Applied Mathematics Letters 23 (2010) 676–680

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations☆

Jiqin Deng *, Lifeng Ma

School of Mathematics and Computational Science, Xiangtan University, Hunan 411105, PR China

ARTICLE INFO

Article history:
Received 17 July 2009
Received in revised form 29 January 2010
Accepted 11 February 2010

Keywords:
Caputo fractional derivative
Fractional integral
Existence and uniqueness
Banach contraction principle

ABSTRACT

In this paper, by using the fixed point theory, we study the existence and uniqueness of initial value problems for nonlinear fractional differential equations and obtain a new result.

1. Introduction

In this paper, we consider the existence and uniqueness of solutions of the following initial value problems:

$$D^{\alpha}x(t) = f(t, D^{\beta}x(t)), \quad 1 \geq t > 0, \tag{1.1}$$

$$x^{(k)}(0) = \eta_k, \quad k = 0, 1, \dots, m-1 \tag{1.2}$$

where $m-1 < \alpha < m, n-1 < \beta < n$ $(m, n \in \mathbf{N}, m-1 \geq n)$, D^{α} are the αth Caputo fractional derivatives and $f \in C([0, 1] \times \mathbf{R})$.

The initial and boundary value problems for nonlinear fractional differential equations arise from the study of models of viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc (see [1–3]). Therefore, they have received much attention. For the most recent works for the existence and uniqueness of solutions of the initial and boundary value problems for nonlinear fractional differential equations, we mention [4–15,1–3,16]. But in the obtained results, for the existence, the nonlinear term f needs to satisfy the condition: there exist functions $p, r \in C([0, 1], [0, \infty))$ such that for $1 \geq t \geq 0$ and each $u \in \mathbf{R}$,

$$|f(t, u)| \leq p(t)|u| + r(t) \tag{1.3}$$

and for the uniqueness, the nonlinear term f needs to satisfy the condition: there exist functions $p,\ r \in C([0, 1], [0, \infty))$ such that for each $1 \geq t \geq 0$ and any $u, v \in \mathbf{R}$,

$$|f(t, u) - f(t, v)| \leq p(t)|u - v| \tag{1.4}$$

such that by using these results, we cannot discuss the existence and uniqueness of solutions of the following simple problems.

☆ Supported by Research Fund of Hunan Provincial Education Department (No. 08A070).

* Corresponding author.
E-mail address: dengjiqin@yahoo.com.cn (J. Deng).

0893-9659/$ – see front matter
doi:10.1016/j.aml.2010.02.007

Example 1.1. Consider the problem

$$D^{3/4}x(t) = \frac{t - [D^{1/4}x(t)]^3}{25(1+t)^4}, \quad 1 \geq t > 0, \tag{1.5}$$

$$x(0) = \frac{1}{4}. \tag{1.6}$$

In this paper, our object is to improve the situation. Our main result is the set of the following theorems.

Theorem 1.1. *Let $n-1 < \beta < \alpha < n$ ($n \in \mathbf{N}$). Assume that*

(H_1) *$f : [0, 1] \times \mathbf{R} \to \mathbf{R}$ is a continuously differentiable function;*
(H_2) *$f(0, 0) = 0$ and $f(t, 0) \neq 0$ on a compact subinterval of $(0, 1]$ and*
(H_3) *there exist $l > 1$, $1 > \gamma > 0$ and $a(t) \in C([0, 1], [0, \infty))$ such that*

$$\frac{1}{\Gamma(\alpha-\beta)} \sup_{0\leq t\leq 1} \int_0^t (t-s)^{\alpha-\beta-1} a(s) \mathrm{d}s \leq 1 - \gamma, \tag{1.7}$$

$$0 < R := \frac{1}{\Gamma(\alpha-\beta)} \sup_{0\leq t\leq 1} \int_0^t (t-s)^{\alpha-\beta-1} |f(s, 0)| \mathrm{d}s < \infty \tag{1.8}$$

and for any $x, y \in C([0, \infty))$ with $0 \leq |x(t)|, |y(t)| \leq \frac{l}{\gamma}R$ for $1 \geq t \geq 0$,

$$|f(t, x(t)) - f(t, y(t))| \leq a(t)|x(t) - y(t)| \quad \text{for } 1 \geq t \geq 0. \tag{1.9}$$

Then (1.1)–(1.2) has a unique solution.

