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# Differential equations with singular fields

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**Abstract**

This paper investigates the well posedness of ordinary differential equations and more precisely the existence (or uniqueness) of a flow through explicit compactness estimates. Instead of assuming a bounded divergence condition on the vector field, a compressibility condition on the flow (bounded Jacobian) is considered. The main result provides existence under the condition that the vector field belongs to *BV* in dimension 2 and *SBV* in higher dimensions.


**Résumé**

Cet article étudie le caractère bien posé d'équations différentielles ordinaires et plus précisémment l'existence (ou l'unicité) d'un flot par des estimations directes de compacité. Une condition de compressibilité sur le flot est supposé au lieu d'une borne sur la divergence du champ de vitesse. Le principal résultat obtenu garantit l'existence sous l'hypothèse que le champ de vitesse est à variations bornées en dimension 2 ou dans l'espace *SBV* en dimensions supérieures.


*MSC:* 34C11; 35L45; 37C10

*Keywords:* ODE's; Singular forces; Stability estimates

## 1. Introduction

This article studies the existence (and secondary uniqueness) of a flow for the equation,

$$\partial_t X(t,x) = b\big(X(t,x)\big), \quad X(0,x) = x. \tag{1.1}$$

The most direct way to establish the existence of such of flow is of course through a simple approximation procedure. That means taking a regularized sequence $b_n \to b$, which enables to solve,

$$\partial_t X_n(t,x) = b_n\big(X(t,x)\big), \quad X_n(0,x) = x, \tag{1.2}$$

by the usual Cauchy–Lipschitz Theorem. To pass to the limit in (1.2) and obtain (1.1), it is enough to have compactness in some strong sense (in $L^1_{\text{loc}}$ for instance) for the sequence $X_n$. Obviously some conditions are needed.

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First note that we are interested in flows, which means that we are looking for solutions $X$ which are invertible: At least $JX = \det d_x X \neq 0$ a.e. (with $d_x X$ the differential of $X$ in $x$ only). So throughout this paper, only flows $x \to X(t,x)$ which are nearly incompressible are considered:

$$\frac{1}{C} \leqslant JX(t,x) \leqslant C, \quad \forall t \in [0,T],\ x \in \mathbb{R}^d. \tag{1.3}$$

If one obtains $X$ as a limit of $X_n$ then the most simple way of satisfying (1.3) is to have,

$$\frac{1}{C} \leqslant JX_n(t,x) \leqslant C, \quad \forall t \in [0,T],\ x \in \mathbb{R}^d, \tag{1.4}$$

for some constant $C$ independent of $n$. Note that both conditions are only required on a finite and given time interval $[0,T]$ since one may easily extend $X$ over $\mathbb{R}_+$ by the semi-group relation $X(t+T,x) = X(t, X(T,x))$. Usually (1.3) and (1.4) are obtained by assuming a bounded divergence condition on $b$ or $b_n$ but this is not the case here.

It is certainly difficult to guess what is the optimal condition on $b$. It is currently thought that $b \in BV(\mathbb{R}^d)$ is enough or (see [12])

**Bressan's compactness conjecture.** *Let $X_n$ be regular ($C^1$) solutions to* (1.2)*, satisfying* (1.4) *and with* $\sup_n \int_{\mathbb{R}^d} |db_n(x)|\,dx < \infty$. *Then the sequence $X_n$ is locally compact in $L^1([0,T]\times\mathbb{R}^d)$.*

From this, one would directly obtain the existence of a flow to (1.1) provided that $b \in BV(\mathbb{R}^d)$ and (1.3) holds. Instead of the full Bressan's conjecture, this article essentially recovers, through a different method, the result of [4] namely under the condition $b \in SBV$.

**Theorem 1.1.** *Assume that $b \in SBV_{\mathrm{loc}}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ with a locally finite jump set* (*for the* $(d-1)$*-dimensional Hausdorff measure*)*. Let $X_n$ be regular solutions to* (1.2)*, satisfying* (1.4) *and such that $b_n \to b$ belongs uniformly to $L^\infty(\mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^d)$. Then $X_n$ is locally compact in $L^1([0,T]\times\mathbb{R}^d)$.*

It is possible to be more precise only in dimension 2:

**Theorem 1.2.** *Assume that $d \leqslant 2$, $b \in BV_{\mathrm{loc}}(\mathbb{R}^d)$. Let $X_n$ be regular solutions to* (1.2)*, satisfying* (1.4) *and such that $b_n \to b$ belongs uniformly to $L^\infty(\mathbb{R}^d) \cap W^{1,1}(\mathbb{R}^d)$ with $\inf_K b_n \cdot B > 0$ for any compact $K$ and some $B \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$. Then $X_n$ is locally compact in $L^1([0,T]\times\mathbb{R}^d)$.*

The proof of the first result is found in Section 7 and the proof of the second in Section 6. After notations and examples in Section 2 and technical lemmas in Section 3, particular cases are studied. In Section 4, a very simple proof is given if $b \in W^{1,1}$. Section 5 studies (1.1) in dimension 1 for which compactness holds under very general conditions (essentially nothing for $b$ and a much weaker version of (1.3)). The final section offers some comments on the unresolved issues in the full $BV$ case.

The question of uniqueness is deeply connected to the existence and in fact the proof of Theorem 1.1 may be slightly altered in order to provide it (it is more complicated for Theorem 1.2). Proofs are always given for the compactness of the sequence but it is indicated and briefly explained after the stated results whether they can also give uniqueness; This is usually the case except for Sections 5 and 6.

The well posedness of (1.1) is classically obtained by the Cauchy–Lipschitz Theorem. This is based on the simple estimate:

$$\big|X(t,x+\delta) - X(t,x)\big| \leqslant |\delta| e^{t\|db\|_{L^\infty}}. \tag{1.5}$$

Notice that a similar bound holds if $b$ is only log-Lipschitz, leading to the important result of uniqueness for the $2d$ incompressible Euler system (see for instance [28] among many other references).

The idea in this article is to get (1.5) for *almost all* $x$. It is therefore greatly inspired by the recent approach developed in [18] (see also [17]), where the authors control the functional:

$$\int_{\mathbb{R}^d} \sup_r \int_{S^{d-1}} \log\left(1 + \frac{|X(t,x+rw) - X(t,x)|}{r}\right) dw\,dx. \tag{1.6}$$

This allows them to get an equivalent of Theorem 1.1 provided $b \in W^{1,p}$ with $p > 1$ (and in fact $db$ in $L \log L$ would work). Here the slightly different functional,

$$\sup_r \int_{\mathbb{R}^d} \int_{S^{d-1}} \log\left(1 + \frac{|X(t, x + rw) - X(t, x)|}{r}\right) dw\, dx, \tag{1.7}$$

is essentially considered. It gives a better condition $b \in SBV$ but has the one drawback of not implying strong differentiability for the flow at the limit.

Other successful approaches of course exist for (1.1). A most important step was achieved in [22] where the well posedness of the flow was obtained by proving uniqueness for the associated transport equation,

$$\partial_t u + b \cdot \nabla u = 0,$$

under the conditions that $b$ be of bounded divergence and in $W^{1,1}$. The crucial concept there is the one of renormalized solutions, namely weak solutions $u$ s.t. $\phi(u)$ is also a solution to the same transport equation. This was extended in [29,27] and [26]; also see [8] where the connection between well posedness for the transport equation and (1.1) is thoroughly analyzed, both for bounded divergence fields and an equivalent to (1.3).

Using a slight different renormalization for the equation $\partial_t u + \nabla \cdot (bu) = 0$, the well posedness was famously obtained in [2] under the same bounded divergence condition and $b \in BV$. This last assumption $b \in BV$ was also considered in [15] and [16]. Still using renormalized solutions, the restriction of bounded divergence was weakened in [4] to an assumption equivalent to (1.3); unfortunately this required $b \in SBV$.

In comparison to [4], Theorem 1.1 is slightly weaker (due to the assumption of bounded jump set for $\mathcal{H}^{d-1}$), except in $2d$ where (1.2) is stronger ($b \in BV$ instead of $SBV$). The main advantage of the approach presented here is that we work directly on the differential equation, giving for instance a very simple and direct theory for $b \in W^{1,1}$. It also provides quantitative estimates for $|X(t,x) - X(t, x+\delta)|$, which is connected to the regularity of the trajectories.

Usually using transport equations for (1.1) does not directly gives estimates like (1.5), (1.6) or (1.7). Some information of this kind can still be derived, for instance by studying the differentiability or approximate differentiability of the flow as in [3,6].

When an additional structure is known or assumed for $b$ (not the case here), the conditions for existence or uniqueness can often be loosen. The typical and most frequent example is the hamiltonian case. For Vlasov equations for instance, uniqueness under a $BV$ condition was obtained earlier and more easily in [7]; it was even derived under slightly less than $BV$ regularity in [25]. In low dimension this structure is especially useful as illustrated in [9] where uniqueness is obtained for Vlasov equations in dimension 2 of phase space for continuous force terms; In [24] for a general hamiltonian (still in $2d$) with $L^p$ coefficients; and in [14] for a continuous $b$ in $2d$ but only a bounded divergence condition instead of a complete hamiltonian structure. In many of those situations an estimate like (1.5) or its variants (1.6) or (1.7) is simply false however and the flow therefore less regular than what can be proved with more regularity.

The issue of which conditions would be optimal is still open for the most part; of course it should also depend on which exact property (uniqueness or something more precise like (1.5)) is looked after. Interesting counterexamples are nevertheless known to test this optimality, from the early [1], to [11] and [21] which indicates that in the general framework $BV$ indeed plays a critical role.

Finally, there are many other ways to look for solutions to (1.1), which are not so relevant here because they do not produce flows. A well-known example is found in [23] where it is noticed that a one-sided Lipschitz condition on $b$ is enough to get either existence or uniqueness (depending on the side) but not both. This is usually not enough to define a flow but can be useful to deal with characteristics for hyperbolic problems (see [19]).

Similarly if one is interested mainly in well posedness for hyperbolic problems, other approaches than renormalization (entropy solutions for instance) exist. We refer to [13] and [20]. Those do not always yield flows, especially where nothing is assumed on the divergence of $b$, but the characteristics for relevant physical solutions are not always flows: See for instance [10] in connection with sticky particles, [31] for the use of Filippov characteristics and [30] for the use of an entropy condition.

## 2. Elementary considerations

### 2.1. Reduction of the problem and notations

First if $b \in L^\infty$, $|X(t,x)-x| \leqslant \|b\|_{L^\infty} t$ and as we look at (1.1) for a finite time, we may of course reduce ourselves to the case of a bounded domain $\Omega$ and assume that $db$ is compactly supported in $\Omega$.

Next note that the time-dependent problem,

$$\partial_t X = b\big(t, X(t,x)\big), \quad X(0,x) = x,$$

can be reduced to (1.1) simply by adding time as a variable. This has for first consequence to increase the dimension by 1, which has no importance for Theorem 1.1 but could matter in low dimension trying to use the results of Sections 5 or 6. Second it would require that $b$ be *SBV* in $x$ and $t$.

