A general tetrakaidecahedron model for open-celled foams

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Abstract

A micro-mechanics model for non-isotropic, open-celled foams is developed using an elongated tetrakaidecahedron (Kelvin model) as the repeating unit cell. Assuming the cell edges possess axial and bending rigidity, the mechanics of deformation of the elongated tetrakaidecahedron lead to a set of equations for the Young’s modulus, Poisson’s ratio and tensile strength of the foam in the principal material directions. These equations are written as a function of the cell edge lengths and cross-section properties, the inclination angle and the strength and stiffness of the solid material. This micro-mechanics model employs an elongated Kelvin model geometry which is more general than that employed by previous authors, as the size and shape of the repeating unit cell are defined by specifying three independent dimensions. As a result, the model accounts for an additional variation in the unit cell shape which is not accounted for in the previous models. The effect of this additional shape parameter on the non-isotropic stiffness and strength behavior is demonstrated and the advantages of this more general micro-mechanics model are illustrated.

Keywords: Foam material; Elastic material; Energy methods; Elongated tetrakaidecahedron; Elongated Kelvin model

1. Introduction

Previous studies on open and closed-cell foams have sought to establish a direct tie between the foam micro-structure and the macro-level foam properties. Through careful consideration of the foam micro-structure and selection of a suitable representative repeating unit, equations for the foam density, elastic constants and strength have been written in terms of the micro-structural dimensions and the physical and mechanical properties of the solid material (Gent and Thomas, 1959; Demet’ev and Tarakanov, 1970; Huber and Gibson, 1988; and Gong et al., 2005a,b).

To represent the foam micro-structure, many of these previous researchers used a tetrakaidecahedron, a 14-sided polyhedron comprised of six quadrilateral and eight hexagonal faces. The tetrakaidecahedron is widely known as the Kelvin foam model, as it was Thomson (1887) who, in his assessment of Plateau’s experiment, identified the tetrakaidecahedron (with slightly curved faces) as the only polyhedron that packs to fill space...
and minimize the surface area per unit volume (Gibson and Ashby, 1997), Zhu et al. (1997), for example, adopted an equi-axed tetrakaidecahedron to develop equations for the foam Young’s modulus, shear modulus and Poisson’s ratio for isotropic, open-celled foams. They assumed that the mechanical behavior of open-celled foams could be simulated by treating the edges of the cell faces as structural elements possessing axial, bending and torsional rigidity. Applying the principle of minimum potential energy to the deformation of the repeating unit, the equations for the foam elastic constants were written in terms of the cell edge length $L$, the edge cross-sectional area $A$, moment of inertia $I$ and polar moment of inertia $J$ and the Young’s modulus $E$ and shear modulus $G$ of the solid material. Using a similar set of assumptions, Warren and Kraynik (1997) developed similar equations for the Young’s modulus, bulk modulus and shear modulus for isotropic, open-celled foams. The more recent model by Gong et al. (2005a) includes the effect of shear deformation and allows for the edge cross-sectional area to vary along the length of the edge.

In many cases, the foam micro-structure is elongated in the rise direction due to the foaming and rising process causing the foam mechanical behavior to be non-isotropic. To treat non-isotropic foams, Dement’ev and Tarakanov (1970), Gong et al. (2005a,b), Ridha et al. (2006) and others have adopted an elongated tetrakaidecahedron (Fig. 1) as the repeating unit cell, deriving equations for the elastic constants and strengths in the principal material directions. An elongated tetrakaidecahedron also packs to fill the space. It contains eight hexagonal faces, two horizontal square faces and four vertical diamond faces. The horizontal square faces have sides of length $b$ and the diamond faces have sides of length $L$. The hexagonal faces have four sides with length $L$ and two sides with length $b$. The inclination angle $\theta$ defines the orientation of the hexagonal faces with respect to the rise direction as well as the obtuse angle of the vertical diamond faces, $2\theta$.

