ELSEVIER

Discrete Applied Mathematics 65 (1996) 387–407

DISCRETE APPLIED MATHEMATICS

Graphs with largest number of minimum cuts

Jenő Lehel [a,1], Frédéric Maffray [b,*], Myriam Preissmann [c]

[a] Computer and Automation Institute, Hungarian Academy of Sciences, Budapest, Hungary
[b] CNRS, Laboratoire LSD2, IMAG, BP 53, 38041 Grenoble Cedex 9, France
[c] CNRS, Laboratoire ARTEMIS, IMAG, BP 53, 38041 Grenoble Cedex 9, France

Received 4 January 1993; revised 19 May 1994

Abstract

Let $\sigma(n,k)$ be the largest number of k-cuts in a k-edge-connected multigraph with n vertices. We determine $\sigma(n,k)$ and characterize extremal multigraphs for every n and k. The same problem is also investigated for graphs with no multiple edges.

1. Introduction

Dinitz et al., in [2], described the structure of minimum cuts of multigraphs: the set of all k-cuts of a graph with edge-connectivity k has a one-to-one mapping onto the set of all minimal cuts of a corresponding "cactus" (the blocks are single edges and cycles). As a corollary, they proved that the vertex set of a graph has a cyclic ordering such that any minimum cut disconnects the graph into components of consecutive vertices. We use here this basic result to investigate further the structure of graphs with maximum number of minimum cuts.

A connected graph G is *k-(edge)-connected*, if any subset of $E(G)$ whose removal disconnects G contains at least k edges. If there are exactly k edges between X and $\widehat{X} = V(G)\backslash X$, then we say that $(X,\widehat{X})$ is a *k-cut*. The *edge-connectivity* of G is the largest k such that G is k-connected; alternately it is the smallest k such that G has a k-cut.

Let $\sigma(n,k)$ be the maximum number of k-cuts in a multigraph of edge-connectivity k with n vertices; and let $\sigma_1(n,k)$ be the maximum number of k-cuts in a simple graph of edge-connectivity k with n vertices. A k-connected graph (resp. k-connected simple graph) is called *extremal* if it has $\sigma(n,k)$ (resp. $\sigma_1(n,k)$) k-cuts.

In [2] the inequality $\sigma(n,k)\leqslant\binom{n}{2}$ was proved, and the cycle on n vertices with edges of multiplicity $k/2$ was exhibited as an example for which this bound is tight when k is even. For k odd, the k-cuts form a nested family, which yields a linear upper bound

*Corresponding author.

[1]Research supported by MRT/MICECO France under grant #058936E12215E.

SSDI 0166-218X(95)00043-7

for $\sigma(n,k)$ in this case. In Section 3, we will show that $\sigma(n,k) = \lfloor 3n/2 \rfloor - 2$, for odd $k > 1$ and for every n. Furthermore, we characterize extremal graphs (Theorem 3.4).

Let us note that the preceding results are already implied by the work of Bixby [1]. Our approach uses similar techniques but more graph theory than Bixby's. It also leads to the new results on simple graphs presented in Section 4, where $\sigma_1(n,k)$ is investigated. A tight upper bound is given for any even $k \geqslant 4$ (Theorem 4.3). We determine $\sigma_1(n,k)$ and characterize extremal graphs for $k = 3$ and $k = 5$ (Theorems 3.4, 4.4 and 4.7). For odd $k > 5$, $\sigma_1(n,k) \geqslant (1 + 2/(k+1))n - O(1)$ follows from a construction, and we prove $\sigma_1(n,k) \leqslant (1 + 4/(k+5))n$ (Theorem 4.9).

In the following section, we give a representation for the structure of all minimum cuts. Let G be a graph with vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$, and let $(X_i, \widehat{X}_i)$, be the minimum cuts of G such that $v_0 \in \widehat{X}_i$, $i = 1, \ldots, p$. First we show that the hypergraph defined on vertices $\{v_1, \ldots, v_{n-1}\}$ with edge set $\{X_1, \ldots, X_p\}$ is an interval hypergraph, then we describe the structure of minimum cuts in terms of the overlap graph of these intervals (Theorem 2.4). This also leads to the above-mentioned corollary of the "cactus" representation of Dinitz et al. Another representation of the minimum cuts was proposed recently by Gabow in [4]. For further reference on related algorithmic results see [5, 7].

2. The structure of minimum cuts

For a fixed integer $k > 0$, let G be a graph with edge-connectivity k. Since we are interested in regarding the k-cuts as vertex subsets rather than edge subsets, we will frequently fix a vertex v_0 of G and, with a slight abuse of terminology, say that $X \subset V$ is a k-cut of G when $(X, \widehat{X})$ is a k-cut with $v_0 \in \widehat{X}$. In this context, a k-cut X will be called *trivial* if $|X| = 1$, i.e., X consists of one vertex of degree k. A nontrivial k-cut X will be called *minimal* if every k-cut Y with $Y \subset X$ is trivial.

We denote by $m_G(xy)$ the multiplicity of an edge xy of G. For disjoint subsets $A, B \subset V(G)$, $m_G(A,B)$ is the total number of edges xy with $x \in A$ and $y \in B$. We simply write $m(A,B)$ omitting index G if no ambiguity occurs. If two k-cuts X, Y have nonempty intersection, then either they are nested (i.e., $X \subset Y$ or $Y \subset X$) or they overlap (i.e., $X \cap Y$, $\widehat{X} \cap Y$ and $X \cap \widehat{Y}$ are nonempty).

It is known (and easy to check) that two k-cuts in a k-connected graph can overlap only for k even. More precisely:

Proposition 2.1. *Let G be a k-connected graph and X, Y be two overlapping k-cuts. Then $m(\widehat{X} \cap \widehat{Y}, X \cap \widehat{Y}) = m(\widehat{X} \cap \widehat{Y}, \widehat{X} \cap Y) = m(X \cap Y, X \cap \widehat{Y}) = m(X \cap Y, \widehat{X} \cap Y) = k/2$. Consequently $X \cup Y$, $X \cap Y$, $X \cap \widehat{Y}$, $\widehat{X} \cap Y$ are k-cuts; moreover,*

$$m(\widehat{X} \cap \widehat{Y}, X \cap Y) = m(\widehat{X} \cap Y, X \cap \widehat{Y}) = 0. \tag{1}$$

Proposition 2.1 is easily proved by counting the number of edges between any two of the above-mentioned four sets. Details are ommited.

The following lemma will be useful.

Lemma 2.2. *If A, B and C are distinct k-cuts, then one of the sets*

$$A_0 = \widehat{A} \cap B \cap C,\ B_0 = A \cap \widehat{B} \cap C \text{ and } C_0 = A \cap B \cap \widehat{C}$$

is empty.

Proof. Assume on the contrary that A_0, B_0 and C_0 are nonempty. Then A, B and C pairwise overlap, and (1) holds for every $X, Y \in \{A, B, C\}$, $X \neq Y$. Consequently, $m(X, Y) = 0$, for every $X, Y \in \{A_0, B_0, C_0, D_0\}$, $X \neq Y$, where $D_0 = \widehat{A} \cap \widehat{B} \cap \widehat{C}$. Since there are at most $3k$ distinct edges defined by the k-cuts A, B and C, one of the cuts A_0, B_0, C_0 and $\widehat{D}_0$ has at most $3k/4 < k$ outgoing edges, a contradiction. □

Let $\mathcal{E}$ be the family of all k-cuts of G and $\mathcal{V} = \{v_1, \ldots, v_{n-1}\}$. Then $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is called the *cut-hypergraph* of G. Using the remark at the beginning of this section, we can easily conclude that $\mathcal{H}$ is *laminar*, i.e., it satisfies

$$A \cup B \in \mathcal{E} \quad \text{for all } A, B \in \mathcal{E} \text{ such that } A \cap B \neq \emptyset. \tag{2}$$

Furthermore, $\mathcal{H}$ satisfies the *strong Helly property:*

$$\bigcap \{E \in \mathcal{E} : |E \cap \{x, y, z\}| \geqslant 2\} \cap \{x, y, z\} \neq \emptyset \quad \text{for every } x, y, z \in \mathcal{V}. \tag{3}$$

To see (3), assume that $x, y \in A \in \mathcal{E}$, $z \in \widehat{A}$ and $x, z \in B \in \mathcal{E}$, $y \in \widehat{B}$, for some $A, B \in \mathcal{E}$. Then by Lemma 2.2, $y, z \in C$ implies $x \in C$, for every $C \in \mathcal{E}$, hence $x \in \bigcap \{E \in \mathcal{E} : |E \cap \{x, y, z\}| \geqslant 2\}$ follows.

