Approximating cost-based abduction is NP-hard

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Abstract

Cost-based abduction (CBA) is an important problem in reasoning under uncertainty. Finding Least-Cost Proofs (LCPs) for CBA systems is known to be NP-hard and has been a subject of considerable research over the past decade. In this paper, we show that approximating LCPs, within a fixed ratio bound of the optimal solution, is NP-hard, even for quite restricted subclasses of CBAs. We also consider a related problem concerned with the fine-tuning of a CBA’s cost function.

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1. Introduction

Cost-based abduction (CBA) [7,8] is an important problem in reasoning under uncertainty. Finding Least-Cost Proofs (LCPs) for CBA systems is known to be NP-hard [8] and has been a subject of considerable research over the past decade. In this paper, we show that approximating LCPs, within a fixed ratio bound of the optimal solution, is NP-hard, even for quite restricted subclasses of CBAs. We also consider a related problem concerned with the fine-tuning of a CBA’s cost function.

Abduction is the process of proceeding from data describing observations or events, to a set of hypotheses, which best explains or accounts for the data [12]. A CBA system is a knowledge representation in which a given world situation is modeled as a 4-tuple \( \mathcal{L} = (\mathcal{H}, \mathcal{R}, c, \mathcal{G}) \), where
• $\mathcal{H}$ is a set of hypotheses or propositions,
• $\mathcal{R}$ is a set of rules of the form
  \[(h_{i1} \land h_{i2} \land \cdots \land h_{in}) \rightarrow h_{ik},\]
  where $h_{i1}, \ldots, h_{in}$ (called the antecedents) and $h_{ik}$ (called the consequent) are all members of $\mathcal{H}$,
• $c$ is a function $c : \mathcal{H} \rightarrow \mathbb{R}^+$, where $c(h)$ is called the assumability cost of hypothesis $h \in \mathcal{H}$ and $\mathbb{R}^+$ denotes the positive reals,
• $\mathcal{G} \subseteq \mathcal{H}$ is called the goal set or the evidence.

The objective is to find the least cost proof (LCP) for the evidence, where the cost of a proof is taken to be the sum of the costs of all hypotheses that must be assumed in order to complete the proof. Any given hypothesis can be made true in two ways: it can be assumed to be true, at a cost of its assumability cost, or it can be proved. If an hypothesis occurs as the consequent of a rule $R$, then it can be proved, at no cost, to be true by making all the antecedents of $R$ true, either by assumption or by proof. If an hypothesis does not appear as the consequent of any rule, then it cannot be proved, it can be made true only by being assumed. The cost of an hypothesis can be $\infty$, which means that it cannot be assumed, it can only be proved. One can assume, without loss of generality, that any hypothesis that appears as the consequent of any rule has an infinite assumability cost. Suppose $x_q$ has a finite assumability cost of $a$, and appears as the consequent of at least one rule. One can add a hypothesis $x'_q$ with assumability cost $a$, set the assumability cost of $x_q$ to $\infty$, and add the rule
  \[x'_q \rightarrow x_q.\]

Therefore, we consider the hypothesis set $\mathcal{H}$ to be partitioned into two subsets: a set of assumable hypotheses $\mathcal{H}^A$, which have finite assumability costs and do not appear as consequents of any rules, and a set of provable hypotheses $\mathcal{H}^P$, which have infinite assumability costs and, hence, can be made true only by being proved.

A feasible solution for the LCP problem is a subset $\phi \subseteq \mathcal{H}^A$ which is sufficient to prove the goal set $\mathcal{G}$. An (optimal) solution to the LCP problem is a minimum-cost feasible solution.

While finding an LCP for an instance of CBA was shown to be NP-hard in 1994 [8], a number of approaches to this problem have been explored. Charniak and Shimony [7,8] presented a best-first heuristic search approach, and Charniak and Husain [6] presented an admissible heuristic for the problem.

Santos [25,26] presented a method for transforming a CBA instance into a set of linear constraints, which could then be solved by 0–1 integer linear programming (ILP). Santos’ operations research (OR) based approach was followed by several others: Ishizuka and Matsuo [13] presented a method called slide down and shift up, which uses a combination of linear programming and nonlinear programming to find near-optimal solutions in polynomial-time; Ohsawa and Ishizuka [20] presented a method called bubble propagation, which also finds near-optimal solutions in polynomial-time; Matsuo and Ishizuka [19] investigated linear and nonlinear programming approaches to CBA and to more general logical reasoning problems such as satisfiability. Santos and Santos [27] presented
sufficient conditions for a CBA instance to be polynomially-solvable based on the idea of
totally unimodular matrices; their work has been extended by Ohsawa and Yachida [21].