With the right result, it would be easy to get rid of this additional time regularity. More precisely assuming that one has a theorem like in Section 6 giving for $b \in BV$:

$$\sup_\delta \int_\Omega \log\left(1 + \frac{|X(t,x+\delta) - X(t,x)|}{r}\right) dw\, dx \leqslant C\big(\|b\|_\infty\big)\left(|\Omega| + \int_{\mathbb{R}^d \times [0,T]} \big(|\partial_t b| + |d_x b|\big)\, dt\, dx\right). \tag{2.1}$$

Take $\varepsilon > 0$ and change variables $X_\varepsilon = X(\varepsilon t, x)$, $b_\varepsilon(t,x) = \varepsilon b_\varepsilon(\varepsilon t, x)$ so that

$$\partial_t X_\varepsilon = b_\varepsilon(t, X_\varepsilon),$$

and (1.3) of course holds for $X_\varepsilon$ on the time interval $[0, T/\varepsilon]$. Now applying (2.1) for $X_\varepsilon$ and letting $\varepsilon$ go to 0, one recovers at the limit an estimate without time derivative:

$$\sup_\delta \int_\Omega \log\left(1 + \frac{|X(t,x+\delta) - X(t,x)|}{r}\right) dw\, dx \leqslant C(0)\left(|\Omega| + \int_{\mathbb{R}^d \times [0,T]} |d_x b|\, dt\, dx\right),$$

so that $b \in L^1([0,T], BV(\mathbb{R}^d))$ is enough. In fact for the two theorems concerned by this remark in this paper (the $W^{1,1}$ case in Theorem 4.1 and the $2d$ case in Theorem 6.1), it is easy to directly modify the proof and obtain $b \in L^1([0,T], W^{1,1}(\mathbb{R}^d))$ for Theorem 4.1 and $b \in L^1([0,T], BV(\mathbb{R}))$ for Theorem 6.1. Unfortunately this is of no use for Theorem 7.1.

Note moreover that taking $\varepsilon = 2/\|b\|_\infty$ one may always assume $\|b\|_{L^\infty} \leqslant 2$ provided $b \in L^\infty$.

Finally, adding one variable even in the time-independent case, it is possible to have $b_1 \geqslant 1$. This will simplify some proofs and is crucial for topological reasons in Section 6.

In summary we may work with $b$ satisfying

$$M = \int_{\mathbb{R}^d} \big|db(x)\big|\, dx < \infty, \qquad \operatorname{supp}|db| \subset \Omega, \qquad \|b\|_{L^\infty} \leqslant 2, \qquad b_1(x) \geqslant 1 \quad \forall x \in \Omega. \tag{2.2}$$

The support is denoted supp, $\mathcal{H}^\gamma$ denotes the $\gamma$-dimensional Hausdorff measure. $C$ will denote any universal constant (possibly depending only on the dimensions $d$ and $\Omega$) and its value may thus change from line to line.

### 2.2. A simple 1$d$ example

As a warm up, study the usual counterexample to Cauchy–Lipschitz in $1d$, namely take,

$$b(x) = \sqrt{|x|}.$$

There are several solutions to (1.1) with starting point $x < 0$. All solutions are the same for some time:

$$X(t,x) = -\big(t/2 - \sqrt{|x|}\big)^2, \quad t \leqslant 2\sqrt{|x|}. \tag{2.3}$$

After $t = 2\sqrt{|x|}$ there are infinitely many possibilities, first

$$X(t,x) = \big(t/2 - \sqrt{|x|}\big)^2, \quad t \geqslant 2\sqrt{|x|}, \tag{2.4}$$

and then for any $t_0 \in [2\sqrt{|x|}, \infty]$,

$$X(t,x) = 0 \quad \text{for } 2\sqrt{|x|} \leqslant t \leqslant t_0, \qquad X(t,x) = (t/2 - t_0)^2, \quad t \geqslant t_0. \tag{2.5}$$

However among all those solutions there is only one which defines a flow (makes $X$ invertible) and it is (2.4). For the point of view followed in this paper, this is the *right* one but there could be situations where it is not the relevant solution (for physical reasons, entropy principles, etc.) and another should be chosen (typically the one corresponding to $t_0 = \infty$).

Note that obviously $\operatorname{div} b$ is not bounded but the solution (2.4) is the only one to satisfy a weaken version of (1.3), namely (5.1) (see Section 5 where the $1d$ case is studied with a result containing this example). This shows that there is a selection principle hidden in (1.3).

### 2.3. *Comments on the compressibility condition* (1.3)

Instead of (1.3), many works rather use the condition that the divergence of $b$ is bounded. Of course if $\operatorname{div} b \in L^\infty$ then (1.3) holds for any solution to (1.1) so the question is only how more general (1.3) is. Two examples are shown to try to investigate this.

First recall that (1.3) may be reformulated in terms of the transport equation. Namely as noticed and widely used in [4], it is equivalent to the existence of a solution $u$ to,

$$\partial_t u + \nabla \cdot (bu) = 0, \qquad \inf u(t=0) > 0, \qquad \sup u(t=0) < \infty,$$

and s.t. for any $t \in [0,T]$,

$$\sup_x u(t,x) \leqslant C \inf_x u(t=0), \qquad \inf_x u(t,x) \geqslant C^{-1} \sup_x u(t=0). \tag{2.6}$$

In general it is very difficult to obtain such bounds without an assumption on the divergence. However say that $b$ is itself computed thanks to an equation: $b(t,x) = \nabla A(u(t,x))$ where $u$ is an already obtained solution to the hyperbolic problem:

$$\partial_t u + \nabla \cdot \big(A(u)\big) = 0.$$

Then the maximum principle for this hyperbolic equation directly gives (2.6). In this case (1.3) is more natural than a bounded divergence hypothesis. It is nevertheless still usually possible to use this latter assumption by adding the time as a dimension and considering the stationary problem. See [8] for a full and general analysis of the connection between transport equations and ODE's.

As a second example, consider what (1.3) implies for $b$ where it is discontinuous. Indeed if $\operatorname{div} b \in L^p$ and $b$ is discontinuous across a regular hypersurface $H$ then it is well known that the normal component $b \cdot \nu$ ($\nu$ being the normal to $H$) *cannot* jump across $H$. This is not true anymore with only (1.3). Take the simple case in dimension 1:

$$b(x) = 1 \quad \text{if } x < 0, \qquad b(x) = 1/2 \quad \text{if } x > 0.$$

Then solving (1.1) gives:

$$X(t,x) = x + t \quad \text{if } t \leqslant -x, \qquad X(t,x) = (t+x)/2 \quad \text{if } t \geqslant -x,$$
$$X(t,x) = x + t/2 \quad \text{if } x > 0,$$

which obviously satisfies (1.3). So it can be seen that the condition (1.3) imposes less constraint on the jumps of $b$.

Note that conversely it can be shown that if $b \in BV$, has a discontinuity along $H$ and denoting $b^-$ and $b^+$ the two traces (see [5]) then (1.3) implies that $b^+ \cdot \nu$ and $b^- \cdot \nu$ have the same sign, and

$$b^+(x) \cdot \nu(x) \geqslant C^{-1} b^-(x) \cdot \nu(x), \quad \text{for } \mathcal{H}^{d-1} \text{ all } x \in H.$$

Indeed take any $x_0 \in H$ s.t. $b^-$ and $b^+$ are approximately continuous at $x_0$ (see [5]). As $H$ is regular, change variable so that around $x_0$, $H$ has equation $x_1 = 0$ (this may modify the constant $C$ in (1.3) which hence depends on $H$). Now consider the domain $\omega_{r,\eta}$ defined by:

$$\omega_{r,\eta} = \{x, -\eta < x_1 < \eta, |x - x_0| \leqslant r\},$$

and its border

$$\omega^{\pm}_{r,\eta} = \{x, x_1 = \pm\eta, |x - x_0| \leqslant r\}, \qquad \omega^0_{r,\eta} = \{x, -\eta < x_1 < \eta, |x - x_0| = r\}.$$

Choose $r$ small enough and a sequence $\eta_n$ by approximate continuity s.t.

$$\frac{1}{|\omega^{\pm}_{r,\eta_n}|} \int\limits_{\omega^{\pm}_{r,\eta_n}} \left|b(x) - b^{\pm}(x_0)\right| d\mathcal{H}^{d-1}(x) \leqslant \frac{1}{C}\left|b^{\pm}_1(x_0)\right|.$$

Finally compute,

$$V(t) = \int \mathbb{I}_{X(t,x)\in\omega_{r,\eta_n}}\, dx.$$

First by simply changing variables and using (1.3) to get:

$$V(t) \leqslant C|\omega_{r,\eta_n}| \leqslant C r^{d-1}\eta_n.$$

And then by,

$$\frac{dV(t)}{dt} = \int \delta\big(X(t,x) \in \omega^0_{r,\eta_n}\big) b\big(X(t,x)\big)\cdot\big(X' - x_0\big)r^{-1}$$
$$+ \int \delta\big(X(t,x) \in \omega^+_{r,\eta_n}\big) b_1\big(X(t,x)\big) - \int \delta\big(X(t,x) \in \omega^-_{r,\eta_n}\big) b_1\big(X(t,x)\big),$$

where $X' = (0, X_2, \ldots, X_d)$. The first term is bounded by $\|b\|_\infty \eta_n$. Assume for instance that $b^+_1(x_0) > 0$ then by change of variables the second term is smaller than $C\mathcal{H}^{d-1}(\omega^+_{r,\eta_n}) b^+_1(x_0) = C r^{d-1} b^+_1(x_0)$. The third term is larger than $r^{d-1} b^-_1(x_0)/C$ (if $b^-_1(x) > 0$). Integrating over $[0, T]$ and taking $\eta_n$ small enough leads to

$$b^+_1(x_0) \geqslant C^{-1} b^-_1(x).$$

## 3. Preliminary results

Two simple lemmas are given here, which will be used frequently in other proofs.

**Lemma 3.1.** *Assume $b \in BV$. There exists a constant $C$ s.t. for any $x$, $y$*

$$\left|b(x) - b(y)\right| \leqslant C \int\limits_{B(x,y)} \left|db(z)\right| \left(\frac{1}{|x - z|^{d-1}} + \frac{1}{|y - z|^{d-1}}\right) dz, \tag{3.1}$$

*where $B(x, y)$ denotes the ball of center $(x + y)/2$ and diameter $|x - y|$.*

**Proof.** This is just an explicit computation: Change coordinates so that $x = (-\alpha, 0, \ldots)$ and $y = -x$. Then take a path $t \in [0, 1/2] \to (-\alpha + 2\alpha t, \alpha t r)$ for any $r$ in the unit ball of $\mathbb{R}^{n-1}$ and take the symmetric path for $t > 1/2$. Then all those paths $\gamma_r$ connect $x$ and $y$ so that

$$\left|b(x) - b(y)\right| \leqslant \int\limits_{\gamma_r} \left|db(z)\right| dl(z).$$

Averaging over $r$ in the ball $B(x, y)$ and changing coordinates get (3.1). □

A slight variant of this (usefull for the *SBV* case in particular) is:

**Lemma 3.2.** *Assume $b \in BV$ and $H$ is a hypersurface, Lipschitz regular of $\mathbb{R}^d$. There exists a constant $C$ and a constant $K$ (depending on $H$) s.t. for any $x$, $y$ locally on the same side of $H$,*

$$\left|b(x) - b(y)\right| \leqslant C \int\limits_{B_K(x,y)\setminus H} \left|db(z)\right| \left(\frac{1}{|x-z|^{d-1}} + \frac{1}{|y-z|^{d-1}}\right) dz, \tag{3.2}$$

*where $B_K(x, y)$ denotes the ball of center $(x + y)/2$ and diameter $K|x - y|$.*

**Proof.** This is just the same idea as before: Consider all paths $\gamma$ connecting $x$ and $y$, of length at most $K|x - y|$ and not crossing $H$. Average over those to get the result.