A hypergraph is called an *interval hypergraph* if there exists a total ordering on its vertex set for which every hyperedge of the hypergraph is an interval. Interval hypergraphs were studied in [3, 6, 8]. In particular it was shown that a hypergraph is an interval hypergraph if and only if it is laminar and satisfies the strong Helly property. Thus we will assume that $L = (v_1, \ldots, v_{n-1})$ is a linear order of the vertices such that every k-cut is a subset of consecutive vertices, that is, every minimum cut is an interval. Remark that by adding v_0 between v_{n-1} and v_1 we can obtain the same cyclic ordering as in [2].

Let $\{X_1, \ldots, X_p\}$ be a family of intervals of L. Two vertices $u, v \in \bigcup_{i=1}^{p} X_i$ are said to be equivalent (with respect to the family) if for all i $(1 \leqslant i \leqslant p)$, $u \in X_i$ if and only if $v \in X_i$. The equivalence classes are called the *atoms* of the family. The *overlap graph* of $\{X_1, \ldots, X_p\}$ is a graph defined on the intervals as vertices, $X_i X_j$ being an edge if and only if X_i and X_j overlap.

As it was discussed in [2], the overlap structure contains the basic information about minimum cuts. This is expressed in the next lemma.

Lemma 2.3. *Let $\mathcal{F}$ be any family of k-cuts and let $\{A_1, \ldots, A_t\}$ be the set of its atoms indexed according to the order L. If $\mathcal{F}$ has connected overlap graph, then $A_p \cup A_{p+1} \cup \cdots \cup A_q$ is a k-cut for every p, q with $1 \leqslant p \leqslant q \leqslant t$.*

Proof. We omit details of the easy induction on $|\mathcal{F}|$ which uses the fact mentioned above that $X \cup Y$, $X \cap Y$, $X \setminus Y$, and $Y \setminus X$ are k-cuts, for every $X, Y \in \mathcal{E}$. □

Let H be the overlap graph of $\mathcal{E}$. If H has no edges, then $\mathcal{E}$ is called a *nested* family, that is, for any pair $X, Y \in \mathcal{E}$, either $X \cap Y = \emptyset$ or one of X and Y contains the other. Notice that this is the case when k is odd.

Let H_i, $i = 1, \ldots, t$, be the connected components of the overlap graph H with $V(H_i) = \mathcal{E}_i$. Set $A_{i,0} = \bigcup\{X \in \mathcal{E}_i\}$, and let $A_{i,1}, \ldots, A_{i,t_i}$ be the atoms of $\mathcal{E}_i$ indexed according to the order L. If $\mathcal{E}_i = \{A_{i,0}\}$, then we say that H_i is isolated or trivial. If H_i is nontrivial, then by Lemma 2.3, H_i consists of all intervals of the form $A_{i,p} \cup A_{i,p+1} \cup \cdots \cup A_{i,q}$, $1 \leqslant p < q \leqslant t_i$, different from $A_{i,0}$. We refer to this fact, that the intervals of H_i form a *full interval system* on their atoms. It is also clear, that the intervals of the set $\{A_{i,j} : 1 \leqslant i \leqslant t, 0 \leqslant j \leqslant t_i\}$ are pairwise nonoverlapping, thus form a nested family. We summarize these results as follows.

Theorem 2.4. *Let G be a k-edge connected graph of order n, and let v_0 be an arbitrary vertex of G. Then $V(G)\setminus\{v_0\}$ has an ordering $(v_1, \ldots, v_{n-1})$ such that every k-cut is an interval on the set $\{v_1, \ldots, v_{n-1}\}$. Moreover, if H is the overlap graph of the k-cuts of G, then its trivial connected components define a nested family, and the k-cuts in each nontrivial connected component form a full interval system on their atoms. For k odd, every connected component of H is trivial.*

In the remaining sections we use the following observation pertaining to the placement of edges of G between atoms. Let $\{A_1, \ldots, A_t\}$ $(t \geqslant 3)$ be the consecutive atoms defined by the k-cuts represented by the vertices of a nontrivial connected component of H. Then $m(A_i, A_{i+1}) = k/2$, for every $1 \leqslant i < t$. This follows from Proposition 2.1 and from the fact that every interior atom A_j $(1 < j < t)$ is the intersection of two overlapping cuts belonging to H, namely, $X = A_{j-1} \cup A_j$ and $Y = A_j \cup A_{j+1}$.

Note that, based on Theorem 2.4, one can easily get the result in [2] for representing minimum cuts by a cactus-like structure. On the other hand, the representation of k-cuts in [2] easily implies Theorem 2.4.

3. Extremal multigraphs

3.1. Multigraphs with odd edge-connectivity

In this subsection we consider graphs of edge-connectivity k, with k odd. Note that $\sigma(n, 1) = n - 1$, and the extremal graphs are the trees. So we may assume $k > 1$. We use the interval representation and the notations introduced in Section 2. In particular, $\mathcal{E}$ denotes the family of intervals corresponding to the minimum cuts of graph G. By Theorem 2.4, the intervals of $\mathcal{E}$ form a nested family, for k odd. In this case there is a further restriction on $\mathcal{E}$.

Lemma 3.1. *Let $A, A_1, \ldots, A_q$ be k-cuts with $A = \bigcup_{i=1}^{q} A_i$ and $A_i \cap A_j = \emptyset$, $1 \leqslant i < j \leqslant q$. Then, for k odd, q is also an odd integer.*

Proof. Obviously,

$$qk = \sum_{i=1}^{q} m(A_i, \widehat{A}_i) = m(A, \widehat{A}) + 2 \sum_{1 \leqslant i < j \leqslant q} m(A_i, A_j) = k + 2 \sum_{1 \leqslant i < j \leqslant q} m(A_i, A_j).$$

Hence $qk - k = k(q-1)$ is even, which implies that q must be odd. □

From this observation one can easily conclude that the maximum number of intervals in the nested family $\mathcal{E}$ is less than $3n/2$. To obtain $\sigma(n,k)$ and the structure of extremal graphs, we need a more accurate count and some definitions.

For $F \subset E(G)$, the *removal* of F results in a partial graph of G we denote by $G - F$; if xy is a multiple edge then $G - \{xy\}$ means the removal of every edge between x and y. The *contraction* of a set $A \subset V(G)$ is the operation which consists in identifying the vertices of A. The graph which results from this operation is denoted by G/A. Notice that contraction does not reduce the edge-connectivity of a graph. Denote by $\sigma(G)$ the number of minimum cuts of a graph G.

Proposition 3.2. *Let $k \geqslant 3$ be odd. If G is a graph of edge-connectivity k and of order n, then*

$$\sigma(G) \leqslant \left\lfloor \frac{3n}{2} \right\rfloor - 2. \tag{4}$$

Moreover, if equality holds in (4) *and $n \geqslant 4$, then G has a k-cut $(A, \widehat{A})$ such that either A or $\widehat{A}$ consists of exactly three vertices of degree k.*

Proof. The inequality is true for $n = 2$ and $n = 3$. Now assume that $n \geqslant 4$ and that (4) holds for graphs of order less than n. If G has trivial cuts only, then (4) follows with strict inequality, for $n \geqslant 4$, thus we may assume that G has nontrivial k-cuts. Let $A \in \mathcal{E}$ be a minimal nontrivial k-cut. Let $G' = G/A$. Then G' is k-connected and has $n' = n - |A| + 1$ vertices. By the minimality of A, G has at most $|A \setminus Q| + \sigma(G')$ minimum cuts, where Q is the set of all vertices of A with degree larger than k. Since $n' < n$, the induction hypothesis entails that $\sigma(G') \leqslant \lfloor 3n'/2 \rfloor - 2$. Here we distinguish between two possibilities.

For $|A| \geqslant 3$, we obtain $|A \setminus Q| + \sigma(G') \leqslant |A| + \lfloor 3(n - |A| + 1)/2 \rfloor - 2 \leqslant \lfloor 3n/2 \rfloor - 2$, with equality only if $|A| = 3$ and $Q = \emptyset$.