Abdelbar [1] showed that methods for cost-based abduction can be used for belief re-
on binary decision diagrams. Den [9] presented a chart-based method for cost-based ab-
duction. Kato et al. [15] investigated a search control mechanism for the A∗ algorithm
for cost-based abduction, and Kato et al. [16] investigated the parallelization of cost-based
abduction with parallel best-first search. Recently, neural network [3], ant colony [4], and
population-oriented simulated annealing approaches [2] to cost-based abduction have also
been explored. In addition, a number of papers on cost-based abduction have been pub-
lished in the Japanese language, e.g., [17,18,22,23].

The complexity of more-general logic-based abduction has been studied by Eiter and
Gottlob [10].

In this paper, we show, in Section 2, that approximating LCP’s, within a fixed ratio \( c \) of
the cost of an optimal solution, is NP-hard for any \( c > 0 \). We show that this result also holds
if the structure of the CBA instance is constrained such that either (i) no hypothesis appears
as both the consequent of one rule and as an antecedent of another rule (i.e., shallow rule
systems), or (ii) all hypotheses have uniform cost, or (iii) each rule has one antecedent.

2. Complexity results

In this section, we show that approximating LCP’s for CBA systems to within a constant
bound is NP-hard. This is followed by some additional results. We begin by defining two
NP-hard problems [11] which will be used in our proofs.

Definition. An instance of the *Exact Cover by 3-Sets* (**X3C**) problem is a pair \((X, C)\),
where \(X\) is a set of elements such that \(|X|\) is divisible by 3, and \(C\) is a collection of
3-element subsets of \(X\). The answer to this “yes–no” instance is “yes” if there exists a
subcollection \(C' \subseteq C\), such that every element of \(x\) is a member of exactly one element of
\(C'\); such a subcollection \(C'\) is called an *exact cover* for \(X\).

Definition. An instance of *set cover* consists of a set \(X\), and a collection \(C\) of subsets of \(X\).
A feasible solution is any subcollection \(C' \subseteq C\) such that every element of \(X\) belongs to
some element of \(C'\). The cost of a feasible solution \(C'\) is equal to its size \(|C'|\). An optimal
solution is a feasible solution of minimum cost.

Our proof will make use of the following theorem on the complexity of approximating
set cover by Bellare et al. [5, Theorem 1.3]:

**Theorem 1.** Let \( c < 0 \) be any constant. Suppose there is a polynomial time algorithm which
approximates the size of the minimum set cover to within \( c \). Then, \( P = NP \).

Now, we present our main result:
Theorem 2. Let $c < 0$ be any constant. Suppose there is a polynomial time algorithm which approximates the LCP for an arbitrary instance of CBA to within $c$. Then, $P = NP$.

Proof. Let $(X, C)$ be an arbitrary instance of set cover. We construct an instance $(\mathcal{H}, \mathcal{R}, c, \mathcal{G})$ of CBA as follows:

1. For each $x \in X$, we construct an hypothesis $h_x \in \mathcal{H}^P$, and define $c(h_x) = \infty$.
2. For each $s \in C$, we construct a hypothesis $h_s \in \mathcal{H}^A$ and define $c(h_s) = 1$.
3. For each $s \in C$, and for every $x \in s$, we construct a rule in $\mathcal{R}$ of the form: $h_s \rightarrow h_x$.
4. We define the goal set $\mathcal{G}$ as $\mathcal{G} = \{h_x \mid x \in X\}$.

We now claim that the given instance $(X, C)$ of set cover has a solution $C'$ of size $k = |C'|$ if and only if the constructed instance $(\mathcal{H}, \mathcal{R}, c, \mathcal{G})$ of CBA has a proof of cost $k$.

Let $\mathcal{H}' \subseteq \mathcal{H}^A$ be a set of $k$ hypotheses sufficient to prove the goal set $\mathcal{G}$. For every $x \in X$, there will be at least one $h_x \in \mathcal{H}'$ such that $x \in s$, and, hence, the set $\{s \mid h_s \in \mathcal{H}'\}$ constitutes a cover for $X$ of size $k$. Conversely, if $C' \subseteq C$ is a cover for $X$ of size $k$, then the set $\{h_x \mid s \in C'\}$ constitutes a proof for $\mathcal{G}$ of cost $|C'| = k$.

