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Codescent objects and coherence

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Dedicated to Max Kelly on the occasion of his 70th birthday

Abstract

We describe 2-categorical colimit notions called codescent objects of coherence data, and lax codescent objects of lax coherence data, and use them to study the inclusion, $T\text{-Alg}_s \to \text{Ps}\text{-}T\text{-}\text{Alg}$, of the 2-category of strict T-algebras and strict T-morphisms of a 2-monad T into the 2-category of pseudo T-algebras and pseudo T-morphisms; and similarly the inclusion $T\text{-Alg}_s \to \text{Lax-}T\text{-Alg}_\ell$, where $\text{Lax-}T\text{-Alg}_\ell$ has lax algebras and lax morphisms rather than pseudo ones. We give sufficient conditions under which these inclusions have left adjoints. We give sufficient conditions under which the first inclusion has left adjoint for which the components of the unit are equivalences, so that every pseudo algebra is equivalent to a strict one.

MSC: 18C15; 18C20; 18D05; 18A30

One of the basic theme's of Max Kelly's book [6] is that ordinary category theory tends to carry over unchanged to the enriched context, provided that one suitably formulates the ordinary category theory. One of the important monoidal categories over which one can enrich is the cartesian closed category **Cat**, and a **Cat**-category is of course just a 2-category. But 2-category theory is not the same thing as **Cat**-category theory, since in 2-category theory one generally has to replace equalities between arrows by suitably coherent isomorphisms, and this "weakening" of equations makes things rather more complicated. One area of ordinary category theory that has quite successfully been adapted to the 2-categorical context is the theory of monads. In this context it has long been argued by Max that one should work "as strictly as

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possible"—that is, with as little weakening as possible—and apply the resulting theory to the weaker, real-life situations. A good example of this philosophy in action is the analysis of two-dimensional monads given in [1]. Here we shall continue this program.

We study certain 2-categorical colimits, called codescent objects and lax codescent objects, which are slight generalizations of the "strict codescent objects" of [22]; and we use these to relate "strict" situations to "weak" ones. Thus it turns out in good situations that if A is a weak algebra, one can form a codescent object or lax codescent object to obtain a strict algebra A' such that, for any other strict algebra B, there is a bijection between weak morphisms from A to B and strict morphisms from A' to B; such a result we call, for the nonce, a "coherence theorem of the first type". Furthermore, it is sometimes the case that A' is equivalent to A; thus we can replace a weak algebra by an equivalent strict one; this we call a "coherence theorem of the second type". Mac Lane's theorem [17] that every monoidal category is equivalent to a strict monoidal category is perhaps the first such example. We give various sufficient conditions under which such coherence theorems can be proved, and compare our conditions to those in other theorems of this nature.

In Section 1 we recall the definitions of the various strict and weak algebras and morphisms. In Section 2 we introduce the codescent objects and lax codescent objects, and give sufficient conditions for proving coherence theorems of the first type. In Section 3 we turn to coherence theorems of the second type, and show that they can be proved under *some* of the sufficient conditions of Section 2. Finally, in Section 4 we compare our results with other known theorems. We also strengthen Power's coherence theorem [19], by adding a theorem of the first type to his theorem of the second type.

It seems appropriate to say a few words about the history of the results in this paper. All the work was essentially done in 1997; the material in Sections 2 and 4.2 was presented to the Australian Category Seminar [14,15] with an indication about how Section 3 would go. I delayed publication at that time because I still hoped to find the "missing" coherence result: that every pseudo T-algebra is equivalent to a strict one if the base 2-category $\mathscr{K}$ is cocomplete and the 2-functor T preserves α-filtered colimits for some regular cardinal α. I feel it is now appropriate to publish for two reasons. First of all, despite several "promising leads", 3 years on I seem to be no closer to proving this theorem than I was in 1997. The second reason is the appearance of [4] (which was written well after the work here was done), in which the main coherence theorems were proved under much more restrictive hypotheses. (To be fair, more is proved in [4] than below—in particular it is shown that a suitable 2-monad T can be replaced by a 2-monad $T_{\dashv}$ of the "Kock–Zöberlein type" with the same algebras and pseudo-algebras—but the intended applications all seem to be results of the kind found in this paper.)

Finally, I am very pleased to dedicate this paper to Max Kelly, who has for so long championed the idea of studying the non-strict via the strict, and from whom I have learned a great deal.

1. Algebras for 2-monads

This section is intended merely to recall definitions and fix terminology, which follows that of [12], our general reference for matters 2-categorical. The exception is *lax algebras* for a 2-monad, which were not defined in [12]; their definition is taken from [20].

Our notion of a 2-monad on a 2-category $\mathscr{K}$ is the strict one: it consists of a 2-functor $T:\mathscr{K} \to \mathscr{K}$ and 2-natural transformations $m:T^2 \to T$ and $i:1 \to T$, satisfying the usual monad equations "on the nose". For such a 2-monad $T=(T,m,i)$, a *lax T-algebra* consists of an object A of $\mathscr{K}$, equipped with a 1-cell $\alpha:TA \to A$ and 2-cells $\alpha:a.Ta \to a.mA$ and $\alpha_0:1 \to a.iA$, satisfying the coherence conditions:

$$
\begin{tikzcd}
 & T^2A \arrow[rr, "Ta"] \arrow[dr, "mA"'] & & TA \arrow[dr, "a"] & \\
T^3A \arrow[ur, "T^2a"] \arrow[dr, "mTA"'] & & TA \arrow[rr, "a"] \arrow[u, phantom, "\Downarrow\alpha" near start, shift left=8ex] & & A \\
 & T^2A \arrow[ur, "Ta"] \arrow[rr, "mA"'] & & TA \arrow[ur, "a"'] \arrow[u, phantom, "\Downarrow\alpha"] &
\end{tikzcd}
=
\begin{tikzcd}
 & T^2A \arrow[rr, "Ta"] \arrow[d, phantom, "\Downarrow T\alpha"] & & TA \arrow[dr, "a"] & \\
T^3A \arrow[ur, "T^2a"] \arrow[dr, "mTA"'] \arrow[rr, "TmA"] & & T^2A \arrow[ur, "Ta"] \arrow[dr, "mA"] \arrow[rr, phantom, "\Downarrow\alpha"] & & A \\
 & T^2A \arrow[rr, "mA"'] & & TA \arrow[ur, "a"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & A \arrow[rrr, bend left, "1"] \arrow[dr, "iA"'] \arrow[drr, phantom, "\Downarrow\alpha_0"] & & & \\
TA \arrow[ur, "a"] \arrow[dr, "iTA"'] & & TA \arrow[rr, "a"] & & A \\
 & T^2A \arrow[ur, "Ta"] \arrow[rr, "mA"'] & & TA \arrow[ur, "a"'] \arrow[ul, phantom, "\Downarrow\alpha"] &
\end{tikzcd}
=
\begin{tikzcd}
 & A \arrow[rrr, bend left, "1"] & & & \\
TA \arrow[ur, "a"] \arrow[drrr, bend left, "1"] \arrow[dr, "iTA"'] & & & & A \\
 & T^2A \arrow[rr, "mA"'] & & TA \arrow[ur, "a"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & TA \arrow[dr, "a"] & \\
TA \arrow[urr, bend left, "1"] \arrow[r, "TiA"'] \arrow[drr, bend right, "1"'] & T^2A \arrow[ur, "Ta"'] \arrow[dr, "mA"] \arrow[u, phantom, "\Downarrow T\alpha_0"] \arrow[rr, phantom, "\Downarrow\alpha"] & & A \\
 & & TA \arrow[ur, "a"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & TA \arrow[dr, "a"] & \\
TA \arrow[ur, bend left, "1"] \arrow[dr, bend right, "1"'] \arrow[rr, phantom, "\Downarrow id"] & & A \\
 & TA \arrow[ur, "a"'] &
\end{tikzcd}
$$

If the 2-cells α and α_0 are invertible, we speak of a *pseudo T-algebra* rather than a lax one, and if they are identities, we call it a *strict T-algebra*, or just a *T-algebra*. We shall not need the notion of *colax T-algebra*, in which the sense of the 2-cells is reversed.