Note that $K$ must be larger than the Lipschitz regularity of $H$: If $H$ is locally given by the equation $f(x) = 0$ then $K \geqslant C\|df\|_\infty$. And if $K$ is chosen like that then the definition of locally on the same side can simply be: There exists a path $\gamma$ connecting $x$ and $y$ without crossing $H$ and of length less than $3K/2|x - y|$. □

Finally let us note that Lemma 3.1 is a more precise version of the well-known bound (used in [18] in particular):

$$\left|b(x) - b(y)\right| \leqslant C|x - y|\big(M|db|(x) + M|db|(y)\big), \tag{3.3}$$

where $M|db|$ is the maximal function of $|db|$. Indeed decomposing $B(x, y)$ into $\bigcup(R(x, 2^{-n}) \cap B(x, y))$ for all $n \geqslant n_0 = -\log_2 |x - y|$, with $R(x, r) = \{z, r/2 \leqslant |z - x| < r\}$, one gets:

$$\int\limits_{B(x,y)} \frac{|db(z)|}{|x-z|^{d-1}}\, dz \leqslant \sum_{n\geqslant n_0} 2^{(d-1)(n+1)} \int\limits_{R(x,2^{-n})} \left|db(z)\right| dz$$
$$\leqslant 2^{d-1} \sum_{n\geqslant n_0} 2^{-n} M\left|db(x)\right| dz \leqslant 2^d |x - y| M|db|(x),$$

recalling that

$$M\left|db(x)\right| = \sup_r r^d \int\limits_{B(x,r)} \left|db(z)\right| dz.$$

## 4. The $W^{1,1}$ case

### *4.1. The result*

Following [18], we define for any $\delta$:

$$Q_\delta(t) = \int\limits_\Omega \log\left(1 + \frac{|X(t,x) - X(t,x+\delta)|}{|\delta|}\right) dx.$$

As $db$ belongs to $L^1$, there exists $\phi \in C^\infty(\mathbb{R}_+)$, with

$$\phi(\xi)/\xi \text{ increasing}, \qquad \frac{\phi(\xi)}{\xi} \to +\infty \quad \text{as } \xi \to +\infty, \tag{4.1}$$

and such that

$$\int\limits_\Omega \phi\big(\left|db(x)\right|\big)\, dx < \infty. \tag{4.2}$$

The main result here is the explicit estimate

**Theorem 4.1.** *Assume that $b$ satisfies* (1.3) *and* (4.2) *then there exists a constant $C$ depending only on $\Omega$ and a continuous function $\psi$ depending only on $\phi$ and with $\psi(\xi)/|\log \xi| \to 0$ as $\xi \to 0$ such that*

$$Q_\delta(t) \leqslant |\Omega| \log 2 + Ct\psi\big(|\delta|\big) \int\limits_\Omega \big(1 + \phi\big(\left|db(x)\right|\big)\big)\, dx.$$

**Remarks.** 1. This of course immediately implies that any sequence of solutions $X_n$ to (1.2) is compact thus proving Bressan's conjecture in the restricted $W^{1,1}$ case.

2. Uniqueness: The proof is identical if one considers two different solutions $X$ and $Y$ to (1.1). Therefore the solution is also unique.

**Proof of Theorem 4.1.** Start by differentiating $Q_\delta$ in time,

$$Q'_\delta(t) \leqslant \int\limits_\Omega \frac{|\partial_t X(t,x) - \partial_t X(t,x+\delta)|}{|\delta| + |X - X_\delta|}\,dx,$$

where $X_\delta$ stands for $X(t,x+\delta)$. As $X$ solves (1.1),

$$Q'_\delta(t) \leqslant \int\limits_\Omega \frac{|b(X(t,x)) - b(X(t,x+\delta))|}{|\delta| + |X - X_\delta|}\,dx,$$

and using Lemma 3.1, one gets:

$$Q'_\delta(t) \leqslant \int\limits_{x,z} \mathbb{I}_{z\in B(X,X_\delta)} \frac{|db(z)|}{|\delta| + |X - X_\delta|}\left(\frac{1}{|z-X|^{d-1}} + \frac{1}{|z-X_\delta|^{d-1}}\right)dx\,dz.$$

Now for any $M$ decompose the integral in $z$ into the domain $E_M$, where $|db(z)| \leqslant M$ and the set $F_M$ where $|db(z)| \geqslant M$,

$$Q'_\delta(t) \leqslant I + II,$$

with

$$\begin{aligned} I &= \int\limits_\Omega\int\limits_{E_M} \mathbb{I}_{z\in B(X,X_\delta)} \frac{|db(z)|}{|\delta| + |X - X_\delta|}\left(\frac{1}{|z-X|^{d-1}} + \frac{1}{|z-X_\delta|^{d-1}}\right)dx\,dz \\ &\leqslant \int\limits_{x,z} \mathbb{I}_{z\in B(X,X_\delta)} \frac{M}{|\delta| + |X - X_\delta|}\left(\frac{1}{|z-X|^{d-1}} + \frac{1}{|z-X_\delta|^{d-1}}\right)dx\,dz \\ &\leqslant M|\Omega|. \end{aligned}$$

On the other hand,

$$\begin{aligned} II &= \int\limits_\Omega\int\limits_{F_M} \mathbb{I}_{z\in B(X,X_\delta)} \frac{|db(z)|}{|\delta| + |X - X_\delta|}\left(\frac{1}{|z-X|^{d-1}} + \frac{1}{|z-X_\delta|^{d-1}}\right)dx\,dz \\ &\leqslant \frac{M}{\phi(M)} \int\limits_{x,z} \phi\big(|db(z)|\big)\left(\frac{1}{(|\delta| + |z-X|)(|z-X|^{d-1})} + \frac{1}{(|\delta| + |z-X_\delta|)(|z-X_\delta|^{d-1})}\right)dx\,dz, \end{aligned}$$

since first as $\phi/\xi$ is increasing, if $|db| > M$ then $|db| \leqslant \phi(|db|)M/\phi(M)$ and second as $z \in B(X,X_\delta)$ then $|z-X| \leqslant |X - X_\delta|$ and $|z - X_\delta| \leqslant |X - X_\delta|$.

Note that $X$ and $X_\delta$ play the same role and in particular the transform $x \to X_\delta = X(t,x+\delta)$ also has a bounded Jacobian. Therefore we change variable in $x$, for the first term in the parenthesis from $x$ to $X$ and for the second from $x$ to $X_\delta$ to find,

$$\begin{aligned} II &\leqslant 2\frac{M}{\phi(M)} \int\limits_{x,z} \phi\big(|db(z)|\big) \frac{1}{(|\delta| + |z-x|)(|z-x|^{d-1})}\,dx\,dz \\ &\leqslant 2\frac{M}{\phi(M)} \log\big(1/|\delta|\big) \int \phi\big(|db(z)|\big)\,dz, \end{aligned}$$

by integrating first in $x$. Combining both estimates, one obtains:

$$Q'_\delta(t) \leqslant \left(M + 2\frac{M}{\phi(M)}\log\big(|\delta|^{-1}\big)\right)\int \big(1 + \phi\big(|db(z)|\big)\big)\,dz.$$

Defining $\psi(|\delta|) = \inf_M M + 2\frac{M}{\phi(M)}\log(|\delta|^{-1})$, this concludes the proof. □

Note that this proof uses a sort of interpolation of $db$ between $L^\infty$ and $L^1$. If instead one uses the result of [18], then it is enough to interpolate between $L\log L$ and $L^1$ and $\psi$ is then defined by,

$$\psi\big(|\delta|\big)=\inf_M \frac{M\log M}{\phi(M)}+2\frac{M}{\phi(M)}\log\big(|\delta|^{-1}\big),$$

provided that $\phi(\xi)\leqslant \xi\log\xi$. This last estimate for $\psi$ is of course much better than the previous one, even though for well posedness (compactness or uniqueness), it does not matter.

*4.2. An example*

The following remark was first made by S. Bianchini. The previous proof shows that if $b\in W^{1,1}$ then for any flow $X$ satisfying (1.3), one may bound,

$$\int_\Omega \frac{|b(X(x))-b(X(x+\delta))|}{|\delta|+|X-X(x+\delta)|}\,dx, \tag{4.3}$$

by $o(\log|\delta|)$. It is then essentially a linear estimate in the sense that in (4.3) one never uses that $b$ and $X$ are connected. The situation below shows that this simple way of controlling (4.3) cannot be extended further to $b\in BV$.

The example is shown in $1d$ but it can of course be extended to any dimension. Simply take for $b$ the Heaviside step function $\mathbb{I}_{x>0}$. As for $X$, fix $n$ and choose:

$$\begin{gathered}X(t,x)=x,\quad \text{if } x\in\big[k/n,(2k+1)/2n\big] \text{ or } x\in\big[-(2k+1)/2n,-k/n\big],\ 1\leqslant k\leqslant n-1\\ X(t,x)=-x,\quad \text{if } x\in\big[(2k+1)/2n,(k+1)/n\big] \text{ or } x\in\big[-(k+1)/n,-(2k+1)/2n\big],\ 1\leqslant k\leqslant n-1,\\ X(t,x)=x,\quad \text{otherwise.}\end{gathered}$$

It is obvious that $X$ satisfies (1.3) (it only swaps the intervals) and choosing $\delta=1/2n$,

$$\int_{-1}^{1} \frac{|b(X(x))-b(X(x+\delta))|}{|\delta|+|X-X(x+\delta)|}\,dx\geqslant C\sum_{k=1}^{n-1}\frac{1}{2n}\frac{n}{k}\geqslant C\log n=C\log|\delta|.$$

To bypass this obstacle, it is necessary to use the dependence of $X$ in terms of $b$ or the fact that $X(0,x)=x$. In dimension 1 for example, it is not possible to pass continuously from $X=x$ at $t=0$ to an $X$ as shown here.

## 5. The $1d$ case

The stationary equation (1.1) in only one dimension is of course a very particular situation. In this case assumption (1.3) is more than enough and no additional regularity is required (just as the divergence controls the whole gradient). In fact it is even too much and one only needs to assume that the image of a nonempty interval remains nonempty

**Definition 5.1** *(Weak compressibility).* $\exists\phi\in C(\mathbb{R}_+)$ with $\phi(\xi)>0$ for any $\xi>0$ s.t. $\forall I$ interval, $\forall t\in[-T,T]$,

$$\big|X(t,I)\big|>\phi(I).$$

Then it is quite straightforward to get:

**Theorem 5.1.** *There exists a strictly increasing $\psi$ with $\psi(0)=0$ s.t. any $X$ limit of solutions to* (1.2) *with $b_n\in W^{1,\infty}_{\text{loc}}$* (*but not necessarily the limit $b$*), *and satisfying* (5.1) *also satisfies*:

$$\big|X(t,x+\delta)-X(t,x)\big|\leqslant\phi(\delta).$$

**Remarks.** 1. This is of course enough to ensure compactness and Bressan's conjecture in this case.

2. The proof relies on the topology of $\mathbb{R}$. It does not give directly any uniqueness result and in particular, $X(t,x+\delta)$ cannot be replaced by another solution $Y$. Of course once the regularity of the solution is obtained it should be possible to then derive uniqueness.

**Proof of Theorem 5.1.** It is enough to prove that the estimate holds for a regular solution of (1.1). First note that as if $I \subset J$, $X(t,I) \subset X(t,J)$ then we may always take $\phi$ increasing in (5.1).

As the solution to (1.1) is unique, the image of the interval $[x_1, x_2]$ is simply the interval $[X(t,x_1), X(t,x_2)]$. This also means that the image by the flow $X(t,.)$ of the interval $[X(t,x), X(t,x+\delta)]$ is the interval $[x, x+\delta]$ so applying (5.1),

$$\delta > \phi\big(\big|X(t,x+\delta) - X(t,x)\big|\big),$$

which gives the result after composition with $\phi^{-1}$. $\square$

## 6. The 2*d* case

The situation in dimension two is more complicated than in dimension one but still very constrained by the topology.