The size and shape of the elongated tetrakaidecahedron is uniquely defined by specifying the value of any three of the cell dimensions: $b$, $L$, $\theta$, $H$ and $D$. The above mentioned authors, however, have developed their equations for the elastic constants and compressive strengths of non-isotropic foams by imposing the restriction on the cell geometry that $b/L = \sqrt{2}/\cos \theta$. This constraint forces the cell shape to be a function of the inclination angle only. Since, from a purely geometrical point of view, $\theta$ and $b/L$ may vary independently, we see no reason for this restriction on the cell geometry, other than to reduce the number of micro-structural measurements required to apply the equations and predict the foam behavior. As such, it is prudent to revisit the formulation of the previous authors and re-derive the equations for the elastic constants and strengths using the most general description of the elongated Kelvin model geometry.

In this paper, we derive the equations for the elastic constants and strengths for non-isotropic, open-celled foams using an elongated tetrakaidecahedron unit cell with a general geometric description, one that is defined with three independent dimensions. The equations are developed following an approach similar to Zhu et al. (1997) and are written in terms of the cell dimensions $b$, $L$ and $\theta$, the edge cross-section properties $A$ and $I$, and the solid material stiffness $E$ and tensile strength $\sigma_{ult}$. In the final section, we demonstrate the effect of the more general unit cell geometry on the non-isotropic mechanical response and strength behavior.

Fig. 1. Elongated tetrakaidecahedron repeating unit cell.
2. Cell aspect ratio

The size and shape of an elongated tetrakaidecahedron are uniquely defined by specifying the value of any three of the five dimensions \( L, b, \theta, H, D \) (see Fig. 1), since the height \( H \) and width \( D \) of the unit cell is related to \( L, b, \theta \) according to

\[
H = 4L \sin \theta \quad \text{and} \quad D = 2L \cos \theta + \sqrt{2b}.
\]  

The cell aspect ratio \( R = H/D \) is therefore

\[
R = \frac{4L \sin \theta}{2L \cos \theta + \sqrt{2b}}. \tag{2}
\]

There is a minimum value of \( \theta \), below which the unit cell in Fig. 1 is no longer elongated in the \( z \)-direction. This minimum value of \( \theta \) is a function of the length ratio \( b/L \), since as \( b/L \) becomes larger, the value of \( \theta \) must become larger in order for \( H > D \) and thus \( R > 1 \). The equation for the minimum \( \theta \) in terms of the ratio \( b/L \) is derived in Appendix A.

3. Foam relative density

The relative density \( \gamma \) is, by definition, the ratio of the foam density to the density of the solid material, \( \gamma = \rho/\rho_s \). The relative density may be written in terms of volumes as \( \gamma = V_s/V \), where \( V_s \) is the volume occupied by solid matter and \( V \) is the total volume of the foam. Using the elongated tetrakaidecahedron shown in Fig. 1 as a representative volume, the total volume is \( V = HD^2 \). The members that form the perimeter of the vertical diamond faces and those of length \( b \) that form the perimeter of the horizontal square faces are shared by the adjacent cells. Thus, they contribute only half their cross-sectional area to the repeating unit. All other members are completely contained within the boundaries of the unit cell. Assuming the edge cross-sectional area \( A \) is the same for all edges and constant along the edge length, then \( V_s = 164L + 84b \). Using the relations in Eq. (1), the relative density may be written

\[
\gamma = \frac{2A(2L + b)}{L \sin \theta [2L \cos \theta + \sqrt{2b}]^2}. \tag{3}
\]

4. Expressions for the foam elastic constants

We seek to develop equations for the foam elastic constants in terms of the micro-structural dimensions \( L, b, \theta \), the edge cross-section area \( A \) and moment of inertia \( I \) and the modulus of the solid material \( E \). For this purpose, we establish the cartesian coordinate system shown in Fig. 2, where the \( z \)-direction is oriented in the rise direction and the \( x \)- and \( y \)-directions are in the plane perpendicular to the rise direction.

For loading in the perpendicular-to-rise direction, we use the repeating unit cell shown in Fig. 2, which represents one-eighth of the tetrakaidecahedron shown in Fig. 1. We consider the deformation of this unit cell under the application of a uni-axial stress in the \( y \)-direction \( \sigma_{yy} \) which results in an extension in the \( y \)-direction and the accompanying contractions in the \( x \)- and \( z \)-directions. Furthermore, due to symmetry of the unit cell, one can easily recognize that the same set of equations apply for loading in the \( x \)-direction, i.e. \( E_x = E_y \), \( v_{xy} = v_{yx} \) and \( v_{xz} = v_{zx} \).

