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Fundamental Study

Dot-depth, monadic quantifier alternation, and first-order closure over grids and pictures

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Abstract

This paper presents results from two different areas. The first area is monadic second-order logic (MSO) over finite structures, in particular over the so-called grids. These are structures whose elements can be arranged as a matrix and which have two binary relations corresponding to vertical and horizontal successors. For this logic, we study the expressive power of the alternation of existential and universal monadic second-order quantifiers, i.e., set quantifiers. In Matz et al. (Information and Computation, LICS' 97, 1999, to appear) it had been shown that these alternations cannot be limited to a fixed number without loss of expressiveness. In this paper, we strengthen this result in several ways. Firstly, we show that there are MSO formulas that have a very restricted form of $k+1$ set quantifiers but are not equivalent to a formula with k quantifiers. Secondly, we show that if we fix the number of such alternations, the expressive power of formulas that start with a block of universal quantifiers differs from the power of those that start with an existential one—this was previously known only for *coloured* grids. Thirdly, we investigate how an additional prefix of first-order (i.e., element) quantifiers influences the expressive power of MSO formulas. The second area that this paper is concerned with is two-dimensional formal language theory. We study how the alternation of (first- and monadic second-order) quantifications, on the one hand, relates to the dot-depth measure of two-dimensional (i.e., picture) languages, on the other hand. That measure is the two-dimensional version of the classical notion of dot-depth for (one-dimensional) starfree word languages. We show that the hierarchy induced by this dot-depth cuts through the hierarchy given by monadic second-order quantifications. In particular, beyond each level of the monadic second-order alternation hierarchy, there is a starfree picture language. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

We give some background about the two areas of this paper in the following two subsections.

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1.1. Quantifier alternation in monadic second-order logic

In MSO over grids, one can quantify over elements as well as over subsets of the universe of a grid. For example, an MSO-formula can express that a grid is a square by asserting that every set of elements (=grid positions) that contains the top-left corner and is closed under "diagonal successors" also contains the bottom-right corner.

MSO-formulas can be classified by the alternation depth of set quantifiers. One speaks of a Σ_k-formula (Π_k-formula) if its prenex normal form has a prefix of k alternating blocks of set quantifiers starting with an existential (universal) block, followed by a first-order kernel. A Δ_k-formula is a formula that is equivalent to both a Σ_k-formula and a Π_k-formula.

The above example can easily be formalized as a Π_1-sentence. (It is also a Δ_1-sentence.)

In [17] it was shown that for the class of grids, Δ_{k+1}-sentences are more expressive than boolean combinations of Σ_k-sentences. The proof of this "strictness of the monadic (second-order quantifier alternation) hierarchy" shows a stronger result, namely that a very limited form of set quantification suffices to exceed any level of the monadic hierarchy in expressiveness: For every k, there is a formula of the form $\exists\bar{x}_1\forall\bar{x}_2\cdots\circlearrowleft\bar{x}_{k+1}\circlearrowleft\bar{X}\varphi$ (where $\bar{x}_1,\dots,\bar{x}_{k+1}$ are tuples of first-order variables, $\bar{X}$ is a tuple of set variables, φ is a first-order formula, and $\circlearrowleft$ is either $\exists$ or $\forall$, depending on whether k is even or odd) that is not equivalent to any Σ_k-formula (see [17, 18, 21].)

This leads us to the definition of the "first-order closure" of a class $\mathscr{F}$ of formulas: A formula in the first-order closure of $\mathscr{F}$ results from a formula in the boolean closure of $\mathscr{F}$ by prefixing with first-order quantifications and negations. The above statement shows that the first-order closure of Σ_1 cannot be captured within any level of the monadic hierarchy, which demonstrates the power of the first-order closure.

On the other hand, in [1], where the first-order closure is introduced, the authors have shown that the expressiveness of Σ_3 cannot be captured within any level of the first-order closure of Σ_1, which demonstrates the weakness of the first-order closure.

In this paper, we show that the expressiveness neither of Δ_{k+2} nor of the first-order closure of Δ_{k+1} can be captured within Σ_{k+1} or the first-order closure of Σ_k. Besides, we show that Σ_k and Π_k differ in expressiveness even for the class of *uncoloured* grids, a questions that had remained open in [17].

Motivation for studying the monadic hierarchy and the first-order closure is given in [1]. The important observation is that the levels of the monadic alternation hierarchy contain properties that are computationally complete for the respective levels of the polynomial hierarchy, which correspond to the levels of the full second-order alternation hierarchy. Thus in some sense, the monadic alternation hierarchy is a "monadic analogue" to the polynomial hierarchy.

But the levels of the polynomial hierarchy are closed under first-order quantifications, whereas the levels of the monadic hierarchy are not. In [1] the authors argue that in order to achieve progress in proving the strictness of the polynomial hierarchy it is promising to study a certain variant of the monadic hierarchy (called "closed hierarchy") that classifies properties by the alternation of set quantifications only, i.e., disregarding the use of first-order quantifications. The first-order closure of the levels of the monadic hierarchy are a first step in this investigation.

1.2. Dot-depth of starfree picture languages

The study of the above questions yields also results about picture languages. Picture languages are the two-dimensional analogue to formal word languages, i.e., sets of matrices (="pictures") over a finite alphabet. There are two partial concatenations defined for pictures, which juxtapose pictures horizontally (or vertically, respectively), provided they have the same height (or width, respectively). The notion of starfreeness carries over from words to pictures easily: A picture language is starfree if it results from atomic pictures by repeated application of these concatenations, union, and complement (in the set of all pictures).

There is a natural correspondence between coloured grids and pictures, so it should be clear when we call a picture language Σ_k-definable, Π_k-definable, etc. In [8] it has been shown that for picture languages, Σ_1 captures the expressiveness of a certain kind of (non-deterministic) automata on pictures, so-called tiling systems. Thus, it is justified to call Σ_1-definable picture languages *recognizable*. See [7] for a survey on research related to this class. One central result (also presented in that paper) is the fact that the class of recognizable picture languages is not closed under complement. Theorem 2.26 shows that this is true even for a unary alphabet.

Other papers that investigate the class of recognizable picture languages are e.g. [5, 9, 11, 20].

From standard automata theory we know that every starfree word language is recognizable. This is not the case for pictures: [14] gave an example for a starfree, non-recognizable picture language, answering a question from [7]. In this paper we even show that for every k, there is a starfree picture language that is not Σ_k-definable. Moreover, we compare quantifier alternation and *dot-depth*. The dot-depth measures the alternation of concatenations and boolean combinations in the definition of starfree picture language and is a adaptation of the corresponding notion for word languages (see [3]).

1.3. What follows

The remainder of the paper is structured as follows. In Section 2 we introduce all the notions and notations needed in this paper, including the formal definition of the quantifier alternation hierarchy.

In Section 2.4 we state all our separation results, namely Theorems 2.16, 2.22, 2.24 and 2.26 and some corollaries. Theorem 2.16 shows that there is a Π_k-definable but not Σ_k-definable starfree picture language. Theorems 2.22 and 2.24 focus on slim grid properties, i.e., ones where the lengths of grids are related to their heights by a fast growing function. Theorem 2.16 states that the maximal growth rate achievable by Σ_k-formulas is the same for a fragment of Σ_k that makes hardly any use of monadic second-order quantifiers. Theorem 2.24 characterizes the above growth rate for Σ_k-formulas with an additional prefix of first-order quantifiers. Theorem 2.26 shows that there is a Π_k-definable but not Σ_k-definable grid property (or picture language over a unary alphabet). Again, very limited use of monadic quantification is needed here.

The proof of the above separation results is split into two parts: The first one (presented in Section 3) consists of showing that particular grid properties or picture languages are expressible by certain formulas or starfree expressions over pictures, whereas the second part (presented in Section 4) shows that they are not expressible by certain other formulas.

Section 3 is structured as follows. Section 3.1 shows some easy inclusion results that relate the dot-depth hierarchy with (certain fragments of the first-order closure of) the monadic alternation hierarchy. Section 3.2 introduces some more notation that is useful for the following three subsections.

Sections 3.3 and 3.4 exhibit particular starfree picture languages (called Num_k) on level k of the dot-depth hierarchy. These picture languages (or rather their corresponding grid properties) are the witnesses for the strictness of the monadic alternation hierarchy, and they are very similar to the grid properties of [17, 18, 21].

In Section 3.5 we exhibit particular grid properties in the first-order closure of the levels of the monadic second-order hierarchy. These picture languages witness, for example, that $FO(\Sigma_{k+1})$ is strictly more expressive than $FO(\Sigma_k)$ for every k. These witnesses are in some sense similar (and depend on) the mentioned picture languages Num_k. In both cases the fundamental idea is to establish iterated counting mechanisms to ensure that pictures (or grids) are very "slim" in the sense that their width is large compared to the height. However, this time these mechanisms are substantially more sophisticated than in the preceding subsections.

Section 4 is structured as follows. In Section 4.1 we recall the automata theoretic technique used in [17, 18]. A picture of height m and width n can be represented by a word of length n whose symbols are columns of height m. The fundamental idea of the mentioned technique is to pass from a Σ_k-formula φ over pictures to a family $(\mathfrak{A}_m)_{m\geqslant 1}$ of finite automata (NFA) on words, where for each m, the NFA $\mathfrak{A}_m$ recognizes exactly those words that represent models of φ that have height m. Furthermore, $\mathfrak{A}_m$ can be chosen with state set size k-fold exponential in m. This asymptotic bound enables to apply standard pumping techniques for finite automata.

Finally, we give a conclusion in Section 5. In Section 5.1 we sum up the technical contributions of this paper.

Status of this work: In this paper I present the essential new contributions of my Ph.D. thesis [16], which contains more results (published in [14, 15, 17]).

2. Pictures, grids, and alternation—definitions and results

In this section we will define the previously mentioned notions formally and observe some easy facts. In Section 2.4 we will state the main results of this paper.

Some basic notations: We write $\mathbb{N}$ for the set of natural numbers (without zero). For every $n, m \geqslant 0$, we write $[n]$ for $\{0, \dots, n-1\}$ and $[n, m]$ for $\{n, n+1, \dots, m\}$.

If a binary relation f is a partial function, we use the usual notations, for example $f(m)$ for "the" element n with $(m,n) \in f$ (if it exists). We write $\operatorname{dom} f$ for the set of all m for which $f(m)$ is defined.

We say that a partial function $f : \mathbb{N} - \rightarrow \mathbb{N}$ "is $g(\mathcal{O}(m))$" if there is a $c \geqslant 1$ such for almost all $m \in \operatorname{dom} f$ we have $f(m) \leqslant g(cm)$. We also use the Θ- and the Ω-notation in the usual meaning.

For a set M, the powerset of M is denoted by $\mathscr{P}(M)$.

2.1. Grids and logic

2.1.1. Grids

Grids are particular relational structures over the signature $\tau_{Grids} = \{S_1, S_2\}$ containing the binary "successor" relation symbols S_1 and S_2. The universe of a grid is of the form $[m] \times [n]$. The *grid of height m and width n* is the τ_{Grids}-structure

$$\underline{[m] \times [n]} := ([m] \times [n], S_1^{m,n}, S_2^{m,n}),$$

where $S_1^{m,n}$ and $S_2^{m,n}$ are the "vertical" and the "horizontal" successor relations on $[m] \times [n]$, containing all pairs $((i,j),(i+1,j))$ and all pairs $((i,j),(i,j+1))$, respectively, from $[m] \times [n]$. Let $\operatorname{size}(\underline{[m] \times [n]}) = (m,n)$.

The expressions *row*, *column*, *top*, *bottom*, etc., are interpreted as in the terminology of matrices; e.g. the leftmost column contains exactly the vertices $(i,0)$ for $0 \leqslant i \leqslant m-1$, and the top row consists of all vertices $(0,j)$ for $0 \leqslant j \leqslant n-1$.

We adjoin unary predicate symbols $X_1, \dots, X_t$ to the signature of grids, obtaining the signature $\tau_{Grids_t} = \tau_{Grids} \cup \{X_1, \dots, X_t\}$. A *t-bit grid* (for some $t \geqslant 0$) is a structure over the signature τ_{Grids_t}, i.e. of the form $R = (\operatorname{dom} R, S_1^R, S_2^R, X_1^R, \dots, X_t^R)$, whose restriction to τ_{Grids} is a grid. So grids are 0-bit grids.

The classes of grids and t-bit grids will be denoted *Grids* and $Grids_t$, respectively.

A different version of grids is obtained via the signature $\tau_{Grids} \cup \{\leqslant_1, \leqslant_2\}$ with binary relation symbols $\leqslant_1, \leqslant_2$. In this context, a grid is considered as the structure

$$([m] \times [n], S_1^{m,n}, S_2^{m,n}, \leqslant_1^{m,n}, \leqslant_2^{m,n}),$$

where $\leqslant_1^{m,n} = \{((i,j),(i',j)) \in ([m] \times [n]) \times ([m] \times [n]) \mid i \leqslant i'\}$, and $\leqslant_2^{m,n} = \{((i,j),(i,j')) \in ([m] \times [n]) \times ([m] \times [n]) \mid j \leqslant j'\}$. That means that $\leqslant_1^{m,n}$ (and $\leqslant_2^{m,n}$) are the reflexive and transitive closure of $S_1^{m,n}$ (and $S_2^{m,n}$, respectively).

2.1.2. Monadic second-order logic over grids

We use $x, y, x_1, \dots$ as first-order variables and $X, Y, \dots$ as monadic second-order variables (i.e., *set variables*). *Atomic formulas* (*over* τ_{Grids}) are of the form $x = y$, Xy, S_1xy, or S_2xy for first-order variables x, y and a set variable X. Their intended meanings are that x, y are equal, y is an element of X, or that y is a vertical (or horizontal, respectively) successor of x.

Monadic second-order formulas over τ (*MSO*-formulas for short) are built up as usual from atomic formulas. Formally, if φ and ψ are *MSO*-formulas, x is a first-order variable, and X is a set variable, then $\varphi \vee \psi$, $\neg\varphi$, $\exists x\varphi$, and $\exists X\varphi$ are *MSO*-formulas, too. We may use the other propositional connectives, like $\to$, $\wedge$, etc., as well as universal quantifications as abbreviations with their usual meaning.

First-order formulas are *MSO*-formulas in which no second-order quantifier occurs. *FO* denotes the class of first-order formulas.

In Section 2.2.2 we will give two extensions of the syntax of formulas.

The set of variables occurring free in a formula φ is denoted $free(\varphi)$.

If $free(\varphi) \subseteq \{X_1,\dots,X_t,x_1,\dots,x_m\}$, we sometimes write $\varphi(X_1,\dots,X_t,x_1,\dots,x_m)$. If R is a grid and $U_1,\dots,U_t$ and $u_1,\dots,u_m$ are subsets and elements, respectively, of the domain of R, then we write $R \models \varphi[U_1,\dots,U_t,u_1,\dots,u_m]$ to indicate that φ is true in R under the assignment that maps X_i to U_i and x_j to u_j for all i,j. If φ is a sentence, i.e., $free(\varphi) = \emptyset$, then we write $Mod_0(\varphi)$ for the set of all grids in which φ is true.

If φ is a formula over τ_{Grids} with $free(\varphi) \subseteq \{X_1,\dots,X_t\}$, we write $Mod_t(\varphi)$ for the set of t-bit grids in which φ is true with the implicitly given assignment to the set variables $X_1,\dots,X_t$. This way, we consider the formula φ as a sentence over τ_{Grids_t}.

If $L = Mod_t(\varphi)$, we say that φ *defines* L. If $\mathscr{F}$ is a set of formulas, then we call a set L of t-bit grids $\mathscr{F}$*-definable* if there is a formula $\varphi \in \mathscr{F}$ with $L = Mod_t(\varphi)$.

2.2. Closures and alternation

We have motivated in Section 1 to measure the complexity of monadic second-order formulas by the alternation of existential and universal set quantifications.

We will continue to define the different levels of the monadic quantifier alternation hierarchy. As mentioned before, we will also be interested in first-order alternation and the interference of these two. Besides, we will introduce the "unary TC-operator" (for the transitive closure) and "the monadic ι-operator" (for unique choice) as syntactic extensions of monadic second-order formulas. Some of our separation results give nontrivial facts about their expressive power in monadic second-order logic.

In order to be able to state and prove all these facts formally, we will have to introduce some more general notations.

Let $\mathscr{F}$ be a class of formulas. Then *co-*$\mathscr{F}$ denotes the class of formulas $\neg\varphi$ with $\varphi \in \mathscr{F}$. The (1) *boolean closure* of $\mathscr{F}$, denoted $B(\mathscr{F})$, (2) *positive boolean closure* of $\mathscr{F}$, denoted $PB(\mathscr{F})$, (3) *existential first-order closure* of $\mathscr{F}$, denoted $\Sigma_1^0(\mathscr{F})$, (4) *universal first-order closure* of $\mathscr{F}$, denoted $\Pi_1^0(\mathscr{F})$, (5) *existential second-order closure* of $\mathscr{F}$, denoted $\Sigma_1(\mathscr{F})$, (6) *first-order closure* of $\mathscr{F}$, denoted $FO(\mathscr{F})$, respectively, is defined as the smallest superclass of $\mathscr{F}$ that is closed under (1) boolean combinations, i.e., $\neg$ and $\vee$. (2) positive boolean combinations, i.e., $\wedge$, $\vee$. (3) existential first-order quantifications and positive boolean combinations, (4) universal first-order quantifications and positive boolean combinations, (5) existential second-order quantifications and positive boolean combinations, (6) existential first-order quantifications and boolean combinations, respectively. In Section 2.2.2 we will introduce two more closure operations on formula classes.

In [17] it is shown how separation results for formula classes involving these closures can be transferred from one class of structures (like grids) to another (say graphs) by the encoding technique (called "strong first-order reductions").

2.2.1. Alternation hierarchies for monadic logic

Now, we are ready to define several alternation hierarchies for monadic logic. One benefit of the notions of "closures" introduced above is that these definitions are quite succinct.

Definition 2.1 (*Alternation hierarchies*). Let $\mathscr{F}$ be a class of formulas. We define

$$\Sigma^0_0(\mathscr{F}) = \mathscr{F} \quad \text{and} \quad \Sigma^0_{k+1}(\mathscr{F}) = \Sigma^0_1(co\text{-}\Sigma^0_k(co - \mathscr{F})),$$

$$\Sigma_0(\mathscr{F}) = \mathscr{F} \quad \text{and} \quad \Sigma_{k+1}(\mathscr{F}) = \Sigma_1(co\text{-}\Sigma_k(co\text{-}\mathscr{F}))$$

for every $k \geqslant 0$. Let $\Pi^0_k(\mathscr{F}) = co\text{-}\Sigma^0_k(co\text{-}\mathscr{F})$ and $\Pi_k(\mathscr{F}) = co\text{-}(\Sigma_k(co\text{-}\mathscr{F})$ for every k. We write Σ_k, and Π_k instead of $\Sigma_k(FO)$, $\Pi_k(FO)$, respectively.

(Note that the re-definition of $\Sigma^0_1(\mathscr{F})$ and $\Sigma_1(\mathscr{F})$ is compatible with the original one, provided we identify formulas of the form φ and $\neg\neg\varphi$.)

Let $k \geqslant 0$. A Σ_k-formula is of the form

$$\exists \bar{X}_1 \neg \exists \bar{X}_2 \cdots \neg \exists \bar{X}_k \varphi$$

for a first-order φ and second-order variable tuples $\bar{X}_1, \ldots, \bar{X}_k$.

The class Σ_k of formulas (as well as the class of properties definable by such formulas) is known as "the kth level of the monadic quantifier alternation hierarchy". There is some arbitrariness in choosing Σ_k to be "the level k". Other possible choices would be Π_k, or $\Sigma_k \cup \Pi_k$, or the boolean closure of Σ_k, or the class Δ_k introduced below. The class of properties definable by formulas in Σ_1, i.e., in the existential fragment of monadic second-order logic, is often called "monadic NP", but we avoid this term because it is not a complexity class.

Definition 2.2. Two formulas φ, ψ are *equivalent over grids* iff $\varphi \leftrightarrow \psi$ is true in all grids under all assignments. For a set $\mathscr{F}$ of formulas we write $\bar{\mathscr{F}}$ for the set of all formulas that are equivalent over grids to a formula in $\mathscr{F}$. Let $\Delta_k = \bar{\Sigma}_k \cap \bar{\Pi}_k$ for every k. The Δ_1-*closure of* $\mathscr{F}$ is given by

$$\Delta_1(\mathscr{F}) = \overline{\Sigma_1(B(\mathscr{F}))} \cap \overline{\Pi_1(B(\mathscr{F}))}.$$

Besides, $\underline{\mathscr{F}}(Grids_t) := \{Mod_t(\varphi) \mid \varphi \in \mathscr{F}, free(\varphi) \subseteq \{X_1, \ldots, X_t\}\}$. In this notation, we omit $Grids_t$ if t is clear from the context.

We sometimes write "Σ_k-formula" instead of "$\bar{\Sigma}_k$-formula" and so on.

2.2.2. The unary TC-operator and the monadic ι-operator

Our proof of the fact that the monadic second-order quantifier alternation hierarchy is strict will show that one can define properties "high" in the hierarchy with a very limited form of set quantifications. In order to formulate this observation precisely, we introduce two extensions of the syntax of formulas which may—in some situations—replace set quantifications.