Let us again consider the functional,

$$Q_\delta(t) = \int_\Omega \log\left(1 + \frac{|X(t,x) - X(t,x+\delta)|}{|\delta|}\right) dy\,dx,$$

where $\delta$ is a fixed vector, for example $\delta = (0, r)$.

In this setting it is possible to obtain the optimal.

**Theorem 6.1.** *Let $X$ be a regular solution to* (1.1) *satisfying* (1.3) *and assume that $b$ satisfies* (2.2) *then there exists a constant $C$* (*depending only on the constant in* (1.3)) *such that*

$$Q_\delta(t) \leqslant |\Omega| \log 2 + C\big(t + |\delta|\big) \int_{\mathbb{R}^d} \big|db(x)\big|\,dx. \tag{6.1}$$

**Remarks.** 1. The proof uses some ideas developed together with U. Stefanelli and C. DeLellis.

2. The assumption $b_1 \geqslant 1$ is *crucial* and deeply connected to the $2d$ topology. Of course the constants 1 and 2 in (2.2) can be changed to $b_1 \geqslant c_1$ and $|b| \leqslant c_2$ but then the constant in (6.1) is modified by $c_2/c_1$ (just a scaling argument). Finally as it is always possible to decompose $\Omega$ into smaller domains, this assumption can be optimized to assuming that there are a finite number of regular domains $\Omega_i$ with $\Omega = \bigcup \Omega_i$ and in each $\Omega_i$ either $b \cdot B > 0$ for a constant $B$ which in turn would give the assumption in Theorem 1.2.

3. This result is optimal as the estimate in (6.1) does not depend at all on $\delta$. I have no idea how to extend it in higher dimensions even for $b \in W^{1,1}$.

4. As in the $1d$ case this implies directly Bressan's compactness conjecture but not uniqueness of the flow. This uniqueness should be deduced in a second step using this estimate.

**Proof of Theorem 6.1.** Compute,

$$\begin{aligned}\frac{d}{dt}\log\left(\frac{|X(t,x) - X(t,x+\delta)|}{|\delta|}\right) &\leqslant \frac{|b(X(t,x)) - b(X(t,x+\delta))|}{|\delta| + |X(t,x) - X(t,x+\delta)|}\\ &\leqslant \frac{|b(X(t_\delta(t,x),x)) - b(X(t,x+\delta))|}{|\delta|} + \frac{|b(X(t,x)) - b(X(t_\delta(x),x))|}{|\delta| + |X(t,x) - X(t,x+\delta)|},\end{aligned}$$

where $t_\delta(t,x)$ is defined as the unique time such that

$$X_1\big(t_\delta(t,x), x\big) = X_1(t, x+\delta).$$

Integrating over $\Omega$ and $[0,t]$, we find that

$$Q_\delta(t) \leqslant I + II,$$

with

$$I=\int_\Omega\int_0^t \frac{|b(X(t_\delta(s,x),x))-b(X(s,x+\delta))|}{|\delta|}\,ds\,dx, \tag{6.2}$$

and

$$II=\int_\Omega\int_0^t \frac{|b(X(s,x))-b(X(t_\delta(s,x),x))|}{|\delta|+|X(s,x)-X(s,x+\delta)|}ds\,dx. \tag{6.3}$$

The first term may be bounded by:

$$I\leqslant \frac{1}{|\delta|}\int_\Omega\int_0^t\int_{X_2(t_\delta,x)}^{X_2(s,x+\delta)} \big|db\big(X_1(s,x+\delta),\alpha\big)\big|\,d\alpha\,ds\,dx.$$

Denote by $\Omega_\delta(x)$ the set of points included between the two trajectories $X(s,x)$ and $X(s',x+\delta)$ for $-t\leqslant s,s'\leqslant t$ (note that as $b_1>0$ each trajectory is indeed a $1d$ manifold). As $b_1\geqslant 1$ then $\partial_t X_1(t,x)\geqslant 1$ and a line of equation $x_1=\alpha$ may cross the trajectory $\{X(s,x),\,s\in\mathbb{R}\}$ only once. Therefore we may describe $\Omega_\delta$ like,

$$\Omega_\delta(x)=\big\{(y_1,y_2)\;\big|\;X_2\big(t(y_1),x\big)<y_2<X_2\big(t_\delta\big(t(y_1),x\big),x+\delta\big)\big\},$$

where $t(\alpha)$ is defined as the unique $t$ such that $X_1(t(\alpha),x)=\alpha$.

Consequently, changing variables, we get that

$$I\leqslant\frac{2}{|\delta|}\int_\Omega\int_{\Omega_\delta(x)}\big|db(y)\big|\,dy\,dx,$$

as the Jacobian of the transform $t\to X_1(t,x)$ is at most 2.

Changing the order of integration, we have:

$$I\leqslant\frac{2}{|\delta|}\int_{\tilde\Omega}\big|db(y)\big|\times\big|\big\{x,\,y\in\Omega_\delta(x)\big\}\big|\,dy,$$

with $\tilde\Omega=\bigcup_x\Omega_\delta(x)$ and therefore $|\tilde\Omega|\leqslant C|\Omega|$ (if $\Omega$ is regular).

On the other hand,

$$\Omega_\delta(x)\cap\Omega_\delta\big((x_1+\alpha,x_2+\beta)\big)=\emptyset,$$

if $|\alpha|>4t+r$ or $|\beta|>r$. This is one point where the two-dimensional aspect is crucial.

Consequently

$$\big|\big\{x,\,y\in\Omega_\delta(x)\big\}\big|\leqslant 2r(4t+r),$$

and

$$I\leqslant 4(4t+r)\int_{\tilde\Omega}\big|db(y)\big|\,dy.$$

Let us now bound $II$. We first obtain:

$$II\leqslant\int_\Omega\int_0^t\frac{2}{|\delta|+|X(s,x)-X(s,x+\delta)|}\left|\int_s^{t_\delta(s,x)}\big|db\big(X(u,x)\big)\big|\,du\right|ds\,dx.$$

Next note that

$$\begin{aligned}\big|X(s,x)-X(s,x+\delta)\big|&\geqslant\big|X_1(s,x)-X_1(s,x+\delta)\big|\\&=\big|X_1(s,x)-X_1\big(t_\delta(s,x),x\big)\big|\geqslant\frac{1}{2}\big|s-t_\delta(s,x)\big|,\end{aligned}$$

so that

$$II \leqslant \int_\Omega \int_0^t \frac{2}{|\delta| + |s - t_\delta(s,x)|} \left| \int_s^{t_\delta(s,x)} \big|db\big(X(u,x)\big)\big|\, du \right| ds\, dx.$$

By Fubini's Theorem, we get:

$$II \leqslant \int_\Omega \int_0^{t+|\delta|} \big|db\big(X(u,x)\big)\big| \int_0^t \frac{\mathbb{I}_{u\in[s,t_\delta(s,x)]}}{r + |s - t_\delta(s,x)|}\, ds\, du\, dx.$$

Note that the convention $[a,b] = [b,a]$ if $a > b$ is used.

First remark that, due to (2.2),

$$\partial_t\big(X_1\big(t_\delta(t,x),x\big)\big) = \partial_t X_1(t, x+\delta) \in [1,2].$$

As such

$$b_1\big(\Phi\big(t_\delta(t,x),x\big)\big) \times \partial_t t_\delta(t,x) \in [1,2],$$

and thanks to (2.2) again:

$$\partial_t t_\delta(t,x) \in [1/2, 2]. \tag{6.4}$$

Hence we define as $s_\delta(u,x)$ the unique $s$ such that

$$t_\delta(s,x) = u.$$

Assume that $s_\delta(u,x) \leqslant u$ (the other case is dealt with in the same manner). Then $u \in [s, t_\delta(s,x)]$ iff $s \in [s_\delta(u,x), u]$.

As long as $u \in [s, t_\delta(s,x)]$, we have that

$$\big|s - t_\delta(s,x)\big| = |s-u| + \big|t_\delta(s,x) - u\big| \geqslant \max\big(|s-u|, \big|t_\delta(s,x) - u\big|\big).$$

Moreover

$$|s-u| \geqslant \big|s_\delta(u,x) - u\big| - \big|s - s_\delta(u,x)\big|,$$

and using (6.4),

$$\big|t_\delta(s,x) - u\big| = \big|t_\delta(s,x) - t_\delta\big(s_\delta(u,x),x\big)\big| \geqslant \frac{1}{2}\big|s - s_\delta(u,x)\big|.$$

So we bound from below, using $|s-u|$ if $|t_\delta(s,x) - u| \leqslant |s_\delta(u,x) - u|/3$ and $|t_\delta(s,x) - u|$ otherwise,

$$\big|s - t_\delta(s,x)\big| \geqslant \frac{1}{3}\big|s_\delta(u,x) - u\big|.$$

Finally this gives that

$$\int_0^t \frac{\mathbb{I}_{u\in[s,t_\delta(s,x)]}}{r + |s - t_\delta(s,x)|}\, ds \leqslant 3 \int_{s_\delta(u,x)}^u \frac{ds}{|s_\delta(u,x) - u|} \leqslant 3.$$

And coming back to $II$ and using (1.3),

$$II \leqslant 3 \int_\Omega \int_0^{t+|\delta|} \big|db\big(X(u,x)\big)\big|\, du\, dx \leqslant 3C\big(t + |\delta|\big) \int_{\mathbb{R}^d} \big|db(y)\big|\, dy,$$

which, summing with $I$, exactly gives the theorem. $\square$

## 7. The *SBV* case

### 7.1. Presentation

We recall the definition of *SBV* (see for example [5, 4.1]).

**Definition 7.1.** $b \in SBV(\mathbb{R}^d)$ iff $db = m + \theta \mathcal{H}^{d-1}|_J$ with $m \in L^1$, $J$ $\sigma$-finite with respect to $\mathcal{H}^{d-1}$, and

$$\int_J |\theta| \, d\mathcal{H}^{d-1} < \infty.$$

For this restricted class of $b$ but now in any dimension, Bressan's compactness conjecture holds and more precisely

**Theorem 7.1.** *Consider a sequence of solutions to* (1.2) *satisfying* (1.4), (2.2) *uniformly in* $n$ *and such that* $b_n \to b \in SBV(\mathbb{R}^d)$ *with* $\mathcal{H}^{d-1}(J) < \infty$. *Then this sequence is compact and more precisely* $\forall \eta$, *there exists a continuous function* $\varepsilon(\delta)$ *with* $\varepsilon(0) = 0$ *and such that* $\forall n$ *large enough,* $\forall \delta' < \delta$, $\forall w \in S^{d-1}$, $\exists \omega$ *(depending on* $\eta$, $n$ *and* $\delta'$*) with* $|\omega| \leqslant \eta$, *and*

$$\forall x \in \Omega \setminus \omega, \ \forall t \leqslant 1, \quad \bigl|X_n(t,x) - X_n\bigl(t, x + \delta' w\bigr)\bigr| \leqslant \varepsilon(\delta). \tag{7.1}$$

**Remarks.** 1. Uniqueness also holds, the proof being the same. It is even easier as there is no need to work with a fixed scale $\delta'$ and the additional assumption $\mathcal{H}^{d-1}(J) < \infty$ is not required (see the more detailed comment below).

2. The function $\varepsilon(\delta)$ strongly depends on the structure of $b$ and in particular on the local regularity of its jump set $J$ (more precisely the Lipschitz norm of $g$ if $J$ has equation $g(x) = 0$ locally). Therefore this result cannot be extended directly to get to $b \in BV$.