If $A=(A,a,\alpha,\alpha_0)$ and $B=(B,b,\beta,\beta_0)$ are lax T-algebras, a *lax T-morphism* from A to B consists of a 1-cell $f:A \to B$ equipped with a 2-cell $\bar{f}:b.Tf \to fa$ satisfying the coherence conditions

$$
\begin{tikzcd}
 & T^2B \arrow[rr, "Tb"] \arrow[dr, "mB"] & & TB \arrow[dr, "b"] & \\
T^2A \arrow[ur, "T^2f"] \arrow[dr, "mA"'] & & TB \arrow[rr, "b"] \arrow[dr, phantom, "\Downarrow\bar{f}"] & & B \\
 & TA \arrow[ur, "Tf"] \arrow[rr, "a"'] & & A \arrow[ur, "f"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & T^2B \arrow[rr, "Tb"] \arrow[dr, phantom, "\Downarrow T\tilde{f}"] & & TB \arrow[dr, "b"] \arrow[d, phantom, "\Downarrow\bar{f}"] & \\
T^2A \arrow[ur, "T^2f"] \arrow[dr, "mA"'] \arrow[rr, "Ta"] & \arrow[d, phantom, "\Downarrow\alpha"] & TA \arrow[ur, "Tf"] \arrow[dr, "a"] & \, & B \\
 & TA \arrow[rr, "a"'] & & A \arrow[ur, "f"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & B \arrow[dr, "iB"] \arrow[drrr, bend left, "1"] & & & \\
A \arrow[ur, "f"] \arrow[dr, "iA"'] & & TB \arrow[rr, "b"] \arrow[dr, phantom, "\Downarrow\bar{f}"] & & B \\
 & TA \arrow[ur, "Tf"] \arrow[rr, "a"'] & & A \arrow[ur, "f"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & B \arrow[drrr, bend left, "1"] & & & \\
A \arrow[ur, "f"] \arrow[dr, "iA"'] \arrow[drrr, bend left, "1"] & \arrow[d, phantom, "\Downarrow\alpha_0"] & & & B \\
 & TA \arrow[rr, "a"'] & & A \arrow[ur, "f"'] &
\end{tikzcd}
$$

If $\bar{f}$ is invertible, we speak of a *pseudo T-morphism*, rather than a lax one, and if $\bar{f}$ is an identity, we call it a *strict T-morphism*. This time, however, *T-morphism* will mean the pseudo notion rather than the strict one. Once again, there is a "colax" version in which the sense of the 2-cell is reversed, but we shall not have cause to consider these.

Finally if $(f, \bar{f})$ and $(g, \bar{g})$ are lax T-morphisms from A to B as above, a *T-transformation* from $(f, \bar{f})$ to $(g, \bar{g})$ is a 2-cell $\rho : f \to g$ in $\mathscr{K}$ satisfying the single condition

$$
\begin{tikzcd}
TA \arrow[r, bend left, "Tf"] \arrow[r, bend right, "Tg"'] \arrow[r, phantom, "\Downarrow T\rho"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\bar{g}"] & TB \arrow[d, "b"] \\
A \arrow[r, bend right, "g"'] & B
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, bend left, "Tf"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\bar{f}"] & TB \arrow[d, "b"] \\
A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\rho"] & B.
\end{tikzcd}
$$

The name T-transformation was introduced in [10]; in [12] these were called *algebra 2-cells*.

We write Lax-T-Alg_ℓ for the 2-category of lax T-algebras, lax T-morphisms, and T-transformations, and T-Alg_ℓ for its full sub-2-category consisting of the strict T-algebras. We write Ps-T-Alg for the 2-category of pseudo T-algebras, T-morphisms, and T-transformations, and T-Alg for its full sub-2-category consisting of the strict T-algebras. Finally we write T-Alg_s for the sub-2-category of all of these, consisting of the strict T-algebras, the strict T-morphisms, and the T-transformations.

2. Codescent objects

Throughout this section $T=(T,m,i)$ will be a fixed 2-monad on a 2-category $\mathscr{K}$. Let $A=(A,a,\alpha,\alpha_0)$ be a lax T-algebra, and let $B=(B,b)$ be a strict T-algebra. To give a lax T-morphism from A to B is to give a 1-cell $f:A\to B$ in $\mathscr{K}$ and a 2-cell $\bar{f}:b.Tf\to fa$ in $\mathscr{K}$ satisfying the two conditions given in Section 1. To give a 1-cell $f:A\to B$ is equivalent to giving a 1-cell $g=b.Tf:TA\to B$ in T-$\mathrm{Alg}_{\mathrm{s}}$, while to give a 2-cell $\bar{f}:b.Tf\to fa$ is equivalent to giving the 2-cell $\bar{g}$ in T-$\mathrm{Alg}_{\mathrm{s}}$ defined by

$$g.mA=b.Tf.mA=b.mB.T^2f=b.Tb.T^2f\xrightarrow{b.T\bar{f}}b.Tf.Ta=g.Ta.$$

To ask that $\bar{f}$ satisfy the condition

$$\begin{tikzcd}
 & T^2B \arrow[rr, "Tb"] \arrow[dr, "mB"'] & & TB \arrow[dr, "b"] & \\
T^2A \arrow[ur, "T^2f"] \arrow[dr, "mA"'] & & TB \arrow[rr, "b"] \arrow[dr, phantom, "\Downarrow\bar{f}"] & & B \\
 & TA \arrow[ur, "Tf"] \arrow[rr, "a"'] & & A \arrow[ur, "f"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & T^2B \arrow[rr, "Tb"] \arrow[dr, phantom, "\Downarrow T\bar{f}"] & & TB \arrow[dr, "b"] & \\
T^2A \arrow[ur, "T^2f"] \arrow[rr, "Ta"] \arrow[dr, "mA"'] & & TA \arrow[ur, "Tf"] \arrow[dr, "a"] \arrow[rr, phantom, "\Downarrow\bar{f}"] \arrow[d, phantom, "\Downarrow\alpha"] & & B \\
 & TA \arrow[rr, "a"'] & & A \arrow[ur, "f"'] &
\end{tikzcd}$$

is equivalent to asking that $\bar{g}$ satisfy [1]

$$\begin{tikzcd}
 & TA \arrow[dr, "g"] \arrow[dd, phantom, "\Downarrow\bar{g}"] & \\
T^2A \arrow[ur, "mA"] \arrow[dr, "Ta"'] & & B \\
 & TA \arrow[ur, "g"'] & \\
 & & \\
 & T^2A \arrow[uu, "mA"] \arrow[dr, "Ta"] \arrow[dd, phantom, "\Downarrow T\alpha"] \arrow[uur, phantom, "\Downarrow\bar{g}"] & \\
T^3A \arrow[uuuu, "mTA"] \arrow[ur, "T^2a"] \arrow[dr, "TmA"'] & & TA \arrow[uuuu, "g"'] \\
 & T^2A \arrow[ur, "Ta"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & TA \arrow[dr, "g"] & \\
T^2A \arrow[ur, "mA"] & & B \\
 & & \\
 & & \\
 & & \\
T^3A \arrow[uuuu, "mTA"] \arrow[dr, "TmA"'] & & TA \arrow[uuuu, "g"'] \arrow[uuuul, phantom, "\Downarrow\bar{g}"] \\
 & T^2A \arrow[uuuuuu, "mA"] \arrow[ur, "Ta"'] &
\end{tikzcd}
\qquad (2.1)$$

while to ask that

$$\begin{tikzcd}
 & B \arrow[dr, "iB"] \arrow[rrrd, bend left, "1"] & & & \\
A \arrow[ur, "f"] \arrow[dr, "iA"'] & & TB \arrow[rr, "b"] \arrow[dr, phantom, "\Downarrow\bar{f}"] & & B \\
 & TA \arrow[ur, "Tf"] \arrow[rr, "a"'] & & A \arrow[ur, "f"'] &
\end{tikzcd}
=
\begin{tikzcd}
 & B \arrow[rrd, bend left, "1"] & & \\
A \arrow[ur, "f"] \arrow[dr, "iA"'] \arrow[drr, bend left, "1"] & & & B \\
 & TA \arrow[r, "a"'] \arrow[u, phantom, "\Downarrow\alpha_0"] & A \arrow[ur, "f"'] &
\end{tikzcd}$$