The unary transitive closure operator: The first one is the "unary transitive closure operator" (TC-operator). If φ is a formula and x, y are first-order variables, then $\mathrm{TC}(x, y, \varphi)xy$ is a formula whose free variables are x, y, and the free variables of φ. (Actually, the occurrences of x and y *inside* φ are bound, and the ones outside the brackets are free and we usually omit them.)

The intended meaning of the formula $\mathrm{TC}(x, y, \varphi)$ is that x and y are in the reflexive and transitive closure of the binary relation given by $\{(x', y') \mid x', y' \text{ satisfy } \varphi\}$.

Formally, if $\varphi(x, y, \bar{Z}, \bar{z})$ is a formula, where $\bar{Z}$ and $\bar{z}$ are first- and second-order variable tuples, respectively, of possibly different lengths, then the satisfaction relation $M \models \mathrm{TC}(x, y, \varphi)[u, v, \bar{W}, \bar{w}]$ holds for two elements u, v, a tuple $\bar{W}$ of subsets, and a tuple $\bar{w}$ of elements of $\operatorname{dom} M$ iff there are elements $u_0, \ldots, u_m \in \operatorname{dom} M$ for some $m \geqslant 0$ such that $u = u_0$, $v = u_m$, and $M \models \varphi[u_i, u_{i+1}, \bar{W}, \bar{w}]$ for all $i \in \{0, \ldots, m-1\}$.

Let $\mathscr{F}$ be a class of formulas. $\mathscr{F}$ is *closed under application of the unary* TC-*operator* iff for every $\varphi \in \mathscr{F}$ and all first-order variables x, y we have that $\mathrm{TC}(x, y, \varphi)$ is in $\mathscr{F}$.

The monadic ι-Operator: The second extension of the syntax of formulas will be described now.

If φ and ψ are formulas and $\bar{X} = (X_1, \ldots, X_t)$ is a tuple of set variables that are free in φ, then "$\psi(\iota\bar{X}(\varphi))$" is meant to be a formula in which the free variables are those of φ and those of ψ but not the variables in $\bar{X}$. It shall express that ψ holds for "the unique" tuple $\bar{X}$ that makes φ true. Of course, this only makes sense in case φ uniquely determines a tuple $\bar{X}$, and this depends on the considered class of structures.

Formally, let $\mathscr{C}$ be a class of structures and let $\bar{Y}$ be a tuple of first- and second-order variables such that $\varphi = \varphi(\bar{X}, \bar{Y})$ and $\psi = \psi(\bar{X}, \bar{Y})$, i.e., every free variable of φ or ψ appears either in $\bar{X}$ or in $\bar{Y}$. Let us write $\bar{X} = \bar{X}'$ to abbreviate the first-order formula $\bigwedge_{i=1}^{t}(\forall x(X_i x \leftrightarrow X_i' x))$.

Assume that the formula $\exists! \bar{X}\varphi$, which abbreviates $\exists\bar{X}\forall\bar{X}'(\varphi(X', \bar{Y}) \leftrightarrow \bar{X} = \bar{X}')$, is true in all $\mathscr{C}$-classes under all assignments; then

$$\psi(\iota\bar{X}(\varphi), \bar{Y})$$

is also a formula with free variables among $\bar{Y}$. (In order to stress that such a formula fulfills the above constraint, we call it an "allowed" formula.)

The satisfaction relation is defined as follows. For a structure M of $\mathscr{C}$ and a tuple $\bar{V}$ (to be assigned to $\bar{Y}$) of subsets and elements of $\operatorname{dom} M$, we have $M \models \psi(\iota\bar{X}(\varphi), \bar{Y})[V]$ iff $M \models \psi[\bar{U}, \bar{V}]$ for the unique tuple $\bar{U}$ for which $M \models \varphi[\bar{U}, \bar{V}]$.

Let $\mathscr{F}$ be a class of formulas and $\mathscr{C}$ be a class of structures over the same signature. Let $\Delta_1^{\mathrm{U},\mathscr{C}}(\mathscr{F})$ be the set of allowed formulas of the form

$$\psi(\iota\bar{X}(\varphi), \bar{Y}),$$

where $\varphi(\bar{X}, \bar{Y})$ is a formula in $B(\mathscr{F})$, the formula $\psi(\bar{X}, \bar{Y})$ is first order, $\bar{X}$ is a tuple of set variables, and $\bar{Y}$ is a tuple of first-order and/or set variables of possibly different length.

2.2.3. Some calculation rules

Some calculation rules may be deduced from the well-known rules of predicate logic. For example, the following inclusions hold for every formula class $\mathscr{F}$:

$$\begin{array}{ll}
\underline{\Delta_k(\mathscr{F})} \subseteq \underline{\Sigma_k(\mathscr{F})} \subseteq \underline{B(\Sigma_k(\mathscr{F}))} & \text{for all } k \geqslant 1. \\
\underline{co\text{-}\Sigma_k(\mathscr{F})} = \underline{\Pi_k(co\text{-}\mathscr{F})} & \text{for all } k \geqslant 0. \\
\underline{\Sigma_1(\Sigma_k(\mathscr{F}))} \subseteq \underline{\Sigma_k(\mathscr{F})} & \text{for all } k \geqslant 0. \\
\underline{\Sigma_k^0(\mathscr{F})} \subseteq \underline{\Sigma_k(\mathscr{F})} & \text{if } k \geqslant 0 \text{ and } FO \subseteq \mathscr{F} = B(\mathscr{F}). \\
\bigcup_{k \geqslant 0} \underline{\Sigma_k^0(\mathscr{F})} = \underline{FO(\mathscr{F})}. &
\end{array}$$

(For the fourth line the crucial point is that first-order quantifications may be replaced by set quantifications relativized by a first-order formula that asserts for a set that it is a singleton.) All of these closure operations are monotone wrt set inclusion. Some of such rules are applied, e.g. in Remark 3.45.

The closure $\Delta_1^{\mathrm{U}}\mathscr{F}$ of a formula class $\mathscr{F}$ may be understood as a particular form of the Δ_1-closure of $\mathscr{F}$, where the membership in both $\Sigma_1(B(\mathscr{F}))$ and $\Pi_1(B(\mathscr{F}))$ is due to the fact that the tuple $\bar{X}$ of sets that is quantified over is determined uniquely by a formula in the boolean closure of $\mathscr{F}$. This is formalized by the following proposition.

Remark 2.3. Let $\mathscr{F}$ be a class of formulas. If $\varphi(\bar{X}, \bar{Y})$ is a formula in $B(\mathscr{F})$ and $\psi(\bar{X}, \bar{Y})$ is a first-order such that $\psi(\iota\bar{X}(\varphi), \bar{Y})$ is allowed, then this formula is equivalent over $\mathscr{C}$ to both of the formulas $\exists\bar{X}(\varphi \wedge \psi)$ and $\forall\bar{X}(\varphi \rightarrow \psi)$, which are in $\overline{\Sigma_1(B(\mathscr{F}))}$ and $\overline{\Pi_1(B(\mathscr{F}))}$, respectively. Thus $\Delta_1(\mathscr{F}) \subseteq \Delta_1^{\mathrm{U}}(\mathscr{F})$.

2.3. Words and pictures

Words: The notations we use for words and formal languages, called "word languages" here, are fairly standard. We recall some of them. Let Γ be an *alphabet*, i.e., a finite set of so-called *letters*. A *word* over Γ is a finite sequence of letters from Γ. The empty word (of length 0) will be denoted by ε. The set of all words (or all non-empty words) over Γ will be denoted by Γ^* (or Γ^+, respectively).

If $w = a_0 \ldots a_{n-1}$ is a word of length n, we write $w\langle i\rangle$ instead of a_i for every $i \in \{0, \ldots, n-1\}$.

The concatenation of words and word languages is defined as usual, and it may be iterated as follows. We let $L^0 = \{\varepsilon\}$ and[1] $L^{k+1} = L \cdot L^k$. Then $L^* = \bigcup_{k \geqslant 0} L^k$ and $L^+ = \bigcup_{k \geqslant 1} L^k$.

If $w = xyz$ for four (possibly empty) words w, x, y, z, then x, y, z are called *prefix*, *infix*, and *suffix*, respectively, of w. For every word language L, we denote by $\mathit{pref}(L)$ (and $\mathit{pref}_+(L)$) the set of all prefixes (and all non-empty prefixes, respectively) of words in L.

One notation that is non-standard is the following. For a word language $L \subseteq \Gamma^*$, the *cyclic closure of* L is given by $\mathit{cycl}(L) = \{vu \mid u, v \in \Gamma^*, uv \in L\}$.

Pictures: The "two-dimensional" analogue to a word is called a *picture*. Many notations for words can be defined similarly for pictures. We pick out some that are useful for our purposes.

Let $(m, n) \in (\mathbb{N} \times \mathbb{N}) \cup \{(0, 0)\}$. Recall that $[n] = \{0, \ldots, n-1\}$. A *picture* over Γ of *size* (m, n) is an $(m \times n)$-matrix over Γ, i.e. a mapping $P : [m] \times [n] \to \Gamma$. For a picture P of size (m, n), we define the *height* of P as $\underline{\bar{P}} = m$, the *width* of P as $|P| = n$, and $\mathit{size}(P) = (m, n)$. We write $P\langle i, j\rangle$ (rather than $P(i, j)$) for the component of P at position (i, j). The *empty picture* is the picture with size $(0, 0)$ and is denoted by ε.

We use the notations $\Gamma^{*,*}$ (and $\Gamma^{+,+}$), for the set of all (or all non-empty, respectively) pictures over Γ. If $m, n \geqslant 1$, then $\Gamma^{m,n}$ denotes the set of all pictures of size (m, n) over Γ. If $m \geqslant 1$ then $\Gamma^{m,+} = \bigcup_{n \geqslant 1} \Gamma^{m,n}$ and $\Gamma^{m,*} = \Gamma^{m,+} \cup \{\varepsilon\}$.

A set of non-empty pictures is called a *picture language*.

Note that unlike in [7] and other publications, the domain of a picture of size (m, n) is of the form $\{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$ rather than of the form $\{1, \ldots, m\} \times \{1, \ldots, n\}$. Another difference is that we do not allow the empty picture in "picture languages". See also Remark 3.1.

For every non-empty picture P over Γ, let $\mathit{top}(P)$ be the word over Γ in the top row of P, i.e., if P is of size (m, n), then $\mathit{top}(P) = P\langle 0, 0\rangle \cdots P\langle 0, n-1\rangle$.

We extend top to a mapping $\Gamma^{+,+} \to \Gamma^+$ in the usual way. As usual, we write $\mathit{top}^{-1}(L)$ for the pre-image of the word language L under this mapping, i.e., $\mathit{top}^{-1}(L)$ is the set of non-empty pictures of Γ whose top row is in L. (We only use this notation when the alphabet Γ is clear from the context.) A picture language of the form $\mathit{top}^{-1}(L)$ (for a word language L) is called *top-pre-image* of L.

Next, we define two partial concatenations for pictures. The first one, called "column concatenation", juxtaposes two pictures next to each other (i.e., concatenates the right column of one picture with the left column of the other) provided they have the same height.

Let P and Q be pictures of size (k, l) and (m, n), respectively. If $k = m \geqslant 1$, then the *column concatenation* of P and Q is defined by

$$P \mathbin{\varobar} Q : [k] \times [l+n] \to \Gamma, (i, j) \mapsto \begin{cases} P\langle i, j\rangle & \text{if } j < l, \\ Q\langle i, j-l\rangle & \text{if } j \geqslant l. \end{cases}$$

[1] Unfortunately, this notation sometimes conflicts with the k-fold Cartesian product, which is considered a different operation. I hope that the reader can always guess what is meant.

Besides, $P \oplus \varepsilon = \varepsilon \oplus P = P$ for every picture P.

These definitions can be extended to sets of pictures as usual, i.e., for $L, M \subseteq \Gamma^{*,*}$ we define $L \oplus M = \{P \oplus Q \mid P \in L, Q \in M\}$. The $\oplus$-symbol is often omitted.

This concatenation can be iterated: For a set $L \subseteq \Gamma^{*,*}$ of pictures we set $L^{1,0} := \{\varepsilon\}$ and $L^{1,k+1} := L^{1,k} \oplus L$. We set $L^{1,*} := \bigcup_{k \geqslant 0} L^{1,k}$ and $L^{1,+} := \bigcup_{k \geqslant 1} L^{1,k}$. Since here the first superscript is always 1, we allow to drop it, thus writing L^k instead of $L^{1,k}$ and so on.

A partial row concatenation, denoted $\ominus$, can be defined similarly: for two pictures P, Q of the same width, $P \ominus Q$ is the picture that results from P by appending Q to the bottom. This row concatenation, however, will be used much less frequently than the column concatenation. Its iteration is defined as follows. For a set $L \subseteq \Gamma^{*,*}$ of pictures we set $L^{0,1} := \{\varepsilon\}$ and $L^{k+1,1} = L^{k,1} \ominus L$. We define $L^{*,1} = \bigcup_{k \geqslant 0} L^{k,1}$ and $L^{+,1} = \bigcup_{k \geqslant 1} L^{k,1}$.

Words vs. pictures: A non-empty picture of size (m,n) over alphabet Γ may be viewed as a word over alphabet $\Gamma^{m,1}$ of length n, the so-called *column word*. The set of all column words of pictures of height m of a given picture language L is called the *height-m fragment of L*. These definitions are made formal in Definition 4.1. We will not identify a picture with its column word, but some frequently used notions like "infix" of a picture, etc., will advocate this view. For example, if P, Q, R are (possibly empty) pictures for which PQR is defined, then Q is an "infix" and P is a "prefix" of PQR. Furthermore, $cycl(L)$ and $pref_+(L)$ denote the cyclic closure and the set of non-empty prefixes, respectively, of a picture language L.

Conversely, every non-empty word is identified with a picture of height 1 over the same alphabet, and the empty word is identified with the empty picture. Consequently, every word language not containing the empty word is also a picture language.

Cyclic: It is sometimes very helpful to think of pictures as if the rightmost column was connected to the leftmost column forming a ring-like stripe. Whenever this ring structure is referred to, we will indicate this by using the word "cyclic". For example, if P, Q, R are pictures for which PQR is defined, then RP will be called a "cyclic infix" of PQR.

Coloured grids vs. Pictures: Recall from Section 2.1 the definition of grids and t-bit grids. To every t-bit grid $([m]\times[n], S_1, S_2, X_1, \ldots, X_t)$ we may associate a non-empty picture P over alphabet $\{0,1\}^t$ of the same size, where $P\langle i,j\rangle = (b_1, \ldots, b_t) \in \{0,1\}^t$ with $\forall s: b_s = 1 \leftrightarrow (i,j) \in X_s$.

Conversely, from every non-empty picture over alphabet $\{0,1\}^t$ of size (m,n) we may extract an assignment of sets $X_1^P, \ldots, X_t^P$ to the set variables $X_1, \ldots, X_t$, namely by defining X_s^P to be the set of positions (i,j) for which the sth component of $P\langle i,j\rangle$ is 1. This way we obtain for every non-empty picture P over $\{0,1\}^t$ the t-bit grid whose associated picture is P.

We allow ourselves to be imprecise when distinguishing t-bit grids from pictures over $\{0,1\}^t$. For example, if P is a picture of size (m,n) over $\{0,1\}^t$ and $\varphi(X_1, \ldots, X_t)$ is a formula over signature τ_{Grids}, then we write $P \models \varphi$ instead of $\underline{[m]\times[n]} \models \varphi[X_1^P, \ldots X_t^P]$.

We write $Mod_t(\varphi)$ for the set of pictures P over $\Gamma=\{0,1\}^t$ with $P\models\varphi$, i.e., the set of pictures associated to models of φ. If L is a picture language over alphabet Γ and $\mathscr{F}$ a class of formulas over grids, then we say that L is $\mathscr{F}$-definable and write $L\in\underline{\mathscr{F}}$ (or $L\in\underline{\mathscr{F}}(\Gamma)$) iff there is an $\mathscr{F}$-formula $\varphi(X_1,\ldots,X_t)$ over τ_{Grids}, i.e., a sentence φ over τ_{Grids_t}, such that $L=Mod_t(\varphi)$.

Binary number representations: For every $m,n\geqslant 0$ with $n\leqslant 2^m-1$ we write $BIN(n,m)$ for the binary representation of length m of the number n, least significant bit first. Conversely, if w is a word of length m over $\{0,1\}$, then $dual(w)=\sum_{i=0}^{m-1}2^i w\langle i\rangle$ is the number represented by w in binary, least significant bit first.

It will be technically convenient to consider the binary representation with least significant bit first, i.e., the other way round than usual.

2.3.1. Different notions of locality

This subsection is concerned with the transfer of the important notions of locality, locally threshold testability, and recognizability from word languages to picture languages. Let us briefly recall some basic facts about these notions in the world of word languages.

Local word languages: A word language L (over some alphabet Γ not containing #) is *local* iff there is a set Δ of words of length 2 over $\Gamma\cup\{\#\}$ such that L is the set of those words w for which all infixes of length 2 of the word $\#w\#$ are in Δ.

Locally threshold testable word languages: For two words u,w we let $occ(u,w)$ be the number of occurrences of w as an infix of u.

A word language L is *locally threshold testable* iff it is a finite union of equivalence classes of equivalence relations $\cong_{d,t}$, where $u\cong_{d,t}v$ holds for two words u,v and two numbers $d,t\geqslant 1$ iff for each word w of length $\leqslant d$ we have $occ(u,w)$ and $occ(v,w)$ are either both $\geqslant t$ or both $<t$.

Intuitively, a word language is locally threshold testable iff membership of a word can be decided by counting the numbers of occurrences of infixes of bounded length up to a fixed threshold.

Local picture languages: Now, we define when to call a picture language "local". These definitions will be compatible with our agreement to identify word languages with picture languages all of whose elements have height 1.

Let P be a picture of size (m,n) over Γ. If $0\leqslant i\leqslant i'\leqslant m-1$ and $0\leqslant j\leqslant j'\leqslant n-1$, we write $P([i,i']\times[j,j'])$ for the picture P' of size $(i'-i+1,j'-j+1)$ over Γ defined by $P'\langle x,y\rangle=P\langle i+x,j+y\rangle$ for every $(x,y)\in[0,i'-i]\times[0,j'-j]$. The picture P' is called a *subblock of P*.

Let # be a new symbol not in Γ. By $\hat{P}$ we denote the picture of size $(m+2,n+2)$ over $\Gamma\cup\{\#\}$ that results from P by surrounding it with the symbol #, i.e.,

$$\hat{P}\langle i,j\rangle=\begin{cases}P\langle i-1,j-1\rangle & \text{if } (i,j)\in[1,m]\times[1,n],\\ \# & \text{if } i\in\{0,m+1\}\vee j\in\{0,n+1\}.\end{cases}$$

A picture language L is *local* if there is a set $\Delta \subseteq (\Gamma \cup \{\#\})^{2,2}$ such that L is the set of non-empty pictures P over Γ such that all (2×2)-subblocks of $\hat{P}$ are in Δ. In that case we write $\mathscr{L}(\Delta) = L$. We call Δ a *local tiling system* and its elements *tiles*.

A picture language is *recognizable* if it is the image of a local picture language under some *alphabet projection*, i.e., under some possibly non-injective renaming of letters. This is one of the various natural adaptations of the notion of recognizability of word languages by nondeterministic finite automata. In our context, recognizable picture languages are important because they are exactly the Σ_1-definable picture languages. See [7] for a survey on this subject.

Two simple examples of local pictures languages follow.

Example 2.4. Consider

$$L = \{P \in \{0,1\}^{+,+} \mid \forall (i,j) \in \operatorname{dom} P : i \neq 0 \rightarrow P\langle i,j\rangle = 0\},$$

$$M = \{P \in \{0,1\}^{+,+} \mid \forall (i,j) \in \operatorname{dom} P : P\langle i,j\rangle = 1 \leftrightarrow i = j\}$$

i.e., L is the set of all non-empty pictures over $\{0,1\}$ that have no 1 in a non-top row, and M is the set of pictures over $\{0,1\}$ that have 1's exactly in the “diagonal” that starts in the upper left corner. Then L and M are local because

$$L = \mathscr{L}\left(\left\{ \begin{matrix} a & c \\ b & d \end{matrix} \in \{0,1,\#\}^{2,2} \,\middle|\, b,d \neq 1 \vee a,c = \# \right\}\right),$$

$$M = \mathscr{L}\left(\left\{ \begin{matrix} a & c \\ b & d \end{matrix} \in \{0,1,\#\}^{2,2} \,\middle|\, d = 1 \leftrightarrow (a = 1 \vee a = b = c = \#) \right\}\right).$$

Remark 2.5. Every local picture language is first-order definable (in the signature with two binary successor relation symbols, but without orderings).

The above remark is a special case of a more general result of [8] that states that the class of first-order definable picture languages coincides with the class of *locally threshold testable* picture languages. Though this notion is not needed in this paper, we introduce it here for the interested reader. It is a straightforward adaptation of the corresponding notion for word languages, see Section 2.3.1.

Definition 2.6 (*Cf. Giammarresi and Restrivo* [6, 7] *and Giammarresi* [8]). For two pictures P, R we let $occ(P,R)$ be the number of occurrences of R as a subblock of P. A picture language L is *locally threshold testable* iff it is a finite union of equivalence classes of equivalence relations $\cong_{d,t}$, where $P \cong_{d,t} Q$ holds for two pictures P, Q and two numbers $d, t \geqslant 1$ iff for each picture R of height and width $\leqslant d$ we have that $occ(P,R)$ and $occ(Q,R)$ are either both $\geqslant t$ or both $< t$.

