KKM-type theorems for best proximity points

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ABSTRACT

Let us consider two nonempty subsets $A, B$ of a normed linear space $X$, and let us denote by $2^B$ the set of all subsets of $B$. We introduce a new class of multivalued mappings \( \{ T : A \rightarrow 2^B \} \), called R-KKM mappings, which extends the notion of KKM mappings. First, we discuss some sufficient conditions for which the set $\bigcap\{ T(x) : x \in A \}$ is nonempty. Using this nonempty intersection theorem, we attempt to prove a extended version of the Fan–Browder multivalued fixed point theorem, in a normed linear space setting, by providing an existence of a best proximity point.

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1. Introduction

The celebrated topological fixed point theorem due to Brouwer states that every continuous function $f : K \rightarrow K$, where $K$ is a closed bounded convex subset of $\mathbb{R}^n$, has at least one $x \in K$ such that $f(x) = x$. This fixed point theorem has many proofs using multivariable calculus, degree theory, and algebraic topology and combinatorial techniques. Several proofs of Brouwer’s theorem can be found in [1,2], and some interesting applications can be found in [3].

Let $X$ be a vector space and $K \subseteq X$ be an arbitrary subset. A multivalued mapping $T : K \rightarrow 2^X$ is said to be a KKM map if co\{ $x_1, \ldots, x_n$ $\}$ $\subseteq \bigcup_{i=1}^n T(x_i)$, for each finite subset $\{ x_1, \ldots, x_n \} \subseteq K$, where co\{ $x_1, \ldots, x_n$ $\}$ denotes the convex hull of $\{ x_1, \ldots, x_n \}$.

In 1929, Knaster et al. [4] proved the following geometric result.

Let $K$ be the set of vertices of an $n$-dimensional simplex $\Delta_n$ in $X = \mathbb{R}^n$ and $T : K \rightarrow 2^X$ be a KKM map with $T(x)$ being compact, for each $x \in K$. Then $\cap\{ T(x) : x \in X \} \neq \emptyset$. The above geometric result is equivalent to the Brouwers fixed point theorem and also equivalent to Sperner's lemma (see [5]). Ky Fan [6] extended the above result to topological vector spaces and gave several interesting applications in fixed point theory, minimax theory, and game theory. Ky Fan proved the following theorem in [6]; we denote this theorem as the KKM principle for our further discussion.

**Theorem 1.1 (KKM Principle [6]).** Let $X$ be a topological vector space, $K \subset X$ be an arbitrary subset, and $T : K \rightarrow 2^X$ be a KKM map. If all the sets $T(x)$, $x \in K$ are closed in $X$, and if one is compact, then

$$\cap\{ T(x) : x \in K \} \neq \emptyset.$$ 

The above theorem has many interesting applications in multivalued fixed point theory, minimax theory, game theory, mathematical economics, and variational inequality. Due to its wide applications, a large number of extended and generalized versions of Theorem 1.1 are available in the literature (see [7]). Let us consider the following multivalued fixed point theorem proved by Browder [8].
**Theorem 1.2** ([8]). Let $K$ be a nonempty compact convex subset of a Hausdorff topological vector space $X$, and let $T : K \to 2^K$, where $2^K$ denotes the set of all subsets of $K$, be a multivalued map such that

1. $T(x)$ is nonempty and convex, for each $x \in K$.
2. $T^{-1}(y) = \{x \in K : y \in T(x)\}$ is open, for each $y \in K$.

Then there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.

Browder proved **Theorem 1.2**, by using the Brouwer’s fixed point theorem and a technique called partition of unity. In 1978, Dugundji and Granas [9] (also see [10]) proved **Theorem 1.2** by invoking the KKM principle technique. Before getting into our main results, let us briefly discuss the notion called best proximity points.