[1] This equation looks prettier if one works with pseudo algebras and uses α^{-1} rather than α; or, alternatively, if one works with colax algebras instead of lax ones, but continues to use lax morphisms.

is equivalent to asking that

$$
\begin{tikzcd}
TA \arrow[r, "1"] \arrow[dr, "TiA"'] & TA \arrow[r, "g"] \arrow[dr, phantom, "\Downarrow \bar{g}"] & B \\
& T^2A \arrow[u, "mA"'] \arrow[r, "Ta"'] & TA \arrow[u, "g"']
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[dr, "TiA"'] \arrow[drr, bend left, "1"] \arrow[dr, phantom, shift left=3, "\Downarrow T\alpha_0"] & & & \\
& T^2A \arrow[r, "Ta"'] & TA \arrow[r, "g"'] & B
\end{tikzcd}
\tag{2.2}
$$

Thus, we have reformulated the definition of lax T-morphism from A to B in terms only of the 2-category T-Alg_s. A similar reformulation is possible when it comes to T-transformations. If $(f_1, \bar{f}_1)$ and $(f_2, \bar{f}_2)$ are lax T-morphisms, and $g_i = b.Tf_i$ and $\bar{g}_i = b.T\bar{f}_i$ are the corresponding 1-cells and 2-cells in T-Alg_s, then to give a 2-cell $\phi : f_1 \to f_2$ in $\mathscr{K}$ is equivalent to giving a 2-cell $\psi = b.T\phi : g_1 \to g_2$ in T-Alg_s. To ask that ϕ be a T-transformation from $(f_1, \bar{f}_1)$ to $(f_2, \bar{f}_2)$ is now to ask that

$$
\begin{tikzcd}
& TA \arrow[dr, "g_2"'] \arrow[dr, bend left, "g_1"] \arrow[dr, phantom, shift left=2, "\Swarrow\psi"] & \\
T^2A \arrow[ur, "mA"] \arrow[dr, "Ta"'] \arrow[rr, phantom, "\Downarrow\bar{g}_2"] & & B \\
& TA \arrow[ur, "g_2"'] &
\end{tikzcd}
=
\begin{tikzcd}
& TA \arrow[dr, "g_1"] & \\
T^2A \arrow[ur, "mA"] \arrow[dr, "Ta"'] \arrow[rr, phantom, "\Downarrow\bar{g}_1"] & & B. \\
& TA \arrow[ur, "g_1"] \arrow[ur, bend right, "g_2"'] \arrow[ur, phantom, shift right=2, "\Downarrow\psi"] &
\end{tikzcd}
\tag{2.3}
$$

Suppose that there is a map $\bar{e} : TA \to A'$ in T-Alg_s equipped with a 2-cell $\bar{e} : e.mA \to e.Ta$ satisfying Eqs. (2.1) and (2.2), with the following universal property. For every pair $(f, \bar{f})$, where $f : TA \to B$ is a map T-Alg_s and $\bar{f} : f.mA \to f.Ta$ is a 2-cell in T-Alg_s which satisfies (2.1) and (2.2), there is a unique arrow $f' : A' \to B$ in T-Alg_s satisfying $f'e = f$ and $f'\bar{e} = \bar{f}$. Furthermore, if $f'_1 f'_2 : A' \to B$ are maps in T-Alg_s and $\phi : f'_1 e \to f'_2 e$ is a 2-cell in T-Alg_s satisfying (2.3), there is a unique 2-cell $\phi' : f'_1 \to f'_2$ satisfying $\phi' e = \phi$. Then there is an isomorphism of categories T-$\mathrm{Alg}_\mathrm{s}(A', B) \cong \text{Lax-}T\text{-}\mathrm{Alg}_\ell(A, B)$, natural in B. If there is such an A' for every lax T-algebra A, this gives a left adjoint to the inclusion of 2-categories T-$\mathrm{Alg}_\mathrm{s} \to \text{Lax-}T\text{-}\mathrm{Alg}_\ell$.

Clearly such an object A' is some kind of (weighted) colimit in T-Alg_s; to investigate the nature of this colimit, we consider a more abstract situation. Let $\mathscr{X}$ be an arbitrary 2-category. We define *lax coherence data* in $\mathscr{X}$ to consist of objects and morphisms as in the following diagram:

$$
\begin{tikzcd}
X_3 \arrow[r, shift left=2, "p"] \arrow[r, "q" description] \arrow[r, shift right=2, "r"'] & X_2 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_1 \arrow[l, "e" description]
\end{tikzcd}
$$

equipped with 2-cells $\delta : de \to 1$, $\gamma : 1 \to ce$, $\kappa : dp \to dq$, $\lambda : cr \to cq$, and $\rho : cp \to dr$. A *lax codescent object* of the lax coherence data is defined to be a pair (x, ξ), where $x : X_1 \to X$ is a 1-cell in $\mathscr{X}$ and $\xi : xd \to xc$ is a 2-cell satisfying

$$
\begin{tikzcd}[column sep=small]
& X_1 \arrow[dr, "x"] & \\
X_2 \arrow[ur, "d"] \arrow[dr, "c"'] \arrow[rr, phantom, "\Downarrow\xi"] & & X \\
& X_1 \arrow[ur, "x"'] & \\
& X_2 \arrow[u, "d"'] \arrow[dr, "c"] & \\
X_3 \arrow[uuu, "p"] \arrow[ur, "r"] \arrow[dr, "q"'] \arrow[rr, phantom, "\Downarrow\lambda"] & & X_1 \arrow[uuu, "x"'] \\
& X_2 \arrow[ur, "c"'] &
\end{tikzcd}
\quad (\Downarrow\rho,\ \Downarrow\xi)
\quad = \quad
\begin{tikzcd}[column sep=small]
& X_1 \arrow[dr, "x"] & \\
X_2 \arrow[ur, "d"] & \Downarrow\kappa \quad \Downarrow\xi & X \\
& & \\
& & \\
X_3 \arrow[uuu, "p"] \arrow[dr, "q"'] & & X_1 \arrow[uuu, "x"'] \\
& X_2 \arrow[uuuuu, "d"] \arrow[ur, "c"'] &
\end{tikzcd}
\tag{2.4}
$$

$$
\begin{tikzcd}[column sep=small]
& & X_1 \arrow[dr, "x"] & \\
X_1 \arrow[r, "e"] & X_2 \arrow[ur, "d"] \arrow[dr, "c"'] \arrow[rr, phantom, "\Downarrow\xi"] & & X \\
& & X_1 \arrow[ur, "x"'] &
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
& X_2 \arrow[dr, "d"] \arrow[d, phantom, "\Downarrow\delta"] & & \\
X_1 \arrow[ur, "e"] \arrow[dr, "e"'] \arrow[rr, "1"] & {} \arrow[d, phantom, "\Downarrow\gamma"] & X_1 \arrow[r, "x"] & X \\
& X_2 \arrow[ur, "c"'] & &
\end{tikzcd}
\tag{2.5}
$$

