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Oscillation and nonoscillation criteria for quasilinear differential equations

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Abstract

Some oscillation and nonoscillation criteria for quasilinear differential equations of second order are obtained. These results are extensions of earlier results of Huang (J. Math. Anal. Appl. 210 (1997) 712–723) and Elbert (J. Math. Anal. Appl. 226 (1998) 207–219).

Keywords: Quasilinear equations; Oscillation; Nonoscillations

1. Introduction and main results

Let us consider the following second order linear differential equation:

$$x'' + q(t)x = 0, \quad t \geqslant t_* > 0, \tag{1}$$

where $q(t) \geqslant 0$ is locally integrable on $[t_*, \infty)$. If a solution $x(t)$ of (1) has arbitrarily large zero, it is called oscillatory, otherwise it is called nonoscillatory. If all nonzero solutions of (1) are oscillatory, then (1) is called oscillatory, otherwise it is called nonoscillatory.

The well-known Hill theorem [2] gives the following global integral criteria:

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Theorem A. *If $q \in L^1[t_0, \infty)$, and*

$$\limsup_{t\to\infty} t \int_t^\infty q(s)\,ds < \frac{1}{4}, \tag{2}$$

then (1) *is oscillatory. Otherwise, if*

$$\liminf_{t\to\infty} t \int_t^\infty q(s)\,ds > \frac{1}{4}, \tag{3}$$

then (1) *is nonoscillatory.*

In 1997, Huang [3] obtained the following interval criteria for the oscillation and nonoscillation of (1):

Theorem B. *If there exists $t_0 \geqslant t_*$ such that for every positive integer n,*

$$\int_{2^n t_0}^{2^{n+1} t_0} q(t)\,dt < \frac{\alpha_0}{2^{n+1} t_0}, \tag{4}$$

where $\alpha_0 = 3 - 2\sqrt{2}$, then (1) *is nonoscillatory.*

If there exists $t_0 \geqslant t_*$ such that for every positive integer n,

$$\int_{2^n t_0}^{2^{n+1} t_0} q(t)\,dt \geqslant \frac{\alpha}{2^n t_0}, \tag{5}$$

where $\alpha > \alpha_0 = 3 - 2\sqrt{2}$, then (1) *is oscillatory.*

In 1998, Elbert [1] generalized Huang's results and obtained the following theorem:

Theorem C. *Assume $t_* \leqslant t_0 < t_1 < t_2 < \cdots < t_n < \cdots$, $t_n \to \infty$. Let*

$$\beta_n = \frac{t_{n+1} - t_n}{t_1 - t_0}, \quad n = 0, 1, \ldots, \tag{6}$$

then $\beta_0 = 1$, $\beta_n > 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$.

If $q(t)$ satisfies the following inequality:

$$(t_{n+1} - t_n) \int_{t_n}^{t_{n+1}} q(s)\,ds \leqslant \alpha_n, \quad 0 \leqslant \alpha_n < 1,\ n = 0, 1, \ldots, \tag{7}$$

and for any sequence $\{z_n\}_{n=0}^{\infty}$ satisfying the following relation:

$$\begin{cases} z_{n+1} = \frac{z_n - \alpha_n}{\theta_n + z_n - \alpha_n}, & n = 1, 2, \ldots, \\ z_0 = 1 \end{cases}$$

we have $0 < z_n < 1,\ n = 1, 2, \ldots,\ \theta_n = \beta_n/\beta_{n+1}$, *then* (1) *is nonoscillatory.*

If $q(t)$ *satisfies the following inequalities*:

$$(t_{n+1} - t_n) \int_{t_n}^{t_{n+1}} q(s)\,ds \geqslant \alpha_n, \quad \alpha_n > 0,\ n = 0, 1, \ldots, \tag{8}$$

and there exists a sequence of numbers $\{u_n\}_{n=0}^{\infty}$ *satisfying the following relation*:

$$\begin{cases} u_{n+1} = \frac{\alpha_{n+1}\theta_n}{\alpha_n}\left(\alpha_n + \frac{u_n}{1-u_n}\right), & n = 0, 1, \ldots, \\ u_1 = 0, \end{cases}$$

with $0 < u_n < 1,\ n = 1, 2, \ldots,\ \theta_n = \beta_n/\beta_{n+1}$, *then* (1) *is oscillatory.*

