ELSEVIER Stochastic Processes and their Applications 61 (1996) 223-248

stochastic processes and their applications

Stochastic dynamics for an infinite system of random closed strings: A Gibbsian point of view

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Received October 1994; revised November 1995

Abstract

We consider the stochastic dynamics of infinitely many, interacting random closed strings, and show that the law of this process can be characterized as a Gibbs state for some Hamiltonian on the path level, which is represented in terms of the interaction. This is done by means of the stochastic calculus of variations, in particular an integration by parts formula in infinite dimensions.

This Gibbsian point of view of the stochastic dynamics allows us to characterize the reversible states as the Gibbs states for the underlying interaction. Under supplementary monotonicity conditions, there is only one stationary distribution, and we prove that there is exactly one Gibbs state.

Keywords: Stochastic dynamics; Interacting strings; Stochastic quantization; Gibbsian measure; Integration by parts formula; Reversible state

0. Introduction

Let $\mathscr{D} \subset \mathscr{H} \subset \mathscr{D}'$ be a Gelfand triple (see Section 1.1). We consider a lattice system of infinitely many random elements in $\mathscr{H}$ evolving under the action of an infinite stochastic gradient system built on a given interaction. To be more precise, let A be some linear negative operator on $\mathscr{H}$ and $h = (h_\Lambda)_{\Lambda \subset \mathbb{Z}^d}$ a Hamiltonian on $\mathscr{H}^{\mathbb{Z}^d}$ in the usual sense of statistical mechanics, both a priori given. We are interested in processes $X = (X_i)_{i \in \mathbb{Z}^d}$, $X_i = (t \to X_{i,t}) \in \mathscr{C}(0, T; \mathscr{H})$, which are weak solutions of the following

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SSDI 0304-4149(95)00086-0

system, which we call the *Langevin equation*:

$$(\mathscr{L}an) \quad \begin{cases} dX_{i,t} = [AX_{i,t} - \frac{1}{2}\,\mathrm{grad}_i\, h_i(X_t)]\,dt + dW_{i,t}, \\ X_0 \overset{(\mathscr{L})}{=} v \end{cases}$$

($i \in \mathbb{Z}^d$, $0 \leqslant t \leqslant T$). Here $(W_i)_{i\in\mathbb{Z}^d}$ is a sequence of independent, $\mathscr{D}'$-valued Brownian motions (see Section 1.1), v a probability law on $\mathscr{H}^{\mathbb{Z}^d}$ and the second equation is an equality in law sense. We have to assume here that h is smooth in the following sense: For each i the gradient of $h_i := h_{\{i\}}$ at $y = (y_j)_{j\in\mathbb{Z}^d}$ with respect to y_i, denoted by $\mathrm{grad}_i\, h_i(y)$, is represented by some element of $\mathscr{H}$, at least when y belongs to some dense subspace E of $\mathscr{H}$, where the processes X_i will live. We denote by Q^v the law on $\mathscr{C}(0, T; E)^{\mathbb{Z}^d}$ of X and by Q_t^v the law on $E^{\mathbb{Z}^d}$ of X_t.

The system ($\mathscr{L}an$) is the basic stochastic time evolution we consider. Our problem is the mathematical foundation of the *Boltzmann–Gibbs hypothesis* for ($\mathscr{L}an$). This contains, besides the problem of existence and uniqueness of solutions of ($\mathscr{L}an$),

(A) the characterization of the reversible distributions of ($\mathscr{L}an$) as the Gibbs states determined by the operator A and the Hamiltonian h;

(B) the evaluation of the number of Gibbs states for A and h; in particular the absence of *phase transition* (existence of more than one Gibbs state);

(C) the ergodicity of the solution process X of ($\mathscr{L}an$), i.e. the convergence of Q_t^v towards an equilibrium state v_e, as $t \to \infty$, for a large class of initial distributions v.

To summarize: The Boltzmann–Gibbs hypothesis states that, for a large class of initial distributions, the time evolution Q_t^v of ($\mathscr{L}an$) converges to a Gibbs state v_e given by A and h.

In the generality we posed the problem above, a proof of the Boltzmann–Gibbs hypothesis is out of reach today.

We now restrict our attention to the following lattice model of an *anharmonic quantum oscillator* which is of importance in quantum statistical mechanics: Let $\mathscr{D} \subset \mathscr{H} \subset \mathscr{D}'$ be the classical Schwartz triple: $\mathscr{D} = \mathscr{C}^\infty(S_\beta)$ and $\mathscr{H} = L_2(S_\beta)$, where S_β is the interval $[0, \beta]$ with 0 and β identified (i.e. S_β is the circle of length β).

On $\mathscr{H}$ we consider the shifted Laplacian operator

$$A = \frac{1}{2}\frac{\partial^2}{\partial u^2} - m^2 \cdot \mathrm{Id} \quad (m > 0). \tag{$*$}$$

Finally the Hamilton function $h = (h_i)_i$ is of ferromagnetic type and is defined by

$$h_i(y) = \psi(y_i) + 2 \sum_{j\in\mathcal{N}(i)} a(j - i)\,\langle y_i, y_j\rangle, \tag{$**$}$$

where $y = (y_i)_i \in \mathscr{H}^{\mathbb{Z}^d}$, the so-called self-potential ψ is defined by some potential function V on $\mathbb{R}$ by

$$\psi(y_i) = \int_{S_\beta} V(y_i(u))\,du, \quad y_i \in \mathscr{H}, \tag{$***$}$$

and $(a(j - i))_{i,j\in\mathbb{Z}^d}$ is some suitable interaction matrix. $\mathcal{N}(i)$ denotes a fixed finite subset of $\mathbb{Z}^d$ for each i and $\langle\, ,\rangle$ the usual inner product in $\mathscr{H}$. We shall show now for this model that, under certain natural assumptions on the parameters of the system, the Boltzmann–Gibbs hypothesis is true in the following precise sense (see Section 4): For a large class of initial states ν, Q_t^ν converges to the unique Gibbs state determined by A and h. This Gibbs state on $\mathscr{H}^{\mathbb{Z}^d}$ describes the so-called Euclidean Gibbs state of a quantum anharmonic system at inverse temperature β with self-interaction ψ and interaction matrix a. See the pioneering work of Albeverio and Høegh-Krohn (1975) and further developments e.g. of Globa and Kondratiev (1990). See Albeverio et al. (1994) for a more detailed description. Let us stress that stochastic dynamics of the type considered here appears in the general stochastic quantization approach to the construction of Euclidean Gibbs measures.

The idea of the proof is the following: We first show (A) in extending the ideas of Cattiaux et al. (1996). There the lattice model of an anharmonic classical oscillator was treated; this is the case $A = 0$ and $\mathscr{H} = \mathbb{R}$. The main tool is to characterize, by means of the stochastic calculus of variations (in particular an integration by parts formula on path spaces), the law Q^ν of X as *Gibbs measures* for some new Hamilton function *on the path space*. This Hamilton function is explicitly represented in terms of A and h. This is our Gibbsian point of view. The Gibbsian description of Q^ν is of independent interest and could furnish more information on the structure of the process X. (The Gibbsian nature of interacting diffusion processes has been pointed out already by Deuschel (1987).)

On the other hand we use the results of Albeverio et al. (1994) who show existence and uniqueness of solutions of ($\mathscr{L}an$), and give a sufficient condition for the existence of exactly one equilibrium state ν_ε which is obtained as the ergodic limit. These results rely on fundamental ideas of Royer (1979) and Sunyach (1975). This solves problem (C). But the states, which are reversible for ($\mathscr{L}an$) and which we described under (A) as Gibbs states for A and h, are invariant for ($\mathscr{L}an$). Therefore there exists exactly one such Gibbs state. This solves problem (B).

We would like to mention that Funaki (1991) obtained in the one-particle case, but for parameter $\beta = +\infty$, results which give the equivalence between reversibility and some new Gibbsian property related to the space parameter $u \in \mathbb{R}$.

1. A variational characterization of the $\mathscr{D}'$-valued Brownian motion and Ornstein–Uhlenbeck process

1.1. Wiener measure

Let us consider a real separable Hilbert space $\mathscr{H}$ and some Gelfand triple $\mathscr{D} \subset \mathscr{H} \subset \mathscr{D}'$ (which will be chosen in the right way in Section 1.2), where $\mathscr{D}$ is, as usual, some nuclear space densely and continuously embedded into $\mathscr{H}$, and $\mathscr{D}'$ is its dual.

For Sections 1–3 we fix some terminal time $T > 0$ and consider the path-space $\mathscr{X} = \mathscr{C}(0, T; \mathscr{D}')$.

(i) We denote by $\pi^y \in \mathscr{P}(\mathscr{X})$, space of probability measures on $\mathscr{X}$, the law of the $\mathscr{D}'$-valued $\mathscr{H}$-cylindrical Brownian motion with initial condition $y \in \mathscr{D}$, defined as follows: if $x.$ is the canonical process on $\mathscr{X}$, for each $\varphi \in \mathscr{D}$, $\langle x., \varphi \rangle$ is a real valued Brownian motion under π^y with variance $t\|\varphi\|^2$, where $\| \ \|$ denotes the norm in $\mathscr{H}$.

(ii) The canonical time projection from $\mathscr{X}$ into $\mathscr{D}' : x \mapsto x_t, t \in [0, T]$, is denoted by p_t.

Functional spaces over $\mathscr{X}$

(a) $W^{1,2}(\mathscr{X})$ is the set of functionals $F \in L^2(\mathscr{X}, \pi^y)$ with L^2-derivative DF defined as follows: $\exists (D_s F(x))_{s,x} \in L^2((0, T) \times \mathscr{X}; \mathscr{H}; \mathrm{d}s \times \pi^y)$ such that

$$\forall g \in L^2(0, T; \mathscr{D}) \quad D_g F(x) = \lim_{\varepsilon \to 0} \varepsilon^{-1} \left[F\left(x + \varepsilon \int_0^{\cdot} g_s \, \mathrm{d}s \right) - F(x) \right]$$

$$= \int_0^T \langle g_s, D_s F(x) \rangle \, \mathrm{d}s.$$

(b) $W^{1,\infty}(\mathscr{X})$ is the subset of $W^{1,2}(\mathscr{X})$ containing the bounded functionals F with bounded derivative, i.e. $\operatorname{esssup}_{s,x} \|D_s F(x)\| < +\infty$.

(c) $\mathscr{W}(\mathscr{X})$ is the subset of $W^{1,2}(\mathscr{X})$ containing the smooth functionals of the form

$$F(x) = f(\langle x_{t_1}, \varphi_1 \rangle, \ldots, \langle x_{t_n}, \varphi_n \rangle),$$

$0 \leqslant t_1 \leqslant \cdots \leqslant t_n \leqslant T$, $\varphi_1, \ldots, \varphi_n \in \mathscr{D}$, $f \, \mathscr{C}^\infty$-function on $\mathbb{R}^n$. For this type of function, the derivative satisfies

$$D_g F(x) = \sum_{i=1}^{n} \int_0^{t_i} \langle g_s, \varphi_i \rangle \, \mathrm{d}s \cdot \partial_i f(\langle x_{t_1}, \varphi_1 \rangle, \ldots, \langle x_{t_n}, \varphi_n \rangle).$$

We know from the stochastic calculus of variations the duality between Skorohod integral and Malliavin derivative (cf. Bismut, 1981, formula (2.2)). In our context, it implies the following equality:

$$\forall F \in W^{1,2}(\mathscr{X}), \ g \in L^2(0, T; \mathscr{D})$$

$$E_{\pi^y}\left(F(x) \int_0^T \langle g_s, \mathrm{d}x_s \rangle\right) = E_{\pi^y}(D_g F(x)),$$

where $\int \langle g_s, \mathrm{d}x_s \rangle$ denotes the real valued stochastic integral.