### 1.1. Best proximity points

Let $(X, d)$ be a metric space and $A$ be a nonempty subset of $X$. Consider a mapping $f : A \to X$. The mapping $f$ is said to have a fixed point in $A$ if the fixed point equation $f(x) = x$ has at least one solution. In metric terminology, we say that $x \in A$ is a fixed point of $f$ if $d(x, f(x)) = 0$. It is clear that the necessary condition for the existence of a fixed point for $f$ is $f(A) \cap A \neq \emptyset$ (but this is not a sufficient condition). If the fixed point equation $f(x) = x$ does not posses a solution, then $d(x, f(x)) > 0$ for all $x \in A$. In such a situation, it is our aim to find an element $x \in A$ such that $d(x, f(x))$ is minimum in some sense. Best proximity point theorems have been explored to find necessary conditions so that the minimization problem

$$\min_{x \in A} d(x, f(x))$$

has at least one solution. To have a concrete lower bound, let us consider two nonempty subsets $A, B$ of a metric space $X$ and a mapping $f : A \to B$. The natural question is whether one can find an element $x_0 \in A$ such that $d(x_0, f(x_0)) = \min \{d(x, f(x)) : x \in A\}$. Since $d(x, f(x)) \geq \text{dist}(A, B)$, the optimal solution to the problem of minimizing the real valued function $x \mapsto d(x, f(x))$ over the domain $A$ of the map $f$ will be the one for which the valued $\text{dist}(A, B)$ is attained. A point $x_0 \in A$ is called a best proximity point of $f$ if $d(x_0, f(x_0)) = \text{dist}(A, B)$. Note that, if $\text{dist}(A, B) = 0$, then the best proximity point is nothing but a fixed point of $f$. Interesting applications of best proximity points can be found in [11–14].

In [15], Basha and Veeramani extended the Browder fixed point theorem (**Theorem 1.2**) by proving the existence of a best proximity point under some suitable conditions. Since the following notation is used in **Theorem 1.3**, let us recall them for our further discussions. Let $A, B$ be nonempty subsets of a normed linear space $X$. Then,

$$\text{Prox}(A, B) = \{(x, y) : x \in A \times B, \|x - y\| = \text{dist}(A, B)\},$$

$$A_0 = \{x \in A : \|x - y\| = \text{dist}(A, B) \text{ for some } y \in B\}$$

and

$$B_0 = \{y \in B : \|x - y\| = \text{dist}(A, B) \text{ for some } x \in A\}.$$

In [16], the authors discussed sufficient conditions which guarantee the nonemptiness of $A_0$ and $B_0$. Also, in [15], the authors proved that $A_0$ is contained in the boundary of $A$. A particular case of the main theorem proved in [15] is the following.

**Theorem 1.3** ([15]). Let $X$ be a normed linear space. Let $A$ be a nonempty, approximately compact and convex subset of $X$ and $B$ be a nonempty, closed and convex subset of $X$ such that $\text{Prox}(A, B)$ is nonempty and $A_0$ is compact. Suppose that the following hold.

1. $T : A \to 2^A$ is a multifunction such that, for every $x \in A_0$, $T(x)$ intersects $B_0$, and, for every $y \in B_0$, the fibre $T^{-1}(y)$ is open.
2. For every open set $U$ in $A$, the set $\cap \{T(u) : u \in U\}$ is convex.

Then there is $x_0 \in A$ such that $\text{dist}(x_0, T(x_0)) = \text{dist}(A, B)$.

The intension of this article is twofold. First, we have attempted to formulate a generalized version of KKM mappings (we call them R-KKM mappings) which fit into best proximity point theory. An analog version of the KKM principle is proved for best proximity point setting. Second, using the R-KKM technique, we attempt to prove some results similar to **Theorem 1.3**.

### 2. Preliminaries

In this section, we introduce some notation and known results which we will use in the ensuing sections. Let $X$ be a normed linear space. The set of all subsets of $X$ is denoted by $2^X$, and $\text{co}(A)$ will denote the convex hull of $A$, where $A \subseteq 2^X$.