The remainder of the proof follows from Theorem 1. Since the cost of an LCP for $(\mathcal{H}, \mathcal{R}, c, \mathcal{G})$ is equal to the minimum cardinality of a set cover for $(X, C)$, the existence of a polynomial-time algorithm for approximating an LCP for an arbitrary instance of CBA within a factor $c > 0$ implies the existence of a polynomial-time algorithm for approximating the size of a set cover within a factor $c > 0$. But this cannot be done unless $P = NP$.  

2.1. Restricted classes

Because the CBA instance constructed in the proof of Theorem 2 has such a simple structure, the theorem will also hold for the following special cases of cost-based abduction:

- Shallow CBA instances, i.e., CBA instances in which no hypothesis appears as the consequent of one rule and as an antecedent of another rule.
- CBA instances in which all hypotheses have uniform cost.
- CBA instances in which each rule has only one antecedent.

On the other hand, the result may not necessarily hold if the number of rules in which a hypothesis appears as an antecedent is constrained. It also may not necessarily hold if the size of the goal is constrained at the same time that the depth of the rule system is constrained.
2.2. Non-constant ratio bounds

It is also possible to extend Theorem 2 to cases where the ratio bound, instead of being constant, is a function of the goal set $G$. Our proof will make use of the following theorem on the complexity of approximating set cover presented in a weaker form in Bellare et al. [5, Theorem 1.4] and strengthened to the form presented here in [24]:

**Theorem 3.** For any constant $c < 1/8$ the following is true. Suppose there is a polynomial time algorithm which approximates the size of the minimum set cover to within $c \log N$ (where $N = |X|$). Then, $P = NP$.

We use this to prove the following corollary:

**Corollary 4.** For any constant $c < 1/8$ the following is true. Suppose there is a polynomial time algorithm which approximates the LCP for an arbitrary CBA instance to within $c \log |G|$. Then, $P = NP$.

**Proof.** In the transformation used in the proof of Theorem 2, the cardinality of $G$ is identical to the cardinality of $X$. Therefore, from Theorem 3, the existence of a polynomial-time algorithm which approximates LCPs to within $c \log |G|$ implies $P = NP$. □

2.3. CBA fine-tuning

Consider the following scenario. Suppose for a given world model, we have a set of hypotheses $H$ (which can be partitioned into $H^A$ and $H^P$), a set of rules and a set of costs, obtained from a domain expert or a machine learning technique. Now, suppose for a given goal set $G$, representing observed evidence, we obtain the (theoretically optimal) LCP for the CBA system. However, we are later able to determine the real-world, true explanation for $G$ and find it to be inconsistent with the LCP. In general, suppose we have a training set of goal sets and their respective true explanations. Can we find a new cost function $c'$, which does not differ from the initial cost function $c$ by more than a specified deviation $B$, that will make the LCP of each specified goal set consistent with the specified true explanation?

Unfortunately, it turns out that this problem is NP-hard even if the training set consists of a single goal set and its true explanation. We formalize this as follows:

**Definition.** Let $H^A$ be the set of assumable hypotheses for some CBA system $(H, R, c, G)$. A partial explanation $\phi' \subseteq H^A$ is any subset of the assumable hypotheses, and may not necessarily be sufficient to prove the goal set. If a partial explanation is sufficient to prove the goal, then it is considered an explanation.

**Definition.** An instance of bounded-deviation CBA cost refinement (BDCCR) consists of a tuple $(H, R, c, T, B)$, where:

- $H$ is a set of hypotheses, partitioned into a set $H^A$ of assumable hypotheses, and a set $H^P$ of provable hypotheses.
• \( \mathcal{R} \) is a set of rules of the usual CBA format.
• \( c \) is an \textit{initial} cost function \( c : \mathcal{H} \mapsto \mathbb{R}^+ \).
• \( T = \{ (\mathcal{G}_i, \phi'_i) \} \) is a training set of pairs of goals and partial explanations.
• \( B \) is a real number, which will act as an upper bound on the allowed deviation between the new cost function \( c' \) and the initial cost function \( c \), where deviation is computed as:
\[
\sum_{h \in \mathcal{H}} |c(h) - c'(h)|, \tag{5}
\]
where \(| \cdot |\) denotes the absolute value function.