We now show that equality (1) is in fact characteristic for Brownian motion.

Theorem 1. *Let* $y \in \mathscr{D}$, $\rho \in \mathscr{P}(\mathscr{X})$, $\rho(x, p_0(x) = y) = 1$, *with the following integrability property*: $\forall g \in \mathscr{D}, \forall t \geqslant 0 \ E_\rho(|\langle x_t, g \rangle|) < +\infty$. *If the equality*

$$E_\rho\left(F(x) \int_0^T \langle g_s, \mathrm{d}x_s \rangle\right) = E_\rho(D_g F) \tag{2}$$

holds for every $F \in \mathscr{W}(\mathscr{X})$ *and* g *step function from* $[0, T]$ *in* $\mathscr{D}$, *then* ρ *is equal to* π^y.

Proof. Let us remark that for step functions g, the stochastic integral in (2) is well defined. Let $\rho \in \mathscr{P}(\mathscr{X})$. ρ is uniquely determined by its initial condition and by the following functional:

$$\hat{\rho}: g \mapsto E_\rho\left(\exp\left\{\mathrm{i}\int \langle g_s, \mathrm{d}x_s\rangle\right\}\right)$$

for g step function, i.e.

$$g_s = \sum_{i=1}^{n} 1_{]t_{i-1},t_i]}(s)\varphi_i, \quad 0 = t_0 \leqslant \cdots \leqslant t_n \leqslant T, \ \varphi_i \in \mathscr{D}.$$

Parallel to Roelly and Zessin (1993, Theorem 1.2) (where the finite-dimensional case is treated), if ρ satisfies (2), we can compute $\hat{\rho}(\lambda g)$, $\lambda \in \mathbb{R}$, as a solution of the following differential equation:

$$\frac{\partial}{\partial \lambda}\hat{\rho}(\lambda g) = -\lambda \int \|g_s\|^2\,\mathrm{d}s \cdot \hat{\rho}(\lambda g), \quad \hat{\rho}(0) = 1.$$

It implies

$$\hat{\rho}(g) = \exp -\frac{1}{2}\int \|g_s\|^2\,\mathrm{d}s,$$

which characterizes ρ as a Wiener measure. □

1.2. Ornstein–Uhlenbeck process

We are now interested in some characterization of the Ornstein–Uhlenbeck (O–U) process as a unique solution of some integral equation like (2). This process is in our application in Sections 3 and 4, the reference linear process which will be perturbed by some non-linear interaction.

Let us now choose the space $\mathscr{D}$ and $\mathscr{D}'$ of the Gelfand triplet in such a way that $-A$, some fixed linear positive self-adjoint operator on $\mathscr{H}$, satisfies $-A(\mathscr{D}) \subset \mathscr{D}$ (then $-A(\mathscr{D}') \subset \mathscr{D}'$) and $\mathrm{e}^{tA}\mathscr{D} \subset \mathscr{D}$. (This is possible, cf. Berezanski and Kondratiev, 1988.)

Theorem 2. *Let ρ be the law of the following* O–U *process on $\mathscr{X}$.*

$$\mathrm{d}x_t = Ax_t\,\mathrm{d}t + \mathrm{d}W_t, \quad 0 \leqslant t \leqslant T, \ x_0 \in \mathscr{H}, \tag{3}$$

where W is a Brownian motion with values in $\mathscr{D}'$. ρ is the unique probability on $\mathscr{H}$ with initial condition x_0 for which the following equality holds for every $F \in W^{1,\infty}(\mathscr{X})$ and g step function in $\mathscr{D}$:

$$E_\rho(F(x)\int_0^T \langle \tilde{g}_s, \mathrm{d}x_s - Ax_s\,\mathrm{d}s\rangle) = E_\rho(D_g F). \tag{4}$$

Here

$$\tilde{g} = -\int_0^{\cdot} Ag_r\,\mathrm{d}r.$$

Proof. In the direct direction, let us prove that the law of the solution of (3) satisfies (4): By definition of the derivative $D_g F$,

$$E_\rho(D_g F) = E_\rho\left(\lim_{\varepsilon\to 0} \varepsilon^{-1}\left[F\left(x + \varepsilon\int_0^\cdot g_s\,\mathrm{d}s\right) - F(x)\right]\right).$$

Let us note x^ε the shifted process $x + \varepsilon\int_0^\cdot g_s\,\mathrm{d}s$. x^ε solves the following stochastic differential equation:

$$\begin{aligned} \mathrm{d}x_t^\varepsilon &= Ax_t^\varepsilon\,\mathrm{d}t + \varepsilon\overbrace{\left(g_t - \int_0^t Ag_s\,\mathrm{d}s\right)}^{\tilde{g}_t}\mathrm{d}t + \mathrm{d}W_t, \quad 0 \leqslant t \leqslant T, \\ x_0^\varepsilon &= x_0. \end{aligned} \tag{5}$$

Since $-A$ has regularizing properties, when the initial condition of (3) (respectively of (5)) belongs to $\mathscr{H}$, the law of the solution of (3) (resp. of (5)) is carried by $C(0, T; \mathscr{H})$.

The difference between the drift in (5) and the drift in (3) is the fixed deterministic function $\varepsilon\tilde{g}$. Since $\sup_{s\in[0,T]} \|\tilde{g}_s\| < +\infty$,

$$M_t^\varepsilon = \exp\left(\int_0^t \varepsilon\langle\tilde{g}_s, \mathrm{d}W_s\rangle - \tfrac{1}{2}\varepsilon^2\int_0^t \|\tilde{g}_s\|^2\,\mathrm{d}s\right), \quad 0 \leqslant t \leqslant T,$$

is a martingale with expectation 1 and the law of x^ε under ρ is absolutely continuous with respect to the law of x under ρ with density M_T^ε (Da Prato and Zabczyk, 1992, Theorem 10.14). Then

$$E_\rho(D_g F) = E_\rho\left(\lim_{\varepsilon\to 0} \varepsilon^{-1}(M_T^\varepsilon - 1)\cdot F(x)\right).$$

By using the Taylor–Lagrange formula $\mathrm{e}^x = 1 + x + (x^2/2)\mathrm{e}^{\theta x}$, one gets that $\varepsilon^{-1}(M_T^\varepsilon - 1)$ converges in $L^1(\rho)$ towards $\int_0^T \langle\tilde{g}_s, \mathrm{d}W_s\rangle$ and then we are done:

$$\begin{aligned} E_\rho(D_g F) &= E_\rho\left(F(x)\int_0^T \langle\tilde{g}_s, \mathrm{d}W_s\rangle\right) \\ &= E_\rho\left(F(x)\int_0^T \langle\tilde{g}_s, \mathrm{d}x_s - Ax_s\,\mathrm{d}s\rangle\right). \end{aligned}$$

Reciprocally, let ρ satisfy (4), and note $\tilde{x}$ the process defined by $\tilde{x} = x - \int_0^\cdot Ax_s\,\mathrm{d}s$. Eq. (4) becomes, for functional of the form $F(x - \int^\cdot Ax_s\,\mathrm{d}s)$,

$$\begin{aligned} &E_\rho\left(F\left(x - \int_0^\cdot Ax_s\,\mathrm{d}s\right)\cdot\int_0^T \langle\tilde{g}_s, \mathrm{d}\tilde{x}_s\rangle\right) = E_\rho(D_g[F(\tilde{x})]) = E_\rho(D_{\tilde{g}}F(\tilde{x})) \\ &\Leftrightarrow E_\rho\left(F(\tilde{x})\int_0^T \langle\tilde{g}_s, \mathrm{d}\tilde{x}_s\rangle\right) = E_\rho(D_{\tilde{g}}F(\tilde{x})) \end{aligned}$$

for each $F \in W^{1,\infty}(\mathscr{X})$ and g step function in $\mathscr{D}$.

To apply Theorem 1, we have to prove that the class of functions $\tilde{g} = g - \int_0^{\cdot} Ag_r\,dr$, g step function with values in $\mathscr{D}$ is dense in $L^2(0, T; \mathscr{D})$. This is true: The equation $\tilde{g} = g - \int_0^{\cdot} Ag_r\,dr$ is a Volterra equation, and due to the non-negativity of $-A$, it is reversible. Then, by Theorem 1, the process $\tilde{x}$ is, under ρ, a Wiener process, or, equivalently, x is, under ρ, a solution of the SPDE (3). □

2. Gibbs measures on $\Omega = \mathscr{C}(0, T; \mathscr{D}')^{\mathbb{Z}^d}$

We recall the definition of Dobrushin–Lanford–Ruelle of Gibbs measures on the infinite product of an abstract polish space X (cf. for example Georgii, 1988). On the product space $X^{\mathbb{Z}^d}$, we define the canonical filtration $(\mathscr{F}_\Lambda)_{\Lambda \text{finite}, \Lambda \subset \mathbb{Z}^d}$. $\mathscr{F}_\Lambda$ is generated by the spatial canonical projections $(\text{pr}_i)_{i\in\Lambda}$ defined by $\text{pr}_i(x) = x_i$, $x = (x_i)_{i\in\mathbb{Z}^d}$. More generally, we denote by pr_Λ the spatial projection into $X^\Lambda : x \mapsto x_\Lambda = (x_i)_{i\in\Lambda}$. For $x_\Lambda \in X^\Lambda$ and $x_{\Lambda^c} \in X^{\Lambda^c}$, we note $x_\Lambda x_{\Lambda^c}$ the element $\bar{x}$ of $X^{\mathbb{Z}^d}$ such that: $\text{pr}_\Lambda(\bar{x}) = x_\Lambda$ and pr $x_{\Lambda^c}(\bar{x}) = x_{\Lambda^c}$.

Definition 3. For an *interaction* $\psi = (\psi_\Lambda; \Lambda \subset \mathbb{Z}^d$ finite subset) where $\psi_\Lambda : X^{\mathbb{Z}^d} \to \mathbb{R}$ are $\mathscr{F}_\Lambda$-measurable functions such that $\sum_{\Lambda'\cap\Lambda\neq\emptyset} |\psi_{\Lambda'}| < +\infty$ for every finite Λ, one defines the Hamilton function $H^\psi = (H^\psi_\Lambda)_{\Lambda\subset\mathbb{Z}^d}$ by $H^\psi_\Lambda = \sum_{\Lambda'\cap\Lambda\neq\emptyset} \psi_{\Lambda'}$.