**Definition 1.** Let $A, B$ be nonempty subsets of a metric space $X$. Then the pair $(A, B)$ is said to be a proximal pair if, for each $(x, y) \in A \times B$, there exists $(\bar{x}, \bar{y}) \in A \times B$ such that $\|x - y\| = \|\bar{x} - \bar{y}\| = \text{dist}(A, B)$.

Note that a pair $(A, B)$ is a proximal pair if and only if $A = A_0$ and $B = B_0$. Now let us define the notion of R-KKM mappings.

**Definition 2.** Let $(A, B)$ be a nonempty proximal pair of a normed linear space $X$. A multivalued mapping $T : A \to 2^B$ is said to be a R-KKM map if, for any $\{x_1, \ldots, x_n\} \subseteq A$, there exists $y_1, \ldots, y_n \in B$ with $\|x_i - y_i\| = \text{dist}(A, B)$, for all $i = 1, \ldots, n$, such that $\text{co}(y_1, y_2, \ldots, y_n) \subseteq \bigcup_{i=1}^n T(x_i)$.

Note that, if $T : A \to 2^B$ is a R-KKM map, then $\text{dist}(T(x_i), T(x_j)) = \text{dist}(A, B)$ for any $x_i \in A$. Also, if $A = B$, then the definition of R-KKM mappings reduces to that of KKM mappings. A subset $C$ of a normed linear space $X$ is said to be finitely closed if $C \cap L$ is closed for every finite-dimensional subspace $L$ of $X$. 
3. Main results

**Theorem 3.1.** Let $(A, B)$ be a nonempty proximal pair in a normed linear space $X$ and $T : A \to 2^B$ be an R-KKM map such that $T(x)$ is finitely closed, for all $x \in A$. Then the family $\{T(x) : x \in A\}$ has finite intersection property.

**Proof.** Assume that there is a finite subset $\{x_1, \ldots, x_n\}$ of $A$ such that $\cap_{i=1}^n T(x_i) = \emptyset$. Consider the finite-dimensional subspace $L = \text{span}\{y_1, \ldots, y_n\}$ of $X$, where $\{y_1, \ldots, y_n\} \subseteq B$ with $\|x_i - y_i\| = \text{dist}(A, B)$, for all $i = 1, \ldots, n$ and $\text{co}\{y_1, \ldots, y_n\} \subseteq \bigcup_{i=1}^n T(x_i)$. Fix $K = \text{co}\{y_1, \ldots, y_n\}$. Clearly, $K \subseteq L$. Since $T(x_i) \cap L$ is closed in $L$, for each $i \in \{1, \ldots, n\}$, we conclude that $\text{dist}(y, T(x_i) \cap L) = 0$ iff $y \in T(x_i) \cap L$, for any $y \in L$. Define a map $\lambda : K \to \mathbb{R}$ by

$$\lambda(y) = \sum_{i=1}^n \text{dist}(y, T(x_i) \cap L).$$

Since $\cap_{i=1}^n T(x_i) = \emptyset$, $\lambda(y) \neq 0$ for each $y \in K$. Now, let us define a map $f : K \to K$ by

$$f(y) = \frac{1}{\lambda(y)} \sum_{i=1}^n \text{dist}(y, T(x_i) \cap L) \cdot y_i.$$

It is clear that $f$ is a well-defined and continuous map from $K$ into $K$. Since $K$ is a closed bounded convex subset of a finite-dimensional space $L$, by invoking Brouwer’s fixed point theorem, $f$ has a fixed point in $K$; i.e., there exists $y_0 \in K$ such that $f(y_0) = y_0$.

Take $I = \{i \in \{1, \ldots, n\} : \text{dist}(y_0, T(x_i) \cap L) > 0\}$. Then $y_0 \not\in \bigcup_{i \in I} T(x_i)$. But $f(y_0) \in \text{co}\{y_i : i \in I\} \subseteq \bigcup_{i \in I} T(x_i)$, which is a contradiction to the fact that $f(y_0) = y_0$. Hence the family $\{T(x) : x \in A\}$ has finite intersection property. \qed

As an immediate consequence, we can prove the following.