Cyclically local picture languages: Recall that the word “cyclic” indicates that one should imagine the rightmost column of a picture as connected to the leftmost column.

With this intuition, it is straightforward to deduce a definition of "cyclically local picture language" and the like.

Definition 2.7 (*Cyclic subblocks*). Let P and Q be pictures. Q is a *cyclic subblock of* P iff it is a subblock of some picture in $cycl\{P\}$. A *properly cyclic subblock* of P is a picture of the form $P([i,i'] \times [j', n-1]) \oplus P([i,i'] \times [0,j])$, where $0 \leqslant i \leqslant i' \leqslant \underline{\bar{P}}$ and $0 \leqslant j < j' \leqslant |P|$.

Obviously, a picture Q is a cyclic subblock of P iff it is a subblock or a properly cyclic subblock. For example, the words *ca* as well as *ab* are (as (1×2)-pictures) cyclic subblocks of *abc*, but the word *abca* is not.

Let P be a picture of size (m,n) over Γ and # a new symbol not in Γ. By $\tilde{P}$ we denote the picture of size $(m+2,n)$ over $\Gamma \cup \{\#\}$ that results from P by attaching one row of #'s to the top and one to the bottom, i.e.,

$$\tilde{P}\langle i,j\rangle = \begin{cases} P\langle i-1,j\rangle & \text{if } (i,j) \in [1,m] \times [0,n-1], \\ \# & \text{if } i \in \{0,m+1\},\ j \in [0,n-1]. \end{cases}$$

A picture language L is *cyclically local* if there is a set $\Delta \subseteq (\Gamma \cup \{\#\})^{2,2}$ such that L is the set of non-empty pictures P over Γ such that Δ *cyclically tiles* $\tilde{P}$, i.e., such that all cyclic 2×2-subblocks of $\tilde{P}$ are in Δ.

The next example illustrates the definition of cyclic locality. The idea of a counting mechanism, which plays a crucial rôle here, will be important later on (see Definition 3.15).

Example 2.8. For a picture $C \in \{0,1\}^{m,1}$, we have that $dual(C^\top)$ is the number represented by C in binary, where the least significant bit is at the top.

Let L be the language of non-empty pictures P over $\{0,1\}$ for which $dual(C'^\top) = dual(C^\top) + 1 \bmod 2^{\underline{\bar{P}}}$ for every two cyclically successive columns C, C' of P. Then L is a cyclically local picture language. To see this, let Δ be the set of cyclic (2×2)-subblocks of

$$\begin{matrix} \# & \# & \# & \# & \# & \# & \# & \# \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ \# & \# & \# & \# & \# & \# & \# & \# \end{matrix}$$

Then L is indeed the set of pictures P for which Δ cyclically tiles $\tilde{P}$.

We conclude with two simple propositions, whose proofs are left to the reader.

Proposition 2.9. *Let Γ be an alphabet. If L is a local (or cyclically local) word language, then $top^{-1}(L)$ is a picture language with the respective property.*

Proposition 2.10. *Let $\alpha : \Gamma \to \Omega$ be a alphabet projection. If $L \subseteq \Omega^{+,+}$ is a local (or cyclically local, or locally threshold testable, respectively) picture language, then so is $\alpha^{-1}(L)$.*

2.3.2. Alternation hierarchies for starfree picture languages

Starfreeness: The class of *starfree* word languages is the smallest class of word languages (over a fixed alphabet) that contains all finite word languages and is closed under concatenation, union, and complement (relative to the set of all words over the fixed alphabet).

This notion can be easily transferred to picture languages. The set of *starfree picture languages over Γ* (denoted $SF(\Gamma)$) is the smallest class of picture languages over alphabet Γ that contains all finite picture languages and is closed under both concatenations $\ominus$ and $\oplus$ as well as under union and complement (relative to the set of all non-empty pictures over Γ).

Since the set of pictures of height 1 over a fixed alphabet is a starfree picture language, the notion of starfreeness is compatible with our identification of words with pictures of height 1.

The following proposition from [14] gives an example of a starfree picture language.

Example 2.11. Let *Corners* be the set of non-empty pictures P over $\{0,1\}$ such that whenever $P\langle i,j\rangle = P\langle i',j\rangle = P\langle i,j'\rangle = 1$ then also $P\langle i',j'\rangle = 1$. (Intuitively: Whenever three corners of a rectangle carry a 1, then also the fourth one does.) *Corners* is a starfree picture language.

To see this, let $L := \bigcup (w(\{0,1\}^{*,*})x) \ominus (\{0,1\}^{*,*}) \ominus (y(\{0,1\}^{*,*})z)$, where the union ranges over all quadruples $(w,x,y,z) \in \{0,1\}^4$ such that $wxyz \in 1^*01^*$. Then L is the set of all pictures over $\{0,1\}$ such that exactly one of its four corners carries an 0.

Clearly, $(\{0,1\}^{*,*}) \ominus ((\{0,1\}^{*,*})L(\{0,1\}^{*,*})) \ominus (\{0,1\}^{*,*})$ is the complement of *Corners*, so *Corners* is starfree. Note that the use of the empty picture is non-essential—one may eliminate the use of $\{0,1\}^{*,*}$ from the above definitions, resulting in slightly more complex expressions.

In [14] it is shown that *Corners* is not recognizable.

There is a certain relation between the starfree word languages and first-order logic: the starfree word languages are exactly those which are definable by a formula in first-order logic with the built-in order relation. This indicates that "starfreeness" is a robust notion for word languages.

However, [24] shows that the notion of "starfreeness" is not as robust for picture languages. The author gives an example of a non-starfree picture language that is definable in first-order logic with the two partial successor relations.

It turns out that, nevertheless, the class of starfree picture languages is rich enough to contain picture languages beyond arbitrarily high levels of the monadic hierarchy, which is one of our main results. The proof shows that there is a certain relation between the alternation of quantifiers (or, equivalently, of existential quantifiers and negations) on one side and the alternation of concatenation and negations on the other. The investigation of this relation is prepared now.

Alternation: Recall from the introduction that we plan to prove all of our separation results uniformly, both those that involve starfreeness and those that deal with monadic second-oder logic only.

We define an alternation hierarchy for starfree picture languages. Later we will prove that this hierarchy "cuts through" the monadic alternation hierarchy in the sense that in level $k+1$ of this alternation hierarchy for starfree picture languages, there is a picture language that is not on level k of the monadic alternation hierarchy.

Let $\mathscr{L}$ be a class of picture languages over some fixed alphabet Γ. The $\oplus$-*closure*, the $\oplus\ominus$-*closure*, the $\cup\cap$-*closure*, and the *boolean closure of* $\mathscr{L}$ are defined as the smallest superset of $\mathscr{L}$ that is closed under $\oplus$-concatenations, or under both $\ominus$- and $\oplus$-concatenations, or under both union and intersection, or under both union and complement (relative to $\Gamma^{+,+}$), respectively. We denote these classes by $\oplus$-cl($\mathscr{L}$), $\oplus\ominus$-cl($\mathscr{L}$), $\cup\cap$-cl($\mathscr{L}$), and $B(\mathscr{L})$, respectively.

The following definition is a straightforward adaptation of the dot-depth hierarchy of starfree word languages.

Definition 2.12 (*Dot-depth hierarchy*). *dot-depth*$_0(\Gamma)$ is the class of all finite or cofinite picture languages over Γ, and

$$dop\text{-}depth_{k+1}(\Gamma) = B(\oplus\ominus\text{-cl}(dot\text{-}depth_k(\Gamma)))$$

for every $k \geqslant 0$.

Intuitively, a picture language over alphabet Γ is in *dot-depth*$_k(\Gamma)$ iff it can be constructed from finite picture languages by k alternations of (row-/column-) concatenation on one side and boolean combinations on the other side.

The dot-depth hierarchy of word languages is a well studied hierarchy (see e.g. [3, 19]), which is strict and exhausts the class of starfree word languages. The strictness of the dot-depth hierarchy of picture languages can be deduced easily from the strictness of the dot-depth hierarchy of word languages because the dot-depth levels are compatible with our identification of words with pictures of height 1.

Every level of the dot-depth hierarchy is closed under boolean combinations, which is not the case for the monadic hierarchy. So if we want to deduce also facts like "$\underline{\Sigma_k} \neq \underline{\Pi_k}$" from the result that our alternation hierarchy of starfree picture languages "cuts through" the monadic hierarchy, we need a version of this alternation hierarchy that gives reasonable distinctions between Σ- and Π-branches.

I suggest the following. (Recall the definitions from the previous page.)

Definition 2.13 (*Local alternation hierarchy*). Let Γ be an alphabet.

Let $\Sigma_0^{\mathrm{loc}}(\Gamma)$ denote the smallest class of picture languages over Γ that contains all local and all cyclically local picture languages and all *top*-pre-images of locally threshold testable word languages, and that is closed under boolean combinations.

For every $k \geqslant 0$, let

$$\Sigma_{k+1}^{\mathrm{loc}}(\Gamma) = \cup\cap\text{-cl}(\varobar\text{-cl}(B(\Sigma_k^{\mathrm{loc}})))(\Gamma).$$

For every $k \geqslant 0$, let $\Pi_k^{\mathrm{loc}}(\Gamma)$ denote the class of picture language that contains all complements of languages in $\Sigma_k^{\mathrm{loc}}(\Gamma)$, and $\Delta_k^{\mathrm{loc}}(\Gamma) = \Sigma_k^{\mathrm{loc}}(\Gamma) \cap \Pi_k^{\mathrm{loc}}(\Gamma)$.

We omit the explicit mentioning of Γ if it is clear from the context. Σ_k^{loc} is called the kth level of the *local alternation hierarchy*.

Note that in this hierarchy, the row concatenation is completely disregarded. Thus it certainly does not exhaust the starfree picture languages. For example, the singleton picture language $\{(\begin{smallmatrix} a \\ a \end{smallmatrix})\}$ is in none of the Σ_k^{loc}, as one easily verifies by induction on k.

There are several ways to modify the definition of this hierarchy in order to make it look more familiar. Firstly, one could choose to replace the $\varobar$-closure in the definition of Σ_k^{loc} by the $\varobar\ominus$-closure. Then the resulting hierarchy would exhaust the class of starfree picture languages.

Secondly, level zero could be modified to contain also all first-order definable (i.e., all locally threshold testable) picture languages.

A third possibility is to modify the recursive definition in such a way that every level of the hierarchy is closed under positive boolean combinations *as well as* $\varobar$- and $\ominus$-concatenations.

The most important reason for the above definition is that by considering this hierarchy, we can state (and prove) the strongest separation results we have: clearly, all of the above modifications would make the levels larger and thus our separation results weaker.

Proposition 2.14. $B(\Sigma_k^{\mathrm{loc}})(\Gamma) \subseteq$ *dot-depth*$_{k+1}(\Gamma)$ *for every* $k \geqslant 0$.

This proposition is shown in Section 3.1.1. In Section 3.1.2, Theorem 3.4, we will show that $\Sigma_k^{\mathrm{loc}} \subseteq \underline{\Sigma_k^0(\Delta_1^{\mathrm{U}})} \subseteq \underline{\Sigma_k}$ for every $k \geqslant 1$.

Syntax vs. semantics: Since this paper concentrates on expressiveness results, I chose not to introduce unnecessary syntactic notions. However, the reader who is used to such notions as “regular” or “starfree expressions” may easily fill this gap. There are a few remarks on the lengths of starfree expressions in Paragraph 3.3.1.

2.4. Separation results

In this subsection I present all separation results we have. The proofs of these results will be presented in the following sections.

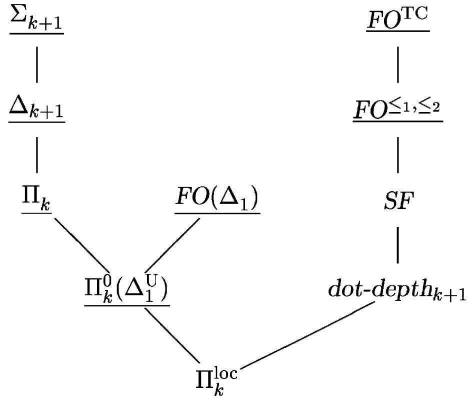


Fig. 1. Picture Languages Separated from $\underline{\Sigma_k}$. In this diagram, line indicate (not necessarily proper) inclusions, all of which are easy. For the $k \geqslant 1$ under consideration, the bottom-most class—and therefore all the others, are not contained in $\underline{\Sigma_k}$, provided that the underlying alphabet has at least two symbols.

2.4.1. Separation results for pictures

Recall that if Γ is an alphabet of the form $\{0,1\}^t$, we write $\underline{\Sigma_k}(\Gamma)$ for the class of $\underline{\Sigma_k}$-definable picture languages over Γ.

In [17], the following separation results have been shown (not all of them stated explicitly):

Fact 2.15. *For every* $k \geqslant 1$, *the following classes contain a picture language over* $\{0,1\}^{2k}$ *that is not in* $\underline{\Sigma_k}$ *(see Fig.* 1*):*
- $\underline{FO^{\leqslant_1,\leqslant_2}}$ *and thus* $\underline{FO^{\mathrm{TC}}}$;
- $\underline{\Pi_k^0(\Delta_1)}$ *and thus also the superclasses* $\underline{FO(\Delta_1)}$ *and* $\underline{\Pi_k}$, $\underline{\Sigma_{k+1}}$, *and* $\underline{\Delta_{k+1}}$.

The two definability parts (one for $\underline{FO^{\leqslant_1,\leqslant_2}}$ and one for $\underline{\Pi_k^0(\Delta_1^{\mathrm{U}})}$) of the above proof are due to Schweikardt. They require two very similar constructions with subtle differences and do not give any results about starfree picture languages. The first contribution of this paper is the following theorem, which says that the local alternation hierarchy for picture languages cuts through the monadic second-order quantifier alternation hierarchy for picture languages. It is shown in Section 3.3 (Corollary 3.25) and Section 4.1 (Corollary 4.3).

Theorem 2.16. *For every* $k \geqslant 1$,

$$\Pi_k^{\mathrm{loc}}(\{0,1\}^{2k}) \not\subseteq \underline{\Sigma_k}(\{0,1\}^{2k}),$$

i.e., there is a starfree picture language over alphabet $\{0,1\}^{2k}$ *in the complement of the kth level of local alternation hierarchy which is not in the kth level of the monadic second-order quantifier alternation hierarchy.*

Since $\Pi_k^{\mathrm{loc}} \subseteq \mathit{dot\text{-}depth}_{k+1} \subseteq SF \subseteq \underline{FO^{\leqslant_1,\leqslant_2}}$ and $\Pi_k^{\mathrm{loc}} \subseteq \underline{\Pi_k^0(\Delta_1^{\mathrm{U}})}$, this reproves and extends Fact 2.15 and shows the following.

Corollary 2.17. *For every $k \geqslant 1$, the following classes contain a picture language over $\{0,1\}$ that is not in Σ_k:*
- *$\mathit{dot\text{-}depth}_{k+1}$ and thus SF, $\underline{FO^{\leqslant_1,\leqslant_2}}$, and $\underline{FO^{\mathrm{TC}}}$;*
- *$\underline{\Pi_k^0(\Delta_1^{\mathrm{U}})}$ and thus $\underline{FO(\Delta_1)}$ as well as $\underline{\Pi_k}$, $\underline{\Sigma_{k+1}}$, and $\underline{\Delta_{k+1}}$.*

The above implies in particular that $SF \nsubseteq \underline{\Sigma_1}$, which has first been shown in [14], solving an open problem from [7]. It had even been unknown whether every regular picture language is in $\underline{\Sigma_1}$, where "regular" means that it results from finite picture languages in a finite number of applications of row-/column-concatenations, boolean combinations, and the Kleene-like iteration of row-/column-concatenations. These questions are somewhat natural to ask because in the one-dimensional case, $\underline{\Sigma_1}$ reflects the notion of recognizability by non-deterministic automata, so that by the Kleene Theorem, the classes of regular and Σ_1-definable classes of word languages coincide.

Note that in Fact 2.15, Theorem 2.16, and Corollary 2.17, there is no fixed bound on the alphabet size for which the respective picture language classes can be separated. However, with standard encoding techniques, this can be repaired, except for the class Π_k^{loc}. In [16], I give a variant of the local alternation hierarchy, the *generalized local alternation hierarchy*, which enables this encoding in the following sense: If the local alternation hierarchy is replaced by the generalized variant, then the levels become larger, but Theorem 3.1 and Proposition 2.14 remain true, and Theorem 2.16 remains true even if the alphabet $\{0,1\}^{2k}$ is replaced by $\{0,1\}$.

In Section 2.4.2 we shall summarize the separation results we have for pictures over a one-letter alphabet.

2.4.2. Separation results for (non-coloured) grids

The investigation of starfree picture languages over a singleton alphabet does not make much sense because the class of starfree picture languages over a singleton alphabet is very poor—in particular, every starfree language over a one-letter alphabet is Σ_1-definable, see e.g. [12]. However, there are non-trivial separations concerning the expressiveness of certain fragments of monadic second-order logic. In this context it is more canonical to refer to "grids" (i.e., particular finite structures) rather than "pictures over a trivial alphabet".

The growth technique: Recall that for a set L of grids, size(L) is the set of all (m,n) such that the grid of size (m,n) is in L.

Our separation results for the class of non-coloured grids are of two types. The first one is concerned with classes L of grids where size(L) is a function, i.e., for every m there is exactly one grid of height m in L. Separations of the first type are then witnessed by k-fold exponential functions for increasing k. This has been called "the growth technique" in [1].

The second type of separation results is by classes of grids L where size(L) is a non-functional relation of $\mathbb{N}$. The separation $\underline{\Pi_k} \nsubseteq \underline{\Sigma_k}$ is witnessed by a set L of grids where the function $m \mapsto \min\{n \mid (m,n) \in \text{size}(L)\}$ is $(k+1)$-fold exponential. This is further explained in Theorem 2.26.

Definition 2.18. A sentence φ over τ_{Grids} *defines* a relation $r \subseteq \mathbb{N}^2$ iff $r = \{(m,n) \mid \underline{[m] \times [n]} \models \varphi\}$. A relation r is $\mathscr{F}$-definable (for a class of formulas $\mathscr{F}$ over τ_{Grids}), if there is a sentence φ that defines r.

Let $f : \mathbb{N} - \to \mathbb{N}$ be a partial function. Then f is called *at most k-fold exponential* if $f(m)$ is $s_k(\mathcal{O}(m))$, where $s_0(m) = m$ and $s_{k+1}(m) = 2^{s_k(m)}$ for every $m \geqslant 1$, $k \geqslant 0$.

We call f *at least k-fold exponential* if f is total and $f(m)$ is $s_k(\Omega(m))$. If f is both at most and at least k-fold exponential, i.e, if f is total and $f(m)$ is $s_k(\Theta(m))$, then we say that f is *k-fold exponential.*

The following notion is convenient to state some of our results succinctly.

Definition 2.19 (*Asymptotic bounds for formula classes*). Let $\mathscr{F}$ be a class of formulas over τ_{Grids}. We say that

- "$\mathscr{F}$ is at most k-fold exponential" iff f is at most k-fold exponential for every $\mathscr{F}$-definable function f;
- "$\mathscr{F}$ is at least k-fold exponential" iff f is at least k-fold exponential for some $\mathscr{F}$-definable function f;
- "$\mathscr{F}$ is k-fold exponential" iff $\mathscr{F}$ is both at most and at least k-fold exponential.

Example 2.20. In [5], the author shows that Σ_1 is singly (i.e. one-fold) exponential. This means that every Σ_1-definable function is at most singly exponential, and there is one particular singly exponential, Σ_1-definable function (namely $m \mapsto 2^m$).

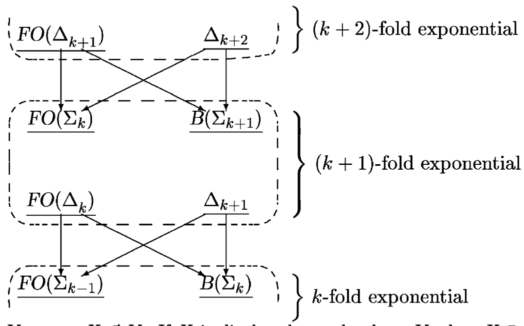
There is a more detailed investigation of the class of Σ_1-definable functions in [5]. It is shown that this class is closed under certain operations and deduced that, for example, every polynomial with non-negative integer coefficients is Σ_1-definable.

In [17], Example 2.20 is extended to higher levels of the monadic alternation hierarchy.

Fact 2.21. *For every $k \geqslant 1$, the formula class $\Delta_1(\Pi^0_{k-1}(\Delta_1))$ is at least k-fold exponential, and thus so are the larger classes $\Delta_k, \Sigma_k, B(\Sigma_k)$.*

On the other hand, $B(\Sigma_k)$ is at most k-fold exponential, thus the above formula classes are k-fold exponential.

We reprove the first statement and extend it to the following, which is the second contribution of this paper (and equal to Corollary 3.33).



$X \to Y$ means $X \not\subseteq Y$. If X is displayed exactly above Y, then $X \supseteq Y$ is valid for trivial reasons. Classes encircled by dashed lines allow for the same rate of growth of functions—nothing is claimed about mutual inclusions!