In order for the unit cell to be a representative repeating unit during deformation, we enforce the symmetry conditions on the member end point displacements:

\[
\begin{align*}
    u_B = u_C &= 0 & u_F = u_G &= \bar{u} \\
    v_C = v_D &= 0 & v_G = v_H &= \bar{v} \\
    w_D = w_F &= 0 & w_B = w_H &= \bar{w}
\end{align*}
\]

where the symbols \( u, v \) and \( w \) denote the displacements in the \( x \)-, \( y \)- and \( z \)-directions, respectively, and the symbols \( \bar{u}, \bar{v} \) and \( \bar{w} \) represent the displacements of the unit cell at the unit cell boundaries (Fig. 3). We also require
that the deformation of the unit cell occurs with no rotation of the member end points. In addition, we note that due to similarity of members $BC$ and $FG$ and the similarity of members $BH$ and $DF$, we have the additional conditions

$$u_H = u_F \quad v_H = v_F \quad w_G = w_B$$

where $v_H$ is the $y$-direction relative displacement of point $H$ with respect to point $B$, $v_H = v_B + v_H^\theta$.

Previous researchers have assumed an edge cross-section with circular, square, equilateral triangular or three-cusp hypocycloid (Plateau Borders) shapes. Note that for any of the four shapes, the moment of inertia of the $L$-length members ($BC$ and $FG$) about the neutral axis parallel to the unit cell $x$-direction $I^L_x$ is equal to the moment of inertia of the $b$-length members ($BH$ and $DF$) about the neutral axis parallel to the unit cell $z$-direction $I^b_z$, as long as the $L$ and $b$-length members have the same cross-section. As a result, the expressions for the elastic constants may be developed using $I^L_x = I^b_z = I$.

The equations for the Young’s modulus and Poisson’s ratios are obtained by applying the minimum potential energy theorem to the unit cell deformation, resulting in

$$E_y = \frac{12EI}{L \sin \theta \left[2L^3 \sin^2 \theta + b^3 + \frac{12I}{A} (2L \cos^2 \theta + b) \right]}.$$  

(4)

Fig. 2. Repeating unit cell for loading in the $y$-direction (perpendicular to rise).

Fig. 3. Unit cell deformation for loading in the $Y$-direction. (a) $Y-Z$ plane, (b) $X-Y$ plane.
and
\[ v_{yx} = \frac{b(Ab^2 - 12I)}{12I(2L \cos^2 \theta + b) + A(2L^3 \sin^2 \theta + b^3)}, \]  
\[ v_{yz} = \frac{(AL^2 - 12I)(2L \cos \theta + \sqrt{2}b) \cos \theta}{2[12I(2L \cos^2 \theta + b) + A(2L^3 \sin^2 \theta + b^3)]}. \]  

(5)

For a more detailed discussion of the derivations which lead to Eqs. (4) and (5), we refer the reader to the report by Sullivan et al. (2007).

The application of a perpendicular-to-rise direction stress \( \sigma_{yz} \) induces both an axial load and bending moments in the \( L \)-length members (BC and FG) and the \( b \)-length members (BH and DF). The maximum bending moment occurs at the member ends, having equal magnitudes, but opposite signs, at opposite ends of the members, \( M_{BC} = -M_{CH}, M_{GF} = -M_{GF}, M_{BH} = -M_{HB}, M_{DF} = -M_{DF} \). Under the application of a perpendicular-to-rise direction stress \( \sigma_{yz} \), the axial force and maximum bending moment in the members of length \( L \) are

\[ N_{BC} = N_{FG} = \sigma_{yz} \left( 2L \cos \theta + \sqrt{2}b \right) L \cos \theta \sin \theta, \]
\[ M_{BC} = M_{GF} = \frac{\sigma_{yz}}{2} \left( L \sin \theta \right)^2 \left( 2L \cos \theta + \sqrt{2}b \right), \]

and the axial force and maximum bending moment in the members of length \( b \) are

\[ N_{BH} = N_{DF} = \frac{\sigma_{yz}}{\sqrt{2}} L \sin \theta \left( 2L \cos \theta + \sqrt{2}b \right), \]
\[ M_{BH} = M_{DF} = \frac{\sigma_{yz}}{2\sqrt{2}} Lb \sin \theta \left( 2L \cos \theta + \sqrt{2}b \right). \]