Theorem 1.2. *Let $n-1 < \beta < n \leq m-1 < \alpha < m$ ($n, m \in \mathbf{N}$). Assume that* (H_1) *and* (H_2) *are satisfied and*

(H_4) *there exist $l > 1$, $1 > \gamma > 0$ and $a(t) \in C([0, 1], [0, \infty))$ such that*

$$\frac{1}{\Gamma(\alpha-n)\Gamma(n-\beta+1)} \sup_{0\leq t\leq 1} \int_0^t (t-s)^{\alpha-n-1} s^{n-\beta} a(s) \mathrm{d}s \leq 1 - \gamma, \tag{1.10}$$

$$0 < R := \sup_{0\leq t\leq 1} \left[\left| \sum_{k=0}^{n-1} \frac{t^k}{k!} \eta_k \right| + \int_0^t (t-s)^{\alpha-n-1} |f(s, 0)| \mathrm{d}s \right] < \infty \tag{1.11}$$

and for any $x, y \in C([0, \infty))$ with $0 \leq |x(t)|, |y(t)| \leq \frac{l}{\gamma}R$ for $1 \geq t \geq 0$,

$$|f(t, x(t)) - f(t, y(t))| \leq a(t)|x(t) - y(t)| \quad \text{for } 1 \geq t \geq 0. \tag{1.12}$$

Then (1.1)–(1.2) has a unique solution.

It is obvious that if Theorem 1.1 holds, then it is easy to see that taking $a(t) = \frac{1}{5(1+t)^4}$, $\gamma = 1/2$ and $l = 2$, the problem (1.5)–(1.6) has a unique solution.

Our work is motivated by the work of [16].

2. Preliminaries

Definition 2.1 (*[16]*). The fractional order integral of order α for the function x is defined as

$$I^\alpha x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} x(s) \mathrm{d}s, \quad 1 \geq t \geq 0$$

where $m-1 < \alpha < m$ and $m \in \mathbf{N}$.

Definition 2.2 (*[12]*). The αth Caputo derivative of x is defined as

$$D^\alpha x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} x^{(m)}(s) \mathrm{d}s, \quad 1 \geq t \geq 0$$

where $m-1 < \alpha < m$ and $m \in \mathbf{N}$.

Definition 2.3 (*[12]*). $u \in C^m[0, 1]$ is called a solution of (1.1)–(1.2) if it satisfies (1.1) and (1.2).

Lemma 2.1 *([15]). Let $m \in \mathbf{N}$, $m-1<\alpha<m$, $u \in C^m[0,1]$ and $v \in C^1[0,1]$, then for $0 \le t \le 1$,*

(2.1.1) $D^\alpha I^\alpha v(t) = v(t)$;
(2.1.2) $I^\alpha D^\alpha u(t) = u(t) - \sum_{i=0}^{m-1} \frac{t^k}{k!} u^{(k)}(0)$;
(2.1.3) $\lim_{t\to 0^+} D^\alpha u(t) = \lim_{t\to 0^+} I^\alpha u(t) = 0$;
(2.1.4) *If $0<\alpha_i \le 1\,(i=1,2,\ldots,n)$ with $\alpha = \sum_{i=1}^n \alpha_i$ such that for each $k \in \{1,\ldots,m-1\}$, there exists $i_k < n$ with $k = \sum_{i=1}^{i_k} \alpha_i$, then*

$$D^\alpha x(t) = D^{\alpha_n} \cdots D^{\alpha_2} D^{\alpha_1} x(t).$$

Lemma 2.2 *([16]). Let $m \in \mathbf{N}$, $m-1<\beta<\alpha<m$, $u \in C^m[0,1]$. Then, for any $k \in \{1,\ldots,m-1\}$ and $0 \le t \le 1$,*

$$D^{\alpha-m+k} x^{(m-k)}(t) = D^\alpha x(t)$$

and

$$D^{\alpha-\beta} D^\beta x(t) = D^\alpha x(t).$$

Lemma 2.3 *([16]). Let $n, m \in \mathbf{N}$, $n-1<\beta<n\le m-1<\alpha<m$ and assume that (H_1) and (H_2) hold. Then, $u \in C^m[0,1]$ is a solution of (1.1)–(1.2) if and only if*

$$u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}\eta_k + \int_0^t \frac{(t-s)^{n-1} v(s)\mathrm{d}s}{(n-1)!}, \quad 0 \le t \le 1$$

where $v \in C[0,1]$ is a solution of the equation

$$v(t) = \sum_{k=0}^{m-n-1} \frac{t^k}{k!}\eta_{n+k} + \frac{1}{\Gamma(\alpha-n)} \int_0^t (t-s)^{\alpha-n-1} f\left(s, \frac{1}{\Gamma(n-\beta)} \int_0^s (s-h)^{n-\beta-1} v(h)\mathrm{d}h\right)\mathrm{d}s, \quad 0 \le t \le 1. \tag{2.1}$$