and where (x, ξ) has the following universal property. Given any other $(y : X_1 \to Y, \eta : yd \to yc)$ satisfying the same equations, there is a unique 1-cell $z : X \to Y$ satisfying $zx = y$ and $z\xi = \eta$. Furthermore, given $z_1, z_2 : X \to Y$ and a 2-cell $\zeta_0 : z_1x \to z_2x$ satisfying

$$
\begin{tikzcd}
& X_1 \arrow[dr, "z_2x"'] \arrow[dr, bend left=50, "z_1x"] \arrow[dr, phantom, bend left=25, "\Downarrow\zeta_0"] & \\
X_2 \arrow[ur, "d"] \arrow[dr, "c"'] \arrow[rr, phantom, "\Downarrow z_2\xi"] & & X \\
& X_1 \arrow[ur, "g_2"'] &
\end{tikzcd}
=
\begin{tikzcd}
& X_1 \arrow[dr, "z_1x"] & \\
X_2 \arrow[ur, "d"] \arrow[dr, "c"'] \arrow[rr, phantom, "\Downarrow z_1\xi"] & & X \\
& TA \arrow[ur, "z_1x"] \arrow[ur, bend right=50, "z_2x"'] \arrow[ur, phantom, bend right=25, "\Downarrow\zeta_0"] &
\end{tikzcd}
\tag{2.6}
$$

there is a unique 2-cell $\zeta : z \to z'$ satisfying $\zeta x = \zeta_0$.

These lax codescent objects are a straightforward modification of a construction due to Street [22]. Specifically, if one considers only lax coherence data for which all the 2-cells δ, γ, κ, λ, and ρ are identities; and one considers a universal pair (x, ξ) as above, except that one insists that the 2-cell ξ be invertible, then one obtains the strict codescent objects of [22]. (The word strict distinguishes the 2-categorical construction just described, in which the universal property is expressed using an *isomorphism* of categories, from the corresponding bicategorical one, in which there is only an equivalence; our construction and all the variants described below are of the 2-categorical variety.)

If $A = (A, a, \alpha, \alpha_0)$ is a lax T-algebra, then the objects and arrows

$$\begin{tikzcd} T^3A \arrow[r, shift left=2, "mTA"] \arrow[r, "TmA"] \arrow[r, shift right=2, "T^2a"'] & T^2A \arrow[r, shift left=2, "mA"] \arrow[r, shift right=2, "Ta"'] & TA \arrow[l, "TiA"'] \end{tikzcd}$$

constitute lax coherence data in $T\text{-Alg}_s$ when we equip them with the 2-cells $T\alpha_0 : 1 \to Ta.TiA$ and $T\alpha : Ta.T^2\alpha \to Ta.TmA$ and suitable identity 2-cells. We call this the lax coherence data of the lax T-algebra. The universal property of the lax codescent object of the lax coherence data of a lax T-algebra A is clearly the same as the universal property of the T-algebra A' described above. This object, if it exists, is called the lax codescent object of the lax T-algebra A.

Recall [8] that the *coinserter* of a pair of arrows $u, v : X \to Y$ in a 2-category $\mathscr{X}$ is the universal 1-cell $w : Y \to Z$ equipped with a 2-cell $wu \to wv$, while the *coequifier* of a pair of 2-cells $\sigma, \tau : u \to v$ is the universal 1-cell $w : Y \to Z$ for which $w\sigma = w\tau$; and that both of these universal constructions can be expressed as (weighted) colimits in $\mathscr{X}$. If $T\text{-Alg}_s$ admits coinserters and coequifiers, then for any lax coherence data as above, we can form the coinserter $(w_1 : X_1 \to W_1, \bar{w}_1 : w_1 d \to w_1 c)$ of d and c, then the coequifier $w_2 : W_1 \to W_2$ of the 2-cells $(\bar{w}_1 r)(\bar{w}_1 e)$ and $(w_1\gamma)(w_1\delta)$; and finally the coequifier $w_3 : W_2 \to X$ of the 2-cells $(w_2 w_1 \lambda)(w_2 \bar{w}_1 r)(w_2 w_1 \rho)(w_2 \bar{w}_1 p)$ and $(w_2 \bar{w}_1 q)(w_2 w_1 \kappa)$. Now $\xi = w_3 w_2 \bar{w}_1$ exhibits $x = w_3 w_2 w_1$ as the lax codescent object of the lax coherence data. It follows that:

Proposition 2.1. *A* 2*-category* $\mathscr{X}$ *has lax codescent objects for lax coherence data whenever it has coinserters and coequifiers. Similarly, a* 2*-functor preserves such lax codescent objects whenever it preserves coinserters and coequifiers.*

Since lax codescent objects can be constructed as "iterated colimits", they must themselves be colimits. We shall now describe the weight for lax codescent objects. There is a 2-category $\mathscr{D}_l$ for which 2-functors from $\mathscr{D}_l$ to $\mathscr{X}$ are in natural bisection with lax coherence data in $\mathscr{X}$. It can be constructed as the free 2-category on a computed [23]; its underlying category is the free category on the graph

$$\begin{tikzcd} X_3 \arrow[r, shift left=2, "p"] \arrow[r, "q"] \arrow[r, shift right=2, "r"'] & X_2 \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & X_1 \arrow[l, "e"'] \end{tikzcd}$$

and its 2-cells are freely generated by $\delta : de \to 1$, $\gamma : 1 \to ce$, $\kappa : dp \to dq$, $\lambda : cr \to cq$, and $\rho : cp \to dr$. Now the Yoneda embedding $Y : \mathscr{D}_l \to [\mathscr{D}_l^{op}, \mathbf{Cat}]$ provides lax coherence data in $[\mathscr{D}_\ell^{op}, \mathbf{Cat}]$, and we write J_ℓ for its lax codescent object. The usual "Yoneda-like" arguments now give:

Proposition 2.2. *The weighted colimit* $J_\ell * S$ *of a* 2*-functor* $S : \mathscr{D}_\ell \to \mathscr{X}$ *is the lax codescent object of the lax coherence data corresponding to* S.

We have seen that to give a left adjoint to the inclusion $T\text{-Alg}_s \to \text{Lax-}T\text{-Alg}_\ell$ is just to give a lax codescent object for (the lax coherence data of) each lax T-algebra. We immediately deduce:

Lemma 2.3. *If T is a 2-monad for which T-Alg_s admits lax codescent objects then the inclusion T-$\mathrm{Alg}_s \to \mathrm{Lax}$-$T$-$\mathrm{Alg}_\ell$ has a left adjoint. This is the case in particular if T-Alg_s admits coinserters and coequifiers.*

Theorem 2.4. *For a 2-monad T on a 2-category $\mathscr{K}$, the inclusion T-$\mathrm{Alg}_s \to \mathrm{Lax}$-$T$-$\mathrm{Alg}_\ell$ has a left adjoint if any of the following conditions holds:*
(i) *$\mathscr{K}$ admits lax codescent objects and T preserves them;*
(ii) *$\mathscr{K}$ admits coinserters and coequifiers and T preserves them;*
(iii) *$\mathscr{K}$ is cocomplete and T preserves α-filtered colimits for some regular cardinal α.*

Proof. Of course T-Alg_s admits any colimits which exist in $\mathscr{K}$ and are preserved by T; on the other hand, if $\mathscr{K}$ is cocomplete and T preserves α-filtered colimits, then T-Alg_s is cocomplete by [1, Theorem 3.8, Remark 3.14]. □

The preceding theorem is essentially known in the case of condition (iii) provided that $\mathscr{K}$ is locally presentable; this is further discussed at the end of Section 3. Similarly, the following result was proved in [1] in the case of condition (iii).