For $|A| = 2$, Q is nonempty by Lemma 3.1. Thus in this case $|A \setminus Q| + \sigma(G') \leqslant 1 + \lfloor 3(n-1)/2 \rfloor - 2 \leqslant \lfloor 3n/2 \rfloor - 2$ follows. Notice that the inequality is strict for n even. We obtain easily, as a consequence, that if equality holds in (4), then every vertex has degree k, for n even; and every vertex but one has degree k, for n odd.

To finish the proof, we have to verify that in the second case G has a minimal nontrivial cut A with $|A| = 3$. Indeed, if we choose v_0 in the interval representation to be the only vertex of G with degree more than k, then the interval representation excludes v_0 from the cuts, hence $|A| \neq 2$. □

Next we exhibit k-connected graphs with $\lfloor 3n/2 \rfloor - 2$ k-cuts. We shall obtain extremal graphs from smaller ones by "splitting" vertices of degree k into three and including some edges between the new vertices.

Let G be a k-connected graph. We denote by $S(G)$ any graph obtained from G as follows. Let v be a vertex of G of degree k. Let p_1, p_2, p_3 be integers such that $p_1 + p_2 + p_3 = k$. Partition the edges of G incident to v into three sets P_1, P_2, P_3 of size p_1, p_2, p_3, respectively. Remove v and add three new vertices v_1, v_2, v_3. For each edge wv in P_i add an edge wv_i. Add edges between v_1, v_2, v_3 with multiplicity $m(v_1,v_2) = p_3$, $m(v_2,v_3) = p_1$ and $m(v_3,v_1) = p_2$. The operation of deriving $S(G)$ from G will be called *k-splitting* of G at v. We will say that a k-splitting is *legal* if $p_i < k/2$ holds for $i = 1,2,3$. It is easy to check that if a k-splitting is not legal then the resulting graph is not k-connected.

Proposition 3.3. *Let G be a graph of order n and edge-connectivity k. Consider a legal splitting of G at a vertex v of degree k. Then the resulting graph $S(G)$ has edge-connectivity k, order $n+2$, and $\sigma(S(G)) = \sigma(G) + 3$.*

Proof. Consider any cut $(A,\widehat{A})$ of $S(G)$.

First suppose that the cut $(A,\widehat{A})$ does not separate v_1, v_2, v_3 from each other. We may assume without loss of generality that these three vertices are in $\widehat{A}$. Now $A \subset V(G)$ and $(A, V(G)\backslash A)$ is a cut of G. There is an evident one-to-one correspondence between the edges of $(A,\widehat{A})$ in $S(G)$ and the edges of $(A, V(G)\backslash A)$ in G. It follows that $m(A,\widehat{A}) \geqslant k$; moreover every k-cut of G corresponds to a k-cut of $S(G)$.

Second suppose that the cut $(A,\widehat{A})$ does separate the v_i's from each other. Without loss of generality, $v_1 \in A$ and $v_2, v_3 \in \widehat{A}$. Notice that v_1v_2 and v_1v_3 form $p_3 + p_2$ edges between A and $\widehat{A}$. Let p_1' be the number of edges between v_1 and $\widehat{A} \setminus \{v_2, v_3\}$, and p be the number of edges between $A \setminus \{v_1\}$ and $\widehat{A}$. Here we have

$$m(A,\widehat{A}) = p_2 + p_3 + p_1' + p.$$

If $A = \{v_1\}$ then clearly $m(A,\widehat{A}) = k$ (since each v_i is of degree k by the construction of $S(G)$). Now assume that $A' = A - \{v_1\}$ is not empty. Hence $(A', V(G)\backslash A')$ is a cut of G and

$$p_1 - p_1' + p = m(A', V(G) \setminus A') \geqslant k.$$

It follows that

$$p \geqslant k - p_1 + p_1' \geqslant p_2 + p_3,$$

because $k = p_1 + p_2 + p_3$, whence

$$m(A,\widehat{A}) \geqslant 2(p_2 + p_3).$$

The hypothesis that $p_1 < k/2$ and $p_1 + p_2 + p_3 = k$ imply $p_2 + p_3 > k/2$, so $m(A, \widehat{A}) > k$. Consequently, in this second case $(A, \widehat{A})$ is not a k-cut unless $A = \{v_i\}$ for $i = 1, 2, 3$. □

For $k \geqslant 3$, the smallest extremal graphs of edge-connectivity k are: the graph with two vertices and k parallel edges, which we will denote by $P(k)$; and any graph with three vertices, one edge of multiplicity $p < k/2$ and two edges of multiplicity $k - p$, which will be denoted by $Q(k, p)$. Proposition 3.3 shows that one obtains extremal graphs for every n by starting with either $P(k)$ or $Q(k, p)$, $p < k/2$, and by performing a sequence of legal splittings. Hence $\sigma(n, k) = \lfloor 3n/2 \rfloor - 2$ follows for every $n > 1$ and odd $k \geqslant 3$.

Theorem 3.4. *For every $n > 1$ and odd $k \geqslant 3$, $\sigma(n, k) = \lfloor 3n/2 \rfloor - 2$. Moreover, a graph of order $n \geqslant 4$ and edge-connectivity k is extremal if and only if it is obtained from either $P(k)$ or $Q(k, p)$, $p < k/2$, by a sequence of legal splittings.*

Proof. Let G be an extremal graph of edge-connectivity k and order n. Then, by Proposition 3.2, G has a k-cut $A = \{v_1, v_2, v_3\}$, with $d_G(v_j) = k$ $(j = 1, 2, 3)$. Let $p_1 = m(v_2 v_3)$, $p_2 = m(v_3 v_1)$ and $p_3 = m(v_1 v_2)$. Since $m(A, \widehat{A}) = k$, we obtain that $p_1 + p_2 + p_3 = k$. If one of p_1, p_2, p_3 was greater than $k/2$, say $p_1 > k/2$, then $\{v_2, v_3\}$ would be a cut of size $2p_2 + 2p_3 = 2k - 2p_1 < k$, which is not possible. Thus $0 < p_1, p_2, p_3 < k/2$, showing that $G = S(G/A)$. Repeating this argument for G/A, and so on, after $\lfloor n/2 \rfloor - 1$ steps we get the graph $P(k)$ or $Q(k, p)$ for some $p \leqslant k/2$. □

3.2. Multigraphs with even edge-connectivity

For k even, $\sigma(n, k) = \binom{n}{2}$ was proved in [2]. For the sake of completeness we show how this result follows from the interval representation of minimum cuts given in Section 2. Theorem 2.4 shows that the number of intervals on $n - 1$ points of the line, i.e., $\binom{n-1}{2} + n - 1 = \binom{n}{2}$ is an upper bound for the number of minimum cuts of a graph of order n. Using the remark after Theorem 2.4, we conclude that extremal graphs having $\binom{n}{2}$ minimum cuts are unique.

Proposition 3.5. *For $n \geqslant 3$ and k even, we have $\sigma(n, k) = \binom{n}{2}$, and the unique extremal graph is the n-cycle with $k/2$ parallel edges between any two consecutive vertices.*

4. Extremal simple graphs

In this section simple graphs with large number of minimum cuts are investigated. From now on we will assume that $k \geqslant 3$. We are using the representation of the k-cuts of a k-connected graph G by intervals of the set $V = \{v_1, \ldots, v_{n-1}\}$ as described in

Section 2. Our goal is to improve on the general upper bounds of Propositions 3.2 and 3.5.

To see that the number of nontrivial cuts decreases when an upper bound is imposed on the edge multiplicity, we need the following observation.

Lemma 4.1. *If G is a k-connected graph with edge multiplicity at most m and A is a nontrivial k-cut, then $|A| \geqslant k/m$.*

Proof. Since G is k-connected, $d_G(x) \geqslant k$ for every $x \in A$. Thus

$$k|A| \leqslant \sum_{x\in A}\left(\sum_{y\in A} m_G(xy) + \sum_{y\in\widehat{A}} m_G(xy)\right) = m_G(A,\widehat{A}) + 2\sum_{\{x,y\}\subset A} m_G(xy). \tag{5}$$

Since the multiplicity of an edge of G is at most m,

$$\sum_{\{x,y\}\subset A} m_G(xy) \leqslant \binom{|A|}{2} m.$$

Using this inequality together with $m_G(A,\widehat{A}) = k$, (5) implies $|A|(|A|-1)m + k \geqslant k|A|$. Hence $|A| \geqslant k/m$ which concludes the proof of the lemma. □

4.1. Simple graphs with even edge-connectivity

Assume that $k \geqslant 4$ and k is even. First we determine $\sigma_1(n,k)$ for small values of n.

