambda - H)(X)$.  □
Theorem 2.12. If \( k \) satisfies (A’) and (B) and (1.5) holds with \( b > 1 \), then, for \( 0 \leq a \leq b, \lambda \in \mathbb{C} \),

\[
(\lambda - K)^{-1} \in B(X) \iff (\lambda - K^{(a)})^{-1} \in B(X) \iff (\lambda - K)^{-1} \in B(X_a),
\]

so that

\[
\Sigma_X(K) = \Sigma_X(K^{(a)}) = \Sigma_{X_a}(K).
\]

Proof. We have observed already that \((\lambda - K^{(a)})^{-1} \in B(X) \Leftrightarrow (\lambda - K)^{-1} \in B(X_a)\).

If \((\lambda - K)^{-1} \in B(X)\) then \((\lambda - K) \in \Phi(X)\) and has index zero. By Theorem 2.10 it follows that \((\lambda - K) \in \Phi(X_a)\) and has index zero. But, by (2.4), \(\lambda - K : X_a \to X_a\) is injective. Since its index is zero, \(\lambda - K : X_a \to X_a\) is also surjective, so that \((\lambda - K)^{-1} \in B(X_a)\).

Conversely, if \((\lambda - K)^{-1} \in B(X_a)\) then, by Theorem 2.10, \(\lambda - K \in \Phi(X)\) and has index zero. Thus \((\lambda - K)(X)\) is closed and, since \((\lambda - K)(X) \supset (\lambda - K)(X_a) = X_a\), it follows from Theorem 2.11 that \(\lambda - K : X \to X\) is surjective. Since its index is zero, \(\lambda - K : X \to X\) is also injective so that \((\lambda - K)^{-1} \in B(X)\). \(\square\)

3. Results for operator families

In this section we examine the relationship between the point spectrum (set of eigenvalues) and the spectrum for the integral operator \( K \) given by (1.2) when \( k \) satisfies (A) and (B). In the case that (1.9) holds for some \( r \in \mathbb{R} \setminus \{0\} \), i.e.,

\[
T_r K = KT_r,
\]

where \( T_r \) is the translation operator given by (1.12), we have mentioned already in the introduction that

\[
\Sigma_X(K) = \Sigma_X^p(K) \cup \{0\}.
\]

We have also pointed out, with an example, that (3.2) does not hold in general given only that \( k \) satisfies (A) and (B). However, a version of (3.2) holds for families of integral operators satisfying (A) and (B) uniformly and having a translation invariance property to replace (3.1).

Let \( K : \{ k : \mathbb{R}^2 \to \mathbb{C} : k \text{ is measurable and satisfies (A) and (B)} \} \). For \( k \in K \) let \( K_k \) denote the integral operator defined by

\[
K_k \phi(s) = \int_{-\infty}^{+\infty} k(s, t) \phi(t) \, dt, \quad s \in \mathbb{R}.
\]
We consider families $W \subset K$ satisfying the following uniform versions of (A) and (B):

(A$_u$) \[ \sup_{k \in W} \| k \| < \infty, \]

where

\[ \| k \| := \sup_{s \in \mathbb{R}} \int_{-\infty}^{+\infty} |k(s, t)| \, dt. \]

(B$_u$) For all $s \in \mathbb{R},$

\[ \sup_{k \in W} \int_{-\infty}^{+\infty} |k(s, t) - k(s', t)| \, dt \to 0 \]

as $s' \to s.$

We saw in Section 2 that, if $k$ satisfies (A) and (B), then $K$ is $s$-sequentially compact. Similarly [2,15], if $W$ satisfies (A$_u$) and (B$_u$) then \{$K_k : k \in W$\} is collectively $s$-sequentially compact, meaning that

\[ (k_n) \subset W, \quad (\phi_n) \subset X, \quad \sup_n \| \phi_n \|_{\infty} < \infty \quad \Rightarrow \quad (K_{k_n} \phi_n) \text{ has a strictly convergent subsequence.} \]

For $(k_n) \subset K, \ k \in K,$ we will write $k_n \sigma \to k$ if $\sup_n \| k_n \| < \infty$ and, for all $\psi \in L^\infty,$

\[ \int_{-\infty}^{+\infty} k_n(s, t)\psi(t) \, dt \to \int_{-\infty}^{+\infty} k(s, t)\psi(t) \, dt, \]

uniformly on every finite interval. (This is convergence in the $\sigma$-topology of [15].) Call $W \subset K$ $\sigma$-sequentially compact if every sequence $(k_n) \subset W$ has a subsequence which is $\sigma$-convergent to some $k \in W.$ Clearly, if $W \subset K$ is $\sigma$-sequentially compact, then $W$ satisfies (A$_u$). It is shown in [15] that, for $(k_n) \subset K, \ k \in K, \ (\phi_n) \subset X, \ \phi \in X,$

\[ k_n \sigma \to k, \quad \phi_n \sigma \to \phi \quad \Rightarrow \quad K_{k_n} \phi_n \sigma \to K_k \phi, \quad (3.3) \]

where $\sigma$ denotes the strict convergence in $X$ introduced in the last section.