A probability measure Q on $X^{\mathbb{Z}^d}$ is called a (ψ, λ)-Gibbs measure where $\lambda = \otimes_{i\in\mathbb{Z}^d} \lambda_i$, λ_i σ-finite measure on X, if $\forall i \in \mathbb{Z}^d$, for Q-a.s. $x_{\{i\}^c}$

$$Q(dx_i / x_{\{i\}^c}) = \frac{1}{Z_i(x_{\{i\}^c})} \exp - H^\psi_{\{i\}}(x) \lambda_i(dx_i), \tag{6}$$

where $Q(/x_{\{i\}^c})$ is some regular version of the conditional probability $Q(/\mathscr{F}_{\{i\}^c})$ and Z_i is a normalizing constant.

We now extended the characterization of Roelly and Zessin (1993, Theorem 2.9) of a Gibbs measure on $C(0, T; \mathbb{R})^{\mathbb{Z}^d}$ as solution of an equilibrium integral equation in the two following directions:

(α) The paths take values in some infinite dimensional Banach space included in $\mathscr{D}'$, i.e. $\mathbb{R}$ is replaced by $\mathscr{D}'$.

(β) The reference measure on Ω is a product of Ornstein–Uhlenbeck measures, each of them defined by (3) (and no more a product of Wiener measures).

Let us note $\Omega = \mathscr{X}^{\mathbb{Z}^d}$, and $P^y = \otimes_{i\in\mathbb{Z}^d} \rho_i$ where ρ_i is the law on $\mathscr{X}$ of the O–U process satisfying (3) with initial condition $y_i \in \mathscr{H}$.

Functional spaces on Ω

$W^{1,2}(\Omega)$ is the set of functionals $F \in L^2(\Omega, P^y)$ with L^2-derivative $D^i F$ with respect to each coordinate $i \in \mathbb{Z}^d$ defined by

$$\forall g \in L^2(0, T; \mathscr{D}), \quad D^i_g F(\omega) = \lim_{\varepsilon\to 0} \varepsilon^{-1} \left[F\left(\left(\omega_j + \varepsilon\delta_{ij} \int_0^{\cdot} g_s\,ds \right)_{j\in\mathbb{Z}^d} \right) - F(\omega) \right].$$

$W^{1,\infty}(\Omega)$ is defined as in Section 1.1.

A functional F on Ω is called local if it only depends on a finite number of coordinates:

$$\exists \Lambda \text{ finite} \subset \mathbb{Z}^d, \quad F(\omega) = F(\omega_\Lambda) \ \forall \omega \in \Omega.$$

$W^{1,2}_{\text{loc}}(\Omega) = \{F \text{ local on } \Omega, F \in W^{1,2}(\Omega)\}$.

It contains the following subset:

$$\mathscr{W}_{\text{loc}}(\Omega) = \{f(\langle \omega_{i_1,t_1}, \varphi_1 \rangle, \dots, \langle \omega_{i_n,t_n}, \varphi_n \rangle), i_1, \dots, i_n \in \mathbb{Z}^d, t_1 \leqslant \cdots \leqslant t_n, \varphi_1, \dots, \varphi_n \in \mathscr{D}, f\mathscr{C}^\infty \text{ on } \mathbb{R}^n\}.$$

Theorem 4. *Let Q be a probability measure on Ω which is carried by paths with values in $\mathscr{H}$, and such that $Q(\omega, p_0(\omega) = y \in \mathscr{H}^{\mathbb{Z}^d}) = 1$. Suppose*

$$E_Q(|\langle \omega_{i,t}, g \rangle|) < +\infty, \quad g \in \mathscr{D}, 0 \leqslant t \leqslant T, \ i \in \mathbb{Z}^d. \tag{7}$$

Let $\Phi = (\Phi_\Lambda)$ be an interaction on $C(0, T; \mathscr{H})^{\mathbb{Z}^d}$ such that the associated Hamilton function $H = (H^\Phi_\Lambda)$ satisfies:

$\forall i \in \mathbb{Z}^d$, $H_i := H^\Phi_{\{i\}}$ is L^2-differentiable w.r.t. the ith coordinate in each direction $g \in L^2(0, T; \mathscr{D})$. (8)

If Q is a (Φ, P^y)-Gibbs measure such that, for each $g \in L^2(0, T; \mathscr{D})$, $i \in \mathbb{Z}^d$,

$$E_Q(|D^i_g H_i|) < +\infty \quad \forall i \in \mathbb{Z}^d, \tag{9}$$

then $\forall F \in W^{1,\infty}_{\text{loc}}(\Omega)$, $\forall g \in L^2(0, T; \mathscr{D})$, $i \in \mathbb{Z}^d$,

$$E_Q(D^i_g F) = E_Q\left(F \cdot \int_0^T \langle \tilde{g}_s, \mathrm{d}\omega_{i,s} - A\omega_{i,s}\,\mathrm{d}s \rangle\right) + E_Q(F \cdot D^i_g H_i). \tag{10}$$

Reciprocally, if Q satisfies (7), *H satisfies* (8), (9) *and*

$$E_Q(\mathrm{e}^{H_i}) < +\infty \quad \forall i \in \mathbb{Z}^d \tag{11}$$

and if, for each $F \in W^{1,\infty}_{\text{loc}}(\Omega)$ and g step function with values in $\mathscr{D}$, (10) *holds, then Q is a (Φ, P^y)-Gibbs measure on Ω.*

Proof. The method is exactly parallel to Proposition 2.4 and Theorem 2.9 of Roelly and Zessin (1993) where $\mathscr{H} = \mathbb{R}$ and Φ is bounded. We just recall the key ideas and the reader should refer to Roelly and Zessin (1993) (or Cattiaux et al. (1996, Theorem 2.11) when the interaction is unbounded) for details.

If Q is Gibbs, by decomposing the Q-integral of

$$F \cdot \int_0^T \langle \tilde{g}_s, \mathrm{d}\omega_{i,s} - A\omega_{i,s}\,\mathrm{d}s \rangle$$

into the integral w.r.t. the conditioned probability $Q(/\mathscr{F}_{\{i\}^c})$ given by (6) and using Eq. (4) satisfied by ρ_i, we quickly obtain (10).

Reciprocally, applying (10) to functionals of the form $F_i(\omega_i)G(\omega_{\{i\}^c})$ one can check that the finite measure $e^{H_i(\cdot\,\omega_{\{i\}^c})}\,Q(d\omega_i/\omega_{\{i\}^c})$ satisfies the functional equation (4) on $\mathscr{X}$. So it is proportional to ρ_i, with some constant depending just on $\omega_{\{i\}^c}$. □

To conclude this section, we extend Theorem 4 to the case of Gibbs measures Q on Ω with non-deterministic initial condition. The theoretical question is the following: if the initial condition of Q is a Gibbs measure on $\mathscr{H}^{\mathbb{Z}^d}$, and Q conditioned w.r.t. each initial value is Gibbs on Ω, does Q remain a Gibbs measure, and what is the potential? In this full generality, the difficulty comes from an eventual dependence between the initial distribution and the dynamics. It is anyway completely analysed in Cattiaux et al. (1996, Proposition 2.6), when $\mathscr{H} = \mathbb{R}$. We now present a result which will be useful in the third section. The proof is not given since it is a special case of Cattiaux et al. (1996, Proposition 2.6).

Theorem 5. *Let $Q \in \mathscr{P}(\Omega)$ carrying $\mathscr{H}$-valued paths on $[0, T]$. If γ, the initial condition of Q, is a (φ, μ)-Gibbs measure on $E^{\mathbb{Z}^d}$, for E some dense Banach space in $\mathscr{H}$, with associated Hamilton function h, and if $Q(/p_0 = y)$ is, for γ a.s. $y \in E^{\mathbb{Z}^d}$, a (Φ, P^y)-Gibbs measure satisfying the hypothesis* (7), (8), (9) *and also*

$$\forall i \in \mathbb{Z}^d,\ \forall y \in E^{\mathbb{Z}^d} \quad E_Q(e^{H_i}/p_0 = y) = 1, \tag{12}$$

$\forall i \in \mathbb{Z}^d$, the r.v. $\omega_{i,0}$ and $\omega_{\{i\}^c}$ *are independent under the measure*

$$z_i(\omega_{\{i\}^c,0}) \cdot \exp(H_i + h_i \circ p_0)(\omega) \cdot Q, \tag{13}$$

where z_i is the normalizing constant of $\exp - h_i(y) \cdot \mu_i(dy_i)$, *then Q is a $(\Phi + \varphi \circ p_0, P^\mu)$-Gibbs measure on Ω, where $P^\mu = \int P^y \mu(dy)$ is the infinite product of O–U laws, with initial condition $\mu = \otimes_{i \in \mathbb{Z}^d} \mu_i \in \mathscr{P}(E^{\mathbb{Z}^d})$.*

3. SPDE law as a Gibbs measure

We now apply the characterizations of Sections 1 and 2 to exhibit the Gibbsian character of path-measure of a certain class of infinitely many non-linear parabolic stochastic differential equations, whose solution represents a time evolution of random interactive closed strings.

From now on $\mathscr{H} = L^2(S_\beta)$ where S_β is the circle with length β, and we consider the following imbedding: $\mathscr{D} = C^\infty(S_\beta) \subset E = C(S_\beta) \subset \mathscr{H} = L^2(S_\beta) \subset \mathscr{D}'$ (β is the inverse temperature parameter). For simplicity we note $\|\cdot\|$ for $\|\cdot\|_{L^2(S_\beta)}$.

We consider the following infinite system of SPDE on $\mathscr{H}$:

$$\begin{aligned} dX_{k,t} &= dW_{k,t} + AX_{k,t}\,dt - \tfrac{1}{2}\,\text{grad}_k h_k(X_t)\,dt, \\ (X_{k,0})_{k\in\mathbb{Z}^d} &\overset{(\mathscr{L})}{=} \gamma, \quad k \in \mathbb{Z}^d,\ 0 \leqslant t \leqslant T, \end{aligned} \tag{14}$$

where $(W_k)_{k\in\mathbb{Z}^d}$ is a sequence of independent $\mathscr{D}'$-valued $L^2(S_\beta)$-cylindrical Brownian motions, A is a differential operator on $\mathscr{H}$ which represents a *quantum* effect, and is defined by

$$A = \frac{1}{2}\frac{\partial^2}{\partial u^2} - m^2\,\mathrm{Id}, \quad u \in S_\beta,$$

where m is some constant related to the mass of the particles, and h is the Hamilton function on $E^{\mathbb{Z}^d}$, and is given by, for $k \in \mathbb{Z}^d$,

$$h_{\{k\}}(y) := h_k(y) = \langle V(y_k), 1\rangle + 2 \sum_{|j-k|\leqslant R} a(j-k)\,\langle y_j, y_k\rangle$$
$$= \int_{S_\beta} V(y_k(u))\,\mathrm{d}u + 2 \sum_{|j-k|\leqslant R} a(j-k) \int_{S_\beta} y_j(u) y_k(u)\,\mathrm{d}u.$$

It is associated to the following quadratic pair interaction ψ:

$$\psi_{\{i\}}(y) = \langle V(y_i), 1\rangle, \quad \psi_{\{i,j\}}(y) = 2a(j-i)\langle y_j, y_i\rangle, \tag{15}$$
$$\psi_\Lambda = 0 \text{ if } \Lambda \neq \{i\} \text{ nor } \{i,j\},\ y \in E^{\mathbb{Z}^d}.$$

Let us assume the following hypothesis on the parameters of h:

(i) Growth assumption on the self-potential $V \in \mathscr{C}^3(\mathbb{R})$.

$$\begin{aligned}&\exists k > 0\ \exists K \geqslant 1 \quad \forall x \in \mathbb{R}\, |\tfrac{1}{2} V'(x)| + |V''(x)| \leqslant k(1 + |x|^K),\\ &\exists b \in \mathbb{R}, \quad \forall x, y \in \mathbb{R}\ -\tfrac{1}{2}(x-y)(V'(x) - V'(y)) \leqslant b(x-y)^2.\end{aligned} \tag{16}$$

(ii) a, the pair interaction, satisfies:

a is a symmetric matrix $(a(i) = a(-i))$ with finite range interaction, i.e. $|i| > R \Rightarrow a(i) = 0$, and vanishing diagonal: $a(0) = 0$. (17)

Then $\mathrm{grad}_k\, h_k$, the functional derivative of h_k with respect to y_k (mapping from E in E), is equal to

$$\mathrm{grad}_k\, h_k(y) = V' \circ y_k + 2 \sum_{|j-k|\leqslant R} a(j-k) y_j.$$

(iii) $\gamma \in \mathscr{P}(E^{\mathbb{Z}^d})$ is some initial law.