Proposition 4.2. *Assume k is even and at least 4. Then*

$$\sigma_1(n,k) = \begin{cases} n & \text{if } k+1 \leqslant n \leqslant 2k-1, \\ n+1 & \text{if } n = 2k, \\ n+2 & \text{if } n = 2k+1, \\ n+4 & \text{if } n = 2k+2. \end{cases}$$

Proof. In a k-connected graph each vertex has degree at least k, thus $n \geqslant k+1$. By Lemma 4.1, the smallest cardinality of a nontrivial k-cut is k. Hence a graph of order $n < 2k$ has only trivial cuts, implying $\sigma_1(n,k) \leqslant n$ for $k+1 \leqslant n \leqslant 2k-1$. Since k is even, obviously there exists a k-regular graph G of order n for every n. Moreover, for $n \leqslant 2k$, G is k-connected. To see this, assume on the contrary that $m_G(A,\widehat{A}) < k$, for some $A \subset V(G)$ with $|A| \leqslant k$. Then clearly, $d(v) < k$ follows for some $v \in A$, contradicting the k-regularity of G. Hence $\sigma_1(n,k) = n$, for $k+1 \leqslant n \leqslant 2k-1$, and every k-regular graph is extremal.

Observe that if G has two nontrivial overlapping cuts A and B, then $A \cap B \neq \emptyset$, $A \cap \widehat{B} \neq \emptyset$, $\widehat{A} \cap B \neq \emptyset$ and $\widehat{A} \cap \widehat{B} \neq \emptyset$ imply $n \geqslant 2k+2$. Hence $\sigma_1(n,k) \leqslant n+1$ and $\sigma_1(n,k) \leqslant n+2$ follows for $n = 2k$ and $n = 2k+1$, respectively. In the first case, the graph G_0 consisting of two disjoint copies of a k-clique with a perfect matching M between them shows that the bound is tight. In the second case, we obtain an extremal graph G_1 from G_0 by subdividing $k/2$ edges of M and identifying all the subdividing vertices into one. (Note that the obtained extremal graphs are unique in both cases.)

For $n = 2k+2$, let G_2 be the graph obtained from G_1 by subdividing the remaining $k/2$ edges of M and by identifying the new subdividing vertices into one. Clearly, G_2 has $n+4$ k-cuts. If G is a graph without overlapping cuts, then $\sigma(G) \leqslant n+3$. Hence $\sigma_1(n,k) = n+4$, for $n = 2k+2$, concluding the proof of the proposition. □

For $n = r(k+1)$ with $r \geqslant 3$, let $F_{n,k}$ be the simple graph obtained as follows. We start from r vertices $c_0 \ldots, c_{r-1}$. For each i we add k new vertices forming a k-clique Q_i; we link $k/2$ of these new vertices to c_i and the other $k/2$ to c_{i+1} (mod r). Clearly $F_{n,k}$ is a k-regular, k-connected simple graph. To count the k-cuts of $F_{n,k}$ consider the sequence $Q_0, c_1, Q_1, c_2, \ldots, c_{r-1}, Q_{r-1}$ and observe that every nonempty interval in this sequence of $2r-1$ elements forms a k-cut. In addition each vertex in any Q_i is a trivial k-cut, so

$$\sigma(F_{n,k}) = \binom{2r}{2} + rk = \frac{2}{(k+1)^2}n^2 + \frac{k-1}{k+1}n.$$

As we will see, the number of k-cuts of $F_{n,k}$ reaches the upper bound obtained for $\sigma_1(n,k)$.

Theorem 4.3. *Consider an even $k \geqslant 4$ and $n \geqslant 2k+2$. Then*

$$\sigma_1(n,k) \leqslant \frac{2}{(k+1)^2}n^2 + \frac{k-1}{k+1}n, \tag{6}$$

and the bound is tight if $k+1$ divides n.

Proof. Let G be an extremal graph of edge-connectivity k and of order n. We say that G is *decomposable* if there exists a nontrivial nonminimal k-cut A such that there is no k-cut which overlaps A (i.e., for every k-cut C one of $A \subset C$, $C \subset A$ and $A \cap C = \emptyset$ must hold).

Case 1: *G is not decomposable.* Consider the interval representation of the k-cuts of G choosing a vertex of degree k in the role of v_0. To insure the existence of such a vertex, assume on the contrary that all vertices of G have degree at least $k+1$. Let C be a nontrivial minimal k-cut. It is easy to check that C contains at least one edge e (for otherwise there would be too many edges going out of C) and that $G-e$ is k-connected (for otherwise a nontrivial k-cut smaller than C would be found); moreover, every k-cut of G is a k-cut of $G-e$. We can repeat this argument until we obtain a k-connected subgraph of G with one vertex of degree k, contradicting the extremality of G.

The choice of v_0 implies that $\{v_1, \ldots, v_{n-1}\}$ is a k-cut. Hence every vertex is in a (minimal) k-cut.

Let $A_1, \ldots, A_m$ be the minimal nontrivial k-cuts of G and let T be the set of trivial k-cuts not in $\bigcup_{i=1}^m A_i$. By Lemma 4.1, $|A_i| \geqslant k$ for every $1 \leqslant i \leqslant m$. By the remark after Theorem 2.4, one must have $t \leqslant m-1$. Indeed, since G is simple, no consecutive vertices v_j and v_{j+1} ($1 \leqslant j \leqslant n-2$) may belong to T, and moreover, the sequence of atoms does start and end with minimal cuts belonging to $\bigcup_{i=1}^{m} A_i$.

Every vertex different from v_0 either has degree k or is in a minimal nontrivial cut. Hence $A_1, \ldots, A_m, T$ form a partition of $V - v_0$, and $|T| + \sum_{i=1}^{m} |A_i| = n - 1$. Moreover, since G is not decomposable, the nonminimal cuts form a full family of intervals on $(\bigcup_{i=1}^{m} A_i) \cup T$. Thus $\sigma(G) = \binom{t+m}{2} + m + n_k$, where $t = |T|$ and n_k is the number of vertices of degree k different from v_0. Hence $\sigma(G) \leqslant \binom{t+m}{2} + m + n - 1$.

So it suffices to find an upper bound on $\binom{t+m}{2} + m + n - 1$ corresponding to a collection of pairwise disjoint subsets $A_1, \ldots, A_m, T$ of an $n-1$ set, subject to $n - 1 = t + \sum_{i=1}^{m} |A_i|$ with $|T| = t < m$ and $|A_i| \geqslant k$ (regardless of graph realizability constraints). So let us consider an arbitrary such collection.

If $t = m - 1$, then trivially $m \leqslant n/(k+1)$ (with equality only if $|A_i| = k$ for every $1 \leqslant i \leqslant m$). Thus $t + m = 2m - 1 \leqslant 2n/(k+1) - 1$, and

$$\begin{aligned}\binom{t+m}{2} + m + n - 1 &\leqslant \left(\frac{2n}{k+1} - 1\right)\left(\frac{n}{k+1} - 1\right) + \frac{n}{k+1} + n - 1 \\ &\leqslant \frac{2}{(k+1)^2} n^2 + \frac{k-1}{k+1} n,\end{aligned}$$

which proves (6).

Now suppose $t < m - 1$. If $|A_i| \geqslant k + 1$ for some i, then $\binom{t+m}{2}$ increases if a vertex is removed from A_i and is added to T, while m does not change. We can iterate this procedure; if t reaches the value $m - 1$, we can apply the preceding case. So we may assume that each $|A_i|$ is equal to k and that $t < m - 1$.