Corollary 2.5. *If T is a 2-monad for which T-Alg_s has lax codescent objects then the inclusion T-$\mathrm{Alg}_s \to T$-Alg_ℓ has a left adjoint; in particular this is the case if condition* (i), (ii), or (iii) *of Theorem* 2.4 *holds.*

Proof. If the composite of the inclusion $H : T$-$\mathrm{Alg}_s \to T$-Alg_ℓ and the fully faithful inclusion T-$\mathrm{Alg}_\ell \to \mathrm{Lax}$-$T$-$\mathrm{Alg}_\ell$ has a left adjoint, then so does H. □

Of course we can adapt the above arguments to the context of Ps-T-Alg in place of Lax-T-Alg_ℓ. We say that lax coherence data is *coherence data* if all the 2-cells are invertible, and a *codescent object* of coherence data is defined like its lax codescent object, except that all the 2-cells are required to be invertible. One can construct codescent objects using co-isoinserters and coequifiers; the co-isoinserter of 1-cells $u, v : X \to Y$ in a 2-category $\mathscr{X}$ is the universal 1-cell $w : Y \to Z$ equipped with an *invertible* 2-cell $wu \to wv$. In fact co-isoinserters exist whenever coinserters and coequifiers do so [8, Proposition 4.2], and we now have:

Theorem 2.6. *If T is a 2-monad for which T-Alg_s admits codescent objects, then the inclusions T-$\mathrm{Alg}_s \to$ Ps-T-Alg and T-$\mathrm{Alg}_s \to T$-Alg have left adjoints. In particular this is the case when condition* (ii) *or* (iii) *of Theorem* 2.4 *holds.*

There is a 2-category $\mathscr{D}$ which is defined like $\mathscr{D}_\ell$, except that the 2-cells are required to be invertible, and to give coherence data in a 2-category $\mathscr{X}$ is just to give a 2-functor from $\mathscr{D}$ to $\mathscr{X}$. This $\mathscr{D}$ is a locally full sub-2-category of the 2-category Δ' defined in [16]. The Yoneda embedding $Y : \mathscr{D} \to [\mathscr{D}^{\mathrm{op}}, \mathbf{Cat}]$ provides coherence data in $[\mathscr{D}^{\mathrm{op}}, \mathbf{Cat}]$, and we define $J : \mathscr{D}^{\mathrm{op}} \to \mathbf{Cat}$ to be the codescent object of the coherence data in $[\mathscr{D}^{\mathrm{op}}, \mathbf{Cat}]$ corresponding to Y. In direct analogy with Proposition 2.2 we have:

Proposition 2.7. *The weighted colimit $J * S$ of a* 2*-functor $S : \mathscr{D} \to \mathscr{X}$ is the codescent object of the coherence data corresponding to S.*

If we wish to compute lax codescent objects only of *strict* T-algebras, then certain simplifications are possible. One modifies the definition of $\mathscr{D}$, replacing the various 2-cells by *equations* between their domains and codomains. The new 2-category $\mathscr{D}_{\mathrm{s}}$ has no non-trivial 2-cells; in fact it is a subcategory of the opposite of the simplicial category Δ, and so $\mathscr{D}_{\mathrm{s}}^{\mathrm{op}}$ is a subcategory of **Cat**. The objects of $\mathscr{D}_{\mathrm{s}}^{\mathrm{op}}$ are the total orders $\mathbf{1} = \{0\}$, $\mathbf{2} = \{0 < 1\}$ and $\mathbf{3} = \{0 < 1 < 2\}$, and the 1-cells are generated by the order-preserving maps from $\mathbf{1}$ to $\mathbf{2}$, from $\mathbf{2}$ to $\mathbf{3}$, and from $\mathbf{2}$ to $\mathbf{1}$. We use the following names for the generators:

$$
\begin{tikzcd}
\mathbf{1} \arrow[r, shift left=2, "d"] \arrow[r, shift right=2, "c"'] & \mathbf{2} \arrow[l, "e"'] \arrow[r, shift left=3, "p"] \arrow[r, "q"] \arrow[r, shift right=3, "r"'] & \mathbf{3}
\end{tikzcd}
$$

where $d(0) = 0$, $c(0) = 1$, $p(i) = i$, $r(i) = i + 1$, $q(0) = 0$, and $q(1) = 2$; and we write $K : \mathscr{D}_{\mathrm{s}}^{\mathrm{op}} \to \mathbf{Cat}$ for the inclusion. If (A, a) is a T-algebra, we define a 2-functor $S_A : \mathscr{D}_{\mathrm{s}}^{\mathrm{op}} \to T\text{-Alg}_{\mathrm{s}}$ by setting $S_A\mathbf{1} = TA$, $S_A\mathbf{2} = T^2A$, $S_A\mathbf{3} = T^3A$, $S_Ad = mA$, $S_Ac = Ta$, $S_Ae = TiA$, $S_Ap = mTA$, $S_Aq = TmA$, and $S_Ar = T^2a$. The K-weighted colimit of S_A is now the lax codescent object of (A, a).

We leave to the reader the modification of K to deal with codescent objects of strict T-algebras. The relevant colimit is precisely what was called a strict codescent object in [22].

3. Coherence

Over the years, various different types of theorem have been called a "coherence theorem". For an abstract 2-monad T, it seems that the best possible coherence theorem for pseudo-T-algebras would be:

Theorem-Schema. *The inclusion* $T\text{-Alg}_{\mathrm{s}} \to \text{Ps-}T\text{-Alg}$ *has a left adjoint, and the components of the unit are equivalences in* Ps-T-Alg.

Thus every pseudo T-algebra would in particular be equivalent to a strict one, and the inclusion T-Alg $\to$ Ps-T-Alg would be a biequivalence of 2-categories, but rather more than this would also be true. Before investigating conditions under which the Theorem-Schema is true, it is perhaps worth giving an example in which it is false:

Example 3.1. Let A be the full subcategory of **Set** containing only the objects $\mathbb{Z}$ and $\{0\}$; then A has finite products. Let T be the 2-monad on **Cat** whose algebras are the strict monoidal categories, and let $\mathscr{K}$ be the full sub-2-category of **Cat** consisting of all objects of the form T^nA for some $n \in \mathbb{N}$. Then T restricts to a 2-monad S on $\mathscr{K}$, and an algebra or pseudo algebra for S is just an algebra or pseudo algebra for T on an object in $\mathscr{K}$. The Cartesian monoidal structure on A makes it into a pseudo S-algebra;

we claim that it is not equivalent to any strict S-algebra. Since the set of isomorphism classes of objects in A has cardinality 2, and the set of isomorphism classes of objects in T^nA has cardinality strictly greater than 2 if $n > 0$, the only object of $\mathscr{K}$ equivalent to A is A itself. Thus the only way that the pseudo S-algebra A could be equivalent to a strict S-algebra is if there were a strict cartesian monoidal structure on A. An argument attributed to Isbell in [18, p. 160] shows that this is impossible.

In Theorem 2.6, we saw three different sufficient conditions for the existence of a left adjoint to the inclusion $T\text{-Alg}_s \to \text{Ps-}T\text{-Alg}$; of these, condition (ii) is a stronger hypothesis than (i), but is perhaps more familiar; while (iii) is rather different. We now turn to the second part of the Theorem-Schema, the condition that the components of the unit be equivalences. First we observe that a morphism in Ps-T-Alg is an equivalence if and only if its underlying 1-cell in $\mathscr{K}$ is an equivalence; this is straightforward to verify directly, but is also essentially [19, Theorem 3.2], which is there attributed to Kelly [5].