In 2000, Jiang [4] generalized Huang's results for linear equation to the following quasilinear equation:

$$\left(\left|u'(t)\right|^{p-1}u'(t)\right)' + q(t)\left|u(t)\right|^{p-1}u(t) = 0, \tag{9}$$

where $p > 0$ is a constant, and obtained the following results:

Theorem D. *Suppose* $p \leqslant 1$, *if there exists* $t_0 \geqslant 0$ *and* $0 < c < 1$ *such that for every positive integer* n,

$$\int_{2^n t_0}^{2^{n+1}t_0} q(t)\,dt < \frac{\alpha_0}{pc^{p-1}(2^{n+1}t_0)^p}, \tag{10}$$

where $\alpha_0 = 3 - 2\sqrt{2}$, *then* (9) *is nonoscillatory.*

Suppose $p \geqslant 1$, *if there exists* $t_0 \geqslant 0,\ 0 < c < 1$ *and* $\alpha > 2^p + 1 - 2^{1+p/2}$ *such that for every positive integer* n,

$$\int_{2^n t_0}^{2^{n+1}t_0} q(t)\,dt \geqslant \frac{\alpha}{pc^{p-1}(2^n t_0)^p}, \tag{11}$$

then (9) *is oscillatory.*

Recently, Wong [7] generalized the results of Huang, he obtained the following results:

Theorem E. *Let* $\lambda > 1$. *If there exists some* t_0 *such that for every positive integer* n,

$$\int_{\lambda^n t_0}^{\lambda^{n+1}t_0} q(t)\,dt \leqslant \frac{\alpha}{(\lambda - 1)\lambda^{n+1}t_0}, \tag{12}$$

where $\alpha \leqslant (\sqrt{\lambda} - 1)^2$, *then* (1) *is nonoscillatory.*

If

$$\int_{\lambda^n t_0}^{\lambda^{n+1} t_0} q(t)\,dt \geqslant \frac{\alpha}{(\lambda-1)\lambda^n t_0}, \tag{13}$$

where $\alpha > (\sqrt{\lambda}-1)^2$, *then* (1) *is oscillatory.*

In this paper, by using a similar method in [1], we generalize Elbert's results to Eq. (9) and obtained the following results:

Theorem 1. *Suppose* $0 < p \leqslant 1$, *assume* $t_* \leqslant t_0 < t_1 < t_2 < \cdots < t_n < \cdots,\ t_n \to \infty$. *Let* β_n *be given by Theorem C and* $\theta_n = (\beta_n/\beta_{n+1})^p,\ n = 0, 1, \ldots.$

If $q(t)$ *satisfies the following inequality*:

$$(t_{n+1} - t_n)^p \int_{t_n}^{t_{n+1}} q(s)\,ds \leqslant \alpha_n, \quad 0 \leqslant \alpha_n < 1,\ n = 0, 1, \ldots, \tag{14}$$

and there exists a sequence of numbers $\{z_n\}_{n=0}^{\infty}$ *satisfying the following relations*:

$$\begin{cases} z_{n+1} = \frac{z_n - \alpha_n}{\theta_n + z_n - \alpha_n}, & n = 1, 2, \ldots, \\ z_0 = 1, \end{cases}$$

with $0 < z_n < 1,\ n = 1, 2, \ldots,$ *then* (9) *is nonoscillatory.*

If $q(t)$ *satisfies the following inequalities*:

$$(t_{n+1} - t_n)^p \int_{t_n}^{t_{n+1}} q(s)\,ds \geqslant \alpha_n, \quad \alpha_n > 0,\ n = 0, 1, \ldots, \tag{15}$$