Let us remark that if $A = \beta = 0$ one recognizes stochastic gradient systems associated to *classical* lattice systems, a model treated in Cattiaux et al. (1996). Existence and uniqueness of solutions of (14) under assumptions (16) and (17) are solved in Albeverio et al. (1993, Theorem 2). (For the one-particle situation we refer to Funaki (1983) and Iwata (1987). We recall it:

Proposition 6. *If* γ, *the initial condition of* (14), *is equal to* δ_y *with* $y \in Q_{\beta,\infty}^{-p_0} = \{(y_k)_k \in C(S_\beta)^{\mathbb{Z}^d}, \sum_{k\in\mathbb{Z}^d} (1+|k|)^{-2p_0} \|y_k(\cdot)\|_\infty^2 < +\infty\}$, *then there exists under assumptions* (16) *and* (17) *a unique strong generalized solution of the system* (14) *with values in* $Q_{\beta,\infty}^{-p}$, *for each* p *large enough.*

Let us give now our main result:

Theorem 7. *Let $Q \in \mathscr{P}(\Omega)$ with initial condition $\gamma \in \mathscr{P}(Q_{\beta,\infty}^{-p_0})$. Q is the law of the solution of* (14)*, where γ is a (φ, μ)-Gibbs measure if and only if Q is a $(\Phi + \varphi \circ p_0, P^\mu)$-Gibbs measure on Ω, where the Hamilton function H associated with Φ on $C(0, T; E)^{\mathbb{Z}^d}$ has the following expression: $\forall i \in \mathbb{Z}^d$,*

$$\begin{aligned} H_i(\omega) &= \tfrac{1}{2} h_i(\omega_T) - \tfrac{1}{2} h_i(\omega_0) \\ &\quad + \tfrac{1}{4}\int_0^T \sum_{|j-i| \leqslant R} [\langle \operatorname{grad}_j h_i(\omega_s), \operatorname{grad}_j h_j(\omega_s)\rangle - \operatorname{Trace} \operatorname{grad}_j^{(2)} h_i(\omega_s) \\ &\quad - \tfrac{1}{8} \|\operatorname{grad}_j h_i(\omega_s)\|^2]\, \mathrm{d}s \\ &\quad - \int_0^T \sum_{|j-i| \leqslant R} \langle \tfrac{1}{2} \operatorname{grad}_j h_i(\omega_s), A\omega_{j,s}\rangle\, \mathrm{d}s, \end{aligned} \tag{18}$$

where $\operatorname{grad}_j^{(2)} h_i(y)$ *is the linear operator which maps E in E equal to* $\operatorname{grad}_j(\operatorname{grad}_j\ h_i)(y)$.

Proof (*Necessary condition*). Let Q be the law solution of (14).

First step: We show

Proposition 8. *Let $Q \in \mathscr{P}(\Omega)$ be the law of the solution of* (14). *Then Q satisfies the equilibrium equation* (10)*, where the functional H_i is given by* (18).

Proof. As for the proof of Theorem 2, we fix $i \in \mathbb{Z}^d$ and compute $E_Q(D_g^i F)$ using the definition of $D_g^i F$. Let $\tau_{i,\varepsilon}$ be the translation $\omega \in \Omega \to (\omega_j + \varepsilon\delta_{ij} \int_0^{\cdot} g_s\, \mathrm{d}s)_{j \in \mathbb{Z}^d} \in \Omega$. For $F \in W^{1,\infty}(\Omega)$, g step function in $\mathscr{D}$,

$$E_Q(D_g^i F) = E_Q\left(\lim_{\varepsilon \to 0} \varepsilon^{-1}(F(\tau_{i,\varepsilon}\omega) - F(\omega))\right).$$

Since F is smooth, we can exchange limit and Q-integration:

$$E_Q(D_g^i F) = \lim_{\varepsilon \to 0} \varepsilon^{-1} E_Q(F(\tau_{i,\varepsilon}\omega) - F(\omega)). \tag{19}$$

Let $Q^{i,\varepsilon} \in \mathscr{P}(\Omega)$ be the image law of Q under the translation $\tau_{i,\varepsilon}$. $Q^{i,\varepsilon}$ is the law of a solution of the following perturbed version of (14):

$$\begin{aligned} \mathrm{d}X_{k,t} &= \mathrm{d}W_{k,t} + \left(AX_{k,t} - \tfrac{1}{2} V'\left(X_{k,t} - \varepsilon \int_0^t \delta_{ik} g_s\, \mathrm{d}s\right)\right. \\ &\quad \left. - \sum_{|j-k| \leqslant R} a(k-j) X_{j,t}\right) \mathrm{d}t \\ &\quad + \varepsilon\left(\delta_{ik}(g_t - \int_0^t A g_s\, \mathrm{d}s) + a(k-i) \int_0^t g_s\, \mathrm{d}s\right) \mathrm{d}t, \\ &\quad (X_{k,0})_{k \in \mathbb{R}^d} \overset{(\mathscr{L})}{=} \gamma. \end{aligned} \tag{20}$$

The drift coefficients of the system of SPDE (20) satisfy trivially the same smoothness assumptions (16), (17) as the drift of (14). So, following the existence and uniqueness result of Albeverio et al. (1994), the *unique* solution of (20) has $Q^{i,\varepsilon}$ as law. Furthermore, if we note $b^{\varepsilon}_{k,t}$ the difference of the kth drifts in (14) and (20), we remark that $b^{\varepsilon}_{k,t}$ differs from zero only for the (finite number of) coordinates k such that $|k-i| \leqslant R$.

Since

$$Q^{i,\varepsilon}\left(\int_0^T \sum_{|k-i|\leqslant R} \|b^{\varepsilon}_{k,t}\|^2 \,\mathrm{d}t < +\infty\right) = 1$$

(consequence of the polynomial growth of V'-assumption (16) – and the $Q^{i,\varepsilon}$-integrability of every power of $\sup_{t\in[0,T]} \|X_{k,t}\|$), we can apply the result of Appendix A and deduce that $Q^{i,\varepsilon}$ is absolutely continuous with respect to Q and the density process N^{ε}_T is the exponential martingale:

$$N^{\varepsilon}_T = \exp \sum_{|k-i|\leqslant R}\left(\int_0^T \langle b^{\varepsilon}_{k,t}, \mathrm{d}W_{k,t}\rangle - \frac{1}{2}\int_0^T \|b^{\varepsilon}_{k,t}\|^2 \,\mathrm{d}t\right),$$

where

$$b^{\varepsilon}_{k,t} = \varepsilon a(k-i)\int_0^t g_s \,\mathrm{d}s, \quad k \neq i,$$

$$b^{\varepsilon}_{i,t} = -\tfrac{1}{2}\left[V'\left(X_{i,t} - \varepsilon\int_0^t g_s \,\mathrm{d}s\right) - V'(X_{i,t})\right] + \varepsilon\tilde{g}_t$$

($\tilde{g}$ was defined in Theorem 2).

So Eq. (19) becomes

$$E_Q(D^i_g F) = \lim_{\varepsilon\to 0} E_Q(\varepsilon^{-1}(N^{\varepsilon}_T - 1)F)$$

and we have to compute the limit of $\varepsilon^{-1}(N^{\varepsilon}_T - 1)$. By using the Taylor–Lagrange formula, there exists $0 \leqslant \varepsilon(t,u) \leqslant \varepsilon$ such that

$$b^{\varepsilon}_{i,t}(u) = \frac{\varepsilon}{2}\int_0^t g_s(u)\,\mathrm{d}s \cdot V''\left(X_{i,t}(u) - \varepsilon(t,u)\int_0^t g_s(u)\,\mathrm{d}s\right) + \varepsilon\tilde{g}_t(u).$$

Applying now the Taylor formula for the exponential function,

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2}\mathrm{e}^{\Theta x},$$

we obtain that

$$\lim_{\varepsilon\to 0}\varepsilon^{-1}(N^{\varepsilon}_T - 1) = \int_0^T \left\langle \tfrac{1}{2}V''(X_{i,t})\int_0^t g_s \,\mathrm{d}s + \tilde{g}_t, dW_{i,t}\right\rangle + \sum_{|k-i|\leqslant R}\int_0^T \left\langle a(k-i)\int_0^t g_s \,\mathrm{d}s, \mathrm{d}W_{k,t}\right\rangle. \tag{21}$$

The above limit takes place in $L^2(Q)$ since, by (16), V'' has polynomial growth. Then,

$$E_Q(D_g^i F) = E_Q\left[F \cdot \int_0^T \left\langle \tfrac{1}{2} V''(X_{i,t}) \int_0^t g_s \, ds, dW_{i,t} \right\rangle \right.$$

$$+ F \cdot \int_0^T \langle \tilde{g}_t, dX_{i,t} - AX_{i,t}\, dt + \tfrac{1}{2} \operatorname{grad}_i h_i(X_t)\, dt \rangle$$

$$+ F \cdot \int_0^T \sum_{|k-i| \leqslant R} \left\langle a(k-i) \int_0^t g_s \, ds, dX_{k,t} - AX_{k,t} dt \right.$$

$$\left. \left. + \tfrac{1}{2} \operatorname{grad}_k h_k(X_t)\, dt \right\rangle \right] \tag{22}$$

In the second (resp. third) term of the right-hand side we have replaced the stochastic integral w.r.t. dW_i (resp. dW_k) in (21) by the integral w.r.t. $dX_i - AX_i\, dt + \frac{1}{2} \operatorname{grad}_i h_i(X_t) dt$ (resp. $dX_k - AX_k dt + \frac{1}{2} \operatorname{grad}_k h_k(X_t)\, dt$), using that $\tilde{g}_t$(resp. $a(k-i) \int_0^t g_s \, ds$) is a smooth process in the variable $u \in S_\beta$. We would like to do the same for the first. But the difficulty comes from the fact that $u \mapsto V''(X_{i,t}(u))$ is not smoother than $u \mapsto X_{i,t}(u)$; the duality $\langle V''(X_{i,t}), AX_{i,t} \rangle$ then does not have any sense because X_i, is not a $\mathscr{D}(A)$-valued process. In Albeverio (1994) it is proven that $X_{i,t} \in \mathscr{D}(A^\alpha)$ for $\alpha < \frac{1}{2}$ (we would need at least $X_{i,t} \in \mathscr{D}(A^{1/2})$ to compute $\langle A^{1/2} V''(X_{i,t}), A^{1/2} X_{i,t} \rangle$).