Write $\delta = m - t$. So $\delta \geqslant 2$ and $n - 1 = t + mk = m(k+1) - \delta$. Then we get

$$\begin{aligned}&\frac{k-1}{k-1} n = -2m + \frac{2(\delta - 1)}{k+1} + n, \\ &\frac{2}{(k+1)^2} n^2 = 2m^2 - \frac{4m(\delta-1)}{k+1} + \frac{2(\delta-1)^2}{(k+1)^2}.\end{aligned} \tag{7}$$

Furthermore, we have

$$\binom{t+m}{2} + m + n - 1 = 2m^2 - 2m\delta + \frac{\delta^2 + \delta}{2} + n - 1. \tag{8}$$

Now, using (7) and (8), inequality (6) will hold if the following quantity is positive:

$$\begin{aligned}&\frac{2}{(k+1)^2} n^2 + \frac{k-1}{k+1} n - \left(2m^2 - 2m\delta + \frac{\delta^2 + \delta}{2} + n - 1\right) \\ &= \left(2m^2 - \frac{4m(\delta-1)}{k+1} - 2m + \frac{2(\delta-1)^2}{(k+1)^2} + \frac{2(\delta-1)}{k+1} + n\right) \\ &\quad - \left(2m^2 - 2m\delta + \frac{\delta^2+\delta}{2} + n - 1\right) \\ &= m\frac{2(\delta-1)(k-1)}{k+1} - \delta\frac{(\delta+1)(k+1) - 4}{2(k+1)} + \left(\frac{2(\delta-1)^2}{(k+1)^2} + 1 - \frac{2}{k+1}\right).\end{aligned}$$

However, for $\delta \geqslant 2$ and $k \geqslant 4$, the inequality $1 + (2k-2)/(3k-5) < \delta$ holds true, which implies

$$\frac{(\delta+1)(k+1)-4}{2(k+1)} < \frac{2(\delta-1)(k-1)}{k+1}.$$

Using this inequality together with $t = m - \delta \geqslant 0$ we obtain that

$$0 \leqslant (m-\delta)\frac{(\delta+1)(k+1)-4}{2(k+1)} < m\frac{2(\delta-1)(k-1)}{k+1} - \delta\frac{(\delta+1)(k+1)-4}{2(k+1)},$$

and since

$$\frac{2(\delta-1)^2}{(k+1)^2} + 1 - \frac{2}{k+1}$$

is positive, (6) follows (with strict inequality).

Case 2: *G is decomposable.* We use induction on n. By Proposition 4.2, the theorem is true for $n = 2k+2$. Assume that $n > 2k+2$, and (6) holds for $n' < n$. Let A be a k-cut with $k+1 \leqslant |A| \leqslant n-k$ such that for every cut C either $C \subset A$ or $C \subset \widehat{A}$. We replace A by a k-clique and let each of the k edges going into A go to a distinct vertex of the clique. The resulting graph G' is simple, k-connected, and has $n' = n - |A| + k$ vertices. Let $n_1 = |A|$. Set

$$\sigma_1 = |\{C \mid C \subset A, C \text{ is a } k\text{-cut of } G\}|.$$

Then clearly, $\sigma(G) = \sigma_1 + \sigma(G') - k$. Since $n' < n$, it follows by induction that

$$\begin{aligned}
\sigma(G') - k &\leqslant \frac{2}{(k+1)^2}n'^2 + \frac{k-1}{k+1}n' - k \\
&= \frac{2}{(k+1)^2}(n^2 + n_1^2 - 2n_1 n - 2n_1 k + 2nk + k^2) \\
&\quad + \frac{k-1}{k+1}(n - n_1 + k) - k \\
&= \frac{2}{(k+1)^2}n^2 + \frac{k-1}{k+1}n + \frac{2}{(k+1)^2}n_1^2 - \frac{k-1}{k+1}n_1 \\
&\quad - \frac{4}{(k+1)^2}(n_1 n + n_1 k - nk) - \frac{2k}{(k+1)^2}.
\end{aligned}$$

Using a similar counting argument as in Case 1 (details are omitted) we obtain

$$\sigma_1 \leqslant \frac{2}{(k+1)^2}n_1^2 + \frac{k-1}{k+1}n_1.$$

From the upper bounds above we have

$$\begin{aligned}
\frac{2}{(k+1)^2}n^2 + \frac{k-1}{k+1}n - \sigma(G) &= \frac{2}{(k+1)^2}n^2 + \frac{k-1}{k+1}n - (\sigma_1 + \sigma(G') - k) \\
&\geqslant -\frac{4}{(k+1)^2}n_1^2 + \frac{4}{(k+1)^2}(n_1 n + n_1 k - nk) \\
&\quad + \frac{2k}{(k+1)^2}.
\end{aligned} \tag{9}$$

To verify (6) we show that the last line of (9) is nonnegative or equivalently,

$$f(n_1) = n_1^2 - n_1(n+k) + nk - k/2 \leqslant 0.$$

Since $f(k+1) < 0$, and the minimum of $f(x)$ is negative at $x = n + k/2$, the inequality $f(n_1) < 0$ follows from $k + 1 \leqslant n_1 \leqslant n - k$. □

4.2. *Simple graphs with edge-connectivity 3 or 5*

For $k = 3$, extremal multigraphs were characterized in Theorem 3.4. These graphs obviously contain no multiple edges when $n \geqslant 6$ or $n = 4$, and so $\sigma_1(n,3) = \lfloor 3n/2 \rfloor - 2$. It is easy to verify that $\sigma_1(5,3) = 4$.

We now examine the case $k = 5$.

Theorem 4.4.

$$\sigma_1(n,5) = \begin{cases} 6 & \text{for } n = 6 \text{ and } 7, \\ 8 & \text{for } n = 8 \text{ and } 9, \\ \lfloor 3n/2 \rfloor - 4 & \text{for } n \geqslant 10. \end{cases}$$

Proof. Note that in a k-connected graph each vertex has degree at least k, thus any simple 5-connected graph has at least six vertices. By Lemma 4.1, the smallest cardinality of a nontrivial 5-cut is five. Hence a graph of order $n < 10$ can have only trivial 5-cuts, which implies $\sigma_1(n,5) \leqslant n$, for $n = 6$ and 8, and $\sigma_1(n,5) \leqslant n - 1$, for $n = 7$ and $n = 9$.

We exhibit graphs of edge-connectivity 5 showing that these bounds are tight. For $n = 6$, the complete graph K_6 on six vertices satisfies $\sigma(K_6) = 6$. For $n = 7$, the graph G_7 obtained from K_7 by removing three independent edges satisfies $\sigma(G_7) = 6$. For $n = 8$ or $n = 9$, let G_8 be the complement of the chordless cycle on eight vertices and let G_9 be the graph obtained from G_8 by removing four independent edges and adding a ninth vertex adjacent to all the other vertices. Then $\sigma(G_8) = \sigma(G_9) = 8$.

To compute $\sigma_1(n,5)$, for $n \geqslant 10$, we will use the following lemma.

Lemma 4.5. *A multigraph G of edge-connectivity 5 and order $n \geqslant 4$ with the property that $\sigma(G) = \lfloor 3n/2 \rfloor - 2$ contains two nonincident edges of multiplicity at least two.*

Proof. We will proceed by induction on n. It is easy to check that the lemma holds true for $n = 4$ or $n = 5$. Now we assume that the lemma is proved for all graphs with at most $n - 1$ vertices and consider a graph G with n vertices satisfying the hypothesis of the lemma. By Theorem 3.4, G is built via a legal splitting from a 5-connected multigraph H of order $n - 2$ such that $\sigma(H) = \lfloor 3(n-2)/2 \rfloor - 2$. By the induction hypothesis, H must contain two nonincident edges, say ab and cd, of multiplicity at least two. If the splitting leading to G is not at any of the vertices a, b, c, d then ab and cd are nonincident multiple edges in G. So let us assume that a is used in the splitting, i.e., a is replaced by three vertices a_1, a_2, a_3 with two edges between a_1 and

a_2, two edges between a_1 and a_3, and one edge between a_2 and a_3. Now a_1a_2 and cd are the desired pair of edges proving the lemma for G. This completes the proof of the lemma. □

Proof of Theorem 4.4 (Conclusion). For every $n \geqslant 10$, we exhibit 5-connected simple graphs of order n with a number of 5-cuts equal to $\lfloor 3n/2 \rfloor - 4$. For $n = 10 + 2p$, let $\{v_1, \ldots, v_5\}$ and $\{v_6, \ldots, v_{10}\}$ induce two disjoint 5-cliques in G_n, and add the edge v_1v_6. If $p = 0$ (i.e., $n = 10$), add the edges $v_2v_7, v_3v_8, v_4v_9, v_5v_{10}$. If $p > 0$ create two vertices a_i, b_i for each $i = 1, \ldots, p$; add the edges v_2a_1, v_3a_1, v_4b_1, v_5b_1 and v_7a_p, v_8a_p, $v_9b_p, v_{10}b_p$. If $p = 1$ add the edge a_1b_1. If $p > 1$, add edges so that each subset $\{a_i, b_i, a_{i-1}, b_{i+1}\}$ $(i = 1, \ldots, p-1)$ forms a 4-clique. For $n = 10 + 2p + 1$ and $p > 0$, start from the graph G_{10+2p}, subdivide each of the three edges v_2a_1, v_4b_1, v_1v_6 with one vertex and identify these three new vertices. If $p = 0$ $(n = 11)$, do the same operation on the three edges v_1v_6, v_2v_7, v_3v_8 of G_{10}. It is not difficult to check that G_n is a simple graph of edge-connectivity 5 and that $\sigma(G_n) = \lfloor 3n/2 \rfloor - 4$ for all $n \geqslant 10$.