Let the translation operator $T_r^{(2)} : K \to K$ be defined, for $r \in \mathbb{R},$ by

\[ T_r^{(2)} k(s, t) = k(s - r, t - r), \quad s, t \in \mathbb{R}. \]

The following results are shown in [15].
Theorem 3.1. Suppose that $\lambda \neq 0$, $W \subset K$, and:

(i) $W$ satisfies $(B_u)$;
(ii) $W$ is $\sigma$-sequentially compact;
(iii) $T_r^{(2)}(W) = W$, for some $r \in \mathbb{R} \setminus \{0\}$, so that
\[ \{T_rK_k : k \in W\} = \{K_kT_r : k \in W\}; \]
(iv) $\lambda \notin \Sigma_p^X(K_k)$, $k \in W$.

Then $(\lambda - K_k)(X)$ is closed for all $k \in W$ so that $(\lambda - K_k)^{-1} : (\lambda - K_k)(X) \to X$ is bounded. Moreover, these inverse operators are uniformly bounded, i.e.,
\[ \sup_{k \in W} \| (\lambda - K_k)^{-1} \|_X < \infty. \] (3.4)

Suppose that, in addition to (i)–(iv), it holds that
(v) for every $\tilde{k} \in W$ there exists $(k_n) \subset W$ such that $k_n \sigma \to \tilde{k}$ and
\[ \lambda \notin \bigcup_{k \in W} \Sigma^p_X(K_k) \quad \Rightarrow \quad \lambda \notin \Sigma_X(K_{k_n}) \] (3.5)

for each $n$.

Then $(\lambda - K_k)(X) = X$ for all $k \in W$, so that $(\lambda - K_k)^{-1} \in B(X)$, $k \in W$.

We point out that (3.5) certainly holds if $\lambda - K_{k_n}$ is Fredholm of index zero. In particular (for any $\lambda \neq 0$), (3.5) holds if $K_{k_n}$ is compact or, more generally, by Corollary 2.4, if $k_n$ satisfies assumptions (A), (B), and (D).

In terms of the relationship between $\Sigma_p^X(K_k)$ and $\Sigma_X(K_k)$, we obtain the following corollary from the above theorem and Lemma 2.5.

Corollary 3.2. Suppose that conditions (i)–(iii) of Theorem 3.1 are satisfied and that, for every $\tilde{k} \in W$, there exists $(k_n) \subset W$ with $k_n \sigma \to \tilde{k}$ and
\[ \Sigma_X(K_{k_n}) \subset \{0\} \cup \bigcup_{k \in W} \Sigma^p_X(K_k) =: \Sigma^p_X(W), \]
for each $n$. Then
\[ \bigcup_{k \in W} \Sigma_X(K_k) = \Sigma^p_X(W). \]

Combining Corollary 3.2 with Theorem 2.12 we see that, if the conditions of Corollary 3.2 are satisfied, each $k \in W$ satisfies $(A')$, and (1.5) holds for some $b > 1$, then
\[
\bigcup_{k \in W} \Sigma_{X_a}(K_k) = \Sigma_X^P(W),
\]
for \(0 \leq a \leq b\). In the next theorem we will show further that, for \(\lambda \notin \Sigma_X^P(W)\), the uniform bound (3.4) holds with \(X\) replaced by \(X_a\). For this to hold we require that \((A')\) is satisfied uniformly for \(k \in W\), i.e., that the following uniform version of \((A')\) holds:

\((A'_u)\) For some \(\kappa \in L^1\),
\[
|k(s, t)| \leq |\kappa(s - t)|, \quad s, t \in \mathbb{R},
\]
for all \(k \in W\).

To prove the next theorem we require some preliminary extension of the results of Section 2. We remind the reader that, for \(k \in K\), we define \(k^{(a)}\) by (2.3). Analogously to (2.2), we will write \(K^{(a)}_k\) as an alternative notation for the integral operator \(K_k(a)\).

**Lemma 3.3.** If \(W\) satisfies \((A'_u)\) and \((B_u)\) and (1.5) holds for some \(b > 1\), then, for \(0 \leq a \leq b\), \(W_a := \{k^{(a)}: k \in W\}\) satisfies \((A_u)\) and \((B_u)\) and
\[
\sup_{k \in W} \sup_{s \in \mathbb{R}} \int_{[A, A]} |k(s, t) - k^{(a)}(s, t)| \, dt \to 0 \quad \text{as } A \to +\infty.
\]

**Proof.** This is clear from (2.6) and from examining the proofs of Theorems 2.7 and 2.8. \(\square\)

**Lemma 3.4.** Suppose that \((\phi_n) \subset X\), \((k_n) \subset K\),
\[
\sup_n \|k_n\| < \infty,
\]
and
\[
\sup_n \sup_{s \in \mathbb{R}} \int_{[A, A]} |k_n(s, t)| \, dt \to 0
\]
as \(A \to \infty\). Then
\[
\phi_n \to 0 \quad \Rightarrow \quad K_{k_n} \phi_n \to 0.
\]

**Proof.** It is easy to see that
\[
\|K_{k_n} \phi_n\| \leq \sup_n \|k_n\| \sup_{|s| \leq A} |\phi_n(s)| + \sup_n \|\phi_n\| \sup_n \sup_{s \in \mathbb{R}} \int_{[A, A]} |k_n(s, t)| \, dt.
\]
Given any $\epsilon > 0$, the second term can be made $\leq \epsilon / 2$ by choosing $A$ large enough, and then the first term can be made $\leq \epsilon / 2$ by choosing $n$ large enough. Thus $K_{k_n} \phi_n \to 0$. □

We also require the following result on $\sigma$-convergence.