We solve this difficulty applying the ideas and results of Appendix B. Using the Itô formula for the function

$$\left\langle \tfrac{1}{2} V'(X_{i,T}), \int_0^T g_s \, ds \right\rangle$$

we verify that all the terms have a well-defined meaning except the term $\int_0^T \langle V''(X_{i,t}) \int_0^t g_s \, ds, AX_{i,t} \rangle \, dt$. So this last integral has a "global" sense, and we note it *formally* as above. In Appendix B, the value of this integral is also obtained as a limit of similar integrals where the process X is replaced by some polygonal approximation or where A is regularized by some little perturbation from a differential operator with larger order.

By comparing now Eq. (22) with Eq. (10), the functional H_i we are looking for must satisfy Q-a.s.:

$$D_g^i H_i = \sum_{|k-i| \leqslant R} \int_0^T \frac{1}{2} \left\langle \operatorname{grad}_k \operatorname{grad}_i h_i(X_t) \left(\int_0^t g_s \, ds \right), dX_{k,t} - AX_{k,t} dt \right.$$

$$\left. + \tfrac{1}{2} \operatorname{grad}_k h_k(X_t) dt \right\rangle + \int_0^T \langle \tilde{g}_t, \tfrac{1}{2} \operatorname{grad}_i h_i(X_t) \rangle \, dt.$$

By applying the Itô formula once more, now to the function $\frac{1}{2}\langle \mathrm{grad}_i\, h_i(X_T), \int_0^T g_t\, \mathrm{d}t\rangle$

$$
\begin{aligned}
D_g^i H_i = {} & \tfrac{1}{2}\left\langle \mathrm{grad}_i\, h_i(X_T), \int_0^T g_t\, \mathrm{d}t \right\rangle - \frac{1}{4}\int_0^T \left\langle V'''(X_{i,t}), \int_0^t g_s\, \mathrm{d}s \right\rangle \mathrm{d}t \\
& - \sum_{|k-i|\leqslant R} \int_0^T \tfrac{1}{2}\left\langle \mathrm{grad}_i\, \mathrm{grad}_k\, h_i(X_t)\left(\int_0^t g_s\, \mathrm{d}s\right), AX_{k,t} \right\rangle \mathrm{d}t \\
& + \tfrac{1}{4} D_g^i \left(\sum_{|k-i|\leqslant R} \int_0^T \langle \mathrm{grad}_k\, h_i(X_t), \mathrm{grad}_k\, (h_k - \tfrac{1}{2} h_i)(X_t)\rangle\, \mathrm{d}t \right) \\
& - \frac{1}{2}\int_0^T \left\langle \int_0^t Ag_s\, \mathrm{d}s, \mathrm{grad}_i\, h_i(X_t) \right\rangle \mathrm{d}t. \qquad (23)
\end{aligned}
$$

In the last expression each term is well defined. We used at some place the following important equality: for $k \neq i$

$$
\begin{aligned}
\mathrm{grad}_k\, \mathrm{grad}_i\, h_i &= \mathrm{grad}_k\, \mathrm{grad}_i\, h_k \\
&= a(i-k)\,\mathrm{Id}.
\end{aligned}
$$

The last step will be to recognize in the RHS of (23) the derivative with respect to the ith coordinate of some functional (which will be H_i).

The first two terms are easy to identify with

$$
D_g^i\left(\tfrac{1}{2} h_i(X_T) - \tfrac{1}{2} h_i(X_0) - \frac{1}{4}\int_0^T \sum_{|k-i|\leqslant R} \mathrm{Trace}\ \mathrm{grad}_k^{(2)}\, h_i(X_t)\mathrm{d}t\right).
$$

The third and fifth terms are formally equal to the derivative of

$$
-\frac{1}{2}\int_0^T \sum_{|k-i|\leqslant R} \langle \mathrm{grad}_k\, h_i(X_t), AX_{k,t}\rangle\, \mathrm{d}t.
$$

Since this expression is not well defined, we use again the techniques of Appendix B; it then has a sense either as

$$
\begin{aligned}
& \tfrac{1}{2} h_i(X_0) - \tfrac{1}{2} h_i(X_T) + \frac{1}{2}\int_0^T \sum_{|k-i|\leqslant R} \langle \mathrm{grad}_k\, h_i(X_t), \mathrm{d}W_{k,t}\rangle \\
& \quad - \frac{1}{4}\int_0^T \sum_{|k-i|\leqslant R} \langle \mathrm{grad}_k\, h_i(X_t), \mathrm{grad}_k\, h_k(X_t)\rangle\, \mathrm{d}t \\
& \quad + \frac{1}{4}\int_0^T \sum_{|k-i|\leqslant R} \mathrm{Trace}\ \mathrm{grad}_k^{(2)}\, h_i(X_t)\mathrm{d}t
\end{aligned}
$$

(where each term is Q-a.s. well defined) or as a limit of polygonal approximations.

Actually, we can take as the functional H_i the following one (it is defined modulo functionals depending only on $\mathrm{pr}_{\{i\}^c}(X)$ and $p_0(X)$):

$$H_i(X) = \tfrac{1}{2}\,h_i(X_T) - \tfrac{1}{2}\,h_i(X_0)$$
$$-\frac{1}{4}\int_0^T \sum_{|k-i|\leqslant R} \mathrm{Trace}\,\mathrm{grad}_k^{(2)}\,h_i(X_t) - \langle \mathrm{grad}_k\,h_i(X_t),$$
$$(\mathrm{grad}_k\,h_k - \tfrac{1}{2}\,\mathrm{grad}_k\,h_i)(X_t)\rangle\,\mathrm{d}t$$
$$-\frac{1}{2}\int_0^T \sum_{|k-i|\leqslant R} \langle \mathrm{grad}_k\,h_i(X_t), AX_{k,t}\rangle\,\mathrm{d}t.$$

We then obtain the desired Eq. (18), and Proposition 8 is proven.

Second step: We now show that the assumptions of Theorems 4 and 5 are fulfilled. Condition (7) is clearly satisfied by Q^y, the law of the solution of (14) with deterministic initial condition $y \in \mathcal{Q}_{\beta,\infty}^{-p_0}$. Conditions (8) and (9) are satisfied: in the proof of Proposition 8 we computed the value of $D_g^i H_i$ and have shown that it belongs to $L^1(Q)$. Condition (11) is a consequence of (12), which we now prove. We can represent H_i also in the following way: Q^y-a.s.

$$H_i = \sum_{|k-i|\leqslant R}\int_0^T \langle \tfrac{1}{2}\,\mathrm{grad}_k\,h_i(X_t), \mathrm{d}W_{k,t}\rangle - \frac{1}{2}\sum_{|k-i|\leqslant R}\int_0^T \|\tfrac{1}{2}\,\mathrm{grad}_k\,h_i(X_t)\|^2\,\mathrm{d}t$$
$$= M_T^i - \tfrac{1}{2}\,\langle M^i\rangle_T,$$

where M_T^i is a Q^y-real valued martingale.

Therefore $\exp H_i$ is the value taken at time T of the exponential supermartingale (or local martingale) corresponding to M^i, so that

$$E_{Q^y}(\mathrm{e}^{H_i}) \leqslant 1.$$

In fact $(\mathrm{e}^{M_t^i-(1/2)\langle M^i\rangle_t})_{t\leqslant T}$ is even a martingale for the following reason: let $\tilde{Q}^y$ be the law of the unique solution of the following system:

$$\begin{aligned}
&\mathrm{d}X_{i,t} = \mathrm{d}W_{i,t} + AX_{i,t}\,\mathrm{d}t,\\
&\mathrm{d}X_{k,t} = \mathrm{d}W_{k,t} + (AX_{k,t} - \tfrac{1}{2}\,V'(X_{k,t}) - \sum_{\substack{|j-k|\leqslant R\\ j\neq i}} a(k-j)X_{j,t})\,\mathrm{d}t, \quad k\neq i,\\
&X_0 \equiv y.
\end{aligned} \tag{24}$$

We remark that the drift in the second equation of (24) and in (14) differ by $\frac{1}{2}\,\mathrm{grad}_k\,h_i(X_t) = a(k-i)X_{i,t}$. (Existence and uniqueness of solutions for (24) is treated exactly as for (14).) As for the proof of Proposition 8, we use Appendix B to deduce the following assertion: Since $\tilde{Q}^y(\sum_{|k-i|\leqslant R}\int_0^T \|\frac{1}{2}\,\mathrm{grad}_k\,h_i(X_t)\|^2\,\mathrm{d}t < +\infty) = 1$, we have $\tilde{Q}^y|_{\mathcal{F}_T} \ll Q^y|_{\mathcal{F}_T}$ and

$$\frac{\mathrm{d}\tilde{Q}^y|_{\mathcal{F}_T}}{\mathrm{d}Q^y|_{\mathcal{F}_T}} = \exp\left(M_T^i - \tfrac{1}{2}\,\langle M^i\rangle_T\right).$$

In particular $\exp H_i$ has a Q^y-expectation *equal* to 1, which proves hypothesis (12).

By the way, (13) is now almost proven since, under the law

$$\exp(H_i + h_i\circ p_0)(\omega).\,Q = \tilde{Q}^{\omega_0},$$

we see from (24) that the dynamics of $(X_k)_{k \neq i}$ is independent from X_i. Then $X_{\{i\}^c}$ and $X_{i,0}$ are independent as soon as $X_{\{i\}^c,0}$ and $X_{i,0}$ are independent. But, since $Q \circ p_0^{-1}$ is a (φ, μ)-Gibbs measure, $X_{\{i\}^c,0}$ and $X_{i,0}$ are independent with laws $\otimes_{j \neq i} \mu_j$ respectively μ_i under $\exp(H_i + h_i \circ p_0)(\omega) . Q$.

We now finish the proof of the necessary condition of Theorem 7; applying the reciprocal of Theorem 4 to Q, we know that Q^y is a (Φ, P^y)-Gibbs measure for each $y \in \mathscr{Q}_{\beta,\infty}^{-p_0}$ and for Φ associated to the Hamilton function H given by (18). (We do not do Φ explicit because it is more complicated than H and it does not give more information.) Theorem 5 allows to conclude that Q is a $(\Phi + \varphi \circ p_0, P^\mu)$-Gibbs measure.

Sufficient condition. Our assumption is now that $Q \in \mathscr{P}(\Omega)$ is a $(\Phi + \varphi \circ p_0, P^\mu)$-Gibbs measure on Ω.