Let G be a simple graph of edge-connectivity 5 and of order $n \geqslant 10$. We will show that

$$\sigma(G) \leqslant \left\lfloor \frac{3n}{2} \right\rfloor - 4. \tag{10}$$

If G contains only 5-cuts of cardinality 1 or $n-1$ then (10) follows with strict inequality. So we may assume that G has a nontrivial 5-cut and choose a minimal such cut A. By Lemma 4.1, we have $5 \leqslant |A| \leqslant n-5$. Let $G' = G/A$ be the graph obtained from G by contracting A into one vertex. It is clear that G' is a 5-connected graph (possibly with multiple edges) of order $n' = n - |A| + 1 \geqslant 6$. By the minimality of A, G has at most $|A| - q + \sigma(G')$ 5-cuts, where q is the number of vertices of A of degree at least six. We can remark that G' has no nonincident edges of multiplicity at least two, and so by the preceding lemma we have $\sigma(G') \leqslant \lfloor 3n'/2 \rfloor - 3$. Thus

$$\sigma(G) = |A| - q + \sigma(G') \leqslant |A| - q + \left\lfloor \frac{3(n - |A| + 1)}{2} \right\rfloor - 3.$$

Let R denote the right-hand side of the above inequality. For $|A| = 5$ it is clear that $R = \lfloor 3n/2 \rfloor - 4 - q$. If $|A| \geqslant 6$, then we get $R \leqslant |A| - q + 3(n - |A| + 1)/2 - 3 = 3n/2 - |A|/2 - 3/2 - q$, whence $R \leqslant \lfloor 3n/2 \rfloor - 4 - q$. In either case (10) follows. This concludes the proof of the theorem. □

The proof of Theorem 4.4 actually implies that, in an extremal simple graph G for $k = 5$, any minimal nontrivial cut A induces a 5-clique of G. Indeed, for $|A| = 5$ it is clear that A must be a 5-clique; if $|A| = 6$ then $q \neq 0$ (see Lemma 3.1) and the inequality (10) is strict; and the same holds for $|A| \geqslant 7$. Furthermore, the graph G' obtained by contracting a 5-clique of G satisfies $\sigma(G') = \lfloor 3n'/2 \rfloor - 3$, so it is extremal among multigraphs of edge-connectivity 5 having a vertex incident to all the multiple edges. So there exist such graphs for any order greater than or equal to six. We will show how these extremal graphs are obtained.

Proposition 4.6. *Let G' be a multigraph of edge-connectivity 5 and order $n' \geqslant 6$ having a vertex incident to all the edges of multiplicity at least two. If $\sigma(G') = \lfloor 3n'/2 \rfloor - 3$, then G' contains a 5-clique. Moreover, the graph G'' obtained from G' by contracting this clique into one vertex is an extremal multigraph for $k = 5$ (i.e., $\sigma(G'') = \lfloor 3n''/2 \rfloor - 2$, where $n'' = n' - 4$ is the order of G'').*

Proof. For the interval representation of the 5-cuts of G' we choose v_0 as the vertex incident to all edges of multiplicity at least two in G'. We consider in G' a minimal nontrivial cut A. (If G' had no such cut then we would have $\sigma(G') \leqslant n' - 1$, which would contradict the assumption.) Since A does not contain v_0, by Lemma 4.1, we have $|A| \geqslant 5$. Let $G'' = G'/A$. Clearly

$$\lfloor 3n'/2 \rfloor - 3 = \sigma(G') \leqslant |A| - q + \sigma(G''),$$

where q is the number of vertices of A of degree at least six. Since G'' is a 5-connected multigraph, the right-hand side R of the inequality above is smaller than or equal to $|A| - q + \lfloor 3(n' - |A| + 1)/2 \rfloor - 2$.

If $|A| = 5$, then

$$R \leqslant \left\lfloor \frac{3(n' - 4)}{2} \right\rfloor + 3 - q = \left\lfloor \frac{3n'}{2} \right\rfloor - 3 - q,$$

furthermore, A is a 5-clique and $\sigma(G'') = \lfloor 3n''/2 \rfloor - 2$. For $|A| \geqslant 6$, it is easy to check that $R < \lfloor 3n'/2 \rfloor - 3$. □

Let us call *special pair* any two vertices of degree 5 such that every edge of multiplicity at least two is incident to at least one of them. Let v, w be a special pair of G. We call *special splitting* on G any splitting on v into vertices v_1, v_2, v_3 such that $m(v_1v_2) = m(v_1v_3) = 2$, $m(v_2v_3) = 1$, and such that v_1, w is a special pair in the resulting graph.

Theorem 4.7. *Let G be a simple graph of edge-connectivity 5 and order $n \geqslant 10$. If G is extremal it is obtained from an extremal multigraph H of edge-connectivity 5 having a special pair by replacing each vertex of the pair by a 5-clique. Moreover H itself is obtained from either $P(5)$ or $Q(5,2)$ through a sequence of special splittings.*

Proof. Note that a graph obtained from a 5-connected graph by replacing a vertex of degree 5 with a 5-clique is also 5-connected.

Let G be a graph satisfying the hypothesis of the theorem. From the proof of Theorem 4.4 and Proposition 4.6 it follows directly that G contains two disjoint 5-cliques and that the multigraph H obtained by contracting each of these cliques into one vertex is an extremal multigraph having a special pair.

Now we show that any extremal multigraph M of edge-connectivity 5 and order $n \geqslant 4$ having a special pair is obtained via a special splitting from an extremal multigraph M' of edge-connectivity 5 and order $n-2$. Indeed let us consider such a multigraph M, where v, w is a special pair. By Theorem 3.4, we know that M is obtained from an extremal multigraph M' of edge-connectivity 5 and order $n-2$ through a legal splitting at a vertex x, which is replaced by three vertices x_1, x_2, x_3 with edges x_1x_2, x_1x_3, x_2x_3 of multiplicity 2, 2, and 1, respectively.

Since v and w together are incident to all multiple edges in M, it is clear that one of v, w must be equal to x_1, x_2 or x_3. Furthermore, only one of them is among x_1, x_2, x_3, for otherwise, M' would be an extremal multigraph with one vertex incident to all multiple edges. This would contradict Lemma 4.5 in the case $n' = n-2 \geqslant 4$. When $n' = 2$ or $n' = 3$, v and w could both be among x_1, x_2, x_3 only if $n' = 2$, but in this particular case it is easy to see that another choice for v, w in M gives the desired result.

So we can assume without loss of generality that v is one of x_1, x_2, x_3. Actually since v and w are adjacent to all multiple edges of M it must be that $v = x_1$. Now w and x are clearly incident to all the multiple edges in M'.

To finish the proof we just remark that if the graph $Q(5,1)$ is used as the starting graph of a special splitting, the resulting multigraph cannot have a special pair. Hence only $P(5)$ and $Q(5,2)$ can be used as the starting graph of the sequence of special splittings leading to M. □

4.3. *Bounds on $\sigma_1(n,k)$ for odd $k \geqslant 7$*

We will use the following result.