Let us comment more on the assumption $\mathcal{H}^{d-1}(J) < \infty$. A natural idea to try to bypass it would be to truncate $J$ into a set with finite Hausdorff measure and a remainder $J'$. This means that we are approximating $b$ by $b_\gamma$ with $|b - b_\gamma| < \gamma$ and the jump set of $b_\gamma$ is a nice $J_\gamma$ with $\mathcal{H}^{d-1}(J_\gamma) < \infty$ (and in fact it is even $o(\gamma^{-1})$). To give an idea of why this is not directly working, consider step 3 in the proof. Its aim is to control the number of times a trajectory $X_n(t,x)$ comes at a distance $\delta$ of $J_\gamma$. For that the crucial estimate is:

$$\bigl|\bigl\{x, d(x, J_\gamma) < \delta\bigr\}\bigr| \leqslant K\delta \mathcal{H}^{d-1}(J_\gamma).$$

However the constant $K$ in this estimate depends on $J_\gamma$. What is true is that $|\{x, d(x, J_\gamma) < \delta\}|/\delta$ is bounded, *asymptotically* as $\delta \to 0$, by $\mathcal{H}^{d-1}(J_\gamma)$. But if $\delta$ is not small enough, then the constant $K$ can be much larger than 1: Take $J_\gamma$ composed of many small pieces of radius much smaller than $\delta$ for example.

So the only solution is to take $\delta$ small enough for $J_\gamma$. Unfortunately we approximated $b$ so in any case we cannot take scales smaller than $\gamma$ and of course it could very well be that $\delta = \gamma$ is still not small enough.

On the other hand this is a problem only when considering a positive scale. If the aim is only uniqueness, one does not have to choose a scale and following the same steps, it is enough to control the number of times that $X(t,x)$ crosses $J_\gamma$. This number is always bounded directly in terms of $\mathcal{H}^{d-1}(J_\gamma)$. This would make the proof of uniqueness really easier and without the assumption $\mathcal{H}^{d-1}(J) < \infty$, in line with [4].

Finally, before giving the details, let us present the main ideas of the proof. The contributions from the jump part of $db_n$ and from the $L^1$ part will be treated separately and for the $L^1$ part of course similarly as the $W^{1,1}$ case (see Section 4). So focus here only on the jump part and simply assume that $b_n$ is piecewise constant: $b_n(x) = b^-$ for $x_1 < 0$ and $b_n(x) = b^+$ for $x_1 > 0$ with $b^- \neq b^+$.

Take the two trajectories $X_n(t,x)$ and $X_n(t, x + \delta w)$. And assume that initially $x_1 < -\delta$ (if $x_1 > \delta$ then they never see the jump in $b_n$). Until one of the two $X_n$ or $X_{n,\delta} = X_n(t, x + \delta w)$ reaches the hyperplane $x_1 = 0$, their velocity is the same. Assume that $X_{n,\delta}$ touches the hyperplane first and denote $t_1$ the first time in $[0, T]$ when this happens. So

$$\forall t \leqslant t_1, \quad |X_n - X_{n,\delta}| = \delta.$$

As $b_1 \geqslant 1$ and in particular $b^+ \geqslant 1$, $X_{n,\delta}$ will never again pass through $\{x_1 = 0\}$ so that

$$b_n(X_{n,\delta}) = b^+, \quad \forall t \geqslant t_1.$$

However $b^- \geqslant 1$ also so $X_n$ will necessarily touch the hyperplane some time after $t_1$. Denote $t_2$ this time. After $t \geqslant t_2$, $b_n(X_n) = b^+$, and so

$$\left|X_n(t) - X_{n,\delta}(t)\right| = \left|X_n(t_2) - X_{n,\delta}(t_2)\right|.$$

As $\|b\|_\infty \leqslant 2$, this implies that for any $t \in [0, T]$,

$$\left|X_n(t) - X_{n,\delta}(t)\right| \leqslant \delta + 2(t_2 - t_1).$$

Finally at $t_1$, $X_{n,1}(t_1) \geqslant -\delta$ and as $b^- \geqslant 1$ this means that $t_2 \leqslant t_1 + \delta$, enabling us to conclude that

$$\left|X_n(t) - X_{n,\delta}(t)\right| \leqslant \delta + 2\delta, \quad \forall t \in [0, T].$$

Notice that this is only one of two cases: Here the trajectories always cross the jump set but it could happen that they are tangent. So another interesting example occurs when: $b_n(x) = b^-$ for $x_2 < 0$, $b_n(x) = b^+$ for $x_2 = 0$ and $b_2^- = 0$ (and therefore $b_2^+ = 0$ by the compressibility condition (1.4), see also Section 2.3). Now the trajectories never cross $\{x_2 = 0\}$ so if $x$ and $x + \delta w$ are on the side of this hyperplane, $X_n$ and $X_{n,\delta}$ stay on the same side, and hence

$$|X_n - X_{n,\delta}| = \delta.$$

We have problems when they start on different sides and then there is nothing one can do: At time $t$, $|X_n - X_{n,\delta}|$ is of order $t$. However this only happens if $x$ belongs to $\omega = \{-\delta \leqslant x_2 \leqslant \delta\}$, which is why in the theorem we need to exclude some starting points. Here we would simply have:

$$\exists \omega \subset \Omega \text{ with } |\omega| \leqslant C\delta, \quad \forall x \in \Omega \setminus \omega,\ \forall t \in [0, T], \quad |X_n - X_{n,\delta}| \leqslant \delta.$$

In the general *SBV* case, the two situations may occur. So it is necessary to first identify the jump set, the regions where the trajectories typically cross and the regions where they are almost tangent (and all that quantitatively). Next one has to exclude the starting points $x$ which would lead to trajectories passing through the tangent regions, and exclude among the other trajectories the ones that are not typical (*i.e.* they do not cross as fast as they should). And of course a trajectory could very well cross the jump set several times so a bound on that number of times will be required as well.

### 7.2. *Proof of Theorem 7.1*

**Step 1.** Decomposition of $db$, $db_n$.

Fix $\eta$. Through all proof $K$ will denote constants depending on $\eta$ or $b$ and $C$ will be kept for constants depending only on $d$ or $\Omega$.

Decompose $b$ as in the definition:

$$db = m + \theta\mathcal{H}^{d-1}|J.$$

$J$ is countably rectifiable and of finite measure. Therefore decompose $J = H \cup J'$ with $H$ a finite union of rectifiable sets $H_i$ such that

$$\mathcal{H}^{d-1}\left(J'\right) < \eta/C. \tag{7.2}$$

By the definition of the Hausdorff measure (and as $J'$ is countably rectifiable), there exists a covering $J' \subset \bigcup_i B(x_i, r_i)$ s.t. a point of $\mathbb{R}^d$ belongs to at most $C$ balls, and with

$$\sum_i r_i^{d-1} < 2\eta/C.$$

As $b_n$ converges toward $b$, and taking $n \geqslant N$ large enough, one may decompose accordingly $db_n$ as

$$|db_n| = m_n + \sigma_n + r_n, \quad m_n,\ \sigma_n,\ r_n \geqslant 0,$$

with for some $\tilde{\phi}$ with $\tilde{\phi}(\xi)/\xi \to +\infty$ as $\xi \to +\infty$

$$\sup_n \int_\Omega \tilde{\phi}(m_n)\, dx \leqslant C, \tag{7.3}$$

and

$$\operatorname{supp}\sigma_n \subset \{x,\ d(x,H) \leqslant \delta\}, \tag{7.4}$$

with finally

$$\operatorname{supp} r_n \subset \bigcup_i B(x_i, 2r_i). \tag{7.5}$$

Denote $\tilde b_n$ the function equal to $b_n$ on $\Omega \setminus \Omega_r$ with $\Omega_r = \bigcup_i B(x_i, 4r_i)$ and in $B(x_i, 3r_i)$,

$$\tilde b_n = b_n \star L_{r_i}, \quad \text{with } L_r(x) = r^{-d} L(x/r),$$

with $L$ a $C^\infty$ function with total mass 1 and compactly supported in $B(0,1)$. In $B(x_i, 4r_i) \setminus B(x_i, 3r_i)$, choose a linear interpolation between the two values on $\partial B(x_i, 4r_i)$ and $\partial B(x_i, 3r_i)$. One obtains a corresponding decomposition of $|d\tilde b_n|$,

$$|db_n| \leqslant m_n \mathbb{I}_{\Omega_r^c} + \sigma_n \mathbb{I}_{\Omega_r^c} + \tilde r_n + \frac{\mu_n}{r_i},$$

with $\mu_n \leqslant 2$ a bounded function, and

$$\tilde r_n \leqslant \sum_i |db_n| \star L_{r_i} \mathbb{I}_{B(x_i,3r_i)}.$$

By De La Vallée Poussin, there exists $\Phi$, with $\psi(\xi) = \Phi(\xi)/\xi$ increasing and converging to $+\infty$ s.t.

$$\begin{aligned}\int_\Omega \Phi(m_n + \tilde r_n + \mu_n/r_i)\,dx &\leqslant \int_\Omega \Phi(m_n)\,dx + C\sum_i \psi\big(r_i^{-1}\big) r_i^{d-1} \\ &\quad + C\sum_i \psi\big(\|L\|_\infty r_i^{-d}\big) \int_{B(x_i,3r_i)} |db_n|\,dx < \infty,\end{aligned}$$

provided $\phi \leqslant \tilde\phi$ and recalling that

$$\sum_i r_i^{d-1} < \infty, \qquad \sum_i \int_{B(x_i,3r_i)} |db_n|\,dx \leqslant C\int_\Omega |db_n|\,dx.$$

Denote $\omega_r$ the set of $x$ s.t. $\exists t \in [0,T]$ with $X_n(t,x) \in \Omega_r$. Of course

$$|\omega_r| \leqslant \sum_i \big|\{x,\ \exists t\ X_n(t,x) \in B(x_i, 4r_i)\}\big|,$$

and if $X_n(t,x) \in B(x_i, 3r_i)$ then $X_n(s,x) \in B(x_i, 5r_i)$ for $s \in [t - r_i, t + r_i]$, and so

$$\begin{aligned}\big|\{x,\ \exists t\ X_n(t,x) \in B(x_i, 4r_i)\}\big| &\leqslant \frac{1}{r_i}\int_\Omega\int_0^T \mathbb{I}_{X_n(t,x)\in B(x_i,5r_i)}\,dt\,dx \\ &\leqslant \frac{C}{r_i}\int_0^T\int_\Omega \mathbb{I}_{x\in B(x_i,5r_i)}\,dx\,dt \leqslant C r_i^{d-1}.\end{aligned}$$

Hence one has:

$$|\omega_r| \leqslant \eta/8. \tag{7.6}$$

Because of (7.4) and Lemma 3.2 as long as $x$ and $y$ are on the same side of $H$, not in $\omega_r$ and both at distance larger than $\delta$, then

$$\begin{aligned}\big|b_n\big(X_n(t,x)\big) - b_n\big(X_n(t,y)\big)\big| &= \big|\tilde b_n\big(X_n(t,x)\big) - \tilde b_n\big(X_n(t,y)\big)\big| \\ &\leqslant C\int_{B_K(x,y)} \tilde m_n\left(\frac{1}{|x-z|^{d-1}} + \frac{1}{|y-z|^{d-1}}\right)dz,\end{aligned} \tag{7.7}$$

with

$$\int_\Omega \Phi(\tilde m_n)\,dx < \infty.$$

**Step 2.** Decomposition of the trajectories.

Fix $w \in S^{d-1}$. For any $x$ we decompose the time interval $[0,T]$ into segments $]s_i, t_i[$ such that on such an interval $X_n(t,x)$ and $X_{n,\delta} = X_n(t, x+\delta w)$ are both on the same side of $H$ and both at a distance larger than $\delta$ of $H$.

On the contrary on each interval $[t_i, s_{i+1}]$, either $X_n(t,x)$ and $X_{n,\delta} = X_n(t, x+\delta w)$ are on different sides of $H$ or one of them is at a distance less than $\delta$ of $H$.