**Lemma 3.5.** If $W$ satisfies $(A'_u)$, (1.5) holds for some $b > 1$, $(k_n) \subset W$, $k \in W$, and $k_n \sigma \to k$, then $k^{(a)}_n \sigma \to k^{(a)}$, for $0 \leq a \leq b$.

**Proof.** From Lemma 3.3 it follows that $W_a := \{k^{(a)}: k \in W\}$ satisfies $(A_u)$ so that $\sup_n \|k_n\| < \infty$. Also, for all $\psi \in L^\infty$, $\tilde{\psi} := \psi/w_a \in L^\infty$ and

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \left( k^{(a)}_n(s,t) - k^{(a)}(s,t) \right) \psi(t) \, dt \\
= w_a(s) \int_{-\infty}^{+\infty} \left( k_n(s,t) - k(s,t) \right) \tilde{\psi}(t) \, dt \to 0
\end{aligned}
$$

as $n \to \infty$, uniformly on finite intervals, since $k_n \sigma \to k$. Thus $k^{(a)}_n \sigma \to k^{(a)}$. □

**Theorem 3.6.** Suppose that $W$ satisfies $(A'_u)$ and $(B_u)$, that (1.5) holds for some $b > 1$, and that $W$ is $\sigma$-sequentially compact. Then

$$
\Sigma_{Xa}(K_k) = \Sigma_X(K_k), \quad k \in W, \quad 0 \leq a \leq b, \tag{3.6}
$$

and, for $\lambda \not\in \bigcup_{k \in W} \Sigma_X(K_k)$, it holds that

$$
\sup_{k \in W} \| (\lambda - K_k)^{-1} \|_{Xa} < \infty \iff \sup_{k \in W} \| (\lambda - K_k)^{-1} \|_{X} < \infty,
$$

$0 \leq a \leq b$.

**Proof.** That (3.6) holds follows from Theorem 2.12. It follows that if $\lambda \not\in \bigcup_{k \in W} \Sigma_X(K_k)$, then $(\lambda - K_k)^{-1} \in B(X) \cap B(X_a)$, $k \in W$, $0 \leq a \leq b$. By (2.5), also $(\lambda - K^{(a)}_k)^{-1} \in B(X)$, $k \in W$, $0 \leq a \leq b$.

Suppose that $\lambda \not\in \bigcup_{k \in W} \Sigma_X(K_k)$ and

$$
\sup_{k \in W} \| (\lambda - K_k)^{-1} \|_{X} < \infty.
$$

(3.7)

We will show by contradiction that it follows that

$$
\sup_{k \in W} \| (\lambda - K_k)^{-1} \|_{Xa} < \infty, \quad 0 < a \leq b.
$$

(3.8)
Suppose that (3.8) does not hold for some \( a \in (0, b] \). We have that \( \lambda \neq 0 \) by Lemma 2.5. Moreover, there exists \((\tilde{\phi}_n) \subset X_a\) with \( \|\tilde{\phi}_n\|_{X_a} = 1 \) and \((k_n) \subset W\) such that \( \|(\lambda - K_{k_n})\tilde{\phi}_n\|_{X_a} \to 0 \). Setting \( \phi_n := w_a\tilde{\phi}_n \), it follows that
\[
(\lambda - K^{(a)}_{k_n})\phi_n \to 0 \tag{3.9}
\]
(as before \( \to \) denoting norm convergence in \( X \)). Now \( W \) satisfies \( (A'_u) \) and \( (B_u) \) so, by Lemma 3.3, \( W_a := \{k^{(a)}: k \in W\} \) satisfies \( (A_u) \) and \( (B_u) \) and so is collectively \( s \)-sequentially compact. It follows that \( (K^{(a)}_{k_n})\phi_n \) has a subsequence, denoted by itself, such that \( K^{(a)}_{k_n}\phi_n \xrightarrow{s} \lambda \phi \in X \). Since \( W \) is \( \sigma \)-sequentially compact we can choose the subsequence so that also \( k_n \xrightarrow{\sigma} k \in W \) and thus, by Lemma 3.5, also \( k^{(a)}_n \xrightarrow{\sigma} k^{(a)} \). It follows from (3.9) that \( \phi_n \xrightarrow{\sigma} \phi \) and, by (3.3), that \( K^{(a)}_{k_n}\phi_n \xrightarrow{s} K^{(a)}_k\phi \). Thus \( \lambda \phi = K^{(a)}_k\phi \) and, since \( (\lambda - K^{(a)}_k)^{-1} \in B(X) \), it follows that \( \phi = 0 \). Thus \( \phi_n \xrightarrow{s} 0 \) and, applying Lemmas 3.3 and 3.4, we deduce that
\[
(K_{k_n} - K^{(a)}_{k_n})\phi_n \to 0.
\]
Thus, from (3.9), we have \( (\lambda - K_{k_n})\phi_n \to 0 \). But this contradicts (3.7).