Fig. 2. Separation results for grids.

Theorem 2.22. *For every $k \geqslant 1$, the formula class $\Delta_1^{\mathrm{U}}(\Pi_{k-1}^0(\Delta_1^{\mathrm{U}}))$ is at least k-fold exponential* (*and thus*, *by the above fact*, *also k-fold exponential*).

In [1], the first-order closure has been introduced and the following has been shown.

Fact 2.23. *$FO(\Sigma_1)$ is at most two-fold exponential.*

We reprove and extend the above theorem the following way, which is the third contribution of this paper (consequence of Theorems 3.44 and 4.6).

Theorem 2.24. *Let $k \geqslant 1$. The formula class $\Pi_2^0(\Delta_1^{\mathrm{U}}(\Pi_{k-1}^0(\Delta_1^{\mathrm{U}})))$ is at least $(k+1)$-fold exponential, and thus so are the larger classes $\Pi_2^0(\Delta_k)$ and $FO(\Sigma_k)$.*

On the other hand, the formula class $FO(\Sigma_k)$ is at most $(k+1)$-fold exponential, thus the above, smaller formula classes are $(k+1)$-fold exponential.

Fig. 2 illustrates some of the separation results stated in the above two theorems for grid classes separated by asymptotic growth rates.

Referring to Theorem 2.22, the above may be rephrased as follows: *With respect to the growth rate of functions*, the first-order closure is as powerful as 'one more' alternation in the monadic hierarchy. However, this should be read with some caution because the focus of attention to asymptotic growth rate bounds is quite essential here—it is by no means true that any alternation of first-order quantifications can be replaced by one block of monadic quantifications. See also Corollary 2.25.

It is a challenging unsolved problem to prove that $\underline{\Pi_j^0(\Sigma_k)} \subsetneq \underline{\Pi_{j+1}^0(\Sigma_k)}$ over grids for every j, k. We only know that this holds for 1-bit grids and $k = 1$.

The separations witnessed by fast growing function are summarized in Corollary 2.25.

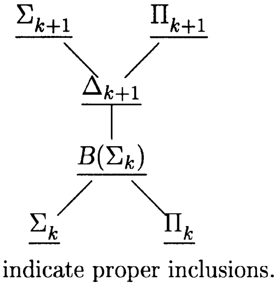


Fig. 3. The monadic hierarchy over grids.

Corollary 2.25. *For every $k \geqslant 0$, the following classes of formulas allow to define a set of grids that is not $\underline{B(\Sigma_k)}$-definable and (if $k \geqslant 1$) neither $\underline{FO(\Sigma_{k-1})}$-definable:*

- $\Delta_1^{\mathrm{U}}(FO^{\leqslant_1,\leqslant_2})$ *and thus* $\Delta_1^{\mathrm{U}}(FO^{\mathrm{TC}})$;
- $\Delta_1^{\mathrm{U}}(\Pi_k^0(\Delta_1^{\mathrm{U}}))$ *and thus* $\Delta_1(FO(\Delta_1))$ *as well as* Π_{k+1} *and* Δ_{k+1};
- $\Sigma_1(\Pi_k^0(\Delta_1^{\mathrm{U}}))$ *and thus* $\Sigma_1(\Pi_k^0(\Delta_1))$ *and* Σ_{k+1};
- *if* $k \geqslant 1$, *also* $\Pi_2^0(\Delta_1^{\mathrm{U}}(\Pi_{k-1}^0(\Delta_1^{\mathrm{U}})))$ *and thus* $\Pi_2^0(\Delta_k)$, $FO(\Delta_k)$, $\Pi_1^0(\Sigma_k)$, $FO(\Sigma_k)$.

The result "$\underline{FO(\Sigma_k)} \subsetneq \underline{FO(\Sigma_{k+1})}$ for the class of grids" may be rephrased informally as follows: The monadic hierarchy quantifier remains strict when one considers the first-order closure of each level. However, this result should not be overestimated because the first-order closure of Σ_k is defined somewhat artificially in the sense that—in contrast to the levels of "closed" hierarchy of [2]—first-order quantifications are added only in the outermost level of second-order quantifications.

Separation results with non-functional relations: While the previous results imply the situation depicted in Fig. 3 for the classes of 1-bit grids, it remained open in [17, 18, 21] how the situation is for the class of non-coloured grids. We know that $\underline{B(\Sigma_k)} \subsetneq \underline{\Delta_{k+1}}$ for the class of grids by Theorem 2.22, but the same theorem tells us that if there is a *function* f witnessing that $\underline{\Sigma_k}$ and $\underline{\Pi_k}$ are incomparable, then the reason is not only the asymptotic growth rate of f.

However, this separation result can be shown using a witness set that is not a functional relation. This shows that Fig. 3 is correct for grids, too, and it implies that the class of recognizable (i.e., Σ_1-definable) picture languages is not closed under complement even in the case of a trivial alphabet. This disproves a conjecture from [13]. Precisely, the following consequence of Theorem 3.44 and Corollary 4.4 is the fourth and last contribution of this paper.

Theorem 2.26. *For every $k \geqslant 1$, there is a $(k+1)$-fold function f such that the relation $\{(m,n) \mid f(m)$ divides $n\}$ is definable in $\Pi_1^0(\Delta_1^{\mathrm{U}}(\Pi_{k-1}^0(\Delta_1^{\mathrm{U}})))$ (and thus in $\Pi_1^0(\Delta_k)$ and Π_k) but not in Σ_k.*

A corollary of the above theorem (for $k = 1$) is that the class of recognizable picture languages is not closed under complement even for a unary alphabet, because this picture language class corresponds to $\underline{\Sigma_1}$ over grids.

2.4.3. Separation results for graphs

In this paper, we restricted our interest to particular finite structures, namely grids. However, this restriction is not essential, because [17] provides an encoding technique that allows us to transfer certain separation results of Corollary 2.17 from coloured grids to other structures, say graphs. This encoding technique, called strong first-order reductions, allows us to transfer separation results that deal with formula classes that result from the class of first-order formulas by repeated application of the closures introduced in Section 2.2.

With those techniques, we can conclude the following from Corollaries 2.17 and 2.25.

Theorem 2.27. *Let $k \geqslant 1$. The following formula classes allow to define (within the class of graphs) a class of connected undirected finite graphs (of bounded degree) that is not definable in Σ_k:*

$$FO^{\mathrm{TC}},\ \Pi^0_{k-1}(\Sigma_1),\ FO(\Sigma_1),\ \Pi_k,\ \Sigma_{k+1},\ \Delta_{k+1}.$$

Besides, the following formula classes allow to define (within the class of graphs) a class of connected undirected finite graphs (of bounded degree) that is not definable in $B(\Sigma_k)$ and neither in $FO(\Sigma_{k-1})$:

$$\Delta_{k+1},\ \Sigma_1(\Pi^0_{k-1}(\Sigma_1)),\ \Pi_1(\Sigma^0_{k-1}(\Pi_1)).$$

The reason why not all of the separation results from Corollary 2.17 carry over to graphs arises from the fact that the class of (structures isomorphic to) grids is itself not first-order definable within the class of τ_{Grids}-structures.

3. Definability results

A separation result typically requires two parts. The first one states that a certain property, say of pictures, is a member of a certain class, e.g., expressible by a certain type of formula. The second part states that it is not a member of some other class.

In this section we collect all the definability results, i.e., all first parts. These are sometimes referred to as "upper bound proofs", but this term is misleading for our purposes because an essential idea for our definability results is to give *lower* bounds on the asymptotic growth rates of functions whose associated sets of grids are definable in a certain fragment.

Our strategy is the following. In Section 3.2 we will introduce some more notation that will be helpful later on. In Subsection 3.3, the most important subsection of this section, we develop a sequence $(Num_k)_{k \geqslant 1}$ of picture languages for which we will prove membership in Π^{loc}_{k-1}. The membership of Num_k in level k of the dot-depth hierarchy as well as its definability in some other fragments of monadic second-order logic we are interested in then follows easily because Π^{loc}_k is a subset of, e.g., $\underline{\Pi_k}$. These and some other easy inclusions will be presented in Section 3.1.2.

In Sections 3.4 and 3.5 we use Num_k to show the additional definability results for Theorems 2.22 and 2.24, respectively.

Talking about formulas: When we talk about formulas, it is often convenient to mix syntax and semantics. For example, when we say "the formula $\varphi(x, y)$ asserts for (a picture P and) a position x that x has some property A, provided that position y has property B", we really mean that for every non-empty picture P and every $u, v \in \operatorname{dom} P$, if v has property B, then $P \models \varphi[u, v]$ iff u has property A.

3.1. Easy inclusion results

In the next four subsections we will state and prove definability results for the local alternation hierarchy. Before we do that, we collect some easy inclusion results that show how to transfer these definability results to some other classes of picture languages.

In Section 3.1.1 we show that level k of the local alternation hierarchy is contained in level $k+1$ of the dot-depth hierarchy.

Section 3.1.2 shows how to transfer these definability results to classes of picture languages as defined by certain fragments of monadic second-order logic. It states e.g. $\Sigma_k^{\mathrm{loc}} \subseteq \underline{\Sigma_k}$.

3.1.1. Dot-depth and the local alternation hierarchy

Now we will prepare the proof of Proposition 2.14, which states that level k of the local alternation hierarchy is contained in level $k+1$ of the dot-depth hierarchy.

Before we do that, let us make an observation that deals with the empty picture.

Remark 3.1. For convenience and because I preferred not to consider "empty grids", picture languages must not contain the empty picture, and complementation is also relative to the set of non-empty pictures.

However, in proofs and examples like Example 2.11 we will frequently use the empty picture and complementation wrt $\Gamma^{*,*}$, so it is necessary to note that this does not affect the definition of the dot-depth hierarchy or the class of starfree picture languages.

Formally, if $\textit{dot-depth}'_k(\Gamma) \subseteq \mathscr{P}(\Gamma^{*,*})$ is defined as in Definition 2.12 but with complementation relative to $\Gamma^{*,*}$, then we have that for every $k \geqslant 0$ and every $L \subseteq \Gamma^{*,*}$ that

$$L \in \textit{dot-depth}'_k(\Gamma) \quad \Leftrightarrow \quad L \backslash \{\varepsilon\} \in \textit{dot-depth}_k(\Gamma)$$

and the same is true for the classes $SF(\Gamma)$ and Σ_k^{loc}, etc. The proof is easy, see e.g. [12, Lemma 3.7]. The essential point is that $\varobar$ and $\ominus$ distribute over $\cup$.

Lemma 3.2. *Every local or cyclically local picture language is of dot-depth* 1.

Proof. It suffices to show that for every (2×2)-picture Q over $\Gamma \cup \{\#\}$, the sets L_Q, M_Q, and N_Q of non-empty pictures P such that Q is a subblock of $\hat{P}$, a properly cyclic subblock of $\tilde{P}$, or a cyclic subblock of $\tilde{P}$, respectively, are of dot-depth one.

So let Q be a 2×2 be a picture over $\Gamma \cup \{\#\}$. It suffices to consider the nine cases that Q is of one of the forms

$$\begin{smallmatrix} a & b \\ c & d \end{smallmatrix},\ \begin{smallmatrix} a & b \\ \# & \# \end{smallmatrix},\ \begin{smallmatrix} \# & b \\ \# & d \end{smallmatrix},\ \begin{smallmatrix} \# & \# \\ c & d \end{smallmatrix},\ \begin{smallmatrix} a & \# \\ c & \# \end{smallmatrix},\ \begin{smallmatrix} a & \# \\ \# & \# \end{smallmatrix},\ \begin{smallmatrix} \# & b \\ \# & \# \end{smallmatrix},\ \begin{smallmatrix} \# & \# \\ c & \# \end{smallmatrix},\ \text{or}\ \begin{smallmatrix} \# & \# \\ \# & d \end{smallmatrix},\ \text{where } a,b,c,d \in \Gamma.$$

(In the other cases, say if $Q = \begin{smallmatrix} a & \# \\ \# & d \end{smallmatrix}$, then Q cannot occur as a subblock of any picture, thus $L_Q = M_Q = N_Q = \emptyset$.)

In the first case,

$$L_Q = \Gamma^{*,*} \ominus (\Gamma^{*,*} Q \Gamma^{*,*}) \ominus \Gamma^{*,*},$$
$$M_Q = \Gamma^{*,*} \ominus \left(\begin{pmatrix} b \\ d \end{pmatrix} \oplus \Gamma^{*,*} \oplus \begin{pmatrix} a \\ c \end{pmatrix} \right) \ominus \Gamma^{*,*}.$$

In the second case,

$$L_Q = \Gamma^{*,*} \ominus (\Gamma^* ab \Gamma^*),$$
$$M_Q = \Gamma^{*,*} \ominus (b \Gamma^* a).$$

In the third case,

$$L_Q = \Gamma^{*,*} \ominus \left(\begin{pmatrix} b \\ d \end{pmatrix} \oplus \Gamma^{*,*} \right) \ominus \Gamma^{*,*},$$
$$M_Q = \emptyset.$$

The other six cases are similar. In every case, the three sets L_Q and M_Q and thus $N_Q = L_Q \cup M_Q$ are of dot-depth one.

The following facts are stated here without proof.

Lemma 3.3. *Every locally threshold testable word language is of dot-depth* 1, *and if L is a word language of dot-depth* 1, *then $top^{-1}(L)$ is a picture language of dot-depth* 1.

Now, we are ready to prove the announced connection between the dot-depth hierarchy and the local alternation hierarchy, i.e., that $B(\Sigma_k^{\mathrm{loc}})(\Gamma) \subseteq \textit{dot-depth}_{k+1}(\Gamma)$ for every $k \geqslant 0$.

Proof of Proposition 2.14. The proof is by induction on k. The case $k = 0$ follows from Propositions 3.2, 3.3 and Lemma 3.3.

So assume the claimed implication is true for some $k \geqslant 0$. Then $B(\Sigma_{k+1}^{\mathrm{loc}}) = B(\cap\cup\text{-cl}(\oplus\text{-cl}(B(\Sigma_k^{\mathrm{loc}})))) = B(\oplus\text{-cl}(B(\Sigma_k^{\mathrm{loc}}))) \subseteq B(\oplus\text{-cl}(\textit{dot-depth}_{k+1})) = \textit{dot-depth}_{k+2}$. $\square$

3.1.2. *Local and monadic alternation hierarchy*

In this section we will show how to pass from a picture language of some given level of the local alternation hierarchy to a formula of monadic second-order logic of some particular shape.

The following is the main result of this section.

Theorem 3.4. *For every $k \geqslant 0$, every picture language in Σ_k^{loc} is definable by a $\Sigma_k^0(\Delta_1^{\mathrm{U}})$-formula.*

Before we turn to the proof of this theorem, let us conclude the following corollary, which will be used in the proofs of Corollaries 3.25 and 3.32.

Corollary 3.5. *Let $k \geqslant 0$. Every picture language in Σ_k^{loc} is definable by a formula in $FO^{\leqslant_1,\leqslant_2}$, in $\Sigma_k^0(\Delta_1^{\mathrm{U}})$, in $\Sigma_k^0(\Delta_1)$, in $FO(\Delta_1^{\mathrm{U}})$, and in $FO(\Delta_1)$. Every picture language in Π_k^{loc} is definable in $\Pi_k^0(\Delta_1)$.*

If $k \geqslant 1$, then every picture language in Σ_k^{loc} (or Π_k^{loc}, respectively) is definable in Σ_k (or Π_k, respectively).

Proof. We have $\Sigma_k^{\mathrm{loc}} \subseteq \underline{FO^{\leqslant_1,\leqslant_2}}$ by Proposition 3.9, and $\underline{\Sigma_k^{\mathrm{loc}}} \subseteq \underline{\Sigma_k^0(\Delta_1^{\mathrm{U}})} \subseteq \underline{FO(\Delta_1^{\mathrm{U}})} \subseteq \underline{FO(\Delta_1)}$ by Proposition 2.3 and Theorem 3.4.

If $k \geqslant 1$, then $\Sigma_k^{\mathrm{loc}} \subseteq \underline{\Sigma_k^0(\Delta_1)} \subseteq \underline{\Sigma_k^0(\Sigma_1)} \subseteq \underline{\Sigma_k}$ and $\Pi_k^{\mathrm{loc}} \subseteq \underline{\Pi_k^0(\Delta_1)} \subseteq \underline{\Pi_k}$ by calculation rules of Section 2.2.3. $\square$

We prepare the proof of Theorem 3.4 with the following definition. For both $i \in \{1,2\}$, let $\leqslant_i$ be the pre-order on $\mathbb{N} \times \mathbb{N}$ defined by $(m_1, m_2) \leqslant_i (n_1, n_2)$ iff $m_i \leqslant n_i$.

Lemma 3.6. *There are first-order formulas $smaller(x, Z)$ and $larger(x, Z)$ such that $smaller(x, Z)$ asserts that $Z = \{z \mid z \leqslant_2 x\}$, and $larger(x, Z)$ asserts that $Z = \{z \mid z \geqslant_2 x\}$.*

The proof is easy.

Throughout Section 3, the following formulas will be useful. The formula $top(x) = \neg\exists y(S_1 yx)$ asserts that position x is in the top row. Similarly, there are first-order formulas $bottom(x)$, $left(x)$, $right(x)$ that assert that x is at the respective border.

Recall the definition of (properly) cyclic subblock from Definition 2.7.

Proposition 3.7. *Let $t \geqslant 1$ and $\Gamma = \{0,1\}^t$. Let Q be a non-empty picture over $\Gamma \cup \{\#\}$. Let $\bar{X}$ be the variable tuple $(X_1, \ldots, X_t)$. There are $\overline{\Delta_1^{\mathrm{U}}}$-formulas has-subblock$_Q(\bar{X})$, has-properly-cyclic-subblock$_Q(\bar{X})$, has-cyclic-subblock$_Q(\bar{X})$ that assert for a picture P that Q is a subblock of $\hat{P}$, a properly cyclic subblock of $\tilde{P}$, or a cyclic subblock of $\tilde{P}$, respectively.*

Proof. Similar to the proof of Lemma 3.2, it suffices to consider the nine cases that Q is of one of the forms

$$\begin{matrix} a & b \\ c & d \end{matrix}, \quad \begin{matrix} a & b \\ \# & \# \end{matrix}, \quad \begin{matrix} \# & b \\ \# & d \end{matrix}, \quad \begin{matrix} \# & \# \\ c & d \end{matrix}, \quad \begin{matrix} a & \# \\ c & \# \end{matrix}, \quad \begin{matrix} a & \# \\ \# & \# \end{matrix}, \quad \begin{matrix} \# & b \\ \# & \# \end{matrix}, \quad \begin{matrix} \# & \# \\ c & \# \end{matrix}, \quad \text{or} \quad \begin{matrix} \# & \# \\ \# & d \end{matrix}, \quad \text{where } a, b, c, d \in \Gamma.$$

Let us consider the first case. Let $\bar{Y}$ be the variable tuple $(Y_1, \dots Y_t)$. We need the following first-order formulas:

- $$stripes(\bar{Y}, \bar{X}) := \left(\bigwedge_{s=1}^{t} \bigwedge_{j=0}^{1} row\text{-}closed(Y_{js}) \right)$$

  $$\wedge \forall x_0, x_1 \left(left(x_0) \wedge S_2 x_0 x_1 \rightarrow \bigwedge_{s=1}^{t} \bigwedge_{j=0}^{1} (X_s x_j \leftrightarrow Y_{js} x_j) \right),$$

  where $row\text{-}closed2(X)$ is a first-order formula that asserts that X is a union of rows. $stripes(\bar{Y}, \bar{X})$ asserts that $Y_s = \{(i, j') \in \operatorname{dom} P \mid (i, j) \in X_s^P\}$ for every $s \leqslant t$, i.e. Y_s is the union of all those rows whose leftmost position is contained in X_s.
- For every $(i, j) \in \operatorname{dom} Q$, the formula

  $$\theta_{ij}(\bar{X}, x) = \bigwedge_{\substack{s \in \{1,\dots,t\} \\ (i,j) \in X_s^Q}} X_s x \wedge \bigwedge_{\substack{s \in \{1,\dots,t\} \\ (i,j) \notin X_s^Q}} \neg X_s x$$

  asserts that, for all s, the sth component of $Q\langle i, j\rangle$ is 1 iff $x \in X_s$.

Let us write $\bar{x}$ for the variable tuple $x_{00}, x_{01}, x_{10}, x_{11}$.

$$has\text{-}subblock_Q(\bar{X})$$

$$:= \exists \bar{x} \left(S_1 x_{00} x_{10} \wedge S_1 x_{01} x_{11} \wedge S_2 x_{00} x_{01} \wedge \bigwedge_{i=0}^{m-1} \bigwedge_{j=0}^{n-1} (\theta_{ij}(X_1, \dots, X_t, x_{ij})) \right)$$

$$has\text{-}properly\text{-}cyclic\text{-}subblock'_Q(\bar{Y}, \bar{X})$$

$$:= S_1 x_{00} x_{10} \wedge right(x_{00}) \wedge \bigwedge_{i=0}^{1} (\theta_{i0}(\bar{X}, x_{i0})) \wedge \bigwedge_{i=0}^{1} (\theta_{i1}(\bar{Y}, x_{i0})),$$

$$has\text{-}cyclic\text{-}subblock'_Q(\bar{Y}, \bar{X})$$

$$:= has\text{-}subblock_Q(\bar{X}) \vee has\text{-}properly\text{-}cyclic\text{-}subblock'_Q(\bar{Y}, \bar{X}).$$

Then the formulas $has\text{-}subblock_Q(\bar{X})$, $has\text{-}properly\text{-}cyclic\text{-}subblock'_Q(\bar{Y}, \bar{X})$, and $has\text{-}cyclic\text{-}subblock'_Q(\bar{Y}, \bar{X})$ assert for a picture P over $\{0, 1\}^t$ that Q is a subblock, a properly cyclic subblock, a cyclic subblock, respectively, of P, provided that for all j, s, the set Y_s is the (uniquely determined) union of all those rows whose leftmost position is contained in X_s.