and for any sequence $\{u_n\}_{n=0}^{\infty}$ *satisfying the following relations*:

$$\begin{cases} u_{n+1} = \frac{\alpha_{n+1}\theta_n}{\alpha_n}\left(\alpha_n + \frac{u_n}{1-u_n}\right), & n = 0, 1, \ldots, \\ u_1 = 0 \end{cases}$$

we have $0 < u_n < 1,\ n = 1, 2, \ldots,$ *then* (9) *is oscillatory.*

Corollary 1. *Let* $\alpha_n = \alpha \in (0, 1),\ \theta_n = \theta \in (0, 1)$ *and* $0 < p \leqslant 1$ *such that*

$$\sqrt{\theta} + \sqrt{\alpha} < 1.$$

Then any nonzero solution of (9) *is nonoscillatory.*

Corollary 2. *Let* $\alpha_n = \alpha > 0,\ \theta_n = \theta \in (0, 1)$ *and* $0 < p \geqslant 1$ *such that*

$$\sqrt{\theta} + \sqrt{\alpha\theta} > 1.$$

Then any solution of (9) *is oscillatory.*

Remark 1. Let $p = 1$, then Theorem 1 reduces to Theorem C, therefore Theorem 1 is a generalization of Theorem C. Comparing Theorem 1 with Theorem D, we find both are generalizations of Theorem C, and neither contains the other.

For the proof of Theorem 1, we need the following lemmas.

Lemma 1. *If $x(t)$ is a nonzero solution of* (9) *satisfying $x(a) = 0,\ x'(\tau) = 0$, where $t^0 \leqslant a < \tau$, then we have*

$$(\tau - a)^p \int_a^\tau q(s)\,ds > 1.$$

Proof. Without loss of generality, we can assume $x(t) > 0,\ t \in (a, \tau)$ and $\tau = \inf\{t > a, x'(t) = 0\}$. Integrating (9) from a to τ and by noticing $x'(\tau) = 0$, we get $x'(a) > 0$ and

$$\big(x'(a)\big)^p = \int_a^\tau q(s)\big(x(s)\big)^p\,ds < \int_a^\tau q(s)\big(x(\tau)\big)^p\,ds < \big(x'(a)\big)^p(\tau - a)^p \int_a^\tau q(s)\,ds,$$

hence we have $(\tau - a)^p \int_a^\tau q(s)\,ds > 1$. $\square$

Lemma 2. *If $a \geqslant 0,\ b \geqslant 0,\ 0 < p \leqslant 1$, then $a^p + b^p \geqslant (a + b)^p$; if $p \geqslant 1$, then $a^p + b^p \leqslant (a + b)^p$.*

Proof. Simple calculation yields above results. $\square$

2. Proof of Theorem 1

The proof of the first part of Theorem 1

Since it is proved in [5,6] that (9) cannot has nonoscillatory and oscillatory nonzero solutions at the same time, we need only to prove that (9) has a nonoscillatory solution. Therefore, we need only to prove the solution $x(t)$ of (9) satisfying initial condition $x(t_0) = 0,\ x'(t_0) > 0$ satisfies $x(t) > 0,\ \forall t > t_0$.

In fact, it follows from (14) that $(t_1 - t_0)^p \int_{t_0}^{t_1} q(s)\,ds < 1$ and Lemma 1, $x'(t) > 0,\ t \in [t_0, t_1]$, that $x(t) > 0$ and by (9), $x(t)$ is concave in (t_0, t_1). Integrating (9) from t_0 to t_1, we get

$$\begin{aligned}\big(x'(t_0)\big)^p - \big(x'(t_1)\big)^p &= \int_{t_0}^{t_1} q(s)\big(x(s)\big)^p\,ds \\ &\leqslant \big(x'(t_0)(t_1 - t_0)\big)^p \int_{t_0}^{t_1} q(s)\,ds \leqslant \alpha_0\big(x'(t_0)\big)^p,\end{aligned}$$

which yields

$$\big(x'(t_1)\big)^p \geqslant \big(x'(t_0)\big)^p - \alpha_0\big(x'(t_0)\big)^p = (1-\alpha_0)\big(x'(t_0)\big)^p > 0. \tag{16}$$