By exactly the same argument we can show that if (3.8) holds for some \( a \in (0, b] \) then (3.7) holds. \( \square \)

Combining Theorems 3.1 and 3.6 we have the following criterion for \( (\lambda - K_k)^{-1} \in B(X_a) \).

**Theorem 3.7.** Suppose that \( \lambda \neq 0 \) and:

(i) \( W \subset K \) satisfies \( (A'_u) \) and \( (B_u) \) and (1.5) holds with \( b > 1 \);
(ii) \( W \) is \( \sigma \)-sequentially compact;
(iii) \( T^2_r(W) = W \) for some \( r \in \mathbb{R} \setminus \{0\} \);
(iv) \( \lambda \notin \Sigma^p_X(K_k), k \in W \);
(v) for every \( \tilde{k} \in W \) there exists \((k_n) \subset W\) such that \( k_n \xrightarrow{\sigma} \tilde{k} \) and
\[
\lambda \notin \bigcup_{k \in W} \Sigma^p_X(K_k) \quad \Rightarrow \quad \lambda \notin \Sigma_X(K_{k_n})
\]
for each \( n \).

Then \( \lambda \notin \Sigma_{X_a}(K_k) \) for \( k \in W, 0 \leq a \leq b \), and
\[
\sup_{k \in W}\|(\lambda - K_k)^{-1}\|_{X_a} < \infty, \quad 0 \leq a \leq b.
\]

We will state this last theorem as a result explicitly about the solvability of the integral equation (1.1).
Corollary 3.8. Suppose that conditions (i)–(v) of Theorem 3.7 are satisfied. Then Eq. (1.1) has exactly one solution \( \phi \in X \) for every \( \psi \in X \) and \( k \in W \), and if \( \psi \in X_a \) for some \( a \in [0, b] \), it holds that \( \phi \in X_a \). Moreover, for \( 0 \leq a \leq b \), there exists a constant \( C_a \), depending only on \( a \) and \( W \), such that, for all \( k \in W \) and \( \psi \in X_a \), the solution \( \phi \) of (1.1) satisfies

\[
|\phi(t)| \leq C_a (1 + |t|)^{-a} \sup_{s \in \mathbb{R}} (1 + |s|)^a |\psi(s)|, \quad t \in \mathbb{R}.
\]

4. An application

We conclude the paper by considering, as an illustration of the power of the results of Section 3, their application to the integral equation

\[
\lambda \phi(s) - \int_{-\infty}^{+\infty} \kappa(s-t)z(t)\phi(t) \, dt = \psi(s), \quad s \in \mathbb{R},
\]

with \( \kappa \in L^1 \), \( z \in L^\infty \). Of course, this is Eq. (1.1) with \( k(s, t) = \kappa(s-t)z(t) \). In operator notation we can abbreviate (4.1) as \( \lambda \phi - K \phi = \psi \), with

\[
K = \mathcal{K}M_z,
\]

where \( M_z \) is the operation of multiplication by \( z \) and \( \mathcal{K} \) is the convolution integral operator defined by

\[
\mathcal{K}\phi(s) = \int_{-\infty}^{+\infty} \kappa(s-t)\phi(t) \, dt, \quad s \in \mathbb{R}.
\]

Our first result illustrates the application of Theorems 3.1 and 3.7. We use extensively the weak* convergence and topology on \( L^\infty = (L^1)^* \). We recall that, for \( (\psi_n) \subset L^\infty \), \( \psi \in L^\infty \), we write \( \psi_n \overset{w^*}{\rightharpoonup} \psi \) if \( (\psi_n) \) converges weak* to \( \psi \), i.e., if

\[
\int_{-\infty}^{+\infty} \psi_n(t)\phi(t) \, dt \to \int_{-\infty}^{+\infty} \psi(t)\phi(t) \, dt, \quad \phi \in L^1.
\]

A useful characterisation of weak* convergence is that

\[
\psi_n \overset{w^*}{\rightharpoonup} \psi \iff \sup_n \|\psi_n\|_\infty < \infty \quad \text{and} \quad \int_{-\infty}^{+\infty} \psi_n(t)\phi(t) \, dt \to \int_{-\infty}^{+\infty} \psi(t)\phi(t) \, dt, \quad \phi \in C^\infty_0(\mathbb{R}),
\]
where \( C^\infty_0(\mathbb{R}) := \{ \phi \in C^\infty(\mathbb{R}) : \phi \) is compactly supported\}. We shall say that \( V \subset L^\infty \) is weak* sequentially compact if every sequence in \( V \) has a subsequence converging weak* to an element of \( V \).