Lemma 2.4 *([16]). Let $n \in \mathbf{N}$, $n-1<\beta<\alpha<n$ and assume that (H_1) and (H_2) hold. Then, $u \in C^n[0,1]$ is a solution of (1.1)–(1.2) if and only if*

$$u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}\eta_k + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} v(s)\mathrm{d}s, \quad 0 \le t \le 1$$

where $v \in C[0,1]$ is a solution of the equation

$$v(t) = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} f(s, v(s))\mathrm{d}s, \quad 0 \le t \le 1. \tag{2.2}$$

3. The proof of main results

We divide the proof of Theorem 1.1 into several steps.
First, we have the following lemma.

Lemma 3.1. *If (H_1)–(H_3) holds, then (2.2) has a unique solution ψ with $\|\psi\| \le lR/\gamma$.*

Proof of Lemma 3.1. Let X be a linear space consisting of all functions $x \in C[0,1]$ such that

$$\sup_{1\ge t\ge 0} |x(t)| < +\infty$$

with the norm

$$\|x\| = \sup_{1\ge t\ge 0} |x(t)|.$$

It follows that X is Banach space.
Define a bounded, convex and closed subset B of X and the operator T as follows:

$$B = \left\{x \in X : |x(t)| \le \frac{lR}{\gamma} \text{ for } 1 \ge t \ge 0\right\} \tag{3.1}$$

and

$$Tx(t) = \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} f(s, x(s))\mathrm{d}s, \quad 1 \geq t \geq 0. \tag{3.2}$$

Then, from (1.7), (1.8) and (3.1)–(3.2), we have that for each $x \in B$,

$$\begin{aligned} |Tx(t)| &= \frac{1}{\Gamma(\alpha-\beta)}\left|\int_0^t (t-s)^{\alpha-\beta-1} f(s, x(s))\mathrm{d}s\right| \\ &\leq \frac{1}{\Gamma(\alpha-\beta)}\left[\left|\int_0^t (t-s)^{\alpha-\beta-1} f(s, 0)\mathrm{d}s\right| + \int_0^t (t-s)^{\alpha-\beta-1}|f(s, x(s)) - f(s, 0)|\mathrm{d}s\right] \\ &\leq R + \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} a(s)|x(s)|\mathrm{d}s \\ &\leq R + (1-\gamma)\frac{lR}{\gamma} = \frac{lR}{\gamma} - (l-1)R, \quad 1 \geq t \geq 0 \end{aligned} \tag{3.3}$$

which yields that

$$T : B \to B, \tag{3.4}$$

and for any $x, y \in B$, from (1.7), (1.9) and (3.1), we have

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq \frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|f(s, x(s)) - f(s, y(s))|\mathrm{d}s \\ &\leq \|x-y\|\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} a(s)\mathrm{d}h\mathrm{d}s \leq (1-\gamma)\|x-y\| \end{aligned}$$

which, together with (3.4), yields that $T : B \to B$ is a contraction. Therefore, from the Banach fixed point theorem, it is easy to see that in B, T has a unique fixed point ψ. The proof is completed. □

Next, we have the following lemma.

Lemma 3.2. *If* (H_1)–(H_3) *holds and* $x(t)$ *is a solution of* (2.2), *then* $x(t) \equiv \psi(t)$ *where* $\psi(t)$ *is as in* Lemma 3.1.

Proof of Lemma 3.2. First, from (3.3), it is easy to see that

$$|\psi(t)| \leq \frac{lR}{\gamma} - (l-1)R, \quad 1 \geq t \geq 0. \tag{3.5}$$

Set

$$T = \inf\left\{t : 1 \geq t \geq 0 \text{ and } |x(t)| > \frac{lR}{\gamma}\right\}.$$

Then, (i) $\left\{t : 1 \geq t \geq 0 \text{ and } |x(t)| > \frac{lR}{\gamma}\right\}$ is empty or (ii) $T = 1$ or (iii) $1 > T > 0$.