Lemma 4.8. *Let G be a simple graph of edge-connectivity $k \geqslant 7$ (k odd or even), and let A and B be distinct k-cuts of G with $B \subset A$. Then either $|A \setminus B| \leqslant 2$ or $|A \setminus B| \geqslant k-1$.*

Proof. Clearly, $d_G(x) \geqslant k$ for every $x \in A \setminus B$. Set $t = |A \setminus B|$. By counting the edges that go out of $A \setminus B$, and using that A and B are k-cuts, we obtain

$$\begin{aligned} tk &\leqslant t(t-1) + m_G(A \setminus B, \widehat{A}) + m_G(A \setminus B, B) \\ &\leqslant t(t-1) + m_G(A, \widehat{A}) + m_G(B, \widehat{B}) = t(t-1) + 2k. \end{aligned}$$

It is easy to see that the resulting inequality $t^2 - t(k+1) + 2k \geqslant 0$, under the condition $k \geqslant 7$, is satisfied only if $t \leqslant 2$ or $t \geqslant k-1$, which concludes the proof of the lemma. □

The construction in this subsection shows that

$$\sigma_1(n,k) \geqslant \left(1 + \frac{2}{k+1}\right) n - \mathrm{O}(1),$$

for every odd $k \geqslant 3$. Let H be a 3-connected, 3-regular simple graph of order $2N$ with $3N-2$ minimum cuts as in Theorem 3.4. We choose H such that it has a maximal

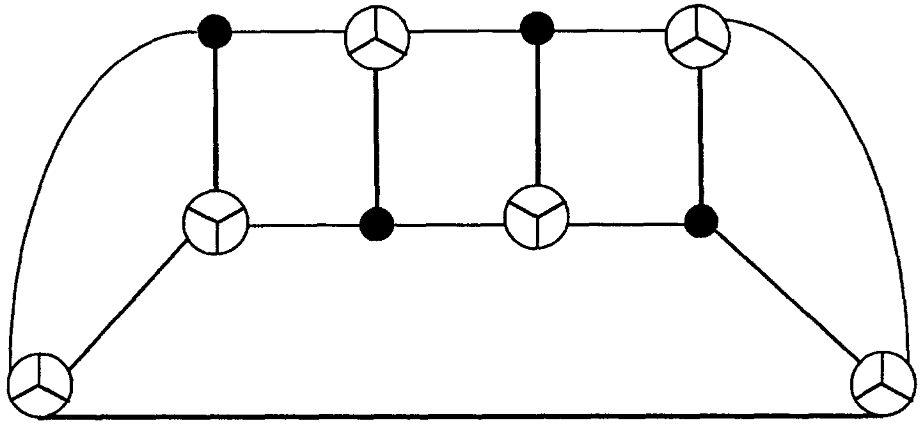

Fig. 1.

independent set S containing $N-1$ vertices (see Fig. 1 for $N=5$). Let F be a perfect matching of H. (It is not difficult to check that F exists; actually the graphs in Theorem 3.4 are Hamiltonian.) To obtain a k-regular graph G first replace every vertex $u \in V(H) \setminus S$ with a k-clique $K(u)$, then partition each such clique into equal (or almost equal) subsets (see Fig. 1).

For $k=3t$, we redefine the three edges of H incident to any $u \in V \setminus S$ as follows. If $x \in S$ then edge ux is replaced with t edges going from x to a t-subset of $K(u)$. If $x \notin S$ then ux is replaced with t independent edges between a t-subset of $K(u)$ and a t-subset of $K(x)$. To get a k-regular graph each t-subset of a k-clique is used just once.

For $k=3t \pm 2$, first we replace the edges of F with t-stars and t independent edges as in the previous case, then the edges not in F are replaced with $(t \pm 1)$-stars and $t \pm 1$ independent edges.

It is easy to check that the resulting graph G is a k-connected k-regular simple graph with $n = N-1+k(N+1) = (k+1)N+k-1$ vertices. Furthermore,

$$\sigma(G) = (3N-2)+k(N+1) = n+2N-1 = \left(1+\frac{2}{k+1}\right)n - \mathrm{O}(1).$$

Hence we have a lower bound on the value of $\sigma_1(n,k)$. Now we give an upper bound.

Theorem 4.9. *If k is odd and $k \geqslant 7$ then*

$$\sigma_1(n,k) \leqslant \left(1+\frac{4}{k+5}\right)n.$$

Proof. Let G be a simple graph of order n and edge-connectivity k, with $\sigma(G) = \sigma_1(n,k)$.

First recall that G has a vertex of degree k. (The proof of this fact is the same as in Theorem 4.3, Case 1.)

Now let v_0 be a vertex of degree k in G. By Theorem 2.4 we can find an ordering $v_1, \ldots, v_{n-1}$ on $V-\{v_0\}$ such that the k-cuts of G form a nested family of intervals in this ordering. Notice that $V-\{v_0\}$ itself is a k-cut. Let T be the rooted tree representing the Hasse diagram of the inclusion relation on the nested family formed by all the k-cuts and all the $\{v_i\}$'s ($i = 1,\dots,n-1$). More precisely, each $\{v_i\}$ is a leaf of T, the root of T is $V-\{v_0\}$, and the parent of any nonroot node X of T is the smallest k-cut in which X is strictly included.

Let $s(T)$ be the number of internal nodes of T, i.e., nodes that are not leaves. Let r be the number of vertices of G of degree at least $k+1$. So we have

$$\sigma(G) = n - 1 - r + s(T).$$

We want to find an upper bound on the value of $s(T) - r$.

We will say that a subtree S of T is *full* if, for each node X in S, either all or none of the children of X in T are in S. Given a full subtree S of T, a leaf X of S is called *special* if it is not a leaf of T. Now we will modify T so as to obtain a tree T_0 with $n-1$ leaves such that:

(t1) every interior node of T_0 has at least three children;

(t2) if all children of a node are leaves then it has at least k children;

(t3) every full subtree of T_0 with exactly one special leaf has either at most three or at least k leaves;

(t4) $s(T_0) \geqslant s(T) - r$.

Lemmas 4.1 and 4.8 obviously imply that T satisfies (t2) and (t3). Thus, when constructing T_0, our goal is to preserve these properties and to obtain (t1) and (t4).

Observe that the root $V-\{v_0\}$ of T has at least three children (for otherwise one of the two children should be a singleton $\{z\}$ where z has degree at least $k+1$ in G, and the other $V-\{v_0, z\}$ would be a k-cut of G, a contradiction to Lemma 4.1). If every interior node of T has at least three children then it suffices to take $T_0 = T$. Now assume that T has an interior node Y with only two children. Since Y is not the root, it has a parent U. By Lemma 4.1 at least one child of Y is not a singleton (hence a nontrivial k-cut). By Lemma 3.1 one child of Y is not a k-cut, hence is a singleton $\{y\}$ where y has degree at least $k+1$. We contract the edge YU, in other words we delete Y and append its children to U. Note that U has at least three children after the contraction; in fact the nodes with two children in the contracted tree have exactly the same two children as in T. We iterate this procedure until the tree has no more vertex with only two children. It is not difficult to check that the resulting contracted tree T_0 has the four desired properties. (To verify the last one, notice that each contraction from a node Y corresponds to one vertex y of degree at least $k+1$, hence the number of contracted edges is at most r.)

We now prove that

$$s(T_0) \leqslant \frac{4n}{k+5}$$

for all trees T_0 having $n-1$ leaves and satisfying properties (t1)–(t3), with $k \geqslant 7$ and $n \geqslant k+1$. Remark that it is true when T_0 is a star, i.e., when all leaves are children of the root, since then $s(T_0) = 1$. To prove it in general we proceed by induction on n. If $n = k+1$, then by (t2) T_0 must be a star and we are done. So we may now assume that $n \geqslant k+2$ and also that T_0 is not a star. Let x_0 be an interior node of T_0 farthest from the root; and let x_1 be the parent of x_0. Let q_1 be the number of children of x_1 that are not leaves. We will distinguish between several cases.

First, assume that $q_1 \geqslant 2$. We build a new tree T_0' from T_0 by removing every descendant of x_1 and adding k leaves at x_1. Clearly the number $n'-1$ of leaves of T_0' is such that

$$n' - 1 \leqslant n - 1 - (q_1 - 1)k$$

and the number of interior nodes of T_0' is $s' = s(T_0) - q_1$. It is not difficult to check that properties (t1)–(t3) are satisfied by T_0'; moreover $n' < n$. So we can apply the induction hypothesis on T_0', which yields $s' \leqslant 4n'/(k+5)$, whence

$$s(T_0) \leqslant \frac{4n}{k+5} - \frac{3q_1k - 4k - 5q_1}{k+5}.$$

Then, since $q_1 \geqslant 2$,

$$k > 5 \geqslant \frac{5q_1}{3q_1 - 4} > 0$$

is true, and $s(T_0) \leqslant 4n/(k+5)$ follows easily.