Given \( \kappa \in L^1 \) and \( V \subset L^\infty \), we will consider families of kernels \( W = \{ k_z : z \in V \} \) where \( k_z(s, t) := \kappa(s-t)z(t), s, t \in \mathbb{R} \). One significance of weak* convergence and weak* sequential compactness for our purposes is the relationship with the \( \sigma \)-convergence introduced in Section 3, expressed in the following lemmas.

**Lemma 4.1** [15, Lemma 3.1]. If \( (z_n) \subset L^\infty, z \in L^\infty, z_n \overset{w^*}{\to} z, \) then \( k_{z_n} \overset{\sigma}{\to} k_z \).

**Lemma 4.2** [15, Lemma 3.2]. If \( V \subset L^\infty \) is weak* sequentially compact, then \( W = \{ k_z : z \in V \} \) satisfies \((A_u)\) and \((B_u)\) and is \( \sigma \)-sequentially compact.

Combining these lemmas with Theorems 3.1 and 3.7 we obtain the following corollary.

**Corollary 4.3.** Suppose \( V \subset L^\infty \) is weak* sequentially compact and \( T_r(V) = V, \) for some \( r \in \mathbb{R} \setminus \{0\} \). Suppose further that \( \lambda \neq 0, \) that \( \lambda \notin \Sigma^p_X(KM_z), \) for \( z \in V, \) and that, for every \( z \in V, \) there exists \( (z_n) \subset V \) such that \( z_n \overset{w^*}{\to} z \) and \( (\lambda - KM_{z_n})(X) = X \) for each \( n \). Then, for \( z \in V, \) \( (\lambda - KM_z)^{-1} \in B(X) \) and

\[
\sup_{z \in V} \| (\lambda - KM_z)^{-1} \|_X < \infty. \tag{4.2}
\]

If also \( \kappa(s) = O(|s|^{-b}) \) as \( |s| \to \infty, \) for some \( b > 1, \) then \( (\lambda - KM_z)^{-1} \in B(X_a), \) for \( z \in V \) and \( 0 < a \leq b, \) and

\[
\sup_{z \in V} \| (\lambda - KM_z)^{-1} \|_{X_a} < \infty. \tag{4.3}
\]

**Proof.** By Lemma 4.2, \( W := \{ k_z : z \in V \} \) satisfies conditions (i) and (ii) of Theorem 3.1, and \( W \) also clearly satisfies conditions (iii)–(v). Thus Theorem 3.1 applies to give that \( (\lambda - KM_z)^{-1} \in B(X), z \in V, \) and the bound (4.2). If also \( \kappa(s) = O(|s|^{-b}) \) as \( |s| \to \infty, \) for some \( b > 1, \) then \( W \) also satisfies conditions (i)–(v) of Theorem 3.7. Thus Theorem 3.7 applies to give that \( (\lambda - KM_z)^{-1} \in B(X_a), z \in V, 0 < a \leq b, \) and the bound (4.3). \( \square \)

Combining Lemmas 4.1 and 4.2 with Corollary 3.2 and Theorem 3.6, we obtain the following statement about the spectra of the operators \( KM_z \).

**Corollary 4.4.** Suppose \( V \subset L^\infty \) is weak* sequentially compact, \( T_r(V) = V, \) for some \( r \in \mathbb{R} \setminus \{0\}, \) and \( \kappa(s) = O(|s|^{-b}) \) as \( |s| \to \infty. \) Suppose also that for every \( \tilde{z} \in V \) there exists \( (z_n) \subset V \) such that \( z_n \overset{w^*}{\to} \tilde{z} \) and
\[ \Sigma_X(KM_{\mathcal{z}_n}) \subset \{0\} \cup \bigcup_{z \in \mathcal{V}} \Sigma_X^p(KM_z). \]

Then, for \(0 \leq a \leq b\),
\[ \bigcup_{z \in \mathcal{V}} \Sigma_{X_a}(KM_z) = \{0\} \cup \bigcup_{z \in \mathcal{V}} \Sigma_X^p(KM_z). \]

We finish the paper with a yet more specific application. For \(Q \subset \mathbb{C}\) let 
\(L_Q := \{\phi \in L^\infty : \phi(s) \in Q, \text{ for almost all } s \in \mathbb{R}\}\). It is shown in [7] that \(V := L_Q\) is weak\(^*\) sequentially compact iff \(Q\) is compact and convex. Whatever the choice of \(Q\), clearly \(T_r(V) = V, r \in \mathbb{R}\). Further, we can satisfy the remaining conditions on \(V\) of Corollary 4.4 in a variety of ways. For example, given \(z \in \mathcal{V}\), choose \((z_n) \subset \mathcal{V}\) so that \(z_n(s) = z(s), -n \leq s < n, \text{ and so that } z_n\) is \(2n\)-periodic.