Claim 1. $x(t) > 0$, $\forall t \in [t_1, t_2]$. Otherwise, let $\tau^* = \inf\{t \in (t_1, t_2) \mid x(t) = 0\}$, then by Rolle's theorem, there exists $t = a^* \in (t_1, \tau^*)$ such that $x'(a^*) = 0$. As $a^* \in (t_1, \tau^*) \subset [t_1, t_2]$, it follows from Lemma 1 that

$$\alpha_1 > (t_2 - t_1)^p \int_{t_1}^{t_2} q(s)\,ds \geqslant (\tau^* - a^*)^p \int_{a^*}^{\tau^+} q(s)\,ds > 1,$$

which contradicts the assumption $\alpha_1 < 1$. Claim 1 is thus proved and we have

$$x(t) > 0, \quad t \in [t_1, t_2]. \tag{17}$$

Next we prove the following inequalities by using mathematical induction:

$$\big(x'(t_n)\big)^p \geqslant \frac{z_n}{\beta_n^p}\left(\sum_{i=0}^{n} \beta_i x'(t_i)\right)^p, \tag{18}$$

$$\big(x'(t_{n+1})\big)^p \geqslant \big(x'(t_n)\big)^p - \frac{\alpha_n}{\beta_n^p}\left(\sum_{i=0}^{n} \beta_i x'(t_i)\right)^p, \tag{19}$$

$$x(t) > 0, \quad t \in [t_{n+1}, t_{n+2}], \tag{20}$$

where z_n is defined in Theorem 1.

The case $n = 0$ follows from (16) and (17). Assume (18)–(20) hold for $0, 1, \dots, n$. We show that (18)–(20) hold also for $n+1$. As $z_{n+1} > 0$, it follows from (18) and (19) that

$$\big(x'(t_{n+1})\big)^p \geqslant \frac{z_n - \alpha_n}{\beta_n^p}\left(\sum_{i=0}^{n} \beta_i x'(t_i)\right)^p > 0.$$

Let $a = \sum_{i=0}^{n} \beta_i x'(t_i)$, $b = \beta_{n+1} x'(t_{n+1})$. From Lemma 2, for $0 < p \leqslant 1$, we get

$$\left(\beta_n^p + \frac{\beta_n^p}{z_n - \alpha_n}\right)\big(x'(t_{n+1})\big)^p \geqslant \left(\sum_{i=0}^{n} \beta_i x'(t_i)\right)^p + \beta_{n+1}^p\big(x'(t_{n+1})\big)^p,$$

or equivalently,

$$\big(x'(t_{n+1})\big)^p \geqslant \frac{z_{n+1}}{\beta_{n+1}^p}\left(\sum_{i=0}^{n+1} \beta_i x'(t_i)\right)^p. \tag{21}$$

By (9) and (20), $x(t)$ is concave on $[t_{n+1}, t_{n+2}]$, hence $x'(t)$ is nonincreasing and satisfies for $t \in [t_{n+1}, t_{n+2}]$

$$x(t) \leqslant x(t_{n+1}) + x'(t_{n+1})(t - t_{n+1}) \leqslant x(t_{n+1}) + x'(t_{n+1})(t_{n+2} - t_{n+1}).$$

Integrating (9) over $[t_{n+1}, t_{n+2}]$, we get

$$\big(x'(t_{n+1})\big)^p - \big|x'(t_{n+2})\big|^{p-1}x'(t_{n+2}) = \int_{t_{n+1}}^{t_{n+2}} q(s)\big(x(s)\big)^p\,ds$$
$$\leqslant \big|x(t_{n+1}) + x'(t_{n+1})(t_{n+2} - t_{n+1})\big|^p \int_{t_{n+1}}^{t_{n+2}} q(s)\,ds.$$