To prove that Q is the law of the solution of (14) we proceed in an analogous way as in Cattiaux et al. (1996), so we give just a sketch of the proof.

Step 1: By definition of the Gibbs property, for each Λ finite subset of $\mathbb{Z}^d$, Q is absolutely continuous with respect to $\otimes_{i \in \Lambda} \rho_{\mu_i} \otimes (Q \circ \mathrm{pr}_{\Lambda^c}^{-1})$ where ρ_{μ_i} is the O–U law with initial distribution $\mu_i \in \mathscr{P}(E)$.

In particular, for $\Lambda = \{i\}$, $Q \circ \mathrm{pr}_i^{-1}$ is absolutely continuous with respect to ρ_{μ_i}, which implies that, for each $\bar{g} \in \mathscr{D}$, there exists unique $(\mathscr{F}_t)$-adapted $\mathscr{H}$-valued process b_i such that, under Q,

$$B_{i,t} = \langle X_{i,t}, \bar{g} \rangle - \langle X_{i,0}, \bar{g} \rangle - \int_0^t \langle AX_{i,s}, \bar{g} \rangle \, \mathrm{d}s - \int_0^t \langle b_{i,s}, \bar{g} \rangle \, \mathrm{d}s, \quad t \in [0, T],$$

is a real Brownian motion (cf. Métivier 1982, Theorem 30.3).

Step 2: *Identification of* b_i. We remark that b_i is the unique process for which $B_{i,t}$ is a Q-martingale. If we verify this martingale property for $b_{i,r} = -\frac{1}{2} \operatorname{grad}_i h_i(X_r)$ the process b_i is identified. Thus it remains to prove the following equality: $\forall s \in [0, T]$ $\forall F_s$ smooth, $\mathscr{F}_s$-measurable,

$$E_Q\left(F_s \cdot \int_0^T \langle \tilde{g}_r, \mathrm{d}X_{i,r} - AX_{i,r} \, \mathrm{d}r \rangle \right) = E_Q\left(F_s \cdot \int_0^T \langle \tilde{g}_r, -\tfrac{1}{2} \operatorname{grad}_i h_i(X_r) \rangle \, \mathrm{d}r \right),$$

where $\tilde{g}_r = 1_{[s,t]}(r) \cdot \tilde{g} \in L^2(0, T; \mathscr{D})$. But, from Theorem 4, since Q is Gibbs it satisfies (10), where we can take $F = F_s$. It implies that the last equation is equivalent to

$$E_Q(D_g^i F_s - F_s \cdot D_g^i H_i) = E_Q\left(F_s \cdot \int_0^T \langle \tilde{g}_r, -\tfrac{1}{2} \operatorname{grad}_i h_i(X_r) \rangle \, \mathrm{d}r \right), \tag{25}$$

where g is the unique element of $L^2(0, T; \mathscr{D})$ such that

$$\tilde{g}_r = g_r - \int_0^r A g_r \, \mathrm{d}\tau.$$

In (25), the term $D_g^i F_s$ disappears for the following two reasons: because the Volterra equation induces a 1–1 correspondence between g and $\tilde{g}$, if $\tilde{g}_r \equiv 0$ for $r \in [0, s]$, then $g_r \equiv 0$, $0 \leqslant r \leqslant s$; secondly, the differential operator D^i is local in time, that is F_s is $\mathscr{F}_s$-measurable and $\sup g \cap [0, s] = \emptyset$ imply $D_g^i F_s = 0$.

Therefore (25) is equivalent to

$$E_Q\left(F_s\cdot\left(\int_0^T \langle \tilde{g}_r, -\tfrac{1}{2}\operatorname{grad}_i h_i(X_r)\rangle \,\mathrm{d}r + D_g^i H_i\right)\right) = 0. \tag{26}$$

We now use $D_g^i H_i$ in the form (23) and apply the Itô formula to the function $\langle \frac{1}{2}\operatorname{grad}_i h_i(X_T), \int_0^T g_r \,\mathrm{d}r\rangle$, to obtain Q-a.s.

$$D_g^i H_i = M_T - M_s + \int_0^T \langle \tfrac{1}{2}\operatorname{grad}_i h_i(X_r), \tilde{g}_r\rangle \,\mathrm{d}r$$

for some martingale M. So, using one more time that $\tilde{g}_r = g_r \equiv 0, 0 \leqslant r \leqslant s$, we deduce that (26) is satisfied.

Step 3: For each finite subset $\Lambda \subset \mathbb{Z}^d$, the Brownian motions $(B_i)_{i\in\Lambda}$ are independent: since $Q \circ \mathrm{pr}_\Lambda^{-1} \ll \otimes_{i\in\Lambda} \rho_{\mu_i}$, $(\langle X_i, \bar{g}\rangle)_{i\in\Lambda}$ is a card Λ-dimensional Q-Brownian motion B_Λ with drift. By the uniqueness of the semi-martingale decomposition we have equality in distribution of B_Λ and $(B_i)_{i\in\Lambda}$. Therefore $(B_i)_{i\in\Lambda}$ are independent.

Step 4: Identification of $Q \circ p_0^{-1}$ as a (φ, μ)-Gibbs measure: independently of the initial condition of Q, one deduces from the above steps 1–3 that e^{H_i} is a Q-martingale. Then it is simple to compute that the projection at time 0 of a $(\Phi + \varphi \circ p_0, P^\mu)$-Gibbs measure is a (φ, μ)-Gibbs measure. The proof of Theorem 7 is complete. □

Remarks. (i) The Hamilton function H can be reformulated in a more suggestive way than we did in (18). Consider the infinitesimal generator $\mathscr{L}$ of the solution process X of (14), defined for $x \in \mathscr{H}^{\mathbb{Z}^d}$ and a smooth local function f on $\mathscr{H}^{\mathbb{Z}^d}$ by

$$\mathscr{L}f(x) = \sum_{k\in\mathbb{Z}^d} \tfrac{1}{2}\operatorname{Trace}\operatorname{grad}_k^{(2)} f(x) + \langle \operatorname{grad}_k f(x), Ax_k - \tfrac{1}{2}\operatorname{grad}_k h_k(x)\rangle.$$

Then we have formally

$$H_i(\omega) = \tfrac{1}{2} h_i(\omega_T) - \tfrac{1}{2} h_i(\omega_0) - \int_0^T \mathscr{L}(\tfrac{1}{2} h_i)(\omega_s)\,\mathrm{d}s - \tfrac{1}{2}\int_0^T \sum_{|j-i|\leqslant R} \|\tfrac{1}{2}\operatorname{grad}_j h_i(\omega_s)\|^2 \,\mathrm{d}s.$$

In the formulation of the theorem we preferred the more detailed formulation, because the term $\mathscr{L}(\frac{1}{2} h_i)(\omega_s)$ includes the problematic terms $\langle \frac{1}{2}\operatorname{grad}_k h_i(\omega_s), A\omega_{k,s}\rangle$, which are not pointwise well defined. Note that the proof used also the following martingale representation:

$$H_i = M_T^i - \tfrac{1}{2}\langle M^i\rangle_T,$$

where M^i is the martingale introduced above.

(ii) In Definition 3 of Gibbs measures, we take as reference measure λ a product measure: it is the most natural one. Without any difficulty one can generalize to a measure with dependent projections. Then, in Theorem 7, we could take as reference measure (free field) the interacting Ornstein–Uhlenbeck process corresponding to the quadratic part of the Hamilton function. The Hamilton function H with respect to this other free field would be simpler (since it is just concerned with the non-linearity of h).

(iii) It is not necessary to assume that the dynamics of the system (14) is of gradient type to obtain the Gibbsian nature of the law of the solution. In this case, stochastic integrals cannot be eliminated form the Hamilton function. But this generalization makes sense only if we know the existence and uniqueness of nice solutions for non-gradient type SPDE systems.

One important interpretation of Theorem 7 is the one-to-one correspondence between Gibbs measures on $E^{\mathbb{Z}^d}(= C(S_\beta)^{\mathbb{Z}^d})$ as initial distribution and some Gibbs measures on the path-space Ω. We have the following criterium on phase transition.

Proposition 9. *For the infinite system of SPDE* (14), *there is phase transition at time* 0 *if and only if there is a global phase transition on the path-space level.*

4. Consequences for the reversible case

We now give an application of Theorem 7 to the study of the reversibility of the stochastic system (14).

Theorem 10. *Let Q be the law of the solution of the SPDE* (14), *with initial distribution $\gamma \in \mathscr{P}(\mathscr{Q}_{\beta,c}^{-p_0})$. γ is reversible for the system – i.e. Q is invariant under time reversal: $\omega(t) \mapsto \omega(T-t)$ – if and only if γ is a (ψ, μ)-Gibbs measure where ψ is the pair interaction on $E^{\mathbb{Z}^d}$ defined by* (15), *$\mu = \bigotimes_{i \in \mathbb{Z}^d} \mu_i$, and μ_i is, for each $i \in \mathbb{Z}^d$, the law of a centred Gaussian field on E with covariance operator $-\frac{1}{2}A^{-1}$.*

Proof. (i) Let us show that if γ is a (ψ, μ)-Gibbs measure, then Q is a time reversible invariant. By Theorem 7, Q is Gibbs with P^μ as reference measure and the following Hamilton function: $\forall i \in \mathbb{Z}^d$, $\omega \in \Omega$,

$$H_i(\omega) + h_i(\omega_0) = \tfrac{1}{2}\, h_i(\omega_T) + \tfrac{1}{2}\, h_i(\omega_0) + \int_0^T K_i(\omega_s)\, \mathrm{d}s - \int_0^T \sum_{|j-i| \leqslant R} \langle \tfrac{1}{2} \operatorname{grad}_j h_i(\omega_s), A\omega_{j,s} \rangle\, \mathrm{d}s,$$

where K_i is defined by the third term in the RHS of (18). The above expression is clearly invariant under time reversal.

Then the law Q^- of the time reversed is a Gibbs measure with the same Hamilton function as Q but with respect to the reversed reference measure $(P^\mu)^-$. But the Gaussian field μ_i is the unique reversible distribution for the O–U process solution of (3). Thus $(P^\mu)^- = P^\mu$ and Q^- is a $(\Phi + \psi \circ p_0, P^\mu)$-Gibbs measure exactly as Q is. By Theorem 7, it implies that Q^- is the law solution of (14), and to obtain $Q = Q^-$ we just have to identify the initial conditions $Q \circ p_0^{-1} = Q \circ p_0^{-1} = \gamma$ and $Q^- \circ p_0^{-1}$.

Let us denote $\gamma_T = Q^- \circ p_0^{-1} = Q \circ p_T^{-1}$. By construction, $Q^- \circ p_{T/2}^{-1} = Q \circ p_{T/2}^{-1}$. We denote this distribution by $\gamma_{T/2}$. But Q (resp. Q^-) is Markovian. Then γ_T (resp. γ) is the law at time $T/2$ of the process under Q (resp. Q^-) with initial condition $\gamma_{T/2}$. This implies $\gamma_T = \gamma$.