Then \(z_n \xrightarrow{w^*} z\) and \(k_{z_n}\) satisfies (A), (B), and (1.9), with \(r = 2n\), so that [13] \(\Sigma_X(KM_{z_n}) = \{0\} \cup \Sigma_X^p(KM_{z_n})\). Alternatively, choose \(z_n\) so that \(z_n(s) = z(s), |s| \leq n, \text{ and so that } z_n(s) = q \in Q, \text{ otherwise. Then, setting } \tilde{z}(s) = q, s \in \mathbb{R}, k_{z_n} - k_{\tilde{z}}\) satisfies (A), (B), and (C), so that \(KM_{z_n} - KM_{\tilde{z}}\) is compact on \(X\). Also, \(KM_{\tilde{z}} = qK\) so that \(k_{\tilde{z}}\) satisfies (A), (B), and (1.9), for all \(r \in \mathbb{R}\), so that [13]
\[ \Sigma_X(KM_{\tilde{z}}) = \{0\} \cup \Sigma_X^p(KM_{\tilde{z}}). \]

Thus
\[ \Sigma_X(KM_{z_n}) \subset \Sigma_X^p(KM_{z_n}) \cup \Sigma_X(KM_{\tilde{z}}) \subset \{0\} \cup \Sigma_X^p(KM_{z_n}) \cup \Sigma_X^p(KM_{\tilde{z}}) \subset \{0\} \cup \bigcup_{z \in \mathcal{V}} \Sigma_X^p(KM_z). \]

We see that all the conditions on \(V\) of Corollary 4.4 are satisfied by the choice \(V := L_Q\), if \(Q\) is compact and convex. In view of Lemma 4.2 it follows that, provided (1.5) holds for some \(b > 1\), the conditions of Theorem 3.6 are also satisfied. Thus, applying Theorems 4.3 and 3.6, we have the following result.

**Corollary 4.5.** Suppose \(\kappa \in L^1\) and \(\kappa(s) = O(|s|^{-b})\) as \(|s| \to \infty\), for some \(b > 1\), \(Q \subset \mathbb{C}\) is compact and convex, and \(\lambda \neq 0\). Then the following statements are equivalent, for \(0 \leq a \leq b\):

(i) For every \(z \in L_Q\) the equation
\[ \lambda \phi(s) = \int_{-\infty}^{+\infty} \kappa(s-t)z(t)\phi(t) \, dt, \quad s \in \mathbb{R}, \tag{4.4} \]
has only the trivial solution in \(X\).

(ii) For every \(z \in L_Q\) the equation
\[ \lambda \phi(s) = \psi(s) + \int_{-\infty}^{+\infty} \kappa(s-t)z(t)\phi(t)\,dt, \quad s \in \mathbb{R}, \quad (4.5) \]

has exactly one solution \( \phi \in X_a \) for every \( \psi \in X_a \) and, for some constant \( C > 0 \) depending only on \( \lambda, \kappa, a, \) and \( Q \), \( \|\phi\|_{X_a} \leq C \|\psi\|_{X_a} \).

A boundary integral equation to which this result can be applied is discussed in [9,19]. Let \( U \) denote the upper half-plane \( U := \{x = (x_1, x_2) \in \mathbb{R}^2: x_2 > 0\} \), and set \( \Gamma := \partial U = \{(x_1, 0): x_1 \in \mathbb{R}\} \). Consider the following two-dimensional acoustics problem which has been used as a model of outdoor sound propagation over an inhomogeneous flat ground surface [10]. A time harmonic \( e^{-i\omega t} \) time dependence) monopole source is situated at \( x_0 \in U \) and noise propagates through a homogeneous medium of sound speed \( c \), wavenumber \( k = \omega/c \), in the upper half-plane \( U \) above the boundary \( \Gamma \) which is characterised by an impedance boundary condition. The surface impedance is a function of position on \( \Gamma \) (and usually also a function of the angular frequency, \( \omega \)). Let \( \beta(x) \) denote the ratio of the surface admittance at \( x \in \Gamma \) to the admittance of the medium of propagation. In the equations below \( u(x) \) denotes the complex velocity potential at \( x \in \overline{U} \).

This problem can be formulated as a boundary value problem for the Helmholtz equation. As part of the formulation we require a radiation condition to express mathematically the idea that the acoustic field should be outgoing, travelling away from the surface \( \Gamma \). For \( h \in \mathbb{R} \) let \( \Gamma_h := \{(x_1, h): x_1 \in \mathbb{R}\} \), and let \( U_h := \{(x_1, x_2): x_1 \in \mathbb{R}, x_2 > h\} \) denote the half-plane above \( \Gamma_h \). The radiation condition we shall impose is that proposed in [9], and termed the \textit{upwards propagating radiation condition} in [14] where its properties are explored. Let \( \Phi(x, y) := (i/4)H_0^{(1)}(k|x-y|), \) \( x, y \in \mathbb{R}^2, x \neq y \), denote the standard fundamental solution of the Helmholtz equation, where \( H_0^{(1)} \) is the Hankel function of the first kind of order zero. Then we shall say that \( u \) satisfies the \textit{upward propagating radiation condition} in \( U \) if, for some \( h \geq 0 \) and \( \phi \in L^\infty(\Gamma_h) \),

\[ u(x) = 2 \int_{\Gamma_h} \frac{\partial \Phi(x, y)}{\partial y_2} \phi(y)\,ds(y), \quad x \in U_h. \quad (4.6) \]

The boundary value problem we consider is thus the following one.