By the Lagrange mean value theorem and the meaning of β_n, θ_n, we get

$$x(t_{n+1}) = \sum_{i=0}^{n}\big[x(t_{i+1}) - x(t_i)\big] = \sum_{i=0}^{n}(t_{i+1} - t_i)x'\big(t_i^*\big)$$
$$\leqslant \sum_{i=0}^{n}(t_{i+1} - t_i)x'(t_i) = (t_1 - t_0)\sum_{i=0}^{n}\beta_i x'(t_i),$$

where $t_i < t_i^* < t_{i+1},\ i = 0, 1, \ldots, n.$

We have therefore

$$\big(x'(t_{n+1})\big)^p - \big|x'(t_{n+2})\big|^{p-1}x'(t_{n+2}) \leqslant (t_1 - t_0)^p\left(\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p \int_{t_{n+1}}^{t_{n+2}} q(s)\,ds$$
$$= \left(\frac{t_{n+2} - t_{n+1}}{\beta_{n+1}}\right)^p \left(\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p \int_{t_{n+1}}^{t_{n+2}} q(s)\,ds \leqslant \frac{\alpha_{n+1}}{\beta_{n+1}^p}\left(\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p.$$

That is,

$$\big|x'(t_{n+2})\big|^{p-1}x'(t_{n+2}) \geqslant \big(x'(t_{n+1})\big)^p - \frac{\alpha_{n+1}}{\beta_{n+1}^p}\left(\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p. \tag{22}$$

Since $z_{n+2} > 0$ implies $z_{n+1} > \alpha_{n+1}$, and by (21), we get

$$\big|x'(t_{n+2})\big|^{p-1}x'(t_{n+2}) \geqslant \frac{z_{n+1} - \alpha_{n+1}}{\beta_{n+1}^p}\left(\sum_{i=0}^{n+1}\beta_i x'(t_i)\right)^p > 0,$$

which implies $x'(t_{n+2}) > 0$. Lemma 1 and the inequality

$$(t_{n+3} - t_{n+2})\int_{t_{n+2}}^{t_{n+3}} q(s)\,ds < \alpha_{n+2} < 1$$

implies that $x(t) > 0,\ t \in [t_{n+2}, t_{n+3}]$, which, together with (21) and (22), completes the induction step. This shows that $x(t) > 0$ holds for all $t > t_0$, hence $x(t)$ is a nonoscillatory solution of (9). This completes the proof of the first part of Theorem 1. □

The proof of the second part of Theorem 1

If the result of the second part of Theorem 1 is not true, then we can without loss of generality assume that there exists a nonoscillatory solution $x(t)$ of (9) such that for all $t \geqslant t_0,\ x(t) > 0$. From (9) we see that $x''(t) \leqslant 0$ and $x'(t)$ is nonincreasing and $x(t)$ is concave for all $t > t_0$.

We show next that $x'(t) \leqslant 0$ can never hold for $t > t_0$. In fact, integrating (9) over $[t_n, t_{n+1}]$ for $n \geqslant m$, we get from (15)

$$\begin{aligned} \left|x'(t_n)\right|^{p-1} x'(t_n) - \left|x'(t_{n+1})\right|^{p-1} x'(t_{n+1}) &= \int_{t_n}^{t_{n+1}} q(s)\big(x(s)\big)^p\, ds \\ &\geqslant \big(x(t_n)\big)^p \int_{t_n}^{t_{n+1}} q(s)\, ds > 0, \end{aligned}$$

hence

$$x'(t_0) > x'(t_1) > \cdots > x'(t_n) > x'(t_{n+1}) > \cdots > 0. \tag{23}$$

By using the Lagrange mean value theorem again and by the definition of β_n, θ_n, we obtain

$$\begin{aligned} x(t_n) &= x(t_0) + \sum_{i=1}^{n} \big[x(t_i) - x(t_{i-1})\big] \\ &> \sum_{i=1}^{n} (t_i - t_{i-1}) x'\big(t_i^*\big) \geqslant (t_1 - t_0) \sum_{i=1}^{n} \beta_{i-1} x'(t_i), \end{aligned}$$