Notice that of course $t_i$ and $s_i$ depend on $x$, $\delta$, $b$ and $\eta$ (as $H$ depends on $\eta$), and

$$[0,T] = \bigcup_{i\leqslant n} ]s_i, t_i[ \cup [t_i, s_{i+1}].$$

The transition between the two intervals (at $t_i$ as well as at $s_i$) is when one of $X_n(t,x)$ or $X_n(t,x+\delta w)$ is exactly at distance $\delta$ of $H$, because for the two of them to be either on both side or on the same side of $H$, one of them has to cross $H$ and thus to come first at a distance $\delta$.

The idea of the rest of the proof is to bound $|X - X_\delta|$ on $]s_i, t_i[$ with arguments similar to the $W^{1,1}$ case. On $[t_i,\ s_{i+1}]$, the bound will be obtained simply by controlling $s_{i+1} - t_i$. This will be enough provided that the number of such intervals $n$ (depending on $x$ again) is not too large.

**Step 3.** Bound on the number of intervals.

At $s_i$ and $t_i$ either $X_n(t,x)$ or $X_{n,\delta} = X(t, x+\delta w)$ is at distance $\delta$ of $H$. Therefore as $\|b\|_\infty \leqslant 2$, on the intervals $[s_i - \delta, s_i + \delta]$ and $[t_i - \delta, t_i + \delta]$, again either $X_n$ or $X_{n,\delta}$ is at distance less than $3\delta$ of $H$.

So if there are $n$ such intervals for $x$ then either $X_n$ stays at distance less than $3\delta$ of $H$ for a total time of at least $n\delta$ or $X_{n,\delta}$ does.

Now denote by $\omega^1$ the set of all $x$ such that $X_n(t,x)$ stays distance less than $3\delta$ of $H$ for a total time of at least $n\delta$. Obviously

$$\int_0^T\int_{\omega^1} \mathbb{I}_{d(X_n(t,x),H)\leqslant 3\delta}\,dx\,dt \geqslant n\delta\left|\omega^1\right|.$$

But on the other hand, changing variables,

$$\begin{aligned}\int_0^T\int_{\omega^1} \mathbb{I}_{d(X_n(t,x),H)\leqslant 3\delta}\,dx\,dt &\leqslant \int_0^T\int_{\Omega} \mathbb{I}_{d(X_n(t,x),H)\leqslant 3\delta}\,dx\,dt \\ &\leqslant \int_0^T\int \mathbb{I}_{d(x,H)\leqslant 3\delta}\,dx\,dt \leqslant KT\delta\mathcal{H}^{d-1}(H) < KT\delta,\end{aligned}$$

as the measure of $H$ is finite but depends on $\eta$. Thus one finds:

$$\left|\omega^1\right| \leqslant K/n.$$

Choose $n = 1/(8K\eta)$ so that except a set of measure less than $\eta/4$ (and equal to $\omega^1 \cup (\omega^1 - \delta w)$), no trajectory is decomposed on more than $n$ intervals.

**Step 4.** Control on $[t_i, s_{i+1}]$, definitions.

First of all, take any $z \in H$. Denote by $b_-(z)$ the trace of $b$ on one side of $H$ and $b_+(z)$ on the other side (again orientation is chosen locally and does not have to hold globally for $H$). For simplicity, we assume that $b_-(z) \cdot \nu(z) \geqslant 0$ (with $\nu$ the normal to $H$ chosen according to the orientation).

Now fix some $\alpha$ to be chosen later.

As $b$ is approximately continuous (see [5]) at $z$ on either side of $H$ one has:

$$r^{-d}\big|\big\{x \in B_\pm(z,r),\ \big|b(x) - b_\pm(z)\big| \geqslant \alpha\big\}\big| \to 0,$$

where $B_\pm(z,r)$ is the subset of $B(z,r)$ which is on the $-$ or $+$ side of $H$.

It is even possible to be more precise. Take a function $\tilde\eta(r) \to 0$ as $r \to 0$ with $\tilde\eta \leqslant \eta/(6\mathcal{H}^{d-1}(H))$. Denote for $r \leqslant 1/2$,

$$\mu(z,r)_\pm = \big|\big\{x \in B_\pm(z,r),\ \big|b(x) - b_\pm(z)\big| \geqslant \alpha\big\}\big|,$$

and

$$\tilde\mu(z,r)_\pm = \big\{x \in \mu(z,r)_\pm,\ d(x,H) \geqslant r\tilde\eta/K\big\},$$

with $K$ large enough with respect to the Lipschitz constant of $H$. Assume that $|\mu(z,r)_-| \geqslant r^d\tilde\eta$, then necessarily $|\tilde\mu(z,r)_-| \geqslant r^d\tilde\eta/2$; For any $x \in \tilde\mu(z,r)_-$, one has

$$\alpha \leqslant \big|b(x) - b_-(z)\big| \leqslant \int_0^1 \big|db\big(\theta x + (1-\theta)z\big)\big|\,d\theta.$$

Integrate this inequality over $\tilde\mu$ to find:

$$\int_0^1 \int_{B(z,r)\ \text{s.t.}\ d(x,H)\geqslant r\tilde\eta/K} \big|db\big(\theta x + (1-\theta)z\big)\big|\,dx\,d\theta \geqslant C\alpha r^d\tilde\eta/2.$$

Now denote $H_r$ the set of $z$ such that $|\mu(z,r)_\pm| \geqslant r^d\tilde\eta$. Integrate the last inequality on $H_r$ on the right-hand side and on $H$ on the left-hand side to obtain:

$$\int_H\int_0^1 \int_{B(z,r)\ \text{s.t.}\ d(x,H)\geqslant r\tilde\eta/K} \big|db\big(\theta x + (1-\theta)z\big)\big|\,dx\,d\theta\,dz \geqslant C\alpha r^d\tilde\eta\mathcal{H}^{d-1}(H_r)/2.$$

Let us bound the integral on the left-hand side. As the case $\theta \geqslant 1/2$ is easy, assume $\theta < 1/2$ and change variables locally around $H$ such that $H$ has equation $x_1 = 0$. The integral is then dominated by:

$$K \int_{z\in H}\int_0^{1/2}\int_{B(z,r),x_1<-r\tilde\eta/K} \big|db\big(\theta x_1, \theta x' + (1-\theta)z\big)\big|\,dx\,d\theta\,dz$$
$$\leqslant 2^{d-1}K \int_{z\in\tilde H}\int_0^{1/2}\int_{B(z,r),x_1<-r\tilde\eta/K} \big|db(\theta x_1, z)\big|\,dx\,d\theta\,dz,$$

with $\tilde H = \{\theta x' + (1-\theta)z,\ z \in H,\ x \in B(z,r)\}$ and denoting $x = (x_1, x')$. Then changing variable from $\theta$ to $\theta x_1$, one may finally bound by:

$$2^{d-1}K \int_{z\in\tilde H}\int_{-r/2}^0 \big|db(\theta,z)\big| \int_{-r<x_1<-r\tilde\eta} \frac{dx_1}{\tilde\eta r}\,d\theta\,dz \leqslant \frac{K}{\tilde\eta}\int_{z\in\tilde H}\int_{-r/2}^0 \big|db(\theta,z)\big|\,d\theta\,dz.$$

Note that this last integral converges to 0 as $r \to 0$ and hence conclude that by choosing $\tilde\eta$ large enough in terms of $r$, one can ensure that

$$|H_{r'}| \leqslant \chi(r'), \qquad \chi(r') \to 0 \quad \text{as } r' \to 0, \qquad \chi(r') \leqslant \eta/C \quad \forall r' \leqslant r.$$

In addition from the strong convergence $b_n \to b$, take $n_0$ s.t. $\forall n \geqslant n_0$, $\forall z \in H \setminus H_r$,

$$r'^{-d}\big|\big\{x \in B_\pm\big(z, r'\big),\ \big|b_n(x) - b_\pm(z)\big| \geqslant \alpha\big\}\big| \leqslant 3\tilde\eta/2 \leqslant \eta/\big(4\mathcal{H}^{d-1}(H)\big) \quad \forall|\delta| < r' < r. \tag{7.8}$$

This is the second point where $n$ has to be large enough in terms of $\delta$.

Remark eventually that one may always assume that $r \leqslant \alpha$ (unfortunately not the opposite way) and $(\alpha r)^d < \eta/C$.

**Step 5.** Control on $[t_i, s_{i+1}]$, non-crossing trajectories.

We start by taking apart the trajectories which do not cross $H$. So define:

$$H_0 = \big\{z_i \notin H_r,\ b_-(z_i)\cdot\nu(z_i) \leqslant 2\alpha\big\}, \quad \text{and}$$

$$\omega^2 = \Big\{x,\ \exists t \in [0,T] \text{ s.t. } X_n(t,x) \in \bigcup_{z\in H_0} B_-(z,\alpha r) \cup B_+(z,\alpha r)\Big\},$$

$$\omega^3 = \big\{x,\ \forall t \in [0,T] \text{ s.t. } d\big(X_n(t,x), H\big) < \alpha r/2, \text{ one has } B(X_n,\alpha r) \cap (H \setminus H_r) = \emptyset\big\}.$$

Note that by (1.4) (see Section 2.3 in the introduction), if $b_-(z)\cdot\nu(z) \leqslant 2\alpha$ on a neighborhood of the border then $b_+\cdot\nu(z) \leqslant K\alpha$ (with $K$ function of the constant in (1.3) and the Lipschitz bound on $H$). Therefore in the previous definition of $H_0$, one only needs to put $b_-$ (in any case the proof could work the same by essentially ignoring what occurs on the side where $b\cdot\nu \geqslant 0$, except in a band of width $\delta$).

Bound $\omega^3$ first. Simply notice that if

$$\Omega^3 = \big\{x \text{ with } d(x,H) < \alpha r/2, \text{ s.t. } B(x,\alpha r) \cap (H \setminus H_r) = \emptyset\big\},$$

then for $x \in \Omega^3$, $\mathcal{H}^{d-1}(B(x,\alpha r) \cap H_r) \geqslant C\alpha^{d-1}r^{d-1}$ and as $\mathcal{H}^{d-1}(H_r) \leqslant \eta/C$, and

$$\big|\Omega^3\big| \leqslant \alpha r\eta/C, \qquad \big|\Omega^3_r\big| = \big|\big\{x,\ d\big(x,\Omega^3\big) \leqslant \alpha r\big\}\big| \leqslant \alpha r\eta/C.$$

As $\omega^3 = \{x,\ \exists t \in [0,T]\ X_n(t,x) \in \Omega^3\}$, and if $X_n(t,x) \in \Omega^3$ then $X_n(s,x) \in \Omega^3_r$ for $s \in [t, t+\alpha r]$,

$$\alpha r\big|\omega^3\big| \leqslant \int_0^T\int_\Omega \mathbb{I}_{X_n(t,x)\in\Omega^3_r}\,dx \leqslant C\int_0^T\int_{\mathbb{R}^d} \mathbb{I}_{x\in\Omega^3_r}\,dx \leqslant T\alpha r\eta/C,$$

by change of variable and the bound on $|\Omega^3_r|$. This implies:

$$\big|\omega^3\big| \leqslant T\eta/C. \tag{7.9}$$

Let us now bound $\omega^2$, decompose,

$$\omega^2 = \omega^{2,1} \cup \omega^{2,2},$$

with $\omega^{2,1}$ the subset of all $x \in \omega^2$ s.t. the trajectory $X_n(t,x)$ stays at least a time interval $r/2$ at a distance less than $K\alpha r$ of $H$. Notice that

$$\int_{\omega^{2,1}}\int_0^T \mathbb{I}_{d(X_n,H)\leqslant K\alpha r}\,dt\,dx \geqslant \big|\omega^{2,1}\big|\frac{r}{2}.$$

On the other hand by (1.4),

$$\int_{\omega^{2,1}}\int_0^T \mathbb{I}_{d(X_n(t,x),H)\leqslant K\alpha r}\,dt\,dx \leqslant \int_0^T\int_{\mathbb{R}^d} \mathbb{I}_{d(x,H)\leqslant K\alpha r}\,dx\,dt \leqslant KT\alpha r\mathcal{H}^{d-1}(H).$$

Therefore

$$\big|\omega^{2,1}\big| \leqslant \eta/8,$$

provided that

$$CTK\mathcal{H}^{d-1}(H)\alpha \leqslant \eta/8. \tag{7.10}$$

There remains $\omega^{2,2}$ which is made of those $x \in \omega^2$ staying a time less than $r/2$ at a distance less than $K\alpha r$ of $H$. Denote $t_0$ s.t. $X_n(t_0, x) \in B_-(z, \alpha r)$ for some $z$ (and of course the same analysis holds for the $+$ side).