**Problem (P).** Given \( x_0 \in U \) (the position of the acoustic source), \( k > 0 \) (the wavenumber), and \( \beta \in L^\infty(\Gamma) \) (the relative surface admittance), find \( u \in C(\overline{U} \setminus \{x_0\}) \cap C^2(U \setminus \{x_0\}) \) such that:

(i) \( \Delta u + k^2 u = -\delta_{x_0} \) in \( U \), in a distributional sense, where \( \delta_{x_0} \) is the 2D delta distribution supported at \( x_0 \);

(ii) \( u \) satisfies the impedance boundary condition,
\[
\frac{\partial u}{\partial x^2} + ik\beta u = 0 \quad \text{on } \Gamma,
\] (4.7)

in the weak sense explained in [9];

(iii) \( u \) is bounded in the strip \( U \setminus U_h \), for every \( h > 0 \);

(iv) \( u \) satisfies the upward propagating radiation condition in \( U \).

It follows from [19, Example 3.7] that \( u \) satisfies Problem (P) iff \( u|_{\Gamma} \in BC(\Gamma) \) and

\[
u(x) = G(x, x_0) + ik \int_{\Gamma} G(x, y) (\beta(y) - 1) u(y) \, ds(y), \quad x \in \overline{U}.
\] (4.8)

In this equation \( G(\cdot, x_0) \) is the solution to Problem (P) in the case \( \beta \equiv 1 \), the function \( G \) given explicitly by

\[
G(x, y) := \Phi(x, y) + \Phi(x, y') + P(k(x - y')) , \quad x, y \in \overline{U}, \ x \neq y,
\] (4.9)

where \( y = (y_1, y_2), \ y' = (y_1, -y_2) \) and

\[
P(x) := \frac{e^{i|x|}}{\pi} \int_0^{\infty} \frac{t^{-1/2} e^{-|x|t(1 + \gamma(1 + it))}}{\sqrt{t - 2it(1 + \gamma)}} \, dt, \quad x \in \overline{U},
\] (4.10)

with \( \gamma = x_2/|x| \) and the square root taken such that both real and imaginary parts are non-negative. Since (see [9]) \( P \in C(\overline{U}) \), and in view of the asymptotic behaviour of the Hankel function for small argument, it follows that, for some \( C > 0 \),

\[
|G(x, y)| \leq C(1 + |\log |x - y||), \quad x, y \in \overline{U}, \ 0 < |x - y| \leq 1,
\] (4.11)

while it is shown in [12] that

\[
|G(x, y)| \leq C(1 + x_2)(1 + y_2)(1 + |x - y|)^{-3/2},
\]
\[
x, y \in \overline{U}, \ |x - y| \geq 1.
\] (4.12)

The bounds (4.11) and (4.12) ensure that the integral in (4.8) is well-defined.

Equation (4.8), restricted to \( \Gamma \), is a boundary integral equation which takes the form (4.5) with \( \lambda = 1 \), \( z(s) = i(1 - \beta((s, 0)))) \), \( \phi(s) = u((s, 0)) \), \( \psi(s) = G((s, 0), x_0) \), \( \kappa(s) = kG((s, 0), (0, 0)) \), \( s \in \mathbb{R} \). Using the bounds (4.11) and (4.12) we see that \( \kappa \in L^1 \), with \( \kappa(s) = O(|s|^{-3/2}) \) as \( |s| \to \infty \), and that \( \psi \in C(\mathbb{R}) \) with \( \psi(s) = O(|s|^{-3/2}) \) as \( |s| \to \infty \), in other words, \( \psi \in X_{3/2} \). It is shown in [9] that, with these definitions of \( z \) and \( \kappa \), the homogeneous equation (4.4) has only the trivial solution in \( X = BC(\mathbb{R}) \) if \( \beta \in L^\infty(\Gamma) \) satisfies ess inf \( \Re \beta > 0 \). Thus, if \( P \) is a compact, convex subset of the right-hand complex plane, Eq. (4.4) has only the trivial solution in \( X \) if the essential range of \( \beta \) is contained in \( P \), i.e., if \( z \in L_Q \), where \( Q := \{i(1 - w) : w \in P\} \). Thus the conditions of Corollary 4.5 are satisfied and (i) in Corollary 4.5 holds. It follows that (ii) in Corollary 4.5 holds,
for $0 \leq a \leq 3/2$. Thus we have the following result concerning the solvability of Eq. (4.8).

**Theorem 4.6.** Suppose that, for some $\epsilon > 0$, $\Re \beta(y) \geq \epsilon$ for almost all $y \in \Gamma$. Then Eq. (4.8) has exactly one solution with $u|_{\Gamma} \in BC(\Gamma)$. Further, for the solution of (4.8) it holds that $u(x) = O(|x|^{-3/2})$ as $|x| \to \infty$ with $x \in \Gamma$.