Second, assume that $q_1 = 1$ and that x_1 has at least k children. We build a new tree T_0' from T_0 by removing all children of x_0. The number $n'-1$ of leaves of T_0' satisfies

$$n' - 1 \leqslant n - k$$

and the number of interior nodes is $s' = s(T_0) - 1$. Again it is not difficult to check that T_0' satisfies properties (t1)–(t3) and $n' < n$. Applying the induction hypothesis on T_0' yields $s' \leqslant 4n'/(k+5)$ from which $s(T_0) \leqslant 4n/(k+5)$ is easily derived.

Now assume that $q_1 = 1$ and x_1 has at most $k-1$ children. Property (t3) on the full subtree formed by x_1 and its children implies that x_1 has at most three children. By (t1) the node x_1 has exactly three children, which are x_0 and two leaves of T_0. Actually we can assume that this is the case for the parent of every interior node farthest from the root. If x_1 is the root then it is easy to see that $s(T_0) = 2$ and $n - 1 \geqslant k+2$, so $s(T_0) \leqslant 4n/(k+5)$. Now let x_2 be the parent of x_1 and q_2 be the number of children of x_2 that are not leaves.

Assume for now that $q_2 = 1$, i.e., x_1 is the only nonleaf child of x_2. Property (t3) on the full subtree formed by x_2, its children and the children of x_1 (where x_0 is the unique special leaf) implies that x_2 has at least $k-3$ children different from x_1. We build a new tree T_0' from T_0 by removing all the descendants of x_2 and adding k leaves at x_2. It is not difficult to check that T_0' satisfies properties (t1)–(t3). Moreover its number of interior nodes is $s' = s(T_0) - 2$, and its number of leaves is $n'-1 \leqslant n-k+1$. Applying the induction hypothesis on T_0' we get $s' \leqslant 4n'/(k+5)$, from which $s \leqslant 4n/(k+5)$ follows easily.

Now assume that $q_2 \geqslant 2$. Let p_1 be the number of children of x_2 whose children are all leaves of T_0; let $p_2 = q_2 - p_1$. So p_2 is the number of children of x_2 that are of the same type as x_1. We build a new tree T_0' from T_0 by removing all the descendants of x_2 and adding k leaves at x_2. It is not difficult to check that T_0' satisfies properties (t1)–(t3). The number of interior nodes of T_0' is $s' = s(T_0) - 2p_2 - p_1$, and the number of its leaves is $n' - 1 \leqslant n - 1 - p_2(k+2) - p_1 k + k$. Applying the induction hypothesis on T_0' we get $s' \leqslant 4n'/(k+5)$, from which

$$s(T_0) \leqslant \frac{4n}{k+5} - \frac{k(2p_2 + 3p_1 - 4) - 2p_2 - 5p_1}{k+5}$$

follows easily. We want to check that the term following the minus sign in the preceding inequality is nonnegative or, equivalently, that

$$p_2(2k-2) \geqslant p_1(5-3k) + 4k.$$

Notice that the left-hand side of this inequality is always nonnegative, and the right-hand side is negative whenever $p_1 \geqslant 2$. If $p_1 = 1$ then $p_2 \geqslant 1$, since $q_2 \geqslant 2$, and in that case the desired inequality is also true. If $p_1 = 0$ the desired inequality fails only if $p_2 = 2$. In that case we observe that, by (t1), the node x_2 must have a third child which is a leaf. Consequently the number of leaves of T_0' can be estimated more tightly as

$$n' - 1 \leqslant n - 1 - 2(k+2) + k - 1.$$

Now the inequality $s' \leqslant 4n'/(k+5)$ directly becomes $s(T_0) \leqslant 4n/(k+5)$. □

We now exhibit graphs which show that the bound obtained in Theorem 4.9 is sharp when $k = 7$ and $k = 9$. These graphs will be built using a recursive construction that we call *i-box* and that we now explain for $k = 7$. A 0-box is a clique with seven vertices, and at each vertex there is an incident edge hanging out of the box. These seven edges are divided into two batches of three plus a “solitary” edge. Given two vertex-disjoint i-boxes X and Y with seven edges hanging out of each of them, each of these two sets of seven edges being divided into two batches of three plus a solitary edge, we obtain an $(i+1)$-box Z as follows.

- Add five vertices x_1, x_2, y_1, y_2 and z. Add edges x_1x_2, y_1y_2, x_1y_1, as well as x_1z, x_2z, y_1z, y_2z.
- Connect the first batch of edges hanging out of X (resp. Y) to x_1 (resp. y_1), and the second batch to x_2 (resp. y_2). The solitary edges hanging out of X and Y are connected to z.
- At each of x_1, x_2, y_2 add one incident edge hanging out of Z; these three new edges will form the first batch of Z. At each of y_1, y_2, x_2 add one incident edge hanging out of Z; these three edges will form the second batch of Z. At z add one new incident edge hanging out of Z, which will be the solitary edge of Z.

We obtain a simple graph G_i by taking the i-box, adding a 7-clique and connecting each vertex of that clique to one edge hanging out of the i-box. By contracting each 0-box of G_i into one vertex one gets a multigraph which can be obtained from $P(7)$ through a sequence of legal 7-splittings, thus G_i is 7-connected. Moreover G_i has $n_i = 3\cdot 2^{i+2} + 2$ vertices and a number of 7-cuts equal to $\sigma(G_i) = n_i + 2^{i+2} - 3$ (this is because, given an $(i+1)$-box Z with the above notation, each of X, $X \cup \{x_1, x_2\}$, Y, $Y \cup \{y_1, y_2\}$ and Z form a 7-cut; moreover every vertex of G_i is of degree 7). So $\sigma(G_i) = 4n_i/3 - o(n_i)$, which is asymptotically equal to the upper bound in Theorem 4.9 when $k = 7$.

For $k = 9$ the construction of the i-boxes is slightly different. The 0-box is a 9-clique. The nine edges hanging out of an i-box are divided into two batches of four plus one solitary edge. The construction of the $(i+1)$-box is as above except that we also add the edges x_1y_2 and x_2y_1, and we also add two new edges incident at z and hanging out of Z; one of them is included in the first batch and the other one in the second batch of Z. We obtain a simple graph G_i' by taking an i-box for $k = 9$, adding a 9-clique and connecting each vertex of that clique to one edge hanging out of the i-box. Again it is not very difficult to check as above that G_i' is a 9-connected graph with $n_i' = 7 \cdot 2^{i+1} + 4$ vertices and a number of 9-cuts equal to $n_i' + 2^{i+2} - 3$, achieving equality asymptotically in Theorem 4.9 when $k = 9$.

For larger values of k we could neither generalize the idea of i-boxes nor find any construction that would imply equality in Theorem 4.9. In fact we conjecture that there exists a function $f(k)$ such that, for every odd $k \geqslant 11$ we have $\sigma_1(n,k) = f(k) \cdot n \pm o(n)$, and with $f(k) < 1 + 4/(k+5)$; but $f(k)$ seems to be very difficult to calculate precisely.

Acknowledgements

We are grateful to the referees for their careful reading of the paper, for making several interesting remarks, and for correcting a few errors.

The first author thanks the support and the hospitality of Laboratoire ARTEMIS during his stay at the University of Grenoble in 1991.

References

[1] R.E. Bixby, The minimum number of edges and vertices in a graph with edge connectivity n and mn-bonds, Networks 5 (1975) 253–298.

[2] E.A. Dinitz, A.V. Karzanov and M.V. Lomonosov, A structure of all the minimum cuts of a graph, in: Studies in Discrete Optimization (Nauka, Moscow, 1976) 290–306 (in Russian).

[3] P. Duchet, Propriété de Helly et problèmes de représentations, problèmes combinatoires et théorie des graphes, in: Proceedings Int. Vol. CNRS 260 (Orsay, 1976), Paris (1978) 117–118.

[4] H.N. Gabow, Applications of a poset representation to edge connectivity and graph rigidity, Report CU-CS-545-91, University of Colorado at Boulder (1991).

[5] A.V. Karzanov and E.A. Timofeev, Efficient algorithm for finding all minimal edge cuts of a non-oriented graph, Kibernetika 2 (1986) 156–162.
[6] J. Lehel, Helly-hypergraphs and abstract interval structures, Ars Combin. 16-A (1983) 239–253.
[7] H. Nagamochi, Z. Sun and T. Ibaraki, Counting the number of minimum cuts in multiple undirected graphs, IEEE Trans. Reliability 40 (1991) 610–614.
[8] A. Tucker, A structure theorem for the consecutive ones property, J. Combin. Theory 12 (1972) 153–162.