We suppose that $T = (T, m, i)$ is a 2-monad on a 2-category $\mathscr{K}$ admitting codescent objects, and that T preserves them; recall that this will be the case if $\mathscr{K}$ admits coinserters and coequifiers, and T preserves these. Let $A = (A, a, \alpha, \alpha_0)$ be a pseudo T-algebra, and let $e : TA \to A'$ and $\bar{e} : e.mA \to e.Ta$ exhibit $A' = (A', a')$ as its codescent object. Since T preserves codescent objects, so does the forgetful 2-functor $U_s : T\text{-Alg}_s \to \mathscr{K}$, thus e and $\bar{e}$ exhibit A' as a codescent object in $\mathscr{K}$. Now $(a, TA \to A, \alpha : a.mA \to a.Ta)$ satisfies (2.4) and (2.5), and so there is a unique 1-cell $q : A' \to A$ satisfying $qe = a$ and $q\bar{e} = \alpha$. Thus $qe.iA = a.iA \cong 1$; we shall show that $e.iA.q \cong 1$, so that the component $e.iA$ of the unit of the adjunction is indeed an equivalence. To construct the isomorphism $e.iA.q \cong 1$, it will suffice, by the two-dimensional aspect of the universal property of the codescent object A', to construct an invertible 2-cell $\psi : e \to e.iA.qe$ satisfying (2.6). But

$$e = e.mA.iTA \xrightarrow{\bar{e}.iTA} e.Ta.iTA = e.iA.a = e.iA.qe$$

is such a 2-cell, and so we conclude:

Theorem 3.2. *If T is a 2-monad on a 2-category $\mathscr{K}$ admitting codescent objects, and T preserves them, then the inclusion* $T\text{-Alg}_s \to \text{Ps-}T\text{-Alg}$ *has a left adjoint, and the components of the unit are equivalences in* Ps-T-Alg. *In particular this is the case if $\mathscr{K}$ has coinserters and coequifiers, and T preserves these.*

We shall see in Theorem 4.4 a weaker sufficient condition under which the theorem holds.

We shall now outline briefly other known results along the lines of the Theorem-Schema:

- The Theorem-Schema was proved in [3] in the case where $\mathscr{K}$ is the 2-category of based topological categories—that is, the 2-category $\mathscr{V}$-**Cat** where $\mathscr{V}$ is the monoidal category of pointed topological spaces—and T is a 2-monad induced by a *braided* **Cat**-*operad* in the sense of that paper. (Note, however, that "lax" is used in [3] to mean what is here called "pseudo", and that the pseudo/lax algebras are defined in terms of the operad structure rather than the monad structure, so that the "lax algebras" of [3] are only equivalent to the pseudo algebras used here.) We shall discuss this class of 2-monads further in Section 4.1.

- The Theorem-Schema was proved in [4] in the case where $\mathscr{K}$ admits coinserters and coequifiers, but also comma objects and pullbacks and $\mathscr{K}$ furthermore satisfies exactness conditions between the limits and colimits; still further, T is required to preserve both the limits and the colimits, and the multiplication and unit of the monad are required to be cartesian natural transformations. In fact, under these conditions a new 2-monad $T_{\dashv}$ was constructed on a new 2-category $\mathscr{K}_{\dashv}$, in such a way that the algebras, pseudoalgebras, algebra morphisms and so on, for the first 2-monad were the same as those for the second; furthermore this new 2-monad was one of the Kock–Zöberlein type [13], or "lax-idempotent" in the language of [10].
- Part of the Theorem-Schema was proved in [19] in the case where $\mathscr{K}$ is $\mathbf{Cat}^X$ or $\mathbf{Cat}_g^X$ for a small set X, and T preserves "bijections on objects"; here $\mathbf{Cat}_g$ denotes the sub-2-category of $\mathbf{Cat}$ containing all the objects and 1-cells, but only those 2-cells which are invertible. Specifically, it was proved for such a T that every pseudo T-algebra is equivalent to a strict one. We shall see in Section 4.2 below that under these assumptions the entire Theorem-Schema is in fact true.
- In [1], it was proved that the inclusion $T\text{-Alg}_s \to T\text{-Alg}$ has a left adjoint if $\mathscr{K}$ is cocomplete and T preserves α-filtered colimits for some regular cardinal α. It was further proved that if $q : T' \to T$ is a strict morphism of 2-monads on $\mathscr{K}$, then the induced map $q^* : T\text{-Alg}_s \to T'\text{-Alg}_s$ has a left adjoint. But if $\mathscr{K}$ is locally α-presentable then, as is sketched in [1, Section 6.4], there is a 2-monad T' which preserves α-filtered colimits, and a strict morphism $q : T' \to T$ for which $T'\text{-Alg} = \text{Ps-}T\text{-Alg}$, and the composite of $q^* : T\text{-Alg}_s \to T'\text{-Alg}_s$ and the inclusion $J : T'\text{-Alg}_s \to T'\text{-Alg}$ is the inclusion $T\text{-Alg}_s \to T'\text{-Alg}$. But each of J and q^* has a left adjoint hence so does their composite. Thus the result of Theorem 2.6 follows if $\mathscr{K}$ is locally presentable and T preserves α-filtered colimits. A similar analysis is possible in relation to Theorem 2.4. Whether the entire Theorem-Schema holds under these assumptions—and more generally when $\mathscr{K}$ is assumed only to be cocomplete—is the major unsolved problem in the subject.

4. Hypotheses

In this section we discuss the hypothesis that $\mathscr{K}$ admits and T preserves codescent objects, and compare it to the hypotheses of [3,19].

4.1. 2-monads induced by braided Cat-operads

In this section we shall briefly sketch how the hypotheses of [3] compare to ours. To simplify things, we shall work with the 2-category $\mathbf{Cat}$ of ordinary categories, rather than those enriched in pointed topological spaces, as considered in [3]. Write

$\mathscr{B}$ for the braid category; this has objects the natural numbers, with hom-set $\mathscr{B}(n,m)$ equal to the Artin braid group on n strings if $n=m$, and empty otherwise. A braided **Cat**-operad consists of a functor $C:\mathscr{B}^{\mathrm{op}}\to\mathbf{Cat}$ equipped with certain extra structure; in fact $\mathscr{B}^{\mathrm{op}}$ is isomorphic to $\mathscr{B}$, but it is convenient to use $\mathscr{B}^{\mathrm{op}}$ rather than $\mathscr{B}$. Since **Cat** is a braided monoidal category, for each object A of **Cat** there is an essentially unique functor $R_A:\mathscr{B}\to\mathbf{Cat}$ which preserves the monoidal structure and the braiding; it sends the object n to A^n. There is an induced 2-functor $T:\mathbf{Cat}\to\mathbf{Cat}$ whose value on the object A is the weighted colimit $C*R_A$; it can be calculated as a certain colimit

$$TA=\sum_n Cn\times A^n/\sim$$

in **Cat**. Thus T is the composite

$$\mathbf{Cat}\xrightarrow{R}[\mathscr{B}^{\mathrm{op}},\mathbf{Cat}]\xrightarrow{C*-}\mathbf{Cat},$$

where R is the evident 2-functor sending A to R_A. Since $C*-$ is cocontinuous, T will preserve whatever colimits which R preserves. Now colimits in $[\mathscr{B}^{\mathrm{op}},\mathbf{Cat}]$ are computed pointwise, so that R will preserve whatever colimits are preserve by the 2-functors $\mathbf{Cat}\to\mathbf{Cat}:A\mapsto A^n$ for $n\in\mathbb{N}$. Thus T will preserve all colimits which commute in **Cat** with finite products. (This argument is essentially that of [11, Theorem 4.1].)

Although codescent objects need not commute with finite products, there is an important class of codescent objects which do commute with finite products, and these are enough to prove the Theorem-Schema. Before describing these, we shall first provide a necessary and sufficient condition for a class of weighted colimits to commute in **Cat** with finite products.

Lemma 4.1. *Let $H:\mathscr{C}^{\mathrm{op}}\to\mathbf{Cat}$ be a 2-functor with small domain. Then the 2-functor $H*-:[\mathscr{C},\mathbf{Cat}]\to\mathbf{Cat}$ preserves finite products if and only if it preserves finite products of representables.*

Proof. To say that $H*-$ preserves finite products of representables is to say that it preserves the terminal object and that it preserves binary products of representables. Of course if $\mathscr{C}^{\mathrm{op}}$ has a terminal object, then $H*-$ preserves the terminal object if and only if H does so.