The significance of this theorem is that it gives us information on the asymptotic behaviour at infinity of the solution $u$ on $\Gamma$. In the case $\beta \equiv 1$ we already have this information in the bound (4.12), obtained from explicit estimates of the exact solution rather than functional analytic arguments. Clearly, (4.12) also estimates $u(x)$ for $x \in U$; we have, in the case $\beta \equiv 1$, that

$$u(x) = O((1 + x_2)|x|^{-3/2}) \text{ as } |x| \to \infty,$$

with $x \in \overline{U}$, uniformly in $x/|x|$. Combining Theorem 4.6 with the bound (4.12) we can show this same asymptotic behaviour (4.13) whenever $\text{ess inf} \Re \beta > 0$.

**Corollary 4.7.** Suppose that, for some $\epsilon > 0$, $\Re \beta(y) \geq \epsilon$ for almost all $y \in \Gamma$. Then Problem (P) has exactly one solution, $u$, and, for some constant $C > 0$,

$$|u(x) - G(x, x_0)| \leq C(1 + x_2)(1 + |x|)^{-3/2}, \quad x \in \overline{U}. \quad (4.14)$$

**Proof.** Let $C$ denote a generic positive constant throughout this proof. We have observed already that Problem (P) is equivalent to Eq. (4.8). Thus, that Problem (P) has exactly one solution, namely the unique solution of Eq. (4.8) with $u|_{\Gamma} \in BC(\Gamma)$, follows from Theorem 4.6. From Theorem 4.6 we also have that the solution of Eq. (4.8) satisfies

$$|u(x)| \leq C(1 + |x|)^{-3/2}, \quad x \in \Gamma.$$

From Eq. (4.8) and the bounds (4.11) and (4.12) it thus follows that

$$|u(x) - G(x, x_0)| \leq CI(x) + CJ(x), \quad x \in \overline{U}, \quad (4.15)$$

where, for $x \in \overline{U}$,

$$I(x) := (1 + x_2) \int_{\Gamma} (1 + |x - y|)^{-3/2}(1 + |y|)^{-3/2} ds(y), \quad (4.16)$$

$$J(x) := (1 + x_2) \int_{\Gamma} f(|x - y|)(1 + |y|)^{-3/2} ds(y),$$

and

$$f(r) := \begin{cases} 1 - \log r, & 0 < r \leq 1, \\ 0, & r > 1. \end{cases}$$
For $r > 0$ and $x = (x_1, x_2) \in \overline{U}$, let $I_r(x) := \{ (y_1, 0) : |y_1 - x_1| < r \}$. Clearly, $J(x) = 0$ for $x_2 > 1$, while, for $0 \leq x_2 \leq 1$,

$$J(x) \leq C(1 + |x_1|)^{-3/2} \int_{I_1(x)} f(|x - y|) ds(y)$$

$$= C(1 + |x_1|)^{-3/2} \int_{-1}^{1} f(\sqrt{x_2^2 + t^2}) dt.$$

Thus

$$J(x) \leq C(1 + x_2) (1 + |x_1|)^{-3/2}, \quad x \in \overline{U}. \quad (4.17)$$

To show the same bound for $I(x)$ we write the integral (4.16) as the sum of integrals $I_1(x)$ and $I_2(x)$ over $I_{|x_1|/2}(x)$ and $I \setminus I_{|x_1|/2}(x)$, respectively. Since $|y| \geq |x_1|/2$ for $y \in I_{|x_1|/2}(x)$ we have that

$$I_1(x) \leq C(1 + x_2) (1 + |x_1|)^{-3/2} \tilde{I}(x), \quad x \in \overline{U}, \quad (4.18)$$

where

$$\tilde{I}(x) := \int_{I_{|x_1|/2}(x)} (1 + |x - y|)^{-3/2} ds(y) = 2 \int_{0}^{|x_1|/2} (1 + \sqrt{t^2 + x_2^2})^{-3/2} dt.$$

Since $\tilde{I}(x) \leq 2 \int_{0}^{\infty} (1 + t)^{-3/2} dt$, it follows from (4.18) that

$$I_1(x) \leq C(1 + x_2) (1 + |x_1|)^{-3/2} \quad (4.19)$$

if $|x| \leq 1$ or $0 \leq x_2 \leq |x_1|$. Since

$$\tilde{I}(x) \leq 2 \int_{0}^{\infty} (t^2 + x_2^2)^{-3/4} dt = 2x_2^{-3/2} \int_{0}^{\infty} (1 + s^2)^{-3/4} ds,$$

it follows from (4.18) that (4.19) holds also if $|x| \geq 1$ and $x_2 \geq |x_1|$, so that (4.19) holds for all $x \in \overline{U}$. As $|x - y|^2 \geq x_2^2 + x_1^2/4 \geq |x|^2/4$ for $x \in \overline{U}$ and $y \in I \setminus I_{|x_1|/2}(x)$,

$$I_2(x) \leq C(1 + x_2) (1 + |x|)^{-3/2} \int_{I} (1 + |y|)^{-3/2} ds(y), \quad x \in \overline{U}. \quad (4.20)$$

Combining (4.15), (4.17), (4.19), and (4.20), the bound (4.14) follows. \qed
References