(6)

(7)

For loading in the \( z \)-direction, it is more convenient to use the unit cell shown in Fig. 4. We define the unit cell displacements relative to point C and impose the member end point displacements

\[ u_C = v_C = w_C = 0 \quad u_D = -u_E = -\bar{u} \]
\[ v_A = -v_B = \bar{v} \quad w_A = w_B = \bar{w} \]
\[ w_D = w_E = -\bar{w}. \]

Application of the minimum potential energy theorem to the unit cell deformation leads to

\[ E_z = \frac{24EI \sin \theta}{L^2 \left[ \cos^2 \theta + \frac{12I \sin^2 \theta}{AL^2} \right] \left[ \sqrt{2L \cos \theta + b} \right]^2}. \]

(8)
and

\[ v_{x_2} = v_{y_2} = \frac{\sqrt{2}L^2(2L \cos \theta - 12I) \cos \theta \sin^2 \theta}{[12IL \sin^2 \theta + A L^2 \cos^2 \theta] \sqrt{2L \cos \theta + b}}. \]  

(9)

Defining the stiffness ratio as \( R_E = E_z/E_y \), then Eqs. (1), (2), (4) and (8) lead to

\[ R_E = \frac{R^2}{4} \left[ \frac{2 \sin^2 \theta + (b/L)^3 + \frac{12L}{AL^2} (2 \cos \theta + b/L)}{[\cos^2 \theta + \frac{12L}{AL^2} \sin^2 \theta]} \right]. \]  

(10)

Under the application of a rise direction stress \( \sigma_{zz} \), the axial force and maximum bending moment in the members of length \( L \) are

\[ N_{BC} = \frac{\sigma_{zz} \sin \theta}{2} \left( \sqrt{2L \cos \theta + b} \right)^2, \]

\[ M_{BC} = -\frac{\sigma_{zz} \cos \theta}{4} \left( \sqrt{2L \cos \theta + b} \right)^2. \]

(11)

Given the restrictions on the member end point displacements, the members of length \( b \) undergo only rigid body motion (Fig. 5) and are therefore unstressed under the application of a rise direction stress. The negative sign appears in the second expression in (11), since the bending moment in struts \( BC \) due to \( \sigma_{zz} \) are opposite in sign to the bending moment produced by \( \sigma_{yy} \).

5. Expressions for the foam tensile strength in the principal material directions

We assume that foam tensile failure occurs when the applied stresses produce a maximum tensile stress in any of the edges which is equal to the ultimate tensile strength of the solid material, that is when \( \sigma_{\text{max}} = \sigma_{\text{ult}} \). Tensile failure in the \( y \)-direction may occur due to failure of either the edges of length \( L \) or the edges of length \( b \). Using \( \sigma_{\text{max}} = (N/A) \pm (M/S) \), where \( S \) is the edge section modulus, and the equations for \( N_{BC} \) and \( M_{BC} \) in Eqs. (6), the ultimate strength of the foam in the \( y \)-direction, based on failure of the edges of length \( L \), is given as

\[ \sigma_{\text{ult}}^{y_y} = \frac{\sigma_{\text{ult}}}{\left[ \frac{L \cos \theta \sin \theta}{A} + \frac{L^2 \sin^2 \theta}{2S_x} \right] \left[ 2L \cos \theta + \sqrt{2b} \right]} \].  

(12)

Fig. 5. Unit cell deformation for loading in the Z-direction. (a) Y–Z plane, (b) X–Y plane.
In (12), $S^L_x$ is the section modulus for the members of length $L$ bending about the section neutral axis which is parallel to the unit cell $x$-direction. Likewise, using the equations for $N_{BH}$ and $M_{BH}$ in (7), the ultimate tensile strength of the foam in the $y$-direction, based on failure of the edges of length $b$, is

$$\sigma_{yy}^{\text{ult}} = \frac{\sigma^{\text{ult}}}{\left[ \frac{L \sin \theta + L b \sin \theta}{2 \sqrt{2} S^L_x} \right] \left[ 2 L \cos \theta + \sqrt{2} b \right]},$$

where $S^L_x$ is the section modulus for the members of length $b$ bending about the section neutral axis which is parallel to the unit cell $z$-direction. Note that for equilateral triangular or three-cusp hypocycloid cross-sections, $S^L_x \neq S^L_z$.