If this is so then the average over the interval $[t_0, t_0 + r/2]$,

$$\frac{2}{r}\int_{t_0}^{t_0+r/2} \nu(X_n)\cdot b_n\big(X_n(t,x)\big)\,dt \geqslant (K-1)\alpha.$$

Taking $K$ large enough with respect to the Lipschitz bound on $H$, this implies:

$$\frac{2}{r}\int_{t_0}^{t_0+r/2} \nu(z)\cdot b_n\big(X_n(t,x)\big)\,dt \geqslant K\alpha,$$

and consequently

$$\frac{2}{r}\int_{t_0}^{t_0+r/2} \big|b_n\big(X_n(t,x)\big) - b_-(z)\big|\,dt \geqslant K\alpha.$$

As $1 \leqslant b_1 \leqslant 2$, notice that on the time interval $[t_0, t_0 + r/2]$ then $X_n(t,x)$ stays inside the ball $B(z,r)$ and moreover the length of the path is of order $r$. Consequently, denoting $\omega_z^{2,2}$ the subset of $\omega^{2,2}$ corresponding to the ball $B(z,r)$, one has by change of variable (and again the use of (1.4))

$$\frac{1}{r^d}\big\{x \in B_-(x),\ \big|b_n(x) - b_-(z)\big| \geqslant \alpha\big\} \geqslant K\big|\omega_z^{2,2}\big|,$$

which finally gives $\omega_z^{2,2} \leqslant \eta r^d/K$ and summing over all $z = z_i$

$$\big|\omega^2\big| \leqslant \big|\omega^{2,1}\big| + \big|\omega^{2,2}\big| \leqslant \eta/8 + \eta\sum_i r^d/K \leqslant \eta/4, \tag{7.11}$$

by choosing $K$ large enough.

**Step 6.** Control on $[t_i, s_{i+1}]$, the crossing trajectories.

We now only have to take into account the trajectories passing through. Consider for example:

$$H_1 = \big\{z \notin H_r,\ b_-(z)\cdot\nu(z) \geqslant 2\alpha\big\}, \quad \text{and}$$
$$\omega^0 = \Big\{x,\ \exists t \in [0,T] \text{ s.t. } X_n(t,x) \in \bigcup_{z\in H_1} B_-(z,\alpha r)\Big\}.$$

The same holds if one uses $b_+(z_i)\cdot\nu(z) < -2\alpha$ in the definition and note again that by (1.3) one has necessarily one or the other as the case $|b_\pm\cdot\nu(z)| \leqslant 2\alpha$ was taken care of in the previous step.

As $|H_{r'}| \to 0$, there exists an extracted sequence $r_k \to 0$ (with $r_0 = r$) s.t. for any $0 < \gamma < 1$,

$$\sum_k r_k < \infty, \qquad \sum_k |H_{r_k}| < \infty, \quad \text{and so} \quad \sum_{k=0}^{\infty} |H_{r_k}| \leqslant \eta/K,$$
$$\sum_k \tilde\eta(r_k) < \infty, \qquad \sum_{k=0}^{+\infty} \tilde\eta(r_k) \leqslant \eta/K. \tag{7.12}$$

For any $\delta$, denote $k_\delta$ s.t.

$$\sum_{k\leqslant k_\delta} \frac{\delta}{r_k} \leqslant \eta/K. \tag{7.13}$$

Of course $r_{k_\delta} > \delta$ but note that $k_\delta \to +\infty$ as $\delta \to 0$.

Denote $\Omega_+$ the set of $y \in \bigcup_{z\in H} B_+(z,r)$ s.t. $d(y,H) > \delta$ and $\Omega_- = \bigcup_{z\in H} B_+(z,r) \setminus \Omega_+$.

For any scale $r_k > C\delta$ with $k \geqslant 1$, define $\omega_k$ as the set of $x$ s.t.

(i) $\exists t_0 \in [0,T]$ with $d(X_n(t_0,x),H) \leqslant r_k\alpha/K$, $K \geqslant 6$,
(ii) one never has $X_n(t,x) \in \Omega_+$ for any $t \in [t_0, t_0 + r_k/3]$.

By (i) and $\|b_n\|_\infty \leqslant 2$, one has that $X_n(t,x)$ remains in the same ball $B(X_n(t_0,x), r_k/2)$ for all $t \in [t_0, t_0 + r_k/K]$ (for $K \geqslant 4$). The intersection of this ball with $H$ has diameter larger than $r_k/3$ as $d(X_n(t_0,x),H) \leqslant r_k\alpha/6$.

Accordingly decompose $\omega_k = \omega_k^1 \cup \omega_k^2$ with $\omega_k^1$ the set of $x \in \omega_k$ s.t.

$$B\big(X_n(t_0,x), r_k/2\big) \cap (H \setminus H_{r_k}) = \emptyset,$$

and $\omega_k^2 = \omega_k \setminus \omega_k^1$.

Start by bounding $\omega_k^1$. Define,

$$\Omega_k^1 = \big\{x,\ B(x, r_k/2) \cap (H \setminus H_{r_k}) = \emptyset\big\}.$$

Just as in the previous step,

$$\big|\Omega_k^1\big| \leqslant C\mathcal{H}^{d-1}(H_{r_k}) r_k, \qquad \big|\tilde\Omega_k^1\big| = \big|\big\{x, d\big(x, \Omega_k^1\big) \leqslant r_k\big\}\big| \leqslant C\mathcal{H}^{d-1}(H_{r_k}) r_k.$$

On the other hand $X_n(t,x)$ stays inside $\tilde\Omega_k^1$ for all the interval $[t_0, t_0 + r_k/K]$ so again,

$$\big|\omega_k^1\big| r_k/K \leqslant \int_0^T\!\!\int_\Omega \mathbb{I}_{X_n(t,x)\in\tilde\Omega_k^1}\,dx\,dt \leqslant \int_0^T\!\!\int_{\mathbb{R}^d} \mathbb{I}_{x\in\tilde\Omega_k^1}\,dx\,dt \leqslant C\mathcal{H}^{d-1}(H_{r_k}) r_k,$$

which gives the desired bound

$$\big|\omega_k^1\big| \leqslant K\mathcal{H}^{d-1}(H_{r_k}). \tag{7.14}$$

Now for $\omega_k^2$, notice that for any $x \in \omega_k^2$ there exists $z \in B(X_n(t_0,x), r_k/2) \cap (H \setminus H_{r_k})$. Hence we may take $N$ points $z_i \in H \setminus H_{r_k}$ with

$$N \leqslant K\mathcal{H}^{d-1}(H)/r_k^{d-1},$$

and such that for any $x \in \omega_k^2$, there is $z_i \in B(X_n(t_0,x), r_k/K)$.

On the interval $[t_0, t_0 + r_k/K]$, $X_n$ is in $\Omega_-$. Denote $H^\delta = \{x, d(x,H) \leqslant \delta\}$, decompose again $\omega_k^2$ into $\omega_k^{2,\delta} \cup \tilde\omega_k^2$ with $\omega_k^{2,\delta}$ the set of $x$ such that $X_n$ stays more than a total time $\Delta = \alpha r_k/K$ in $H^\delta$. Bound

$$\big|\omega_k^{2,\delta}\big| \alpha r_k/K \leqslant \int_0^T\!\!\int_\Omega \mathbb{I}_{d(X_n(t,x),H)<\delta}\,dx\,dt \leqslant K\delta\mathcal{H}^{d-1}(H).$$

Thus

$$\big|\omega_k^{2,\delta}\big| \leqslant K\delta\mathcal{H}^{d-1}(H)/r_k. \tag{7.15}$$

For $x \in \tilde\omega_k^2$, denote by $I$ the subset of $[t_0, r_k/K]$ s.t. $X_n(t,x) \notin H^\delta$. As $X_n$ never reaches $\Omega_+$ and stays in $H^\delta$ at most $\alpha r_k/K$ one has:

$$\frac{1}{|I|}\int_I b_n\big(X_n(t,x)\big)\cdot\nu(X_n)\,dt < \frac{K}{r_k}(\delta + \alpha r_k/K + 2\alpha r_k/K) < \alpha/3.$$

Moreover for any $t \in [t_0, t_0 + r_k/K]$, one has $|X_n(t,x) - z_i| \leqslant r_k/K$ and as $\nu$ is Lipschitz (at least around $H$), then

$$\frac{1}{|I|}\int_I b_n\big(X_n(t,x)\big)\cdot\nu(z_i)\,dt < \alpha/2,$$

and

$$\frac{1}{|I|}\int_I \mathbb{I}_{|b_n(X_n(t,x))-b_\pm(z_i)|\geqslant\alpha}\,dt > 1.$$

Put $\tilde\omega_k^i = \{x\in\tilde\omega_k^2,\ X(t_0,x)\}\in B(z_i,r_k/K)$. By (7.8) and integrating along the trajectories (as we did many times before), we deduce that

$$\left|\tilde\omega_k^i\right| \leqslant K\tilde\eta(r_k)r_k^{d-1}.$$

Summing over $i$,

$$\left|\tilde\omega_k^2\right| \leqslant K\tilde\eta(r_k). \tag{7.16}$$

*Summary and conclusion of the estimate*

Define:

$$\tilde\omega = \omega_r\cup\omega^1\cup\omega^2\cup\omega^3\cup_k\omega_k, \qquad \bar\omega = \tilde\omega\cup(\tilde\omega-\delta w).$$

By the previous computations (end of steps 1, 3, (7.9) and (7.11) in step 5 and (7.14), (7.15), (7.16) in step 6),

$$|\bar\omega| \leqslant \eta/4+\eta/4+T\eta/C+\eta/4+K\sum_k\big(|H_{r_k}|+\delta/r_k+\tilde\eta(r_k)\big)\leqslant 7\eta/8, \tag{7.17}$$

by (7.12) and (7.13).