Let $R,S:\mathscr{C}\to\mathbf{Cat}$ be given. We must show that $H*(R\times S)\cong H*R\times H*S$. Write (R,S) for the evident 2-functor $\mathscr{C}\times\mathscr{C}\to\mathbf{Cat}$ defined on objects by $(R,S)(A,B)=RA\times SB$; then $R\times S=(R,S)\Delta$, where $\Delta:\mathscr{C}\to\mathscr{C}\times\mathscr{C}$ is the diagonal.

There are natural isomorphisms

$$\begin{aligned} H*R\times H*S &\cong \left(\int^B HB\times RB\right)\times\left(\int^C HC\times SC\right)\\ &\cong \int^B\int^C HB\times RB\times HC\times SC\\ &\cong \int^{B,C}(HB\times HC)\times(RB\times SC) \end{aligned}$$

$$
\begin{aligned}
&\cong \int^{B,C} (H * \mathscr{D}(B,-) \times H * \mathscr{D}(C,-)) \times (R,S)(B,C) \\
&\cong \int^{B,C} H * (\mathscr{D}(B,-) \times \mathscr{D}(C,-)) \times (R,S)(B,C) \\
&\cong \int^{B,C,D} HD \times \mathscr{D}^2((B,C),\Delta D) \times (R,S)(B,C) \\
&\cong \int^{D} HD \times (R,S)(\Delta D) \\
&\cong \int^{D} HD \times (R \times S)(D) \\
&\cong H * (R \times S)
\end{aligned}
$$

coming variously from the Yoneda lemma, the cartesian closedness of **Cat**, and the fact that $H * -$ preserves binary products of representables. $\square$

Remark 4.2. The special case where $\mathscr{C}$ has finite products is well known; the lemma then amounts to the fact that $H * -$ preserves finite products if and only if H does so. See for instance [2] or [9].

Let $\mathscr{G}$ be the graph

$$
\begin{tikzcd}
1 \arrow[r, shift left=4, "\delta_0"] \arrow[r, shift right=4, "\delta_1"'] & 2 \arrow[l, "\sigma_0"'] \arrow[r, shift left=8, "\delta_0"] \arrow[r, "\delta_1"] \arrow[r, shift right=8, "\delta_2"'] & 3 \arrow[l, shift right=4, "\sigma_0"'] \arrow[l, shift left=4, "\sigma_1"]
\end{tikzcd}
$$

whose vertices and edges are to seen as objects and arrows in the simplicial category $\varDelta$ Let $\mathscr{D}_r$ be the 2-category whose opposite, $\mathscr{D}_r^{\mathrm{op}}$, has as underlying category the free category on $\mathscr{G}$, and has a unique invertible 2-cell between any parallel pair of arrows which become equal in $\varDelta$. Thus $\mathscr{D}_r^{\mathrm{op}}$ is a full sub-2-category of the 2-category $\varDelta'$ defined in [16], and $\mathscr{D}_r$ contains the 2-category $\mathscr{D}$ of Section 2 as a locally full sub-2-category. Now to give a 2-functor $\mathscr{D}_r \to \mathscr{X}$ is to give coherence data in $\mathscr{X}$ with, in the notation of Section 2, further 1-cells $u, v : X_2 \to X_3$ and further invertible 2-cells $pu \to 1$, $qu \to 1$, $ru \to ec$, $pv \to ed$, $qv \to 1$, and $rv \to 1$; such that “all diagrams of 2-cells commute”. We say that coherence data in $\mathscr{X}$ is reflexive if it is equipped with the further structure required to give a 2-functor $\mathscr{D}_r \to \mathscr{X}$.

Proposition 4.3. *Codescent objects of reflexive coherence data commute in* **Cat** *with finite products.*

Proof. We briefly sketch two possible proofs. One could use Lemma 4.1 to show that the 2-functor $[\mathscr{D}_r, \mathbf{Cat}] \to \mathbf{Cat}$ which computes the codescent object of reflexive coherence data preserves finite products; preservation of the terminal object is obvious, so that it would suffice to show that the 2-functor preserves binary products of representables. Since $\mathscr{D}_r$ has only three objects, there are not too many things to check.

Alternatively, one could modify the "3-by-3 lemma" of [11, Section 2]. Reflexive coherence data is more complicated than a reflexive 2-cell, but the argument is essentially the same. □

Theorem 4.4. *The* Theorem-Schema *holds if $\mathscr{K}$ has codescent objects for reflexive coherence data, and T preserves them. In particular this is the case if T is the* 2-*monad induced by a braided* **Cat**-*operad.*

Proof. It will suffice to show that the coherence data of a pseudo T-algebra is reflexive. One takes u to be $TiTA$ and v to be T^2iA and makes the obvious choices for 2-cells. That these choices are coherent follows from the results of [16, Section 4]. □

Example 4.5. Any strongly-finitary 2-monad [9,11] preserves codescent objects of reflexive coherence data by Proposition 4.3 and the argument of [11, Theorem 4.1], so that the Theorem-Schema holds for such 2-monads.

4.2. 2-monads which preserve bijections on objects

In this section we discuss the hypotheses of [19], and show that under these hypotheses the entire Theorem-Schema can be proved. In [19], Power gave a very simple proof that every pseudo T-algebra is equivalent to a strict one, in the case of a 2-category $\mathscr{K}$ which admits a factorization system like the (bijective-on-objects = *bo*, fully-faithful = *ff*) one on **Cat**, and where the collection of bijective-on-objects maps is closed under the action of T.

A factorization system on a 2-category $\mathscr{K}$ is a factorization system $(\mathscr{E},\mathscr{M})$ on the underlying category $\mathscr{K}_0$ of $\mathscr{K}$, for which if $e:A \to B$ is in $\mathscr{E}, m:C \to D$ is in $\mathscr{M}, \sigma:s \to s':A \to C$, and $\tau:t \to t':B \to D$ satisfy $\tau e = m\sigma$, not only are there unique $r,r':B \to C$ satisfying $re = s$, $r'e = s'$, $mr = t$, and $mr' = t'$, but there is also a unique $\rho:r \to r'$ satisfying $pe = \sigma$ and $mp = \tau$. The extra property of the *bo*-*ff* factorization system on **Cat** is that, for any invertible 2-cell

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "s"'] \arrow[dr, phantom, "\swarrow\alpha"] & B \arrow[d, "t"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

with $e \in \mathscr{E}$ and $m \in \mathscr{M}$ there is a unique pair (r,β) with $re = s$, and $\beta:t \to mr$ an invertible 2-cell satisfying $\beta e = \alpha$. Such a factorization system has been called an *enchanced factorization system* by Max Kelly [7].

The abstract form of Power's result is:

Theorem 4.6 (Power). *If $\mathscr{K}$ is a* 2-*category with an enhanced factorization system $(\mathscr{E},\mathscr{M})$ with the property that if $j \in \mathscr{M}$ and $jk \cong 1$ then $kj \cong 1$, and if T is a* 2-*monad on $\mathscr{K}$ for which $Tf \in \mathscr{E}$ whenever $f \in \mathscr{E}$, then every pseudo T-algebra is equivalent to a strict one.*

For simplicity, we restrict ourselves to the case where $\mathscr{K}$ is **Cat** with the *bo-ff* enhanced factorization system, and consider the hypothesis, for a 2-monad T, that $Tf \in \mathscr{E}$ whenever $f \in \mathscr{E}$. For any functor $f:A \to B$, we can form the comma object

$$\begin{tikzcd} K \arrow[r, "d"] \arrow[d, "c"'] \arrow[dr, phantom, "\swArrow\lambda"] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

of f. There is then a unique functor $e:A \to K$ for which $de = ce = 1$ and λe is the identity. Furthermore the functors $d, c:K \to A$ constitute the underlying graph of a category object

$$\begin{tikzcd} L \arrow[r, shift left=2, "p"] \arrow[r, "q"] \arrow[r, shift right=2, "r"'] & K \arrow[r, shift left=2, "d"] & A \arrow[l, "e"'] \end{tikzcd}$$

in **Cat**, which we can think of as lax coherence data via suitable choices of identity 2-cells; we call this lax coherence data the *congruence* of f. Explicitly, an object of K consists of objects a and a' in A and a morphism $\beta : fa \to fa'$ in B, while an object of L consists of objects a, a', and a'' in A, and morphisms $\beta : fa \to fa'$ and $\beta' : fa' \to fa''$ in B.