where $t_{i-1} < t_i^* < t_{i+1},\ i = 1, 2, \ldots, n$, and by (15), we get

$$\begin{aligned} \big(x'(t_n)\big)^p - \big(x'(t_{n+1})\big)^p &> (t_1 - t_0)^p \left(\sum_{i=1}^{n} \beta_{i-1} x'(t_i)\right)^p \int_{t_n}^{t_{n+1}} q(s)\, ds \\ &= \frac{(t_{n+1} - t_n)^p \int_{t_n}^{t_{n+1}} q(s)\, ds}{\beta_n^p} \left(\sum_{i=1}^{n} \beta_{i-1} x'(t_i)\right)^p \\ &\geqslant \frac{\alpha_n}{\beta_n^p} \left(\sum_{i=1}^{n} \beta_{i-1} x'(t_i)\right)^p, \end{aligned}$$

which implies the following two inequalities:

$$\frac{\alpha_n}{\beta_n^p} \left(\sum_{i=1}^{n} \beta_{i-1} x'(t_i)\right)^p < \big(x'(t_n)\big)^p \tag{24}$$

and

$$\big(x'(t_{n+1})\big)^p < \big(x'(t_n)\big)^p - \frac{\alpha_n}{\beta_n^p} \left(\sum_{i=1}^{n} \beta_{i-1} x'(t_i)\right)^p. \tag{25}$$

Let $u_0, u_1, \ldots$ be given by Theorem 1, then we prove the following claim:

Claim 2. For $n = 1, 2, \ldots,$ we have

$$u_n\big(x'(t_n)\big)^p < \big(x'(t_n)\big)^p \leqslant \frac{\alpha_n}{\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p \tag{26}$$

and

$$0 < u_n < 1. \tag{27}$$

It is easy to see that (26), (27) hold for $n = 0, 1$. Assume (26), (27) hold for $0, 1, \ldots, n$, we show next that they hold also for $n + 1$.

From (24) and (26), we know

$$\big(x'(t_n)\big)^p - \frac{\alpha_n}{\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p \leqslant (1-u_n)\big(x'(t_n)\big)^p.$$

By (25)–(27), we obtain

$$\begin{aligned}\big(x'(t_{n+1})\big)^p < (1-u_n)\big(x'(t_n)\big)^p &= \frac{(1-u_n)}{u_n}u_n\big(x'(t_n)\big)^p \\ &\leqslant \frac{(1-u_n)\alpha_n}{u_n\beta_n^p}\left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p,\end{aligned}$$

that is,

$$\frac{u_n\beta_n^p}{(1-u_n)\alpha_n}\big(x'(t_{n+1})\big)^p \leqslant \left(\sum_{i=1}^{n}\beta_{i-1}x'(t_i)\right)^p. \tag{28}$$

Adding $\beta_n^p(x'(t_{n+1}))^p$ to both sides of (28) and then multiplying both sides of the obtained inequality by $\alpha_{n+1}/\beta_{n+1}^p$, applying the result of Lemma 2 for $p \geqslant 1$, and using the definition of u_{n+1} in Theorem 1, we obtain from (24) (replace n by $n+1$) the following inequalities:

$$u_n\big(x'(t_{n+1})\big)^p < \frac{\alpha_{n+1}}{\beta_{n+1}^p}\left(\sum_{i=1}^{n+1}\beta_{i-1}x'(t_i)\right)^p < \big(x'(t_{n+1})\big)^p. \tag{29}$$

Hence Claim 2 is proved.

Now it follows from (29) and definition of u_{n+1} that $0 < u_{n+1} < 1$. This completes the induction step and this also implies that (26), (27) hold for any $n \in N$. But this contradicts the assumption of Theorem 1. Therefore the second part of Theorem 1 is thus proved. □

The following example shows that there exists $p > 1$, $q(t) > 0$ satisfying the conditions of the first part of Theorem 1, but Theorem 1 does not hold.