Now for $x\notin\bar\omega$, we have for any $t\in[s_i,t_i]$

$$\left|b(X_n)-b(X_{n,\delta})\right| \leqslant C\int_{B_K(X_n,\ X_{n,\delta})}\tilde m_n\left(\frac{1}{|X_n-z|^{d-1}}+\frac{1}{|X_{n,\delta}-z|^{d-1}}\right)dz.$$

As in the proof of Theorem 4.1, we define:

$$\psi\big(|\delta|\big) = \inf_M M - 2K\frac{M}{\phi(M)}\log\big(|\delta|\big).$$

And we obtain:

$$\begin{aligned}
&\int_{\bar\omega^c}\sum_i\log\left(\frac{\delta+|X_n(t_i,x)-X_{n,\delta}(t_i,x)|}{\delta+|X_n(s_i,x)-X_{n,\delta}(s_i,x)|}\right)\\
&\quad\leqslant\int_{\omega^c}\sum_i\int_{s_i}^{t_i}\frac{|b(X_n)-b(X_{n,\delta})|}{|\delta|+|X_n-X_{n,\delta}|}\,dt\,dx\\
&\quad\leqslant C\int_\Omega\int_0^T\int_{B_K(X_n,X_{n,\delta})}\tilde m_n\left(\frac{1}{|X_n-z|^{d-1}}+\frac{1}{|X_{n,\delta}-z|^{d-1}}\right)dz\leqslant\psi\big(|\delta|\big).
\end{aligned}$$

So up to the first time $t_1$, one has that

$$|X_n-X_{n,\delta}|\leqslant\delta\exp\big(8\psi\big(|\delta|\big)n/\eta\big),$$

except for $x\in\omega_1^4\cup\bar\omega$ with

$$\left|\omega_1^4\right|\leqslant\eta/8n.$$

Now by induction let us prove that up to time $t_i$,

$$|X_n-X_{n,\delta}|\leqslant\delta_i,$$

except for $x\in\bar\omega\cup\omega_i^4$ with

$$\left|\omega_i^4\right|\leqslant i\eta/8n,$$

and $\delta_i$ defined through

$$\delta_{i+1} = (\delta_i + r_{k_i}/K)\exp\big(8n\psi(\delta_i + r_{k_i}/K)/\eta\big), \quad r_{k_i} = \inf\{r_k, \alpha r_k/K > \delta + \delta_i\}. \tag{7.18}$$

This is true for $i = 1$. So assume it is still true up to $t_i$ and study what happens on $[t_i, t_{i+1}]$. First of all on $[t_i, s_{i+1}]$: at $t_i$ assume for instance that $d(X_{n,\delta}, H) = \delta$ and take $k_i$ s.t. $r_{k_i} = \inf\{r_k, \alpha r_k/K > \delta + \delta_i\}$. Then necessarily $d(X_n(t_i), H) \leqslant \alpha r_{k_i}/K$. As $x \notin \bar\omega$, by step 6, one has that $s_{i+1} \leqslant t_i + r_{k_i}/K$. Therefore

$$\big|X_n(s_i) - X_{n,\delta}(s_i)\big| \leqslant \delta_i + r_{k_i}/K.$$

Now on $[s_{i+1}, t_{i+1}]$, simply write,

$$\begin{aligned}
&\int\limits_{\bar\omega^c\cap\omega_i^4} \log\left(\frac{\delta_i + r_{k_i}/K + |X_n(t_{i+1},x) - X_{n,\delta}(t_{i+1},x)|}{\delta_i + r_{k_i}/K + |X_n(s_{i+1},x) - X_{n,\delta}(s_{i+1},x)|}\right)\\
&\quad\leqslant \int\limits_{\omega^c\cap\omega_i^4}\int\limits_{s_{i+1}}^{t_{i+1}} \frac{|b(X_n) - b(X_{n,\delta})|}{\delta_i + r_{k_i}/K + |X_n - X_{n,\delta}|}\,dt\,dx\\
&\quad\leqslant C\int\limits_{\Omega}\int\limits_0^T\int\limits_{B_K(X_n,X_{n,\delta})} \tilde m_n\left(\frac{1}{|X_n - z|^{d-1}} + \frac{1}{|X_{n,\delta} - z|^{d-1}}\right)dz\\
&\quad\leqslant \psi(\delta_i + r_{k_i}/K).
\end{aligned}$$

Therefore for $t \leqslant t_{i+1}$, one has that

$$|X_n - X_{n,\delta}| \leqslant \delta_{i+1} = (\delta_i + r_{k_i}/K)\exp\big(8n\psi(\delta_i + r_{k_i}/K)/\eta\big),$$

except for $x \in \bar\omega \cup \omega_{i+1}^4$, with $\omega_i^4 \subset \omega_{i+1}^4$, and

$$\big|\omega_{i+1}^4 \setminus \omega_i^4\big| \leqslant \eta/8n \quad \text{s.t.} \quad \big|\omega_{i+1}^4\big| \leqslant (i+1)\eta/8n.$$

Finally for any $t \in [0, T]$, one has that

$$|X_n - X_{n,\delta}| \leqslant \delta_n,$$

for any $x \notin \omega$ with $\omega = \bar\omega \cup \omega_n$ and thus

$$|\omega| \leqslant \eta.$$

This concludes the proof once one notices that $\delta_n \to 0$ as $\delta \to 0$ since $\psi(\delta)/|\log\delta| \to 0$.

*Compactness*

Let us just briefly indicate how to obtain the compactness from the estimate on $X_n - X_{n,\delta}$.

Notice first that in the previous proof, $n$ had to be taken large enough in terms of $\delta$. So if for a given $n$, one considers $\delta' \ll \delta$ instead of $\delta$, it is not true that $|X_n - X_{n,\delta'}|$ may be bounded by $\varepsilon(\delta')$. Instead the best that can be done is nothing as long as $|X_n - X_{n,\delta'}| \leqslant \delta$ and bound as before once $|X_n - X_{n,\delta'}| \geqslant \delta$. So in the end $|X_n - X_{n,\delta'}|$ is only bounded by $\varepsilon(\delta)$.

For the compactness of $X_n$, recalling that $X_n$ is uniformly Lipschitz in time, we apply the usual criterion saying that $X_n$ is compact in $L^1([0,T]\times\Omega)$, iff

$$\forall\gamma,\ \forall w,\ \exists\delta \quad \text{s.t.} \quad \forall\delta' < \delta, \qquad \sup_n \int\limits_0^T\int\limits_\Omega \big|X_n(t,x) - X_n\big(t, x + \delta' w\big)\big|\,dx\,dt < \gamma.$$

So fix $\gamma$ and $w$. First choose $\eta$ s.t.

$$\eta T \max_\Omega |x| < \gamma/4.$$

Now apply the previous quantitative estimate to obtain a function $\varepsilon(\delta)$ (recall that this depends on $\eta$). Choose $\delta_1$ with $\varepsilon(\delta_1) < \gamma/(2|\Omega|T)$. This gives $n_1$ s.t. for any $n \geqslant n_1$ and any $\delta < \delta_1$,

$$|X_n - X_{n,\delta}| \leqslant \varepsilon(\delta_1) < \gamma/\big(2|\Omega|T\big), \quad \forall x \in \Omega \setminus \omega_\delta \text{ with } |\omega_\delta| \leqslant \eta.$$

Now for $n < n_1$ as $X_n$ is regular (not uniformly in $n$ but there are only $n_1$ indices $n$ now), choose $\delta_2$ s.t. for any $n < n_1$ and any $\delta < \delta_2$,

$$|X_n - X_{n,\delta}| < \gamma/\big(|\Omega|T\big).$$

Consequently take $\delta = \min(\delta_1, \delta_2)$. For any $n$ and any $\delta' < \delta$, if $n < n_1$ then simply,

$$\int_0^T\int_\Omega \big|X_n(t,x) - X_n\big(t, x+\delta' w\big)\big|\,dx\,dt \leqslant T|\Omega| \sup_x |X_n - X_{n,\delta}| < \gamma.$$

And if $n \geqslant n_1$, then decompose,

$$\int_0^T\int_\Omega \big|X_n(t,x) - X_n\big(t, x+\delta' w\big)\big|\,dx\,dt = \int_0^T\int_{\omega_{\delta'}} + \cdots + \int_0^T\int_{\Omega\setminus\omega_{\delta'}} \cdots$$
$$\leqslant 2T|\omega_{\delta'}| \max_\Omega |x| + T|\Omega| \sup_{\Omega\setminus\omega_{\delta'}} |X_n - X_{n,\delta'}| < \gamma.$$

Hence the compactness criterion is indeed satisfied.

## 8. The *BV* case

I do not know how to perform a rigorous proof in the full *BV* case so the purpose of this section is only to try to explain what can be done and where are the problems. Accordingly most technical details are omitted.

The aim here is the uniqueness of the flow, which turns out to be much simpler than the compactness (as in the *SBV* case). So consider two solutions $X(t,x)$ and $Y(t,x)$ to (1.1), both of them satisfying (1.3) and (2.2). We can define the set of points where $X$ and $Y$ start being different and as both are flows (semi-groups) then it comes to,

$$F = \big\{x \,\big|\, \exists t_n \to 0 \text{ s.t. } X(t_n,x) \neq Y(t_n,x)\big\}.$$

If $\mathcal{H}^{d-1}(F) = 0$ then everything is fine as almost no trajectory passes through $F$ and uniqueness holds for a.e. initial data $x$. So we may assume that $\mathcal{H}^{d-1}(F) > 0$.

Note first that the Lebesgue measure of $F$ is necessarily 0 because nonuniqueness may occur only where $db \notin L^1$ and this set has vanishing Lebesgue measure.

On an interval $[t_0, s_0]$, $|X(t,x) - Y(t,x)|$ passes from $\delta$ to $2\delta$ (the infimum is $\delta$ and the maximum larger than $2\delta$) then by Lemma 3.1,

$$\int_{t_0}^{s_0}\int_{B(X,2\delta)} \big|db(z)\big| \geqslant 1/C.$$

Therefore defining:

$$F_\varepsilon = \big\{x, d(x,F) \leqslant \varepsilon\big\},$$

one has for any $\varepsilon > 0$ that

$$\int_{F_\varepsilon} \big|db(z)\big| \geqslant 1/C.$$

It implies that some mass of $db$ is concentrated on $\bar{F}$ (the closure of $F$) or $\int_{\bar{F}} |db(z)| > 0$. As $|db|$ is the distributional derivative of a *BV* function, it cannot concentrate mass on a purely unrectifiable set. Therefore if $\bar{F}$ has a purely unrectifiable set $F_0$ then almost no trajectory $X$ or $Y$ crosses $F_0$. So we may reduce ourselves to the case where $\bar{F}$ does not contain any such set (by considering $\bar{F} \setminus F_0$).

Take $H \subset \bar{F}$, rectifiable and with $0 < \mathcal{H}^{d-1}(H) < \infty$. Denote $\nu(x)$ the normal on $H$ and look at the part of $H$ where the trajectories $X$ and $Y$ are not tangent,

$$H_0 = \{x \in H, b(x) \cdot \nu(x) \neq 0\}.$$

If $\mathcal{H}^{d-1}(H_0) > 0$ then necessarily the set,

$$\Omega_H = \{x, \exists t\ X(t,x) \in H \text{ or } Y(t,x) \in H\},$$

has nonzero Lebesgue measure. Consequently, by the same arguments as before,

$$\int_H |db(z)| > 0,$$

and this implies that $\mathcal{H}^{d-1}(J \cap H) > 0$ where $J$ is the jump set of $b$. This case was dealt with before as it is exactly the *SBV* situation.

So considering $\tilde{F} = \bar{F} \setminus J$ instead of $\bar{F}$, the only remaining situation is where for any rectifiable $H \subset \tilde{F}$ with $\mathcal{H}^{d-1}(H) < \infty$, then

$$\mathcal{H}^{d-1}(\{x \in H, b(x) \cdot \nu(x) \neq 0\}) = 0,$$

or for $\mathcal{H}^{d-1}$ $x$ in $\tilde{F}$, one has $\nu(x) \cdot b(x) = 0$ but still $\Omega_{\tilde{F}} = \{x, \exists t\ X(t,x) \in \tilde{F} \text{ or } Y(t,x) \in \tilde{F}\}$ has nonzero Lebesgue measure. This is the case which cannot be handled. Note that, rather unsurprisingly, the structure of the problem here is very similar to the one faced in [4].

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