(ii) Reciprocally, let γ be a reversible initial condition for the system (14). We will show that γ is a Gibbs measure (since it solves an integration by parts formula). By Proposition 8, Q satisfies the equilibrium equation (10). Firstly, by taking the test functional equal to $F(X_t) = f(\langle X_{j1}, \varphi_1\rangle, \ldots, \langle X_{jn,t}, \varphi_n\rangle, g_r(u) = 1_{[s,t]}(r)\bar{g}(u), 0 \leqslant s < t \leqslant T$, $\bar{g} \in \mathscr{D}$ and f a C^∞ function on $\mathbb{R}^n$, we obtain

$$(t-s)E_Q(\langle \mathrm{grad}_i\, F(X_t), \bar{g}\rangle) = E_Q(F(X_t)\int_s^T \langle \tilde{g}_r, \mathrm{d}X_{i,r} - AX_{i,r}\,\mathrm{d}r\rangle) + E_Q(F(X_t)D_g^i H_i), \tag{27}$$

where $\mathrm{grad}_i\, F(y) = \sum_{k,j_k=i} \partial_k f(\langle y_{j1}, \varphi_1\rangle, \ldots, \langle y_{jn}, \varphi_n\rangle)\varphi_k$.

Now choosing the test functional equal to $F(X_s)$ we obtain in (10):

$$0 = E_Q(F(X_s)\int_s^T \langle \tilde{g}_r, \mathrm{d}X_{i,r} - AX_{i,r}\,\mathrm{d}r\rangle) + E_Q(F(X_s)D_g^i H_i) \tag{28}$$

By summing (27) and (28), we obtain

$$E_Q(\langle \mathrm{grad}_i\, F(X_t), \bar{g}\rangle) = \frac{1}{t-s}E_Q((F(X_t)+F(X_s))\int_s^T \langle \tilde{g}_r, \mathrm{d}X_{i,r} - AX_{i,r}\mathrm{d}r\rangle) + E_Q((F(X_t)+F(X_s))\frac{1}{t-s}D_g^i H_i). \tag{29}$$

By stationarity of Q, the left-hand side is equal to $E_\gamma(\langle \mathrm{grad}_i\, F(\cdot), \bar{g}\rangle)$. Let us replace $\tilde{g}$ by its definition in the time integral of the RHS. It then becomes

$$\int_s^t \langle \bar{g}, \mathrm{d}X_{i,r}\rangle - \int_s^t \langle \bar{g}, AX_{i,r}\rangle\,\mathrm{d}r - \int_s^t (r-s)\langle A\bar{g}, \mathrm{d}X_{i,r} - AX_{i,r}\,\mathrm{d}r\rangle$$
$$- \int_t^T (t-s)\langle A\bar{g}, \mathrm{d}X_{i,r} - AX_{i,r}\,\mathrm{d}r\rangle$$
$$=: (1)+(2)+(3)+(4).$$

The contribution of (1) in (29) is zero since the dynamics is reversible:

$$E_Q(F(X_t)\langle \bar{g}, X_{i,t} - X_{i,s}\rangle) = E(F(X_s)\langle \bar{g}, X_{i,s} - X_{i,t}\rangle) = -E(F(X_s)\langle \bar{g}, X_{i,t} - X_{i,s}\rangle).$$

In the limit $s \to t$, the contribution of (2) in (29) becomes

$$\lim_{s\to t} -E_Q\left((F(X_t)+F(X_s))\frac{1}{t-s}\int_s^t \langle \bar{g}, AX_{i,r}\rangle\,\mathrm{d}r\right)$$
$$= -2E_Q(F(X_t)\langle \bar{g}, AX_{i,t}\rangle)$$
$$= -2E_\gamma(F(y)\langle \bar{g}, Ay_i\rangle).$$

The interchange of expectation and limit is allowed here.

The contribution of (3) in (29) is zero, since

$$\lim_{s\to t} E_Q(F(X_t) + F(X_s)) \int_s^t \frac{r-s}{t-s} \langle A\bar{g}, \mathrm{d}X_{i,r} - AX_{i,r}\,\mathrm{d}r\rangle) = 0.$$

The contribution of (4) in (29) is then

$$-E_Q(2F(X_t)\langle A\bar{g}, \int_t^T \mathrm{d}X_{i,r} - AX_{i,r}\,\mathrm{d}r\rangle),$$

which vanishes when we let t tend to T.

Finally the last term of (29) becomes

$$\lim_{t\to T}\lim_{s\to t} E_Q((F(X_t) + F(X_s))\frac{1}{t-s} D_g^i H_i).$$

But, using Proposition 2.5 in Föllmer (1986) in the first equation and (23) in the second equation yields, for a.s.t,

$$\begin{aligned}
&\lim_{s\to t} E_Q((F(X_t) + F(X_s))\frac{1}{t-s} D_g^i H_i) \\
&\quad = E_Q(2F(X_t)\langle g_t, D_t^i H_i)\rangle) \\
&\quad = E_Q(2F(X_t)\langle \bar{g}, E(D_t^i H_i/\mathscr{F}_t)\rangle) \\
&\quad = E_Q\left(F(X_t)\left(\langle \bar{g}, \operatorname{grad}_i h_i(X_t)\rangle - \int_t^T \langle A\bar{g}, \operatorname{grad}_i h_i(X_r)\rangle\,\mathrm{d}r\right)\right),
\end{aligned}$$

which tends, when t goes to T, to

$$\begin{aligned}
&E_Q(F(X_T)\langle \bar{g}, \operatorname{grad}_i h_i(X_T)\rangle) \\
&\quad = E_\gamma(F(y)\langle \bar{g}, \operatorname{grad}_i h_i(y)\rangle).
\end{aligned}$$

To summarize γ satisfies the following functional equation:

$$\int \langle \bar{g}, \operatorname{grad}_i F(y)\rangle\, \gamma(\mathrm{d}y) = \int F(y)\langle \bar{g}, -2Ay_i + \operatorname{grad}_i h_i(y)\rangle \gamma(\mathrm{d}y) \tag{30}$$

for each $i \in \mathbb{Z}^d$, F smooth cylindrical function on $C(S_\beta)^{\mathbb{Z}}$ and $\bar{g} \in \mathscr{D}$.

Let us prove finally that this characterizes γ as Gibbs measure: If we take a test function F of the form

$$F(y) = F_i(y_i)G(y_{\{i\}^c}),$$

then (30) implies that $\gamma(\mathrm{d}y_i/y_{\{i\}^c})$ satisfies γ a.s.

$$\int \langle \bar{g}, \operatorname{grad}_i F_i(y_i)\rangle \gamma(\mathrm{d}y_i/y_{\{i\}^c}) = \int F_i(y_i)\langle \bar{g}, 2Ay_i + \operatorname{grad}_i h_i(y)\rangle \gamma(\mathrm{d}y_i/y_{\{i\}^c})$$

for each F_i smooth cylindrical function on $C(S_\beta)$ and $\bar{g} \in \mathscr{D}$.

Therefore, the measure $\tilde{\gamma}_i(\mathrm{d}y_i) = \exp h_i(y) \cdot \gamma(\mathrm{d}y_i/y_{\{i\}^c})$ satisfies, for each smooth F_i and $\bar{g}$, the equation

$$\int \langle \bar{g}, \mathrm{grad}_i\, F_i(y_i) \rangle \tilde{\gamma}_i(\mathrm{d}y_i) = -\int F_i(y_i) \langle \bar{g}, 2Ay_i \rangle \tilde{\gamma}_i(\mathrm{d}y_i)$$

which characterizes $\tilde{\gamma}_i$, up to a normalizing constant, as the centred Gaussian measure on $C(S_\beta)$ with covariance operator $\frac{1}{2} A^{-1}$ (cf. for example Berezanski and Kondratiev, 1988). This completes the proof. □

Theorem 10 shows the equivalence between *reversibility* and *Gibbsian* property for the initial distribution of the system (14). It is the generalization for this quantum system of Doss and Royer's result (1978) for classical systems. The next natural but much more delicate question is the following: is every *stationary or time invariant distribution a Gibbs* one? In classical context, the answer is positive under some assumptions (cf. Fritz (1982), using entropy arguments). For general quantum systems, there is no answer, but we can give one in our context when we assume the ergodicity of the system. To this aim, we adjoin the following assumption (31) on the coefficients of (14) in order to assure the monotonicity of the system: Let p be such that the solution of (14) takes its values in $\mathcal{Q}_{\beta,c}^{-p}$. We suppose the mass term m in the operator A is large enough so that

$$l =: m^2 - \frac{1}{2} \sum_{0 < |k| \leqslant R} a^2(k) \cdot \sum_{0 < |k| \leqslant R} (1 + |k|)^{2p} - \tfrac{1}{2} - b > 0. \tag{31}$$

(b is the constant appearing in (16).) We also let the system evolve during unbounded time, that is $T = +\infty$.

Theorem 11. *Let us suppose that the coefficients of the system* (14) *satisfy the assumptions* (16) *and* (31). *Then the system admits a unique invariant distribution* γ_e *on* $\mathcal{Q}_{\beta,r}^{-p} = \{y_k(\cdot) \in L_r(S_\beta), k \in \mathbb{Z}^d$, *such that* $\sum_{k \in \mathbb{Z}^d} (1 + |k|)^{-2p} \|y_k\|_r^2 < +\infty\}$ *for* $r > 2K$. *Furthermore, the set of* (ψ, μ)*-Gibbs measures* (*defined in Theorem* 10) *reduces to one element.*

Proof. (*Uniqueness of the invariant distribution*). This is a direct consequence of the ergodic behaviour of the system (14) proven in Albeverio et al. (1994) under the monotonicity assumption (31), the proof of which is based on the exponential loss of memory: If X_t and $\tilde{X}_t$ are two solutions of (14) with initial condition X_0 resp. $\tilde{X}_0$, then

$$\|X_t - \tilde{X}_t\|_{\mathcal{Q}_{\beta,r}^{-p}} \leqslant \mathrm{e}^{-lt} \|X_0 - \tilde{X}_0\|_{\mathcal{Q}_{\beta,r}^{-p}}, \quad \forall t \geqslant 0.$$

Uniqueness of the (ψ, μ)*-Gibbs measures.* Let us first remark that such Gibbs measures exist (see, for example, Park and Yoo, 1994, Theorem 2.7). By Theorem 10 each (ψ, μ)-Gibbs measure is reversible on each finite time interval $[0, T]$, and then invariant. Its support is $\mathcal{Q}_{\beta,\infty}^{-p_0} \subset \mathcal{Q}_{\beta,r}^{-p}$ for every r and $p > p_0$. By the previous result, we deduce that it is unique. □

The result shows how stochastic dynamics can be useful for the study of properties of Gibbs states: the monotonicity assumption (31) assures the stability of the system, which in turn implies the uniqueness of Gibbs states associated to the potential which defines the dynamics.

Furthermore, Theorem 11 completes the proof of the Boltzmann–Gibbs hypothesis for the model of an anharmonic quantum oscillator, under assumption (31): in Albeverio et al. (1994) the system was proven to be ergodic but the limit was not identified. We now know that the ergodic limit is the Euclidean Gibbs state, belonging to the given inverse temperature and interaction.

Appendix A

We would like to thank Jean Jacod for his help concerning the results which follow.