Thus,

$$has\text{-}properly\text{-}cyclic\text{-}subblock_Q(\bar{X})$$

$$:= has\text{-}properly\text{-}cyclic\text{-}subblock'_Q(\iota \bar{Y}(stripes(\bar{Y}, \bar{X}), \bar{X}))$$

is a $\overline{\Delta_1^{\mathrm{U}}}$-formula that asserts that Q is a properly cyclic subblock of P, as required. An analogous construction works for $has\text{-}cyclic\text{-}subblock_Q$.

This completes the proof for the first case, i.e., that the boundary symbol # does not appear in Q. The other eight cases require minor modifications, which we sketch now.

Let us consider the second and fourth case, i.e., that Q is of the form

$$\begin{matrix} a & b \\ \# & \# \end{matrix} \quad \text{or} \quad \begin{matrix} \# & \# \\ a & b \end{matrix},$$

respectively, with $a, b \in \Gamma$. In these cases, we remove the variables x_{10}, x_{11} (or the variables x_{00}, x_{11}, respectively) from the variable tuple $\bar{x}$ and add $bottom(x_{00})$ (or $top(x_{10})$, respectively) as a conjunct in the scope of the existential quantifier in the definition of $has\text{-}subblock_Q$. The other two formulas are modified similarly.

Let us consider the cases Q is of the form

$$\begin{matrix} \# & a \\ \# & b \end{matrix} \quad \text{or} \quad \begin{matrix} a & \# \\ b & \# \end{matrix},$$

respectively. Then we remove the variables $x_{00}, x_{1,0}$ (or $x_{0,1}, x_{1,1}$, respectively) from the variable tuple $\bar{x}$ and add $left(x_{01})$ (or $right(x_{0,0})$, respectively) in the definition of $has\text{-}subblock_Q$. The formula $has\text{-}properly\text{-}cyclic\text{-}subblock'_Q$ can be chosen such that it is always false.

The remaining four cases (where all but one position of Q are #) are even simpler. This completes the proof of Proposition 3.7. □

Proposition 3.8. *Every picture language in* Σ_0^{loc} *is definable in* Δ_1^{U}.

Proof. Since $\underline{\Delta_1^{\text{U}}}$ is closed under boolean combinations, it suffices to consider the cases firstly of those picture languages that are defined by demanding the non-occurrence of a particular (2×2)-picture as a (cyclic or ordinary) subblock and, secondly, of *top*-preimage of locally threshold testable word languages.

The first type of picture language is in $\underline{\Delta_1^{\text{U}}}$ by Proposition 3.7.

Every locally threshold testable word language is known to be first-order definable (in the signature with one successor relation symbol S but without ordering). A first-order sentence φ in this signature translates easily (by replacing S with S_2 and relativizing quantifications to *top*) to a first-order sentence φ' in our signature τ_{Grids} in such a way that φ' is true for those non-empty pictures whose top row fulfills φ. This completes the proof. □

A proof for the following easy fact can be found in [24].

Proposition 3.9. *Every starfree picture language is definable in* $FO^{\leqslant_1, \leqslant_2}$.

The crucial idea of the proof is that a concatenation can be imitated by an existential first-order quantification, just like in the well-known word language case.

We will do a similar proof for Theorem 3.4. It will be essential to relativize quantifications in some first-order formula, say φ, in such a way that the resulting formula asserts for the picture left (or right) from a given position the same as φ asserts for the whole picture. This idea is made precise by the following definition.

Definition 3.10. Let $\bar{x}$ be the variable tuple $(x_1,\dots,x_n)$. Let $\varphi(\bar{x})$, $\varphi'(\bar{x},z)$ be formulas in the signature τ_{Grids_t}, where $t\geqslant 0$.

φ' *relativizes* φ to $\leqslant_2 z$ iff for all non-empty pictures P over $\{0,1\}^t$, all $(i,j)\in \mathrm{dom}\, P$, and all $x_1,\dots,x_n \leqslant_2 (i,j)$:

$$P \models \varphi'[\bar{x},(i,j)] \;\Leftrightarrow\; P([0,\underline{\bar{P}}-1]\times[0,j]) \models \varphi[\bar{x}].$$

φ' *relativizes* φ to $\geqslant_2 z$ iff for all non-empty pictures P over $\{0,1\}^t$, all $(i,j)\in \mathrm{dom}\, P$, and all $x_1,\dots,x_n \geqslant_2 (i,j)$:

$$P \models \varphi'[\bar{x},(i,j)] \;\Leftrightarrow\; P([0,\underline{\bar{P}}-1]\times[j,|P|-1]) \models \varphi[\bar{x}].$$

A class of formulas $\mathscr{F}$ is $\leqslant_2$-*relativizable* iff for every $\varphi\in\mathscr{F}$ there are $\mathscr{F}$-formulas that relativize φ to $\leqslant_2 z$ and to $\geqslant_2 z$, respectively.

Lemma 3.11. *For all $k\geqslant 0$, the formula class $\Sigma^0_k(\Delta^{\mathrm{U}}_1)$ is $\leqslant_2$-relativizable.*

Proof. We argue by induction on k.

For the induction base $k=0$, let $\psi(\iota\bar{X}(\varphi(\bar{X},\bar{Y}),\bar{Y}))$ be some Δ^{U}_1-formula, i.e., $\varphi(\bar{X},\bar{Y}),\psi(\bar{X},\bar{Y})$ are first-order and $\bar{X}$ is a tuple of set variables, whereas the tuple $\bar{Y}$ may contain first-order as well as set variables. Let Z be a fresh set variable.

Let $\varphi'(\bar{X},Z,\bar{Y})$ and $\psi'(\bar{X},Z,\bar{Y})$ result from φ and ψ, respectively, by relativizing first-order quantifications to Z. For every X_i in the tuple $\bar{X}$, let us write "$X_i\subseteq Z$" instead of $\forall x(X_i x\rightarrow Z)$. Let $\varphi''(\bar{X},Z,\bar{Y},z)=\varphi'\wedge smaller(z,Z)\wedge\bigwedge_i(X_i\subseteq Z)$. Since $\exists!\bar{X}(\varphi)$ is valid, so is $\exists!\bar{X}Z(\varphi'')$. Thus[2] $\psi'(\iota\bar{X}Z(\varphi''),\bar{Y},z)$ is an allowed Δ^{U}_1-formula. It is easy to see that it relativizes ψ to $\leqslant_2 z$. A similar construction can be done to get a Δ^{U}_1-formula that relativizes ψ to $\geqslant_2 z$.

The induction step is simple, using relativization of first-order quantifications by a Δ^{U}_1-formula $\sigma(y,z)$ that asserts that $y\leqslant_2 z$. (Such a formula exists by Lemma 3.6.) A formula that relativizes φ to $\geqslant z$ can be obtained similarly. This finishes the induction and thus the proof.

The reader who is not interested in the formula classes $\Sigma^0_k(\Delta^{\mathrm{U}}_1)$ may prove (similarly) that the classes $\Sigma^0_k(\Delta_1)$ and (if $k\geqslant 1$) also Σ_k are $\leqslant_2$-relativizable.

Now, we can state the lemma that clarifies the mentioned idea that first-order quantifications may replace (column-)concatenations.

Lemma 3.12. *Let $\mathscr{F}$ be a $\leqslant_2$-relativizable class of formulas that contains all first-order formulas. If L_1,L_2 are picture languages (over some alphabet $\{0,1\}^t$) and definable in $\mathscr{F}$, then $L_1 \oplus L_2$ is definable in $\Sigma^0_1(\mathscr{F})$.*

[2] Note that we added z to the list of potentially free variables in ψ.

In particular, if $\mathscr{F}$ is closed under conjunction, disjunction, and existential first-order quantifications, then the set of $\mathscr{F}$-definable picture languages is closed under column concatenation.

Proof. Write $\bar{X}$ for the variable tuple $(X_1,\dots,X_t)$. Let $\varphi_1(\bar{X})$ and $\varphi_2(\bar{X})$ be formulas in $\mathscr{F}$ that define L_1 and L_2, respectively. Let $\varphi_1'(\bar{X},z)$ and $\varphi_2'(\bar{X},z)$ be $\mathscr{F}$-formulas that relativize φ_1 to $\leqslant_2 z$ and φ_2 to $\geqslant_2 z$, respectively. Let $\varphi(\bar{X})=\exists z,z'(S_2zz'\wedge\varphi_1'(\bar{X},z)\wedge\varphi_2'(\bar{X},z'))$. Then φ defines $L_1 \oplus L_2$. $\square$

We conclude this section with the proof of its main result, Theorem 3.4, which states that $\Sigma_k^{\text{loc}} \subseteq \underline{\Sigma_k^0(\Delta_1^{\text{U}})}$.

Proof of Theorem 3.4. We argue by induction on k. The case $k=0$ is Proposition 3.8. Assume $\Sigma_k^{\text{loc}} \subseteq \underline{\Sigma_k^0(\Delta_1^{\text{U}})}$ is true for some k.

Since $\Sigma_{k+1}^{\text{loc}}$ is a subset of the smallest superset of $B(\Sigma_k^{\text{loc}})$ that is closed under union, intersection, and column concatenation, the claimed implication follows from the fact that $\Sigma_{k+1}^0(\Delta_1^{\text{U}}) \supseteq B(\Sigma_k^0(\Delta_1^{\text{U}}))$ and that $\Sigma_{k+1}^0(\Delta_1^{\text{U}})$ is closed under disjunction, conjunction, and (in the sense of Lemma 3.12) column concatenation. $\square$

Note that the proof shows that Theorem 3.4 remains true if the inductive definition of the local alternation hierarchy is modified as follows: $\Sigma_{k+1}^{\text{loc}}$ is the smallest superclass of $B(\Sigma_k^{\text{loc}})$ that is closed under positive boolean combinations *as well as* $\oplus$- and $\ominus$-concatenations.

3.2. Attributed alphabets

In this subsection we will introduce some notation that will be helpful later on. Since this notation might seem strange at first sight, I would like to motivate it.

It will be agreed that it is extraordinarily helpful to allow free variables to be chosen arbitrarily and not only from the “anonymous” set $\{X_1,X_2,\dots\}$. This improves readability; it allows us to indicate a particular relationship between free variables by choosing similar, suggestive names.

This section will be chiefly concerned with starfree picture languages over alphabets of (0–1)-tuples. Suppose we would use only alphabets of the form $\{0,1\}^t$ for some t. This would make notations difficult for the same reasons as in the context of formulas, as I will illustrate in an example. Suppose we want to construct a starfree word language L over $\{0,1\}^3$ from two languages M, N over $\{0,1\}^2$ in such a way that L contains those words over $\{0,1\}^3$ whose letterwise restriction to the first two components is in M whereas the letterwise restriction to the second two ones is in N. Then we would have to write $L=(M\otimes\{0,1\}^*)\cap(\{0,1\}^*\otimes N)$, where the $\otimes$ means “letterwise pairing”. Note that the understanding is further complicated by the fact that what used to be the first component in (symbols of words in) the language N is the second component in L.

The main idea to avoid such complications is to give names to the components of letters, i.e., to have families over some finite index set as alphabet symbols. Such families are called "data base tuples" in data base theory, and the elements of the index set are called "attributes". Suppose in the above example these "attributed" alphabets are chosen in such a way that the second component in M and the first component in N correspond to the same attributes, i.e., μ, v, ξ are different attributes, and M and N are over alphabets $\{0,1\}^{\{\mu,v\}}$ and $\{0,1\}^{\{v,\xi\}}$, respectively. If $a \in \{0,1\}^{\{\mu,v\}}$ and $b \in \{0,1\}^{\{v,\xi\}}$ are two letters, then the *join* $a \bowtie b$ is defined iff $a(v) = b(v)$, namely by $(a \bowtie b)(\mu) = a(\mu)$, and $(a \bowtie b)(v) = a(v) = b(v)$, and $(a \bowtie b)(\xi) = b(\xi)$. This join operation is lifted to an operation $\otimes$ on words and pictures in the usual (i.e., letterwise) way. This operation is in turn lifted to sets of words or pictures, thus we will simply write $L = M \otimes N$ in the above situation.

Formally, let I be a finite set of so-called "attributes". By $\{0,1\}^I$ we denote the set of functions $I \to \{0,1\}$, i.e., $\{0,1\}^I$ is the set of I-indexed families over $\{0,1\}$.

If I,J are attribute sets and $a \in \{0,1\}^I$, $b \in \{0,1\}^J$, and if $a(\mu) = b(\mu)$ for every $\mu \in I \cap J$, then $a \bowtie b$ is the element of $\{0,1\}^{I \cup J}$ with

$$(a \bowtie b)(\mu) = \begin{cases} a(\mu) & \text{if } \mu \in I, \\ b(\mu) & \text{if} \mu \in J. \end{cases}$$

If $a(\mu) \neq b(\mu)$ for some $\mu \in I \cap J$, then "$a \bowtie b$" is not defined.

This (partial) operation is extended to a (total) operation on sets as usual: If $L \subseteq \{0,1\}^I$, $M \subseteq \{0,1\}^J$ then $L \bowtie M = \{a \bowtie b \mid a \in L, b \in M\}$. We will use sets of the form $\{0,1\}^I$ as alphabets. The operation $\bowtie$ is also extended to pictures over such alphabets: If P is a picture over $\{0,1\}^I$ and Q is a picture over $\{0,1\}^J$ of equal size, and if "$(P\langle i,j\rangle) \bowtie (Q\langle i,j\rangle)$" is defined for every position $(i,j) \in \mathrm{dom}\, P$, then $P \otimes Q$ is defined, namely as the picture of size size(P) over $\{0,1\}^{I \cup J}$ with

$$(P \otimes Q)\langle i,j\rangle(P\langle i,j\rangle) \bowtie (Q\langle i,j\rangle)$$

for every $(i,j) \in \mathrm{dom}\, P$. Again, this partial operation is lifted to a total operation on sets as usual.

Some more notation is needed for attributed alphabets. If μ is some attribute and $b \in \{0,1\}$, then $(b)_\mu \in \{0,1\}^{\{\mu\}}$ is defined by $((b)_\mu)(\mu) = b$. This way, $(\cdot)_\mu$ is an alphabet projection from $\{0,1\}$ to $\{0,1\}^\mu$, which is lifted to pictures and picture languages the usual way. If $a \in \{0,1\}^I$ is some I-indexed family and $J \subseteq I$, then $\mathsf{restr}_J(a)$ is the restriction of a to a J-indexed family. Again, $\mathsf{restr}_J(\cdot)$ is an alphabet projection from $\{0,1\}^I$ to $\{0,1\}^J$ and is lifted to pictures and picture languages.

Likewise, we consider for some $\mu \in I$ the alphabet projection $\mathsf{pr}_\mu : \{0,1\}^I \to \{0,1\}$, $a \mapsto a(\mu)$. It can be lifted to pictures, i.e., if P is a picture over $\{0,1\}^I$ and $\mu \in I$, then we write $\mathsf{pr}_\mu(P)$ for the picture of the same size over $\{0,1\}$ with $(\mathsf{pr}_\mu(P))\langle i,j\rangle = (P\langle i,j\rangle)(\mu)$ for every position (i,j) of P, and analogously for picture languages.

Remark 3.13. Let L, M be picture languages over alphabets $\{0,1\}^I$ and $\{0,1\}^J$, respectively, and let $\Omega = \{0,1\}^{I \cup J}$.

Then $L \otimes M = (L \otimes \Omega^{++}) \cap (M \otimes \Omega^{++})$ and $L \otimes \Omega^{++} = \{P \in \Omega^{++} \mid \mathsf{restr}_I(P) \in L\}$.

With these identities one easily deduces that the class of local picture languages is closed under $\otimes$ from the fact that it is closed under inverse projections (Proposition 2.10) and intersection.

We sometimes write $(0)_I$ for the letter $a \in \{0\}^I$.

3.3. Counting cyclically—the central definability results

In this subsection we shall introduce the sequence $(Num_k)_k$ of those picture languages that witness almost all non-inclusion results of this paper. The crucial point is that the pictures of these languages are very "slim" in the sense that the pictures are very wide compared to their height.

For every $k \geqslant 1$, we define the k-fold exponential function $f_k : \mathbb{N} \to \mathbb{N}$ as follows.

$$f_1(m) = 2^m,$$

$$f_{k+1}(m) = f_k(m) 2^{f_k(m)}.$$

The pictures of Num_k will be of size $(m, f_k(m))$ for $m \geqslant 1$.

Throughout this section, we consider the attributes num-1, num-2, …, end-1, end-2, … . For every $k \geqslant 1$ the attribute set I_k is fixed as $I_k = \{num\text{-}1, \ldots, num\text{-}k, end\text{-}1, \ldots, end\text{-}k\}$.

The picture language Num_k will be defined as a certain picture language over alphabet $\{0,1\}^{I_k}$. For every $l \leqslant k$, the restriction of a picture in Num_k to attribute num-l will establish a counting mechanism that cyclically enumerates binary numbers of length $f_l(m)$. The projection to attribute end-l will mark the ends of these binary number representations.

We start with the following auxiliary definition.

Definition 3.14. For every $k \geqslant 1$, let $OnlyTop_k$ be the set of non-empty pictures P over $\{0,1\}^{\{num\text{-}k, end\text{-}k\}}$ such that $\mathsf{pr}_{num\text{-}k}(P)\langle i,j\rangle = \mathsf{pr}_{end\text{-}k}(P)\langle i,j\rangle = 0$ for all $(i,j) \in \mathrm{dom}\, P$ with $i \neq 0$.

In other words, $OnlyTop_k = (L)_{num\text{-}k} \otimes (L)_{end\text{-}k}$ for the local picture language $L = (\{0,1\}^+ \ominus \{0\}^{**})$ of Example 2.4. As a consequence of that example, $OnlyTop_k$ is a local picture language.

Now we will define, for every $k, m \geqslant 1$, a picture P_{km} of height m over alphabet $\{0,1\}^{I_k}$, and then the picture language Num_k over $\{0,1\}^{I_k}$.

Definition 3.15. For every $m \geqslant 1$ let P_{1m} be the picture of size $(m, f_1(m))$ over $\{0,1\}^{I_1}$ for which

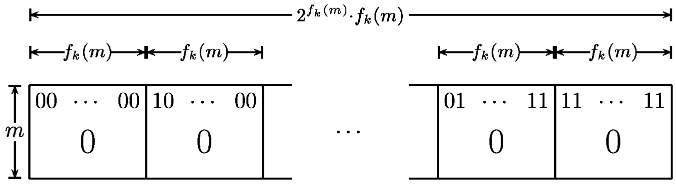


Fig. 4. The picture $\mathsf{pr}_{num\text{-}k+1}(P_{k+1,m})$.

- $\mathsf{pr}_{num\text{-}1}(P_{1m})$ is the picture over $\{0,1\}$ whose jth column holds the binary representation of j for every $j \in \{0,\ldots,2^m-1\}$ (least significant bit at the top), (cf. Example 2.8),
- $\mathsf{pr}_{end\text{-}1}(P_{1m})$ is the picture over $\{0,1\}$ such that for all $(i,j) \in \operatorname{dom} P = [m] \times [2^m]$

$$\mathsf{pr}_{end\text{-}1}(P_{1m})\langle i,j\rangle = 1 \Leftrightarrow \forall j' \geqslant j\colon \mathsf{pr}_{num\text{-}1}(P_{1m})\langle i,j'\rangle = 1.$$

For every $k \geqslant 1$, $m \geqslant 1$, let $P_{k+1,m}$ be the picture of size $(m, f_{k+1}(m))$ over alphabet $\{0,1\}^{\{num\text{-}k+1,\, end\text{-}k+1\}}$ such that
- $\mathsf{pr}_{num\text{-}k+1}(top(P_{k+1,m})) = BIN(0, f_k(m)) \cdots BIN(2^{f_k(m)-1}, f_k(m))$ is the word of length $f_{k+1}(m)$ that consists of the concatenation of the reverse binary representations (of length $f_k(m)$) of the numbers $0,\ldots,2^{f_k(m)}-1$,
- $\mathsf{pr}_{end\text{-}k+1}(top(P_{k+1,m})) = 0^{f_{k+1}(m)-1}1$, and
- $P_{k+1,m} \in OnlyTop_{k+1}$.

For every $k \geqslant 1$, let

$$Num_k = \bigcup_{m \geqslant 1} \{P_{1m}\}^+ \otimes \cdots \otimes \{P_{k-1,m}\}^+ \otimes \{P_{k,m}\}.$$

The picture $\mathsf{pr}_{num\text{-}1}(P_{1,3})$ is displayed as (1). As an illustration of Definition 3.15, Fig. 4 displays the projection of $P_{k+1,m}$ to its attribute $num\text{-}k+1$. Note that the length of $P_{k+1,m}$ is indeed $2^{f_k(m)} \cdot f_k(m) = f_{k+1}(m)$.