Eq. (12) will always yield a lower estimate of the $y$-direction ultimate strength than Eq. (13) provided the foam micro-structure is such that

$$2 S^L_x \cos \theta + A L \sin \theta > \sqrt{2} S^L_z + A b / \sqrt{2}. \quad (14)$$

If this condition is met, the edges with length $L$ will fail at a lower applied stress $\sigma_{yy}$ than the edges with length $b$.

Using (11), the ultimate tensile strength of the foam in the $z$-direction is

$$\sigma_{zz}^{\text{ult}} = \frac{\sigma^{\text{ult}}}{\left[ \frac{\sin \theta + L \cos \theta}{2 A L} \right] \left[ \sqrt{2} L \cos \theta + b \right]^2}. \quad (15)$$

Assuming the perpendicular-to-rise direction strength is limited by failure of the members of length $L$, then the strength ratio in tension, $R_\sigma = \sigma_{zz}^{\text{ult}} / \sigma_{yy}^{\text{ult}}$, can be written using (1), (2), (12) and (15) as

$$R_\sigma = R \left[ \frac{\sin \theta + 2 S^L_x \cos \theta}{\cos \theta + 2 S^L_x \sin \theta} \right]. \quad (16)$$

6. Stiffness and strength ratios versus $R$, $\gamma$ and the shape parameter $Q$

In addition to the aspect ratio, the shape of an elongated tetrakaidecahedron is dependent on the parameter $Q$, which we define as $Q = b / (L \cos \theta)$. The effect of the value of $Q$ on the unit cell shape is illustrated in Fig. 6, where two tetrakaidecahedrons are drawn to have the same aspect ratio, but their values of $Q$ are such that

![Fig. 6. Sketch of two elongated tetrakaidecahedrons illustrating the effect of the shape parameter $Q$.](image-url)
Values for the constants $C_i$ Table 1

<table>
<thead>
<tr>
<th></th>
<th>$C_1$</th>
<th>$C_2$</th>
<th>$C_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Circular</td>
<td>$\pi$</td>
<td>$\frac{3}{1}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>Hypocycloid</td>
<td>$\sqrt{3} - \pi/2$</td>
<td>$\frac{20\sqrt{3} - \pi}{2\sqrt{3} - \pi}$</td>
<td>$\frac{60 - 11\sqrt{3} \pi}{24(\sqrt{3} - \pi/2)}$</td>
</tr>
</tbody>
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Using (19), the stiffness ratio is plotted in Fig. 7 as a function of the aspect ratio, the shape parameter $Q$ and the relative density. Plateau Border cross-sections were assumed. The experimental results reported by a number of previous researchers are also included. Note that the stiffness ratio is a stronger function of the shape parameter $Q$ than the relative density $c$ and that the dependence on $c$ decreases as the value of $Q$ decreases. It is also surprising to find that the stiffness ratio $R_E$ may not equal unity at $R = 1$, which is the case when $Q \neq \sqrt{2}$.

We note that although most of the experimental results seem to fall near the curves where $Q = \sqrt{2}$, quite a few deviate considerably from these curves. Although it may be argued that these deviations are due to exper-
imental scatter, we suggest that they may be attributed to a foam microstructure where the representative unit cell has a shape such that $Q \neq \sqrt{2}$.

Using (20), the tensile strength ratio is plotted in Fig. 8 as a function of the aspect ratio $R$, the shape parameter $Q$ and the relative density $\gamma$. Again, Plateau Border cross-sections were assumed. The strength ratio is also a much stronger function of the shape parameter $Q$ than the relative density $\gamma$. The experimental results from three studies are included in Fig. 8. In comparing the experimental results with the curves drawn from (20), it appears, once again, that restricting the repeating unit cell shape to $Q = \sqrt{2}$ may be ill-advised as some of the experimental results deviate considerably from the $Q = \sqrt{2}$ curves. Finally, we note that the plot of the stiffness ratio versus aspect ratio for BX-265 and NCFI24-124 rigid polyurethane foams indicate that their microstructure is such that $Q \approx 1$ and that a plot of their strength ratio versus aspect ratio imply a similar microstructure.