Proposition 4.7. *A functor $f:A \to B$ is bijective on objects if and only if (f, λ) is the lax codescent object of the congruence of f.*

Proof. Let $(g:A \to C,\ \bar{g}:gd \to gc)$ satisfy (2.4) and (2.5). We must show that there is a unique functor $h:B \to C$ satisfying $hf = g$ and $h\lambda = \bar{g}$. Since f is bijective on objects there is clearly a unique way to define h on objects. Since an arbitrary arrow $\beta : fa \to fa'$ in B may be viewed as an object k of K, and we need $h\beta = h\lambda k = \bar{g}k$, there is only one possible way to define h on arrows. The fact that the resulting h preserves composition and identities follows from (2.4) and (2.5). The verification that hf agrees with g on morphisms is left to the reader. □

Remark 4.8. This characterization of bijective-on-object functors is reminiscent of the analysis in [21] of "acute" arrows in **Cat**.

Corollary 4.9. *If T : **Cat** → **Cat** is a 2-functor which preserves lax codescent objects, then T preserves bijections on objects. In particular this is the case when T preserves coinserters and coequifiers.*

We have not investigated the relationship in general between preservation of codescent objects and preservation of lax codescent objects, except for the observation that they are equivalent in the important special case of a 2-category $\mathscr{K}$ in which every 2-cell is invertible, such as in the 2-category $\mathbf{Cat}_g^X$, considered in [19]. It is presumably not true that a 2-functor which preserves bijections on objects must preserve lax codescent objects.

We conclude the paper by proving, under the hypotheses of Power's theorem in the abstract form of Theorem 4.6, the full Theorem-Schema:

Theorem 4.10. *If $\mathscr{K}$ is a 2-category with an enhanced factorization system $(\mathscr{E},\mathscr{M})$ having the property that if $j \in \mathscr{M}$ and $jk \cong 1$ then $kj \cong 1$, and if T is a 2-monad on $\mathscr{K}$ for which $Tf \in \mathscr{E}$ whenever $f \in \mathscr{E}$, then the inclusion $T\text{-Alg}_{\mathrm{s}} \to \text{Ps-}T\text{-Alg}$ has a left adjoint, and the components of the unit of the adjunction are equivalences in* Ps-T-Alg.

Proof. Recall that, for a pseudo-T-algebra $A=(A,a,\alpha,\alpha_0)$, a strict T-algebra equivalent to it was constructed by Power by factorizing $a : TA \to A$ as a map $e : TA \to A'$ in $\mathscr{E}$ followed by a map $j : A' \to A$ in $\mathscr{M}$; then, since $Te \in \mathscr{E}$, the isomorphism $\alpha : je.mA \cong a.Tj.Te$ induces a unique map $a' : TA' \to A'$ and a unique isomorphism $\bar{j} : ja' \to a.Tj$ such that $a'.Te = e.mA$ and $\bar{j}.Te = \alpha$. Then (A',a') is a strict T-algebra, and $(j,\bar{j})$ is an equivalence of pseudo T-algebras from (A',a') to (A,a,α,α_0). One can construct an inverse-equivalence $(k,\bar{k})$ with $k = e.iA$ and

$$\begin{tikzcd}
TA \arrow[r, "TiA"] \arrow[d, "a"'] & T^2A \arrow[r, "Te"] \arrow[d, phantom, "\Downarrow\bar{k}"] & TA' \arrow[r, "Tj"] \arrow[d, "a'"'] \arrow[dr, phantom, "\Downarrow\bar{j}"] & TA \arrow[d, "a"] \\
A \arrow[r, "iA"'] & TA \arrow[r, "e"'] & A' \arrow[r, "j"'] & A
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "TiA"] \arrow[rr, bend right, "1"', ""{name=U}] \arrow[d, "a"'] & T^2A \arrow[r, "Ta"] \arrow[d, phantom, "\Downarrow T\alpha_0"] & TA \arrow[d, "a"] \\
A \arrow[r, "iA"'] \arrow[rr, bend left, "1"] & TA \arrow[r, "a"'] \arrow[u, phantom, "\Downarrow T\alpha^{-1}"] & A.
\end{tikzcd}$$

We shall show that $(k,\bar{k})$ is the component at A of the unit of the desired adjunction. Suppose then that $B=(B,b)$ is a strict T-algebra, and $(f,\bar{f})$ is a T-morphism from A to B. Since $e : TA \to A'$ is in $\mathscr{E}$, and $1 : B \to B$ is in $\mathscr{M}$, there is a unique 1-cell $f' : A' \to B$ and a unique invertible 2-cell $\phi : f' \to fj$ with

$$\begin{tikzcd}
TA \arrow[r, "e"] \arrow[d, "Tf"'] & A' \arrow[r, "j"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Downarrow\phi"] & A \arrow[d, "f"] \\
TB \arrow[r, "b"'] & B \arrow[r, "1"'] & B
\end{tikzcd}
=
\begin{tikzcd}
TA \arrow[r, "a"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\Downarrow\bar{f}^{-1}"] & A \arrow[d, "f"] \\
TB \arrow[r, "b"'] & B.
\end{tikzcd}$$

Now the equality

$$\begin{tikzcd}
T^2A \arrow[r, "Te"] \arrow[d, "mA"'] & TA' \arrow[r, "Tj"] \arrow[d, "a'"'] \arrow[dr, phantom, "\Downarrow\bar{j}"] & TA \arrow[d, "a"] \\
TA \arrow[r, "e"] \arrow[d, "Tf"'] & A \arrow[r, "j"] \arrow[d, "f'"'] \arrow[dr, phantom, "\Downarrow\phi"] & A \arrow[d, "f"] \\
TB \arrow[r, "b"'] & B \arrow[r, "1"'] & B
\end{tikzcd}
=
\begin{tikzcd}
T^2A \arrow[r, "Te"] \arrow[d, "T^2f"'] & TA' \arrow[r, "Tj"] \arrow[d, "Tf'"'] \arrow[dr, phantom, "\Downarrow T\phi"] & TA \arrow[d, "Tf"] \arrow[dr, "a"] & \\
T^2B \arrow[r, "Tb"] \arrow[d, "mB"'] & TB \arrow[r, "1"] \arrow[d, "b"'] & TB \arrow[d, "b"] \arrow[r, phantom, "\Downarrow\bar{f}^{-1}"] & A \arrow[dl, "f"] \\
TB \arrow[r, "b"'] & B \arrow[r, "1"'] & B &
\end{tikzcd}$$

follows from one of the coherence conditions for a T-morphism $(f,\bar{f})$, and implies that $b.Tf' = f'a$, so that f' is a strict morphism of T-algebras. Also $f'k = f'e.iA = b.Tf.iA = b.iB.f = f$. We leave to the reader the verification that $f'\bar{k} = \bar{f}$, so that $(f', id)(k, \bar{k}) = (f, \bar{f})$. The uniqueness of f' is now clear, giving the one-dimensional aspect of the universal property. It remains only to prove the two-dimensional aspect: if $(f,\bar{f})$ and $(g,\bar{g})$ are T-morphisms from A to B, and $\rho : (f,\bar{f}) \to (g,\bar{g})$ is a T-transformation, then there is a unique T-transformation $\rho' : f' \to g'$ with $\rho' k = \rho$. This follows easily from the two-dimensional aspect of the universal property of TA and the fact that $e \in \mathscr{E}$. $\square$

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