The following is the main theorem of this subsection. When this theorem is proved, most of the "hard work" of this section will have been done. Theorem 3.27 follows quite simply and rest of the hard work is for Theorem 3.44.

These three theorems and the easy inclusion results from Section 3.1 provide all definability results needed for the separation results announced in Section 2.4.

Theorem 3.16. *$cycl(Num_k^+)$ is in Π_{k-1}^{loc} for every $k \geqslant 1$.*

Note that this will in particular imply that Num_k is Π_{k-1}^{loc}, which in turn allows us to deduce that the function f_k is Σ_k-definable in the sense of Definition 2.18. Both [18, 21] construct monadic second-order formulas that define such k-fold exponential function inductively over k. (It is f_k in [21] and a very similar function in [18].) However, both of these inductions require the simultaneous construction of two other formulas, which makes the proof difficult to understand.

My point is that considering the picture language $cycl(Num_k^+)$ instead of Num_k makes the induction required for the construction a lot simpler.

Contrary to this simplification, the detour over the local alternation hierarchy complicates the construction of such formulas. However, this detour is non-essential. The reader who is interested in definability results only for monadic second-order logic may extract easily an inductive proof that directly constructs, say, Π_k-formulas. That means that the following may be proved via induction over k:

$$cycl(Num_k^+) \in \begin{cases} \underline{\Delta_1} & \text{if } k = 1, \\ \underline{\Pi_{k-1}} & \text{if } k \geqslant 2. \end{cases}$$

(In the induction step one constructs from a Δ_k-*sentence*(!) for $cycl(Num_k^+)$ a Π_k-sentence for $cycl(Num_{k+1}^+)$.)

The following proposition serves as an induction basis for Theorem 3.16.

Proposition 3.17. *$cycl(Num_1^+)$ is in Π_0^{loc}.*

This proposition is an almost immediate consequence of Example 2.8. The picture language considered in that example is $\mathsf{pr}_{num\text{-}1}(cycl(Num_1^+))$. Nevertheless, it is instructive to carry out the proof in more detail because the verification of counting mechanisms in pictures in $cycl(Num_k^+)$ for $k>1$ will depend on the same ideas.

Definition 3.18. Let Allow be the set of (2×2)-pictures

$$\begin{matrix} a & c \\ b & d \end{matrix}$$

over $\{0,1\}$ for which $((a,c)=(1,0)) \leftrightarrow (b \neq d)$. Let $\mathrm{Forb} = \{0,1\}^{2,2} \backslash \mathrm{Allow}$.

In other words, Allow is the set of (2×2) subblocks pictures that result from column concatenating columns that are successive binary number representations, i.e., Allow the set of cyclic (2×2)-subblocks of the picture

$$\begin{matrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1\,. \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{matrix} \tag{1}$$

For the proof of Proposition 3.17 we need the following lemma, whose proof is left to the reader.

Lemma 3.19 (Cyclic Counting Lemma). *Let $m,n \geqslant 1$ and $x_0,\ldots,x_{n-1}$ be words of length m over alphabet $\{0,1\}$. Then the following are equivalent*:

1. $dual(x_{(j+1) \bmod n}) = (dual(x_j)+1) \bmod 2^m$ *for every* $j \in \{0,\ldots,n-1\}$;
2. • $x_j\langle 0\rangle \neq x_{(j+1) \bmod n}\langle 0\rangle$ *for every* $j \in \{0,\ldots,n-1\}$.

- $\begin{pmatrix} x_j\langle i\rangle & x_{(j+1)\bmod n}\langle i\rangle \\ x_j\langle i+1\rangle & x_{(j+1)\bmod n}\langle i+1\rangle \end{pmatrix} \in \mathit{Allow}$ *for all* $j\in\{0,\dots,n-1\}$,
  $i\in\{0,\dots,m-1\}$.

Proof of Proposition 3.17. Let $\Gamma=\{0,1\}^{I_1}$.

$$\begin{aligned}\varDelta = \Bigg\{ \begin{matrix} \boldsymbol{a} & \boldsymbol{c} \\ \boldsymbol{b} & \boldsymbol{d} \end{matrix} \in \Gamma^{2,2} \;\Bigg|\; \mathsf{pr}_{num\text{-}1}\begin{pmatrix} \boldsymbol{a} & \boldsymbol{c} \\ \boldsymbol{b} & \boldsymbol{d} \end{pmatrix} \in \text{Allow} \\ \wedge(\mathsf{pr}_{end\text{-}1}(\boldsymbol{a})=1 \leftrightarrow \mathsf{pr}_{end\text{-}1}(\boldsymbol{b})=\mathsf{pr}_{num\text{-}1}(\boldsymbol{a})=1)\Bigg\} \\ \cup \Bigg\{ \begin{matrix} \boldsymbol{a} & \boldsymbol{c} \\ \# & \# \end{matrix} \;\Bigg|\; \mathsf{pr}_{end\text{-}1}(\boldsymbol{a})=1 \leftrightarrow \mathsf{pr}_{num\text{-}1}(\boldsymbol{a})=1 \Bigg\} \\ \cup \Bigg\{ \begin{matrix} \# & \# \\ \boldsymbol{d} & \boldsymbol{d} \end{matrix} \;\Bigg|\; \mathsf{pr}_{num\text{-}1}(\boldsymbol{b}) \neq \mathsf{pr}_{num\text{-}1}(\boldsymbol{d}) \Bigg\}.\end{aligned}$$

Let P be a non-empty picture over Γ, say of size (m,n). Let $x_0,x_1,\dots,x_{n-1}$ be the words of length m over Γ such that x_i is the transposed column i of P, i.e.,

$$x_i = P\langle 0,i\rangle \dots P\langle m-1,i\rangle.$$

$\varDelta$ is chosen in such a way that every cyclic (2×2)-subblock of $\tilde{P}$ is in $\varDelta$ iff

- $\mathsf{pr}_{num\text{-}1}(x_0),\dots,\mathsf{pr}_{num\text{-}1}(x_{n-1})$ have the second property of Lemma 3.19 ("the Cyclic Counting Lemma"), and
- for every x among $x_0,\dots,x_{n-1}$, the word $\mathsf{pr}_{end\text{-}1}(x)$ is determined by $\mathsf{pr}_{end\text{-}1}(x)=0^i1^j$ iff j is maximal such that 1^j is a suffix of $\mathsf{pr}_{num\text{-}1}(x)$.

Thus (by Lemma 3.19) Num_1^+ is the set of non-empty pictures P over Γ such that $\varDelta$ cyclically tiles $\tilde{P}$, thus Num_1^+ is cyclically local in particular Π_0^{loc}. $\square$

The observation stated in the above lemma, namely that—speaking informally—counting is a local process, has been one of the two[3] essential improvements of the definability part of the proof of [21] compared to [18] because it made it possible to save quantifier alternations in the monadic second-order formulas that define "slim" picture languages. This observation has been used previously in [5] (see Example 2.20) and it can also be found implicit in [12, Example 2.4]. What is new here is to continue the counting process cyclically.

Proposition 3.17 proves Theorem 3.16 for the case $k=1$. Our aim is to do the proof of Theorem 3.16 by induction on k. We will sketch the strategy for this informally.

[3] The other improvement is to start the induction appropriately, i.e., at $k=1$ with the exponential function f_1 instead of at $k=0$ with the identity. This allows to save one quantifier alternation and, moreover, this allows to describe the counting pattern in $FO^{\leqslant_1,\leqslant_2}$.

Checking membership in $cycl(Num_{k+1}^+)$ for some picture P over $\{0,1\}^{I_{k+1}}$ whose restriction to I_k is in $cycl(Num_k^+)$ means essentially to check that the component corresponding to the attribute $num\text{-}k+1$ enumerates all binary representations of length $f_k(m)$, where m is the height of P. The idea for this is similar to the case $k=1$, namely to exploit the Cyclic Counting Lemma 3.19 in order to verify the counting mechanism locally. The difference is that—unlike in the case $k=1$—corresponding bits of successive binary number representations are no more horizontally neighboured, but at horizontal cyclic distance $f_k(m)$ from each other. The restriction of P to the I_k-components can be used to determine such corresponding positions because if the restriction of P to attribute set I_k is in $cycl(Num_k^+)$, then two positions are at horizontal cyclic distance $f_k(m)$ from each other iff the restriction to I_k-components of the cyclic infix of P between these positions is in $cycl(Num_k)$. Some additional tricks are necessary to control the existence of particular cyclic infixes on the next level of the local alternation hierarchy.

We will carry out the sketched proof formally. For this, we need the following technical lemma. Its essential statement is the equivalence of Items 1 and 2, which states that Num_{k+1}^+ results from Num_k^+ by enlarging the underlying alphabet (i.e. passing from attribute set I_k to I_{k+1}) and then forbidding particular cyclic infixes.

Lemma 3.20. *Let $k\geqslant 1$, let $\Omega=\{0,1\}^{I_{k+1}}$. Let U_1,U_2,U_3 be the word languages over Ω defined as follows*:

$$U_1=\bigcup_{b\in\{0,1\}}(10^*10)_{end\text{-}k}\otimes(\{0,1\}b\{0,1\}^*b)_{num\text{-}k+1}\otimes\Omega^+,$$

$$U_2=\bigcup_{\substack{a\ c\\ b\ d}\in Forb}(0^+10^+\{0,1\})_{end\text{-}k}\otimes(ab\{0,1\}^*cd)_{num\text{-}k+1}\otimes\Omega^+,$$

$$U_3=(0^*1)_{end\text{-}k}\otimes(1^*)_{num\text{-}k+1}\otimes(\sim(0^*1))_{end\text{-}k+1}\otimes\Omega^+$$

$$\cup(0^*1)_{end\text{-}k}\otimes(\sim(1^*))_{num\text{-}k+1}\otimes(\sim(0^*))_{end\text{-}k+1}\otimes\Omega^+.$$

For every $m\geqslant 1$ let bin_m abbreviate $BIN(0,f_k(m))\cdots BIN(2^{f_k(m)-1},f_k(m))$, i.e., the word of length $f_{k+1}(m)$ that consists of the reverse binary representations (of length $f_k(m)$) of the numbers $0,\ldots,2^{f_k(m)}-1$.

The following are equivalent for every $P\in cycl(Num_k^+)\otimes OnlyTop_{k+1}$ of height m:[4]

1. *$P\in cycl(Num_{k+1}^+)$,*
2. *$top(P)$ has no cyclic infix of length $f_k(m)+2$ in $U_1\cup U_2$ and none of length $f_k(m)$ in U_3.*

Proof. Let $P\in cycl(Num_k^+)\otimes OnlyTop_{k+1}$ and m be the height of P.

[4] Recall $OnlyTop_{k+1}$ from Definition 3.14.

P can be cyclically partitioned in blocks of length $f_k(m)$ whose top rows end in a position that carries a 1 for the attribute *end-k*. Let $x_0,\ldots,x_n$ be the sequence of top rows of these blocks.

We apply Lemma 3.19 to the projections $\mathsf{pr}_{end\text{-}k}(x_j)$ (for $j\in\{0,\ldots,n-1\}$. The assertion that $top(P)$ has no cyclic infix of length $f_k(m)+2$ in U_1 (and in U_2, respectively) is equivalent to the first (and second, respectively) subitem of item 2 in Lemma 3.19, which makes sure that the projection of the top row to the attribute *num-k* $+ 1$ establishes the desired counting mechanism.

The additional assertion that $top(P)$ has no cyclic infix of length $f_k(m)$ in U_3 is equivalent to the statement that the sequence of words $x_0,\ldots,x_{n-1}$ can be grouped into sequences of $2^{f_k(m)}$ of these words such that (1) exactly those of these words whose projection to attribute *num-k* $+ 1$ represents the maximal binary number have a 1 in the attribute of *end-k* $+ 1$ of last position, and (2) all other positions have a 0 in the attribute of *end-k* $+ 1$. That makes sure that the projection of the top row to the attribute *end-k* $+ 1$ marks the ends of blocks of length $f_{k+1}(m)$ that are in $cycl(Num^+_{k+1})$. $\square$

A precise proof can be found in [16].

The important point of Lemma 3.20 is that Num^+_{k+1} results from Num^+_k by enlarging the underlying alphabet (i.e., passing from attribute set I_k to I_{k+1}) and then forbidding particular cyclic infixes. The next lemmas show how this “forbidding” can be done on the next level of the alternation hierarchy for starfree picture languages.

Firstly, we remark that it is easy to forbid certain pictures as (non-cyclic) infixes: Suppose M,K are picture languages over Γ. Then the set of all pictures of M that do not have an infix in K is given by $M\backslash(\Gamma^{*,*}K\Gamma^{*,*})$. If M and K are in Π^{loc}_{k-1}, then this set is in Π^{loc}_k.

The situation is more complicated when we deal with *cyclic* infixes because a cyclic infix consists of a prefix and a suffix and it is not possible to control its length etc. as easily as usual infixes.

However, in our particular case (see Lemma 3.20), the languages Num^+_k (which will play the role of M) and also the languages $top^{-1}((U_1\cap\Omega^{f_k(m)+2})\cup(U_2\cap\Omega^{f_k(m)+2}))$ and $top^{-1}(U_3\cap\Omega^{f_k(m)})$ (which will play the rôle of K) have certain properties that can be exploited.

The next definition and proposition state that property of $top^{-1}(U_1\cup U_2)$ and $top^{-1}(U_3)$.

Definition 3.21 (*Suffix–prefix decomposition*). A picture language L over alphabet Γ is Π^{loc}_0-*suffix/prefix-decomposable* (or *sp-decomposable* for short) iff there are $n\geqslant 1$ and picture languages $U_1,\ldots,U_n,V_1,\ldots,V_n\in\Pi^{\mathrm{loc}}_0$ such that for all non-empty pictures P,Q over Γ we have $PQ\in L$ iff there is an $i\leqslant n$ with $P\in V_i$, $Q\in U_i$.

Proposition 3.22. *Let* $a,b\in\{0,1\}$. *The word languages* 10^*, $\{0,1\}b\{0,1\}^*$, 0^+10^*, $0\{0,1\}^*$, $ab\{0,1\}^*$ *are sp-decomposable.*

Proof. To see that $ab\{0,1\}^*$ is sp-decomposable, consider $V_1=\{a\}$, $U_1=b\{0,1\}^*$, $V_2=ab\{0,1\}^*$, $U_2=\{0,1\}^+$. The other cases are even simpler. □

Lemma 3.23 (Cyclic Infix Lemma). *Let L_1, L_2 be sp-decomposable picture languages. Then $cycl((L_1 \cap cycl(Num_k^+))(L_2 \cap cycl(Num_k^+))) \in \Sigma_k^{\mathrm{loc}}$.*

Proof. Choose $U_1,\dots,U_n,V_1,\dots,V_n$ according to Definition 3.21 for L_1, and likewise $U_1',\dots,U_m',V_1',\dots,V_m'$ for L_2. As an abbreviation, let $M=cycl(Num_k^+)$. By the above proposition,

$$\begin{aligned} cycl((L_1\cap M)(L_2\cap M)) = {} & (L_1\cap M)(L_2\cap M)\cup(L_2\cap M)(L_1\cap M)\\ & \cup\left(M\cap\bigcup_{j\leqslant m}(V_j'(L_1\cap M)U_j')\right)\\ & \cup\left(M\cap\bigcup_{i\leqslant n}(V_i(L_2\cap M)U_i)\right), \end{aligned}$$

which is clearly in Σ_k^{loc}. (For the inclusion "$\subseteq$" note that by definition of Num_k we have that for all pictures P,Q,R of equal height we have: $PQR\in cycl(Num_k^*)\wedge Q\in cycl(Num_k^*)\rightarrow PR\in cycl(Num_k^*)$. A precise proof of the latter statement can be found in [16]. □

By Remark 3.13, the above lemma remains true if $\cap$ is replaced by $\otimes$.

In the next lemma we will combine the above lemma and Lemma 3.20 (Items 1 and 2) in order to construct, for a given k, a picture language $K_3\in\Sigma_k^{\mathrm{loc}}$ such that for every picture P in $cycl(Num_k^+)\otimes OnlyTop_{k+1}$ over $\{0,1\}^{I_{k+1}}$ we have $P\in K_3 \Leftrightarrow P\notin Num_{k+1}$. This will suffice to complete the proof of Theorem 3.16.

Lemma 3.24. *Let $k\geqslant 1$ such that $cycl(Num_k^+)\in\Pi_{k-1}^{\mathrm{loc}}(\{0,1\}^{I_k})$. Let $\Omega:=\{0,1\}^{I_{k+1}}$. Let U_1,U_2,U_3 as in Lemma 3.20.*

For every $a,b,c,d\in\{0,1\}$, there are picture languages $L_{bb},L_{abcd},L,K_1,K_2,K_3$ in $\Sigma_k^{\mathrm{loc}}(\Omega)$ such that for all $P\in cycl(Num_k^+)\otimes OnlyTop_{k+1}$ we have

1. *$P\in L_{bb}$ iff $top(P)$ has a cyclic infix of length $f_k(m)+2$ in*

$$(10^*10)_{end\text{-}k}\otimes(\{0,1\}b\{0,1\}^*b)_{num-k+1}\otimes\Omega^+.$$

2. *$P\in L_{abcd}$ iff $top(P)$ has a cyclic infix of length $f_k(m)+2$ in*

$$(0^+10^+\{0,1\})_{end\text{-}k}\otimes(ab\{0,1\}^*cd)_{num\text{-}k+1}\otimes\Omega^+.$$

3. *$P\in L$ iff $top(P)$ has a cyclic infix of length $f_k(m)$ in U_3.*
4. *$P\in K_1$ iff $top(P)$ has a cyclic infix of length $f_k(m)+2$ in U_1.*
5. *$P\in K_2$ iff $top(P)$ has a cyclic infix of length $f_k(m)+2$ in U_2.*
6. *$P\in K_3$ iff $top(P)$ has a cyclic infix of length $f_k(m)+2$ in $U_1\cup U_2$ or one of length $f_k(m)$ in U_3.*

Proof. The first three claim are immediate consequences of Lemma 3.23. We give the choice of L_1 and L_2 in the application of that lemma. The remaining three claims are consequences.

Ad1. Let $b \in \{0,1\}$. Choose

$$L_1 = top^{-1}((10^*)_{end\text{-}k} \otimes (\{0,1\}b\{0,1\}^*)_{num-k+1} \otimes \Omega^+),$$

$$L_2 = top^{-1}((10\{0,1\}^*)_{end\text{-}k} \otimes (\{0,1\}b\{0,1\}^*)_{num-k+1} \otimes \Omega^+),$$

$$L_{bb} = cycl((L_1 \otimes cycl(Num_k^+))(L_2 \otimes cycl(Num_k^+))).$$

By Lemma 3.23 and Proposition 3.22, L_{bb} is in Σ_k^{loc}. It is easy to show that for every $P \in cycl(Num_k^+) \otimes OnlyTop_{k+1}$ we have that $P \in L_{bb}$ iff $top(P)$ has a cyclic infix of the desired form.

Ad 2. For every $a,b,c,d \in \{0,1\}$, choose

$$L_1 = top^{-1}((0^+10^*)_{end-k} \otimes (ab\{0,1\}^*)_{num-k+1} \otimes \Omega^+),$$

$$L_2 = top^{-1}((0\{0,1\}^*)_{end\text{-}k} \otimes (cd\{0,1\}^*)_{num-k+1} \otimes \Omega^+),$$

$$L_{abcd} = cycl((L_1 \otimes cycl(Num_k^+))(L_2 \otimes cycl(Num_k^+))).$$

Again, by Lemma 3.23 and Proposition 3.22, L_{abcd} is in Σ_k^{loc}. Again, the proof that L_{abcd} has the property claimed in Item 1 is easy.

Ad 3. $L = cycl((top^{-1}(U_3) \otimes cycl(Num_k^+))(\Omega^{+,+} \otimes cycl(Num_k^+))) \cup cycl((top^{-1}(U_3) \otimes cycl(Num_k^+)))$ is in Σ_k^{loc} (exploit Lemma 3.23 with $L_1 = top^{-1}(U_3)$ and $L_2 = \Omega^{+,+}$) and has the desired property by similar arguments.

Ad 4,5 and 6. Choose $K_1 = L_{00} \cup L_{11}$, $K_2 = \bigcup_{\begin{smallmatrix} a & c \\ b & d \end{smallmatrix} \in Forb} L_{abcd}$, and $K_3 = K_1 \cup K_2 \cup L$.

Then K_1, K_2, and K_3 are in Σ_k^{loc} and have the desired properties. □

The sixth claim of the above lemma is the key for the following proof of Theorem 3.16, which states that $cycl(Num_k^+)$ is in Π_{k-1}^{loc} for every $k \geqslant 1$.