The question is does there exist a cost function \( c' \) such that

1. the deviation between \( c \) and \( c' \), defined as in Eq. (5), is no greater than \( B \),
2. under the new cost function \( c' \),
\[
\phi'_i \subseteq \phi_i,
\]
where \( \phi_i \) is the LCP for \( \mathcal{G}_i \), for all \( (\mathcal{G}_i, \phi'_i) \in T \).

\textbf{Theorem 5.} The \textit{BDCCR problem} is NP-hard.

\textbf{Proof.} Let \( (X, C) \) be an arbitrary instance of X3C. Let \( q = |X|/3 \). We construct an instance \( (\mathcal{H}, \mathcal{R}, c, T, B) \) of BDCCR as follows:

1. For every element \( x \in X \), construct an hypothesis \( h_x \in \mathcal{H}^P \), with \( c(h_x) = \infty \). For every \( s \in C \), construct an hypothesis \( h_s \in \mathcal{H}^A \), with \( c(h_s) = 100 \). Construct two additional hypotheses \( h_Y \) and \( h_N \) and define:
\[
c(h_Y) = 1, \tag{6}
\]
\[
c(h_N) = q \cdot 100 + 50. \tag{7}
\]
2. For every \( s \in C \) and \( x \in s \), if \( x \in s \), construct a rule:
\[
h_Y \land h_s \rightarrow h_x. \tag{8}
\]
For every \( x \in X \), construct a rule:
\[
h_N \rightarrow h_x. \tag{9}
\]
3. Construct \( T \) as the single element set
\[
T = \{ (\mathcal{G}_1, \phi'_1) \}, \tag{10}
\]
where \( \mathcal{G}_1 = \{ h_x \mid x \in X \} \) and \( \phi'_1 = \{ h_Y \} \).
4. Let \( B = 25 \).

Consider the CBA instance \( \mathcal{L} = (\mathcal{H}, \mathcal{R}, c, \mathcal{G}_1) \). If \( (X, C) \) has an exact cover \( C' \), then the LCP \( \phi_Y \) for \( \mathcal{L} \) will have cost \( q \cdot 100 + 1 \), and will consist of
\[
\phi_Y = \{ h_Y \} \cup \{ h_s \mid s \in C' \}. \tag{11}
\]
If \((X, C)\) has no exact cover, the LCP \(\phi_N\) for \(L\) will have cost \(q \cdot 100 + 50\), and will consist of
\[
\phi_N = \{h_N\}.
\] (12)

Now, let \(c'\) be an arbitrary cost function whose deviation from \(c\), as defined in Eq. (5), does not exceed \(B\). We claim such a cost function \(c'\) exists if and only if an exact cover for \((X, C)\) exists.

1. First, suppose that \(c'\) exists but no exact cover for \((X, C)\) exists. If \(c'\) exists, then the LCP \(\phi_N\) for \((H, R, c', G)\) assumes \(h_Y\). Now, if \(h_N \in \phi_N\), then the fact that \(\phi_N\) is a least cost proof implies that \(h_N\) is the only member of \(\phi_N\). Therefore, \(h_Y \in \phi_N\) implies \(h_N \notin \phi_N\). But, if \(h_N \notin \phi_N\), then
\[
\{s \mid h_s \in \phi_N\}
\]
must constitute an exact cover for \((X, C)\) which contradicts the assumption that \((X, C)\) has no exact cover.

2. On the other hand, suppose \((X, C)\) has an exact cover \(C'\). We can construct \(c'\) by simply assigning \(c' = c\).

Therefore, the answer to \((X, C)\) is affirmative if and only if the answer to \((H, R, c, T, B)\) is affirmative. □

3. Concluding remarks

In this paper, we have shown that approximating LCPs for CBA systems within a fixed ratio bound is NP-hard. We have also shown that the related problem of finding the minimum adjustment to a CBA system’s cost function sufficient to make a goal set’s LCP consistent with what we know to be true is NP-hard.

Charniak and Shimony’s [8] NP-hardness result was published in 1994. Over the past decade, as discussed in an earlier section, there have been a number of different methods proposed for this problem: many of these have been in the general area of OR. There is a need for a comprehensive empirical study which compares the practical run-time behavior of these different approaches. There is also a need for a publicly-accessible standard repository of benchmark instances of cost-based abduction systems, along the lines of the traveling salesman repository TSPLIB (www.iwr.uni-heidelberg.de/groups/compopt/software/TSPLIB95), the satisfiability repository SATLIB (www.satlib.org), and the quadratic assignment problem repository QAPLIB (www.seas.upenn.edu/qaplib).

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