Example 1. It follows from [5] that the necessary and sufficient conditions for any solution of (9) to be nonoscillatory is that there exists continuous function $r(t)$ satisfying the following equation:

$$r(t) = \int_t^\infty q(s)\,ds + p\int_t^\infty \left|r(s)\right|^{1+1/p}\,ds. \tag{30}$$

Let $q(t) = \alpha p/t^{p+1}$, $\alpha \geqslant 1$ is a constant. If (9) has a nonoscillatory solution, then (30) has a solution $r(t)$. Since $\int_t^\infty q(s)\,ds = \alpha/t^p$, we obtain

$$r(t) = \frac{\alpha}{t^p} + p\int_t^\infty \left|r(s)\right|^{1+1/p}\,ds > \frac{\alpha}{t^p} > 0. \tag{31}$$

This implies that

$$r(t) > \frac{\alpha}{t^p} + \frac{\alpha^{1+1/p}}{t^p} = \frac{\alpha + \alpha^{1+1/p}}{t^p}. \tag{32}$$

Substituting (32) into the right side of (31), we obtain

$$r(t) > \frac{\alpha + (\alpha + \alpha^{1+1/p})^{(1+1/p)}}{t^p}. \tag{33}$$

Continuing in this way, we get

$$r(t) > \frac{f_n(\alpha)}{t^p}, \quad n = 0, 1, \ldots, \tag{34}$$

where $f_n(x) = x + (f_{n-1}(x))^{1+1/p}$, $f_1(x) = x + x^{1+1/p}$, $f_0(x) = x$ for $x \geqslant 1$. It is easy to verify the following inequalities:

$$f_{n+1}(x) > f_n(x) > \cdots > f_1(x) > f_0(x) \geqslant 1.$$

The exist therefore two possibilities:

(i) $\lim_{n\to\infty} f_n(x) = \infty$, for all $x \geqslant 1$;
(ii) $\lim_{n\to\infty} f_n(x) = M < \infty$.

If (ii) holds, we have $M = x + M^{1+1/p}$, which is impossible, since $M > 1$. Hence (i) holds, which implies that $r(t) = \infty$ for all $t \gg 1$. A contradiction.

It is easy to verify that for $q(t) = \alpha p/t^{p+1}$, $p > 1$, the conditions of the first part of Theorem 1 hold. In fact, let $t_n = 2^n$, $n = 0, 1, \ldots,$ then $t_{n+1} - t_n = 2^n$, $\beta_n = 2^n$, $\theta_n = \theta = 2^{-n}$ and

$$(t_{n+1} - t_n)^p \int_{t_n}^{t_{n+1}} q(s)\,ds = \alpha\left(1 - \frac{1}{2^p}\right).$$

Hence

$$\alpha_n = \alpha\left(1 - \frac{1}{2^p}\right) < 1, \quad n = 1, 2, \ldots.$$

If we take $1 < \alpha < 2^p/2^{p-1}$, z_n satisfies $0 < z_n < 1$, $n = 1, 2, \ldots$. Then the assumptions of the first part of Theorem 1 are satisfied. But it follows from [5] that any solution of (9) is oscillatory.

The following example shows that for $p \in (0, 1)$, the conclusion of the second part of Theorem 1 may be incorrect.

Example 2. For $0 < p < 1$, $q(t) = cp/t^{p+1}$, where

$$(2^{p/2} - 1)^2 \frac{2^p}{2^p - 1} < c < \frac{2^p}{2^p - 1}.$$

Let $t_n = 2^n$, $\theta_n = \theta = 2^{-p}$, $\alpha_n = \alpha = c(2^p - 1)/2^p < 1$. Then it is easy to verify that $\sqrt{\theta} + \sqrt{\theta\alpha} > 1$. Hence the condition of Corollary 2 holds, but at the same time (14) and $0 < z_n < 1$, $n = 1, 2, \ldots,$ hold. Hence any nonzero solution of (9) is nonoscillatory, which shows that the second part of Theorem 1 may be incorrect for $p \in (0, 1)$.

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