Proof of Theorem 3.16. The proof is by induction. By Proposition 3.17, $cycl(Num_1^+)$ is in Π_0^{loc}, which is the induction basis. So let $k \geqslant 1$ with $cycl(Num_k^+)$ is in Π_{k-1}^{loc}, choose K_3 as in Lemma 3.24. By Lemma 3.20, we have

$$cycl(Num_{k+1}^+) = (cycl(Num_k^+) \otimes OnlyTop_{k+1}) \backslash K_3.$$

Since $cycl(Num_k^+) \in \Pi_{k-1}^{\mathrm{loc}} \subseteq \Pi_k^{\mathrm{loc}}$ and $K_3 \in \Sigma_k^{\mathrm{loc}}$ and $OnlyTop_{k+1} \in \Pi_0^{\mathrm{loc}}$, it follows that $cycl(Num_{k+1}^+)$ is in Π_k^{loc}, which completes the induction. □

The following corollary provides the definability result for Theorem 2.16, stating that $\Pi_k^{\mathrm{loc}}(\{0,1\}^{2k}) \nsubseteq \underline{\Sigma_k}(\{0,1\}^{2k})$ for all $k \geqslant 1$. For a picture language L, we define size(L) as $\{\mathrm{size}(P) \mid P \in L\}$.

Corollary 3.25. *For every k, the languages* $cycl(Num_k^+)$, Num_k^+, *and* Num_k *are in* $\underline{\Pi_{k-1}^0}(\{0,1\}^{I_k})$, *in* $\underline{\Pi_{k-1}^0(\Delta_1^{\mathrm{U}})}$ *in* $\underline{\Pi_{k-1}^0(\Delta_1)}$, *in* $\underline{\Pi_{k-1}}$, *in* $\underline{FO(\Delta_1)}$, *and in* $\underline{FO^{\leqslant_1,\leqslant_2}}$.

Proof. Let $k \geqslant 1$. We first show membership in Π_{k-1}^{loc}. For $cycl(Num_k^+)$, this is Theorem 3.16, and for the other two languages, this follows from

$$Num_k^+ = cycl(Num_k^+) \otimes top^{-1}((\{0,1\}^*1)_{end\text{-}k}),$$

$$Num_k = cycl(Num_k^+) \otimes top^{-1}((0^*1)_{end\text{-}k}),$$

because $\{0,1\}^*1$ and 0^*1 are locally threshold testable word languages, so that their *top*-pre-images are in Π_0^{loc}, and because Π_{k-1}^{loc} is closed under join. Membership in the other listed classes follows from Theorem 3.4 and Corollary 3.5. □

3.3.1. Digression: length of starfree expressions

The notion of "starfreeness" of a picture language may alternatively be introduced via "starfree picture expressions". Starfree picture expressions over alphabet Ω are built as follows: For every $a \in \Omega$, the term a is a starfree picture expression, and if e and f are starfree picture expressions, then so are the terms $(e \oplus f)$, $(e \ominus f)$, $\sim e$, and $(e \cup f)$.

The semantics is defined in a straightforward manner. Then a picture language is starfree iff it is defined by a starfree expression.

With this syntactic notion one may investigate the *length* $|r|$ of an expression r, i.e., the number of symbols in r.

In [22] the author shows the following surprising fact. There is a sequence $(r_k)_{k\geqslant 1}$ of starfree word expressions such that the length of the shortest word matching r_k is not elementary in the length of r_k, i.e., not bounded by $f_i(|r|)$ for any fixed i.

A proof of this fact can also be extracted from the constructions of this subsection. Note that these hardly used the row concatenation $\ominus$. A careful analysis of these constructions—in particular, those of Lemmas 3.23, 3.24 and Theorem 3.16—shows that the height-1 fragment of $cycl(Num_k)$ is defined by a starfree word expression r_k of length singly exponential in k, and we know that the length of the only word matching r_k is $f_k(1)$, i.e., more than a tower of 2's of height k.

Concerning the length of starfree *picture* expressions, one may extract from these proofs that there is a sequence $(e_k)_{k\geqslant 1}$ of expressions such that e_k is of length $2^{\mathcal{O}(k)}$ and defines $cycl(Num_k)$.

3.4. Interrupted counting—more definability results

This subsection contains more definability results, which are need for examples to show that $\underline{\Sigma_k} \subsetneqq \underline{\Delta_{k+1}}$ for the class of coloured grids or the class of graphs.

We will show that $pref_+(Num_k^+)$ is in Π_{k-1}^{loc} for every k. With respect to monadic logic this will in particular imply that the k-fold exponential function f_k is Δ_k-definable (rather than just Σ_k-definable as shown in the previous subsection).

The idea and the justification for the headline of this subsection is that the counting mechanism in Num_k^+ is interrupted somewhere in the middle. In the previous subsection, the validity of the counting mechanism was essentially checked by excluding particular patterns as cyclic infixes. The trick works as well if that counting mechanism is interrupted somewhere, provided we forbid these patterns only as (non-cyclic) infixes. Thus this subsection will be quite analogous to the previous one.

The following lemma, for instance, is the counterpart of the Cyclic Counting Lemma 3.19 for the case of "interrupted counting". Here we benefit from considering binary number representations with least significant bit first.

Lemma 3.26 (Prefix Counting Lemma). *Let* $n \geqslant 0$ *and* $x_0,\ldots,x_n \in \{0,1\}^+$ *with* $m = |x_0| = \cdots = |x_{n-1}| \geqslant |x_n| \geqslant 1$. *Then the following are equivalent*:

1. $dual(x_j) = j \bmod 2^{|x_j|}$ *for all* $j \geqslant n$.
2.
   - $x_0 \in 0^+$,
   - $x_j\langle 0\rangle \neq x_{j+1}\langle 0\rangle$ *for all* $j \in \{0,\ldots,n-1\}$,
   - $\begin{pmatrix} x_j\langle i\rangle & x_{j+1}\langle i\rangle \\ x_j\langle i+1\rangle & x_{j+1}\langle i+1\rangle \end{pmatrix} \in Allow$ *for every* $j \in \{0,\ldots,n-1\}$ *and* $i \in \{0,\ldots,|x_{j+1}|-2\}$.

The proof is as simple as that of Lemma 3.19 and thus omitted.

We wish to show the following counterpart of Theorem 3.16.

Theorem 3.27. $pref_+(Num_k^+) \in \Pi_{k-1}^{\mathrm{loc}}$ *for every* $k \geqslant 1$.

The induction basis is given by the counterpart of Theorem 3.17:

Proposition 3.28. $pref_+(Num_1^k) \in \Pi_0^{\mathrm{loc}}$.

Proof. Let $\Gamma = \{0,1\}^{I_1}$. We will show that $pref_+(Num_1^+)$ is a local picture language. Let $\Delta \subseteq (\Gamma \cup \{\#\})^{2,2}$ as in Proposition 3.17.

Let Δ' be the following subset of $(\Gamma \cup \{\#\})^{2,2}$:

$$\Delta' = \Delta \cup \left\{ \begin{matrix} \# & \boldsymbol{c} \\ \# & \boldsymbol{d} \end{matrix} \,\middle|\, \boldsymbol{c},\boldsymbol{d} \in \{(0)_{num\text{-}1} \bowtie (0)_{end\text{-}1},\ \#\} \right\}$$

$$\cup \left\{ \begin{matrix} \boldsymbol{a} & \# \\ \boldsymbol{b} & \# \end{matrix} \,\middle|\, \mathsf{pr}_{end\text{-}1}(\boldsymbol{a}) = 1 \Leftrightarrow \mathsf{pr}_{end\text{-}1}(\boldsymbol{b}) = \mathsf{pr}_{num\text{-}1}(\boldsymbol{a}) = 1 \right\}$$

$$\cup \left\{ \begin{matrix} \boldsymbol{a} & \# \\ \# & \# \end{matrix} \,\middle|\, \mathsf{pr}_{end\text{-}1}(\boldsymbol{a}) = 1 \Leftrightarrow \mathsf{pr}_{num\text{-}1}(\boldsymbol{a}) = 1 \right\}$$

$$\cup \left\{ \begin{matrix} \# & \# \\ \boldsymbol{b} & \# \end{matrix} \,\middle|\, \boldsymbol{b} \in \Gamma \right\}.$$

Let P be a non-empty picture over Γ, say of size (m,n). For every $i \leqslant n-1$, let $x_i = P\langle 0,i\rangle \dots P\langle m-1,i\rangle$.

Δ' is chosen in such a way that Δ' contains all (2×2)-subblocks of $\hat{P}$ iff

- $\mathsf{pr}_{num\text{-}1}(x_0),\dots,\mathsf{pr}_{num\text{-}1}(x_{n-1})$ fulfill the second property of the Prefix Counting Lemma 3.26, and
- (like in the proof of Proposition 3.17) for every x among $x_0,\dots,x_{n-1}$, the $\mathsf{pr}_{end\text{-}1}(x)$ is determined by $\mathsf{pr}_{end\text{-}1}(x) = 0^i 1^j$ iff j is maximal such that 1^j is a suffix of $\mathsf{pr}_{num\text{-}1}(x)$.

Thus $\mathscr{L}(\Delta') = pref_+(Num_1)$. $\square$

The following is the counterpart of Lemma 3.20.

Lemma 3.29. *Let $k \geqslant 1$, let $\Omega = \{0,1\}^{I_{k+1}}$. Let U_1, U_2, U_3 and bin_m as in Lemma 3.20. Let*

$$U_4 = (pref_+(0^*1)))_{end\text{-}k} \otimes (\sim 0^*)_{num\text{-}k+1} \otimes \Omega^+.$$

The following are equivalent for every $P \in pref_+(Num_k^+ \otimes OnlyTop_{k+1})$ of height m.

1. *$P \in pref_+(Num_{k+1}^+)$;*
2. *$top(P)$ has no infix of length $f_k(m)+2$ in $U_1 \cup U_2$ and none of length $f_k(m)$ in U_3 and no prefix in U_4.*

The proof of this lemma is very similar to the proof of Lemma 3.20, except for the difference that the Prefix Counting Lemma 3.26 is used instead of the Cyclic Counting Lemma 3.19.

The following lemma is analogous to Lemma 3.23.

Lemma 3.30. *Let N such that $cycl(N^+) \in \Pi_{k-1}^{\mathrm{loc}}(\Omega)$. Let $L_1, L_2 \in \Pi_0^{\mathrm{loc}}(\Omega)$. Then there is a picture language L in $\Sigma_k^{\mathrm{loc}}(\Omega)$ such that for a non-empty picture P in $pref_+(N^+)$ we have that P has an infix in $(L_1 \cap cycl(N^+))L_2$ iff $P \in L$.*

Proof. Simply choose $L = \Omega^{*,*}(L_1 \cap cycl(N^+))L_2\Omega^{*,*}$. Then $L \in \Sigma_k^{\mathrm{loc}}$. $\square$

The proof of Lemma 3.30 is much simpler than that of Lemma 3.23 because it is so much simpler to exclude a particular pattern as a (non-cyclic) *infix* than as a *cyclic infix*.

Lemma 3.31. *Let $k \geqslant 1$ and $\Omega := \{0,1\}^{I_{k+1}}$. There is a picture language K_4 in $\Sigma_1^{\mathrm{loc}}(\Omega)$ such that for all $P \in pref_+(Num_k^+) \otimes OnlyTop_{k+1}$ we have $P \in K_4$ iff $top(P)$ has a prefix in U_4, where U_4 is as in Lemma 3.29.*

Proof. Choose $K_4 = top^{-1}(U_4) \oplus \Omega^{*,*}$. Since U_4 is locally threshold testable, K_4 is in $\oplus(\Pi_0^{\mathrm{loc}}) \subseteq \Sigma_1^{\mathrm{loc}}(\Omega)$. $\square$

Now, we are ready to do the proof of the main result of this subsection. It is very analogous to that of Theorem 3.16.

Proof of Theorem 3.27. By induction. The induction basis is provided by Proposition 3.28.

For the induction step, let U_1, U_2, U_3 as in Lemma 3.20, and let $k \geqslant 1$ with $pref_+(Num_k^+) \in \Pi_{k-1}^{\text{loc}}$. Like in Lemma 3.24 one shows (using Lemma 3.30 instead of Lemma 3.23) that there is a $K_3 \in \Pi_k^{\text{loc}}(\{0,1\}^{I_{k+1}})$ such that $P \in K_3$ iff $top(P)$ has an infix of length $f_k(m)+2$ in $U_1 \cup U_2$ or one of length $f_k(m)$ in U_3. Choose K_4 as in the above lemma. By Lemma 3.29, Items 1–2,

$$pref_+(Num_{k+1}^+) = (pref_+(Num_k^+) \otimes OnlyTop_{k+1}) \setminus (K_3 \cup K_4) \in \Pi_k^{\text{loc}}. \qquad \square$$

Like the previous subsection we close this one with a few corollaries that depend on results of Section 3.1.

Corollary 3.32. *Let $k \geqslant 1$, $\Omega = \{0,1\}^{I_k}$. There is a picture languages L in $\Pi_k^{\text{loc}}(\Omega)$, $\underline{\Pi_{k-1}^0(\Delta_1^{\text{U}})}$, in $\underline{\Pi_{k-1}^0(\Delta_1)}$, in $\underline{\Pi_{k-1}}$, in $\underline{FO(\Delta_1)}$, and in $\underline{FO^{\leqslant_1,\leqslant_2}}$, such that*

$$\text{size}(L) = \{(m,n) \mid n = f_k(m)\}.$$

Besides, there are no distinct pictures $P, P' \in L$ with $\text{size}\,P = \text{size}\,P'$.

Proof. Choose $L = pref_+(Num_k^+) \otimes top^{-1}((0^*1)_{end\text{-}k}) \otimes \Omega^+$. Membership of L in Π_{k-1}^{loc} follows from Theorem 3.27 and the fact that the word languages $\{0,1\}^*0, \sim(0^*1)$, and 0^+ are locally threshold testable, so that their top-pre-images are Π_0^{loc}.

Membership in the other listed classes follows from Theorem 3.4 and Corollary 3.5. The additional claim is immediate from the fact that there are no distinct pictures $P, P' \in pref_+(Num_k^+)$ with $\text{size}\,P = \text{size}\,P'$. $\square$

The following provides the first half of the proof of Theorem 2.22, i.e., the fact that for every k, the class $\Delta_1^{\text{U}}(\Pi_{k-1}^0(\Delta_1^{\text{U}}))$ (and hence the other, larger formula classes of that theorem) are at least k-fold exponential.

Corollary 3.33. *Let $k \geqslant 1$. The formula class $\Delta_1^{\text{U}}(\Pi_{k-1}^0(\Delta_1^{\text{U}}))$ is at least k-fold exponential, and thus so are the classes $\Delta_1(\Pi_{k-1}^0(\Delta_1)), \Delta_1(FO(\Delta_1)), \Delta_k, \Delta_1(FO^{\leqslant_1,\leqslant_2})$, and $\Delta_1(FO^{\text{TC}})$.*

Proof. Let $\mathscr{F}$ be one of the formula classes listed in Corollary 3.32. Let $\bar{X}$ be the variable tuple $(X_1, \ldots, X_{2k})$.

Let $\pi : \{0,1\}^{I_k} \to \{0,1\}^{2k}$ be a bijection. By Theorems 3.27 and 3.5 we may choose $\varphi(\bar{X}) \in \mathscr{F}$ such that φ defines $\pi(pref_+(Num_k^+))$. Choose a first-order formula $\psi(\bar{X})$ such that ψ defines $\pi(top^{-1}(0^*1)_{end\text{-}k})$. Then $\psi(\iota\bar{X}\varphi(\bar{X}))$ is (by the additional claim of Corollary 3.32) an allowed $\Delta_1^{\text{U}}(\mathscr{F})$-formula, and it asserts for the grid of size (m,n) that $n = f_k(m)$. $\square$

3.5. Twin counting—yet more definability results

In this subsection we will develop some definability results that allow for more separation results than those of Section 3.3, among them $\underline{\Sigma_k} \neq \underline{\Pi_k}$ over *non-coloured* grids and $\underline{FO(\Sigma_k)} \subsetneq \underline{FO(\Delta_{k+1})}$ for the classes of grids or of graphs.

These definability results depend on some different approach unlike Section 3.4, which was very analogous to Section 3.3. This time we leave the detour of the starfree alternation hierarchy somewhat earlier so that the main theorem states the definability by monadic second-order formulas rather than membership in some level of the starfree alternation hierarchy.

However, a similarity between Sections 3.3 and 3.4, on one hand, and Section 3.5, on the other, is that we construct "slim" picture languages by exploiting counting mechanisms. But this time there are two of them in the same picture and two patterns of the two mechanisms belong together like twins, which explains the title of this subsection.

Recall the definition of the functions $f_k : \mathbb{N} \to \mathbb{N}$.

$$f_1(m) = 2^m,$$

$$f_{k+1}(m) = f_k(m)2^{f_k(m)}.$$

Furthermore, let $g_k(m) = \mathrm{lcm}\{f_k(m)+1, \ldots, 2f_k(m)\}$ for every m and every k. We will show in Proposition 3.50 that g_k is $(k+1)$-fold exponential.

The main result of this subsection is Theorem 3.44. It states that, for every k, there are two picture languages L_k, M_k definable in certain fragments of the first-order closure of Δ_k and such that $\mathrm{size}(L_k) = \{(m,n) \mid g_k(m) \text{ divides } n\}$ and $\mathrm{size}(M_k) = g_k$. The picture language L_k is even definable in Π_k. In Section 4 we will show that L_k and M_k are not definable in Σ_k and neither in $FO(\Sigma_{k-1})$ because g_k is a $(k+1)$-fold exponential function.

This implies the separation results $\underline{FO(\Sigma_k)} \subsetneq \underline{FO(\Delta_k)}$ and $\underline{\Sigma_k} \neq \underline{\Pi_k}$.

3.5.1. Attributed alphabets and free variables

In Section 2.3 we defined how to associate a non-empty picture over $\{0,1\}^t$ to a t-bit grid and then what we call the picture language over $\{0,1\}^t$ *defined by* a formula with free variables among $X_1, \ldots, X_t$. Since now we wish to construct formulas defining picture languages over attributed alphabets, we require the (straightforward) adaptions of these definitions to this case.

Let I be a set of attributes, let $(X_\mu)_{\mu \in I}$ be a tuple of monadic second-order variables. An *I-coloured grid* is a grid expanded by monadic predicates $(X_\mu)_{\mu \in I}$. To every I-coloured grid we associate a non-empty picture over $\{0,1\}^I$. We say that this picture fulfills a formula $\varphi((X_\mu)_{\mu \in I})$ iff the I-coloured grid it is associated to makes (with the implicitly given assignment) φ true. A formula $\varphi((X_\mu)_{\mu \in I})$ *defines* the picture language of those pictures over $\{0,1\}^I$ that fulfil φ.

It would have been a natural idea to let the attributes themselves serve as free set variables, but this conflicts with our conventions to use capital letters for second-order variables.

3.5.2. The twin language

Recall the attribute sets $I_k = \{num\text{-}1, \ldots, num\text{-}k, end\text{-}1, \ldots, end\text{-}k\}$ for every k. Let *act* (for "active") be a fresh attribute not in any of the I_k.

For this section we fix some $k \geqslant 1$.

Let $\imath$ be some bijection defined on $I_k \cup \{act\}$ such that $I_k \cup \{act\}$ and its image under $\imath$ are disjoint. We use $\imath$ also to denote the mapping $\{0,1\}^{I_k \cup \{act\}} \to \{0,1\}^{\imath(I_k \cup \{act\})}$ defined by $\imath(a) : \imath(\mu) \mapsto a(\mu)$ for all $\mu \in I_k \cup \{act\}$. This alphabet projection is further extended to pictures and picture languages in the usual way.

We turn to the definition of the twin language. We proceed in several steps.

$$L_k = Num_k \otimes (1^{+,+})_{act},$$

$$S = (0^{+,1})_{I_k \cup \{act\}}.$$

So L_k is a picture language over alphabet $\{0,1\}^{I_k \cup \{act\}}$ whose restriction to I_k is the counting pattern introduced in Definition 3.15. The picture language S is over the same alphabet and contains all columns (i.e., pictures of length 1) that have, at each position, a 0 in every component.

For all $m \geqslant 1$ and all $j \in \{1, \ldots, f_k(m)\}$ let

$$twin_{k,m,j} = \{P \in (L_k \mathbin{\oplus} S^j) \otimes \imath(S^j \mathbin{\oplus} L_k) \mid \bar{\underline{P}} = m\},$$

$$\begin{aligned} ptwin_{k,m,j} = {} & \{P \in pref_+(L_k) \otimes \imath(S^+) \mid \bar{\underline{P}} = m\} \\ & \cup \{P \in (L_k \mathbin{\oplus} S^+) \otimes \imath(S^j \mid pref_+(L_k)) \mid \bar{\underline{P}} = m\}, \end{aligned}$$

$$xtwin_k = \bigcup_{m \geqslant 1} \left(\left(\bigcup_j twin_{k,m,j} \right)^* \mathbin{\oplus} \left(\bigcup_j ptwin_{k,m,j} \right) \right),$$

$$iso\text{-}xtwin_k = \bigcup_{m \geqslant 1} \left(\bigcup_j (twin_{k,m,j})^* \mathbin{\oplus} ptwin_{k,m,j} \right).$$

These picture languages are over alphabet $\{0,1\}^{\{I_k \cup \imath(I_k) \cup \{act, \imath(act)\}\}}$. Fig. 5 illustrates a member of $twin_{k,m,j}$ (together with two other members of $ptwin_{k,m,j}$). Roughly speaking, a picture in $twin_{k,m,j}$ results by joining two copies (over disjoint alphabets) of the picture P_{km} of height m in Num_k in such a way that these copies may overlap. Additionally, the components associated with the two copies of attribute *act* mark those positions that belong to the respective copy of P_{km}, i.e., where the respective counting process is "active". In the position in the areas to the right and left, in which one counting process is not active, the components corresponding to its copy of the alphabet have a 0. Note that there is no area in which both counting processes are non-active

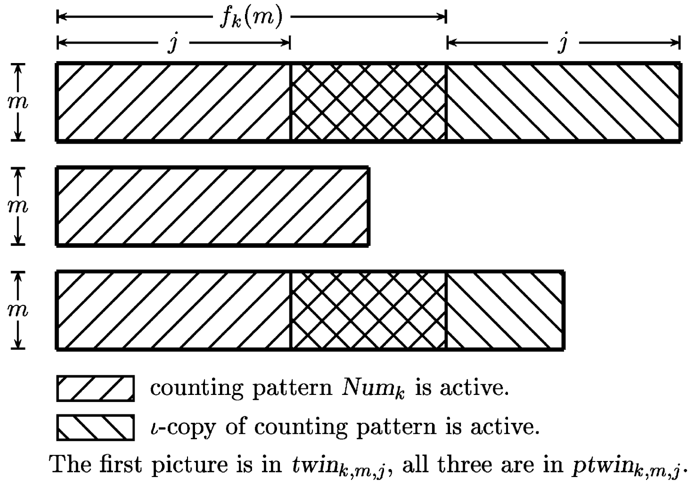


Fig. 5. The twin language.

because a picture of height m in L_k has length $f_k(m)$. Let us call the width j of each of the areas to the right and left, where one counting process is not active, "offset". Note that the width of the picture of height m in $twin_{k,m,j}$ is $f_k(m)+j$.