If (i) or (ii) holds, then, $|x(t)| \leq \frac{lR}{\gamma}$ for $1 \geq t \geq 0$ and from (1.7) and (1.9), we have

$$\begin{aligned} \|x-\psi\| &\leq \left\|\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}|f(s, x(s)) - f(s, \psi(s))|\mathrm{d}s\right\| \\ &\leq \|x-\psi\|\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1} a(s)\mathrm{d}s \\ &\leq (1-\gamma)\|x-\psi\| \end{aligned}$$

which yields that

$$x(t) = \psi(t) \quad \text{for } 1 \geq t \geq 0. \tag{3.6}$$

If (iii) holds, then, $|x(T)| = \frac{lR}{\gamma}$, $|x(t)| \leq \frac{lR}{\gamma}$ for $T \geq t \geq 0$ and from (1.7) and (1.9), we have

$$\begin{aligned} \sup_{0\leq t\leq T}|x(t) - \psi(t)| &\leq \frac{1}{\Gamma(\alpha-\beta)}\sup_{0\leq t\leq T}\int_0^t (t-s)^{\alpha-\beta-1}|f(s, x(s)) - f(s, \psi(s))|\mathrm{d}s \\ &\leq \frac{1}{\Gamma(\alpha-\beta)}\sup_{0\leq t\leq T}\int_0^t (t-s)^{\alpha-\beta-1} a(s)|x(s) - \psi(s)|\mathrm{d}s \\ &\leq (1-\gamma)\sup_{0\leq t\leq T}|x(t) - \psi(t)| \end{aligned}$$

which yields that

$$x(t) = \psi(t) \quad \text{for } T \geq t \geq 0$$

which, together with (3.5), yields that

$$(l-1)R \leq |x(T) - \psi(T)| \leq \frac{1}{\Gamma(\alpha-\beta)} \int_0^T (T-s)^{\alpha-\beta-1} |f(s,\ x(s)) - f(s,\ \psi(s))| \mathrm{d}s = 0$$

which yields that (i) or (ii) holds (note (3.6)). The proof is completed. □

Finally, from Lemmas 2.4, 3.1 and 3.2, it is easy to see that Theorem 1.1 holds.
Similar to Theorem 1.1, we can prove that Theorem 1.2 holds.

References

[1] W. Lin, Global existence theory and chaos control of fractional differential equations, J. Math. Anal. Appl. 332 (2007) 709–726.
[2] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.
[3] K. Diethelm, A.D. Freed, On the solution of nonlinear fractional order differential equations used in the modelling of viscoplasticity, in: F. Keil, W. Mackens, H. Voss (Eds.), Scientific Computing in Chemical Engineering II—Computational Fluid Dynamics and Molecular Properties, Springer-Verlag, Heidelberg, 1999, pp. 217–224.
[4] V. Lakshmikantham, Theory of fractional functional differential equations, Nonlinear Anal. 69 (2008) 3337–3343.
[5] V. Lakshmikantham, A.S. Vatsala, Basic theory of fractional differential equations, Nonlinear Anal. 69 (2008) 2677–2682.
[6] V. Lakshmikantham, A.S. Vatsala, General uniqueness and monotone iterative technique for fractional differential equations, Appl. Math. Lett. 21 (2008) 828–834.
[7] M. Benchohra, J. Henderson, S.K. Ntoyuas, A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, J. Math. Anal. Appl. 338 (2008) 1340–1350.
[8] M. Caputo, Linear models of dissipation whose Q is almost frequency independent (Part II), Geophys. J. R. Astron. Soc. 13 (1967) 529–539.
[9] V. Daftardar-Gejji, H. Jaffari, Analysis of a system of nonautonomous fractional differential equations involving Caputo derivatives, J. Math. Anal. Appl. 328 (2007) 1026–1033.
[10] D. Delbosco, L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, J. Math. Anal. Appl. 204 (1996) 609–625.
[11] W.G. El-Sayed, A.M.A. El-Sayed, On the functional integral equations of mixed type and integro-differential equations of fractional orders, Appl. Math. Comput. 154 (2004) 461–467.
[12] A.A. Kilbas, S.A. Marzan, Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions, Differ. Uravn. 41 (1) (2005) 82–86 (in Russian).
[13] B. Bonilla, M. Rivero, L. Rodriguez-Germa, J.J. Trujillo, Fractional differential equations as alternative models to nonlinear differential equations, Appl. Math. Comput. 187 (2007) 79–88.
[14] O.K. Jaradat, A. Al-Omari, S. Momani, Existence of the mild solution for fractional semilinear initial value problem, Nonlinear Anal. 69 (2008) 3153–3159.
[15] C. Li, W. Deng, Remarks on fractional derivatives, Appl. Math. Comput. 187 (2007) 777–784.
[16] N. Kosmatov, Integral equations and initial value problems for nonlinear differential equations of fractional order, Nonlinear Anal. 70 (2009) 2521–2529.