We show here a version of Girsanov's theorem for the laws of continuous solutions of an SDE with values in Hilbert space, which adapt some well-known results of Jacod and Mémin on solutions of martingale problems in an infinite dimensional Hilbertian situation.

Let $\mathscr{D} \subset \mathscr{H} \subset \mathscr{D}'$ be a Gelfand triple, A an unbounded self-adjoint linear operator on $\mathscr{H}$ with domain in $\mathscr{D}$ and b (resp. $\tilde{b}$), some bounded $\mathscr{H}$-valued (non-linear) term defined on E, dense Banach space in $\mathscr{H}$. We suppose that the following SDE on the probability space $(\Omega, \mathscr{F})$,

$$\mathrm{d}X_t = \mathrm{d}W_t + AX_t + b(X_t)\,\mathrm{d}t, \quad 0 \leqslant t \leqslant T, \quad X_0 = x \in \mathscr{H}, \tag{A.1}$$

$$\mathrm{d}X_t = \mathrm{d}W_t + AX_t + \tilde{b}(X_t)\,\mathrm{d}t, \quad 0 \leqslant t \leqslant T, \quad X_0 = x \in \mathscr{H}, \qquad \text{(resp. A.}\tilde{1}\text{)}$$

with W, a cylindrical Brownian motion on $\mathscr{H}$, admits a unique, continuous E-valued mild solution whose law is denoted by Q (resp. $\tilde{Q}$) (cf. Da Prato and Zabczyk, 1992, Ch. 7).

Theorem A.1. *We have the following equivalence:*

$$\tilde{Q} \ll Q \;\Leftrightarrow\; \tilde{Q}\left(\int_0^T \|\tilde{b}(X_t) - b(X_t)\|^2\,\mathrm{d}t < +\infty\right) = 1.$$

In this case,

$$\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}Q} = \exp\left(\int_0^T \langle \tilde{b}(X_t) - b(X_t), \mathrm{d}W_t\rangle - \frac{1}{2}\int_0^T \|\tilde{b}(X_t) - b(X_t)\|^2\,\mathrm{d}t\right),$$

where the bracket in the first integral denotes the real valued stochastic integral.

Proof. Jacod (1979, Theorem 12.57) proves the above theorem for finite dimensional spaces $\mathscr{H}$, or equivalently for the law of a finite system of real valued SDE:

$$\mathrm{d}X_t^i = \mathrm{d}W_t^i + \beta^i(X_t)\,\mathrm{d}t, \quad i \in I = \{1, \dots, N\}; \quad X_0^i = x_i \in \mathbb{R},$$
$$X_t = (X_t^1, \dots, X_t^N) \in \mathbb{R}^N.$$

One can extend his proof without any difficulty to a countable set of indices I, for example $I = \mathbb{N}$. This infinite system is then equivalent to our Hilbertian situation (A.1) by posing

$X_t^i = \langle X_t, e_i \rangle$, for $(e_i, i \in \mathbb{N})$ an orthonormal basis of $\mathscr{H}$ which belongs to $\mathscr{D}$.

$\beta^i(\cdot) = \langle A + b(\cdot), e_i \rangle$, $W_t^i = \langle W_t, e_i \rangle$.

The density of $\tilde{Q}$ with respect to Q is then

$$\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}Q} = \exp \int_0^T \sum_{i \in \mathbb{N}} (\tilde{\beta}^i - \beta^i)(X_t)\, \mathrm{d}W_{i,t} - \frac{1}{2} \int_0^T \sum_{i \in \mathbb{N}} (\tilde{\beta}^i - \beta^i)^2(X_t)\, \mathrm{d}t$$

$$= \exp \int_0^T \langle (\tilde{b} - b)(X_t), \mathrm{d}W_t \rangle - \frac{1}{2} \int_0^T \|(\tilde{b} - b)(X_t)\|^2\, \mathrm{d}t,$$

which is the desired result. □

With the same method, we can generalize the above theorem to the laws of an infinite system of $\mathscr{H}$-valued SDE. Let us suppose that, for a family of $\mathscr{H}$-valued drifts b_k (resp. $\tilde{b}_k$), $k \in \mathbb{Z}^d$, defined on $E^{\mathbb{Z}^d}$, and a family W_k of independent cylindrical Brownian motions on $\mathscr{H}$, the system

$$\mathrm{d}X_{k,t} = \mathrm{d}W_{k,t} + AX_{k,t} + b_k(X_t)\,\mathrm{d}t, \quad k \in \mathbb{Z}^d, \quad 0 \leqslant t \leqslant T, \tag{A.2}$$

$$X_t = (X_{k,t})_{k \in \mathbb{Z}^d} \in \mathscr{H}^{\mathbb{Z}^d}, \quad X_0 = x \in \mathscr{H}^{\mathbb{Z}^d}$$

(resp. $(\widetilde{\mathrm{A.2}})$) admits a unique continuous solution whose law is denoted by Q (resp. $\tilde{Q}$), probability on $C(0, T; E^{\mathbb{Z}^d})$.

Theorem A.2. *The following equivalence holds:*

$$\tilde{Q} \ll Q \Leftrightarrow \tilde{Q}\left(\int_0^T \sum_{k \in \mathbb{Z}^d} \|\tilde{b}_k(X_t) - b_k(X_t)\|^2\, \mathrm{d}t < +\infty \right) = 1.$$

In this case,

$$\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}Q} = \exp\left(\int_0^T \sum_{k \in \mathbb{Z}^d} \langle (\tilde{b}_k - b_k)(X_t), \mathrm{d}W_{k,t} \rangle - \frac{1}{2} \int_0^T \sum_{k \in \mathbb{Z}^d} \|(\tilde{b}_k - b_k)^2(X_t)\|^2\, \mathrm{d}t \right).$$

Proof. We reduce the problem to a system of real valued SDE indexed by $I = \mathbb{N}^{\mathbb{Z}^d}$ by projecting each $\mathscr{H}$-valued SDE on $\mathbb{R}^{\mathbb{N}}$, exactly as in the last proof, and then use the same argument. □

Appendix B

Let V be a C^2 self-potential on $\mathbb{R}$ satisfying the (usual) growth conditions:

$$\forall x \in \mathbb{R} \quad |V'(x)| \leqslant k(1 + |z|^K), \quad k > 0, \quad K \geqslant 1,$$

$$\forall x, y \in \mathbb{R} \quad -\frac{1}{2}(x - y)(V'(x) - V'(y)) \leqslant b(x - y)^2, \quad b \in \mathbb{R}.$$

Let $-A$ be the linear positive self-adjoint operator on $\mathscr{H} = L^2(S_\beta)$ defined in Section 3 by $A = +\frac{1}{2}(\partial^2/\partial u^2) - m^2\,\mathrm{Id}$, and let X be a strong generalized solution of the $\mathscr{H}$-valued SDE:

$$\mathrm{d}X_t = AX_t\,\mathrm{d}t - \tfrac{1}{2}V'(X_t(\cdot))\,\mathrm{d}t + \mathrm{d}W_t, \quad 0 \leqslant t \leqslant T, \tag{B.1}$$

where W is a cylindrical-Brownian motion on $\mathscr{H}$. Following Funaki (1983) or Iwata (1987), X exists, is unique and, for each t, $X_t \in C(S_\beta)$ a.s.

If f is a $\mathscr{C}^2$ function on $\mathbb{R}$, one can then compute the real valued stochastic integral

$$\int_0^T \langle f'(X_t(\cdot)), \mathrm{d}W_t\rangle. \tag{B.2}$$

Our aim is to represent this stochastic integral in another way, i.e. in terms of the operator A, in order to be able to treat the reversibility of this functional. The first natural idea is to obtain (B.2) by applying the Itô formula to the functional: $X_t \to \int_{S_\beta} f(X_t(u))\,\mathrm{d}u$. Indeed,

$$\int_{S_\beta} (f(X_T(u)) - f(X_0(u)))\,\mathrm{d}u = \int_0^T \langle f'(X_t(\cdot)), \mathrm{d}X_t\rangle + \frac{1}{2}\int_0^T \int_{S_\beta} f''(X_t(u))\,\mathrm{d}u\,\mathrm{d}t,$$

where each term is well defined.

The question is now whether we can replace the integral with respect to the semi-martingale X by some other terms including the integral with respect to W. Formally (X is only a mild and not a strong solution; cf. Da Prato and Zabczyk, 1992, Theorem 7.6)

$$\int_0^T \langle f'(X_t), \mathrm{d}X_t\rangle = \int_0^T \langle f'(X_t), AX_t\rangle\,\mathrm{d}t - \frac{1}{2}\int_0^T \langle f'(X_t), V'(X_t)\rangle\,\mathrm{d}t$$
$$+ \int_0^t \langle f'(X_t), \mathrm{d}W_t\rangle.$$

In the above equation, each term is well defined. The only exception is

$$\int_0^T \langle f'(X_t), AX_t\rangle\,\mathrm{d}t, \text{ because } X_t \text{ (and thus } f'(X_t)) \notin \mathscr{D}(A),$$

So that for fixed t, $\langle f'(X_t), AX_t\rangle$ does not exist.

Therefore, we give a sense to the double integral

$$\int_{[0,T]\times S_\beta} f'(X(u))AX_t(u)\,\mathrm{d}u\,\mathrm{d}t \tag{B.3}$$

and define it as

$$\int_0^T \langle f'(X_t), \mathrm{d}X_t\rangle - \int_0^T \langle f'(X_t), \mathrm{d}W_t\rangle + \int_0^T \frac{1}{2}\langle f'(X_t), V'(X_t)\rangle\,\mathrm{d}t.$$

In conclusion,

$$\int_0^T \langle f'(X_t), \mathrm{d}W_t \rangle = \int_0^T \langle f'(X_t), \mathrm{d}X_t \rangle - \int_0^T \langle f'(X_t), V'(X_t) \rangle \,\mathrm{d}t - \int_0^T \langle f'(X_t), AX_t \rangle \,\mathrm{d}t$$

$$\Leftrightarrow \int_0^T \langle f'(X_t), \mathrm{d}W_t \rangle = \langle f(X_T) - f(X_0), 1 \rangle + \frac{1}{2}\int_0^T \langle f'(X_t)V'(X_t) - f''(X_t), 1 \rangle \,\mathrm{d}t - \int_0^T \langle f'(X_t), AX_t \rangle \,\mathrm{d}t. \quad \text{(B.4)}$$

Let us remark that the equality (B.4) can also be obtained by approximation. Funaki (1983) shows that X, the solution of (B.1), is a scaling limit of processes $(X^n)_{n \in \mathbb{N}}$ which solve polygonal approximations of Eq. (B.1), where the shifted Laplacian A is replaced by the usual finite difference (bounded) operator A^n. A version of (B.4) for X^n is then simple to obtain (there is no problem related to the domain of A^n!) and one easily proves that each term converges to the desired term.

Acknowledgements

The first author thanks D.F.G. for its support. The second author was partially supported by the project "Stochastic Analysis and Statistical Mechanics" of the European Community, which is here greatly acknowledged.

We are grateful to the referees for a fruitful discussion which ameliorated the paper at several points.

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