7. Concluding remarks

Equations for the Young’s modulus, Poisson’s ratio and strength have been derived for non-isotropic, open-celled foams using an elongated Kelvin model repeating unit cell with the most general geometric description. The repeating unit cell is defined by specifying the values of four independent dimensions: three to specify the unit cell size and shape and one to specify the edge cross-section dimension. The equations for the elastic constants and strength were written in terms of the edge lengths and edge cross-section properties, the inclination angle and the strength and stiffness of the solid material.

In closing, it is worth noting that by adopting an elongated Kelvin model with the most general geometry, a more detailed description of the foam microstructure is required in order to apply the resulting equations and predict the foam behavior. More specifically, it is now necessary to obtain four separate physical and mechanical measurements of the foam in order to apply the equations. Aside from this added burden, the model is an improvement over the previous models, since it accounts for an additional variation in the unit cell shape and, as such, it is capable of representing a wider range of foam microstructures.

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Appendix A. Relation between the minimum allowable inclination angle and the length ratio $b/L$ for an elongated tetrakaidecahedron

If the height $H$ and width $D$ of the tetrakaidecahedron are equal, the expressions in (1) lead to

$$2 \sin \theta - \cos \theta = \frac{b}{\sqrt{2L}}.$$  \hspace{1cm} (A1)

For an elongated tetrakaidecahedron ($H > D$), we have the inequality

$$2 \sin \theta - \cos \theta > \frac{b}{\sqrt{2L}}.$$  \hspace{1cm} (A2)

Using the trigonometric identity $\cos \theta = \sqrt{1 - \sin^2 \theta}$, (A1) can be rearranged and rewritten as a second-order polynomial in $\sin \theta$, that is

$$5 \sin^2 \theta - 2\sqrt{2} \frac{b}{L} \sin \theta + \left( \frac{b^2}{2L^2} - 1 \right) = 0.$$  \hspace{1cm} (A3)

The solution to (A3) is obtained using the quadratic formula resulting in the two solutions

$$\sin \theta = \frac{1}{2} \left( \frac{b}{L} \pm \sqrt{\frac{b^2}{L^2} + 1} \right).$$
\[
\sin \theta = \sqrt{2b} + \sqrt{2} \sqrt{\frac{10 - b^2}{L^2}} \quad (A4a)
\]
\[
\sin \theta = \sqrt{2b} - \sqrt{2} \sqrt{\frac{10 - b^2}{L^2}} \quad (A4b)
\]

where both roots are real provided \( b/L \leq \sqrt{10} \) and where the two are identical when \( b/L = \sqrt{10} \). The solutions listed in (A4) are plotted in Fig. A.1.

It is easily shown that (A4a) is the solution to (A1) and that (A4b) is the solution to

\[
2 \sin \theta = -\cos \theta + \frac{b}{\sqrt{2L}}. \quad (A5)
\]

Since (A5) has no physical significance here, we will ignore the latter of the two solutions in (A4).

The plot of (A4a) in Fig. A.1 defines the value of \( \sin \theta \) as a function of the length ratio \( b/L \) for any tetrakaidecahedron with \( H = D \). As such, it defines the lower bound on the inclination angle for all possible elongated tetrakaidecahedrons, since \( \sin \theta \) for any elongated tetrakaidecahedron must lie above the upper curve in Fig. A.1.

We note that, in the range \( 2/\sqrt{5} < \sin \theta < 1 \), (A4a) yields two possible values for the ratio \( b/L \) for each value of \( \sin \theta \). The values of \( b/L > 2\sqrt{2} \), however, violate (A2) since

\[
2 \geq 2 \sin \theta - \cos \theta \quad \text{for all } \theta \leq \pi/2.
\]

Hence, for an elongated tetrakaidecahedron with \( \theta < \pi/2 \), the length ratio \( b/L \) must be less than \( 2\sqrt{2} \).

The lower bound on the inclination angle is therefore given by

\[
\theta > \arcsin \left( \frac{\sqrt{2b}}{5L} - \frac{\sqrt{2} \sqrt{10 - b^2}}{10L^2} \right), \quad (A6)
\]

which is valid over the domain \( 0 < b/L < 2\sqrt{2} \).

References