**Theorem 3.2.** Let $(A, B)$ be a nonempty proximal pair in a normed linear space $X$ and $T : A \to 2^B$ be an R-KKM map. If, for each $x \in A$, $T(x)$ is closed in $X$ and there exists at least one $x_0 \in A$ such that $T(x_0)$ is compact in $X$, then $\cap\{T(x) : x \in A\}$ is nonempty.

Now let us prove the extended version of the Browder fixed point theorem in a normed linear space with best proximity point setting.

**Theorem 3.3.** Let $(A, B)$ be a nonempty compact convex proximal pair in a normed linear space $X$ and $T : A \to 2^B$ be a multivalued map that satisfies the following:

1. for each $x \in A$, $T(x)$ is a convex subset of $B$;
2. for each $y \in B$, $T^{-1}(y)$ is an open empty set.

Then there is $w \in A_0$ such that $\text{dist}(w, T(w)) = \text{dist}(A, B)$.

**Proof.** Let us define $G : B \to 2^A$ such that $G(y) = A \setminus T^{-1}(y)$, $y \in B$. Suppose that $G(y) = \emptyset$ for some $y \in B_0$, that is, $T^{-1}(y) = A$; then $y \in T(x)$, for all $x \in A$. Since $y \in B$, there exists $x_0 \in A$ such that $\|x_0 - y\| = \text{dist}(A, B)$. In particular, $y \in T(x_0)$. Then

$$\text{dist}(A, B) \leq \text{dist}(x_0, T(x_0)) \leq \|x - y\| = \text{dist}(A, B),$$

which implies that $x_0$ is a best proximity point in $A$.

Assume that $G(y)$ is nonempty for each $y \in B$. By the hypothesis we have that $G$ is a nonempty closed valued multimap on $B$. Also, $\cap\{T^{-1}(y) : y \in B\}$ is an open cover for $A$. Then

$$\bigcap_{y \in B} G(y) = \bigcap_{y \in B} \left(A \setminus T^{-1}(y)\right) = A \setminus \bigcup_{y \in B} T^{-1}(y) = \emptyset.$$ 

From **Theorem 3.2**, we conclude that $G$ is not an R-KKM map. Therefore, there exist $\{y_1, \ldots, y_n\} \subseteq B$ and $\{x_1, \ldots, x_n\} \subseteq A$ with $\|x_i - y_i\| = \text{dist}(A, B)$, for all $i = 1, \ldots, n$, such that $\text{co}\{x_1, \ldots, x_n\}$ is not contained in $\bigcup_{i=1}^n G(y_i)$.

Choose $w = \sum_{i=1}^n \lambda_i x_i \in \text{co}\{x_1, \ldots, x_n\}$, but $w \not\in \bigcup_{i=1}^n G(y_i) = A \setminus \bigcap_{i=1}^n T^{-1}(y_i)$. Therefore, $w \in T^{-1}(y_i)$, for all $i = 1, \ldots, n$. That is, $y_i \in T(w)$, for all $i = 1, \ldots, n$. Put $z = \sum_{i=1}^n \lambda_i y_i$. Since $T(w)$ is convex and $y_i \in T(w)$, for each $i = 1, \ldots, n$, we conclude that $z \in T(w)$. Then

$$\text{dist}(A, B) \leq \text{dist}(w, T(w)) \leq \|w - z\| = \left\|\sum_{i=1}^n \lambda_i x_i - \sum_{i=1}^n \lambda_i y_i\right\| = \text{dist}(A, B).$$

Therefore, $\text{dist}(w, T(w)) = \text{dist}(A, B)$. \qed

The preceding result includes the following special case of a fixed point theorem due to Browder [8].

**Corollary 3.1.** Let $X$ be a normed linear space and $A$ be a nonempty compact convex subset of $X$. If $T : A \to 2^A$ is a convex valued multifunction with open fibers, then there exists $x_0 \in A$ such that $x_0 \in T(x_0)$. 
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References