A picture in $ptwin_{k,m,j}$ ("p" standing for "prefix") is of one of two different types (see last two boxes in Fig. 5). Either it results from a non-empty prefix of the picture over alphabet $\{0,1\}^{I_k}$ of height m in Num_k by joining with a picture of zeros over a disjoint copy of that alphabet, or it results from a picture in $twin_{k,m,j}$ by clipping off something—possibly empty—in the rightmost area. In both cases each of the two copies of the "*act*"-component carries a 1 whenever the respective counting process is active. This way, for every $j \in \{1,\dots,f_k(m)\}$ and every $n \in \{1,\dots,f_k(m)+j\}$ there is exactly one picture of size (m,n) in $ptwin_{k,m,j}$.

A picture in $xtwin_k$ is a concatenation of several pictures from $twin_{k,m,j}$ (for the same height m, but possibly different "offsets" j) and one picture from $ptwin_{k,m,j}$.

A picture in $iso\text{-}xtwin_k$ is a picture in $xtwin_k$ where the offsets j are equal in all of the subblocks from $twin_{k,m,j}$ and $ptwin_{k,m,j}$.

The following theorem will imply all other definability results of this subsection.

Theorem 3.34. *$iso\text{-}xtwin_k$ is definable in $\Pi^0_{k-1}(\Delta^{\mathrm{U}}_1)$.*

We will prove Theorem 3.34 for the cases $k=1$ and $k>1$ separately.

How to prove Theorem 3.34, case $k=1$. In this part we assume that $k=1$.

We have reached one of the few points where our notations for attributed alphabets, which are motivated in the beginning of this section, make things more complicated rather than easier. That is why we will sometimes use the following abbreviation. Consider the alphabet $\{0,1\}^{\{act,\iota(act)\}}$. In the following, we will sometimes write $\binom{i}{j}$ instead of $(i)_{act} \otimes (j)_{\iota\{act\}}$ for $i,j \in \{0,1\}$, as if $\{0,1\}^{\{act,\iota(act)\}}$ was equal to the alphabet $\{0,1\}\times\{0,1\}$.

Remark 3.35. Let us use the just explained short-hand notation. Choose the word language M and the picture language M' as follows:

$$M = \left(\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}^+ \begin{pmatrix} 1 \\ 1 \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}^+ \right)^* \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* \right) \Big\backslash \{\varepsilon\},$$

$$M' = top^{-1}(M) \cap \left(\bigcup_a a^{+,1} \right)^+,$$

where the union ranges over $a \in \{0,1\}^{\{act,\iota(act)\}}$. Then M is local, and M' is the set of non-empty pictures over $\{0,1\}^{\{act,\iota(act)\}}$ whose top row is in M and in which two positions have the same letter if they are in the same column. Clearly, M' is local by locality of M. In the above short-hand notation,

$$M' = \left(\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{+,+} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{*,*} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{+,+} \right)^* \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{*,*} \right) \Big\backslash \{\varepsilon\}.$$

Let us use the abbreviation $\mathbf{0} = (0)_{num\text{-}1} \bowtie (0)_{end\text{-}1} \bowtie (0)_{act}$.

Proposition 3.36. *$N = (\mathbf{0}^{*,*} \oplus (L_1 \oplus \mathbf{0}^{+,+})^* \oplus pref(L_1)) \backslash \{\varepsilon\}$ is a local picture language.*

Proof. It is straightforward to construct a local tiling system Δ' defining $pref_+(L_1) = pref_+(Num_1) \otimes ((1)_{act})^{*,*}$ from the one given for $pref_+(Num_1^+)$ in the proof of Proposition 3.28. In order to construct a local tiling system for N, this Δ' can easily modified by removing the tile

$$(1)_{num\text{-}1} \bowtie \underset{\#}{(1)_{end\text{-}1}} \bowtie (1)_{act}, (0)_{num\text{-}1} \bowtie \underset{\#}{(0)_{end\text{-}1}} \bowtie (1)_{act},$$

which allows the counting process wrap around, and adding tiles that allow to reset the counting process via a non-empty sequence of columns of $\mathbf{0}$'s. □

Proposition 3.37. *$xtwin_1$ is local.*

Proof. We claim

$$xtwin_1 = N \otimes \iota(N) \otimes M', \tag{2}$$

where M' is chosen as in Remark 3.35 and N is chosen as in the previous proposition. Then we are finished because the class of local picture languages is closed under intersection (and thus, by Remark 3.13, under join). A precise proof for the above equation can be found in [16]. Informally, we may argue as follows:

Consider the join of two picture languages N and $\iota(N)$. This is the set of pictures that consist of two distinct copies of the counting processes (one in the attributes of I_k and the other in those of $\iota(I_k)$) that are reset independently from each other. Whenever one of the counting processes are "active", i.e., the corresponding columns represent binary numbers in succession, the respective attribute act or $\iota(act)$, respectively, equals

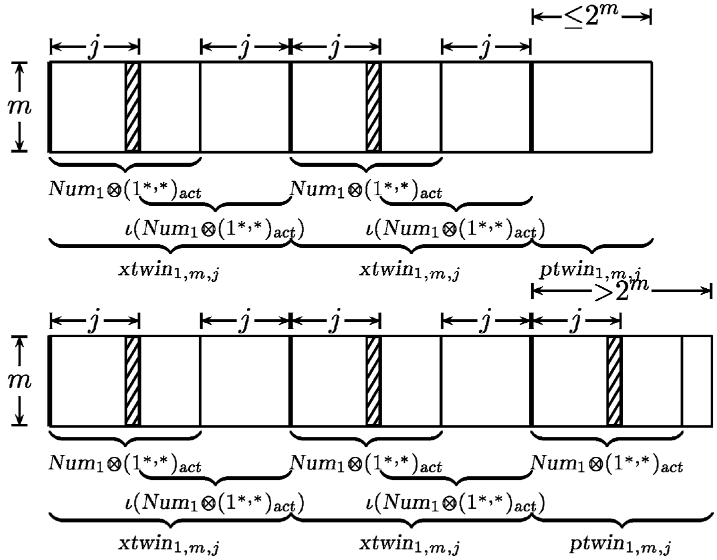


Fig. 6. Two pictures in *iso-xtwin*$_1$.

one. Between the two columns where the counting processes wraps around, i.e., is reset to zero, there is a non-empty sequence of columns that carry a zero in all components of $I_k \cup \{act\}$ or $\iota(I_k \cup \{act\})$, respectively.

By joining M' to this picture language we control the counting mechanisms via the *act*- and $\iota(act)$-attribute to synchronize in the way illustrated in Fig. 6. That means, whenever the counting process represented by the attributes I_k terminates and starts to repeat zero columns, the other copy has to be active already. Whenever the second counting process (represented by the attributes $\iota(I_k)$) terminates and starts to repeat zero columns, the first one has to be restarted immediately.

Note that M' is designed in such a way that it allows for both types of finishing these simultaneous counting processes: Either the first or the second counting process is active in the last column, corresponding to the values $\binom{1}{0}$ or $\binom{0}{1}$ in the attributes $(act, \iota(act))$. $\square$

We use the previous proposition to prove the following, which is the claim of Theorem 3.34 for the case $k = 1$.

Proposition 3.38. *iso-xtwin*$_1$ *is definable in* Δ_1^{U}.

Proof. There is a first-order formula $less(X_{leq}, X_{\iota(act)})$ such that *less* asserts for a grid of height m that

$$X_{leq} = \{(0,j) \mid (0,0), \ldots, (0,j) \notin X_{\iota(act)}\},$$

i.e., that X_{leq} is the set of all top-row positions up to (but not including) the first position in $X_{\iota(act)}$.

There is a first-order formula $maxless(X_{leq}, X_{maxleq})$ that asserts that X_{maxleq} contains those positions that are in a column whose top row position is in X_{leq} and has no right successor in X_{leq}.

There is a first-order formula $offsets(X_{offs}, X_{\iota(act)})$ that asserts that X_{offs} contains those top row positions that are not in $X_{\iota(act)}$ but whose right successor is.

There is a first-order formula $transport(X_{row}, Y, X_{num\text{-}1})$ that asserts that X_{row} is a union of rows with $X_{row} \cap Y = X_{num\text{-}1} \cap Y$.

Choose

$$\varphi(\bar{X}) = less(X_{leq}, X_{\iota(act)}) \wedge maxless(X_{leq}, X_{maxleq})$$

$$\wedge transport(X_{row}, X_{maxleq}, X_{num\text{-}1}).$$

Then for every $I_1 \cup \{\iota(act)\}$-coloured grid R there is exactly one tuple $\bar{X} = (X_{leq}, X_{maxleq}, X_{offs}, X_{row})$ of subsets of $\operatorname{dom} R$ that fulfills φ because $less(X_{leq}, X_{\iota(act)}) \wedge maxless(X_{leq}, X_{maxleq})$ determines the column that the set X_{maxleq} contains (it is the first hatched one in Fig. 6), and thus $transport(X_{row}, X_{maxleq}, X_{num\text{-}1})$ determines X_{row} uniquely.

Choose [5]

$$\psi(\bar{X}, (X_\mu)_{\mu \in (I_1 \cup \iota(I_1) \cup \{act, \iota(act)\})}) = transport(X_{row}, X_{offs}, X_{num\text{-}1}).$$

Then $\psi(\iota\bar{X}(\varphi), (X_\mu))$ is an allowed Δ_1^{U}-formula which asserts for a $(I_1 \cup \iota(I_1) \cup \{act, \iota(act)\})$-coloured grid whose associated picture P is in $xtwin_1$ that P is in $iso\text{-}xtwin_1$ because it asserts that the colouring of those rows that precede the rows in which the second copy of the counting process starts are equal. See Fig. 6.

Since $xtwin_1$ is local by the previous Proposition, there is a first-order formula π that asserts for a $(I_1 \cup \iota(I_1) \cup \{act, \iota(act)\})$-coloured grid that its associated picture is in $xtwin_1$. Thus $\pi \wedge \psi(\iota\bar{X}(\varphi), (X_\mu))$ defines $iso\text{-}xtwin_1$. This formula is $\overline{\Delta_1^{\mathrm{U}}}$ because $\overline{\Delta_1^{\mathrm{U}}}$ is closed under intersection.

This completes the proof of Proposition 3.38 and thus the proof of Theorem 3.34 for the case $k = 1$. $\square$

How to prove Theorem 3.34, case $k > 1$. For the proof of Theorem 3.34 we will need the following proposition, which will be proved in Section 3.5.3. Recall that $(0)_I$ denotes the letter in $\{0,1\}^I$ all of whose components are 0.

Proposition 3.39. *$cycl(L_k \oplus (0)^{*,*}_{I_k \cup \{act\}})$ is in Π^{loc}_{k-1}.*

Proof of Theorem 3.34, case $k \geqslant 2$. We use the following abbreviations: $\mathbf{0} = (0)_{I_k \cup \{act\}}$, the mapping $\varphi : \{0,1\}^{I_k} \to \{0,1\}^{I_k \cup \{act\}}$, $a \mapsto a \bowtie (1)_{act}$, and $\Omega = \{0,1\}^{I_k \cup \iota(I_k) \cup \{act, \iota(act)\}}$, and $L = L_k = Num_k \otimes (1^{+,+})_{act}$.

[5] List of potentially free variables is enlarged for formal reasons. Note that the two tuples are disjoint.

Let M' be the local picture languages of Remark 3.35. For a picture language U over some alphabet Γ we will write $\bar{U}$ for the complement $\Gamma^{+,+}\setminus U$ of U in $\Gamma^{+,+}$.

We wish to show that *iso-xtwin*$_k$ is the set of all non-empty pictures P over Ω that fulfil the following properties:

1. $\mathsf{restr}_{\{act,\iota(act)\}}(P) \in M'$.
2. P has no prefix in

$$M_1 = (1^{+,+})_{act} \otimes \overline{pref_+(L)} \otimes \Omega^{+,+},$$

$$M_2 = (1^{*,*}0^{+,1})_{act} \otimes (\bar{L} \oplus \Gamma^{+,1}) \otimes \Omega^{+,+}.$$

3. P has no infix in

$$N_1 = ((0)_{act} \bowtie \Omega)^{+,1}((1^{+,+})_{act} \otimes \bar{L} \otimes \Omega^{+,+})((0)_{act} \bowtie \Omega)^{+,1},$$

$$N_2 = ((0)_{\iota(act)} \bowtie \Omega)^{+,1}((1^{+,+})_{\iota(act)} \otimes \iota(\bar{L}) \otimes \Omega^{+,+})((0)_{\iota(act)} \bowtie \Omega)^{+,1},$$

$$N_3 = ((0)_{\iota(act)} \bowtie \Omega)^{+,1}((1^{+,+}0^{+,+})_{\iota(act)} \otimes \overline{cycl(\varphi(L)\mathbf{0}^{*,*})} \otimes \Omega^{+,+})$$
$$((1)_{\iota(act)} \bowtie \Omega)^{+,1}.$$

4. P has no suffix in

$$K_1 = ((0)_{act} \bowtie \Omega)^{+,1}((1^{+,+})_{act} \otimes \overline{pref_+(L)} \otimes \Omega^{+,+}),$$

$$K_2 = ((0)_{\iota(act)} \bowtie \Omega)^{+,1}((1^{+,+})_{\iota(act)} \otimes \iota(\overline{pref_+(L)}) \otimes \Omega^{+,+}).$$

Before we verify these properties 1–4, let us note how this implies that *iso-xtwin*$_k \in \underline{\Pi^0_{k-1}(\Delta^{\mathrm{U}}_1)}$. By choice of L and Theorems 3.16, 3.27 and Proposition 3.39, the languages $pref_+(L)$, L, $\iota(L)$, $cycl(\varphi(L)\mathbf{0}^{*,*})$, and $\iota(pref_+(L))$ are Π^{loc}_{k-1}, thus their complements are $\Sigma^{\mathrm{loc}}_{k-1}$ and hence in $\underline{\Sigma^0_{k-1}(\Delta^{\mathrm{U}}_1)}(\Omega)$. Since $\underline{\Sigma^0_{k-1}(\Delta^{\mathrm{U}}_1)}$ is closed under $\otimes$ and (by Lemma 3.12) under column concatenation, the picture languages $M_1, M_2, N_1, N_2, N_3, K_1, K_2$ are in $\underline{\Sigma^0_{k-1}(\Delta^{\mathrm{U}}_1)}$. Since $\underline{\Sigma^0_{k-1}(\Delta^{\mathrm{U}}_1)}$ is closed under union and column concatenation, the picture languages $(M_1 \cup M_2)\Omega^{*,*}$, $\Omega^{*,*}(N_1 \cup N_2 \cup N_3)\Omega^{*,*}$, and $\Omega^{*,*}(K_1 \cup K_2)$ are in $\underline{\Sigma^0_{k-1}(\Delta^{\mathrm{U}}_1)}$. Thus the picture language *iso-xtwin*$_k = (M' \otimes \Omega^{+,+}) \setminus ((M_1 \cup M_2)\Omega^{*,*} \cup \Omega^{*,*}(N_1 \cup N_2 \cup N_3)\Omega^{*,*} \cup \Omega^{*,*}(K_1 \cup K_2))$ is in $\underline{\Pi^0_{k-1}(\Delta^{\mathrm{U}}_1)}$.

It is easy to verify that every picture P in *iso-xtwin*$_k$ fulfils properties 1–4. Let us consider the converse direction. Let P be a non-empty picture over Ω with properties 1–4. Again, we sometimes write $\binom{i}{j}$ instead of $(i)_{act} \otimes (j)_{\iota(act)}$ for every $i, j \in \{0, 1\}$. By property 1 there are $n \geqslant 1$ and pictures $P_1, \ldots, P_n$ such that $P = P_1 \ldots P_n$ and

$$\mathsf{restr}_{\{act,\iota(act)\}}(P_j) \in \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{+,+} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{*,*} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{+,+}$$

for all $j \leqslant n-1$ and

$$\mathsf{restr}_{\{act,\iota(act)\}}(P_n) \in \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{+,+} \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{*,*} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{+,+} \cup \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{+,+}.$$

Thus for every $j \leqslant n$ there are P'_j, P''_j, P'''_j such that
- $P = P'_j P''_j P'''_j$,
- $\mathsf{restr}_{\{act,\iota(act)\}}(P'_j) \in ((1)_{act} \bowtie (0)_{\iota(act)})^{+,+}$,
- $\mathsf{restr}_{\{act,\iota(act)\}}(P''_j) \in ((1)_{act} \bowtie (1)_{\iota(act)})^{*,*}$,
- $\mathsf{restr}_{\{act,\iota(act)\}}(P'''_j) \in ((0)_{act} \bowtie (1)_{\iota(act)})^{*,*}$,
- $P'''_j = \varepsilon \Rightarrow j = n, P''_n = \varepsilon$.

We have that

$$\forall j \leqslant n : P'''_j \neq \varepsilon \Rightarrow P'_j P''_j \in (1^{+,+})_{act} \otimes L \otimes \Omega^{+,+}. \tag{3}$$

For $j = 1$ this follows because P has no prefix in M_2. For $j > 1$ this follows because P has no infix in N_1. We also have

$$\forall j < n : P''_j P'''_j \in \iota((1^{+,+})_{act} \otimes L) \otimes \Omega^{+,+}, \tag{4}$$

because P has no infix in N_2. Besides,

$$P'''_n \neq \varepsilon \Rightarrow P''_n P'''_n \in \iota((1^{+,+})_{act} \otimes pref_{+}(L)) \otimes \Omega^{+,+}, \tag{5}$$

because P has no suffix in K_2. Moreover,

$$P'''_n = \varepsilon \Rightarrow P_n \in (1^{+,+})_{act} \otimes pref_{+}(L) \otimes \Omega^{+,+}. \tag{6}$$

In case $n = 1$ this is true because P has no prefix in M_1. In case $n \geqslant 2$ this is true because P has no suffix in K_1. Finally,

$$\forall j \leqslant n : P'''_j \neq \varepsilon \Rightarrow |P'_j| = |P'_1|. \tag{7}$$

This is shown by induction on j. It is immediate for the case $j = 1$. So let $j < n$ with $|P'_j| = |P'_1|$ and $P'''_{j+1} \neq \varepsilon$. Since P has no infix in N_3 we have that $P''_j P'''_j P'_{j+1} \in cycl(\varphi(L)\ \mathbf{0}^{*,*}) \otimes \Omega^{+,+}$. Since $\mathsf{pr}_{act}(P''_j), \mathsf{pr}_{act}(P'_{j+1}) \in 1^{*,*}$ and $\mathsf{pr}_{act}(P'''_j) \in 0^{*,*}$, this implies $P''_j P'_{j+1} \in cycl(\varphi(L)) \otimes \Omega^{+,+}$, thus $|P''_j| + |P'_{j+1}| = f_k(m)$, where $m = \underline{\bar{P}}$. By (3) it follows $|P'_j| + |P''_j| = f_k(m) = |P''_j| + |P'_{j+1}|$, thus $|P'_{j+1}| = |P'_j|$. So by induction hypothesis $|P'_{j+1}| = |P'_1|$. This completes the induction and thus the proof for (7).

Now we are ready to show that $P \in iso\text{-}xtwin_k$. Choose $j = |P'_1|$. By (3), (4), (7) and choice of j we have that $P_j \in twin_{k,m,j}$ for every $j \leqslant n-1$. If $P'''_n \neq \varepsilon$, then $P_n \in ptwin_{k,m,j}$ by (3), (5), (7) and choice of j; if $P'''_n = \varepsilon$, then $P_n \in ptwin_{k,m,j}$ (independently from the choice of j by (6)).

This implies $P = P_1 \ldots P_n \in (twin_{k,m,j})^* \, ptwin_{k,m,j} \subseteq iso\text{-}xtwin_k$. This completes the proof of Theorem 3.34. $\square$

3.5.3. Inserting a sequence of zero columns

In this section we will prove Proposition 3.39. First we shall sketch informally what has to be done. Let $\alpha(Num_k) = (Num_k \otimes (1^{+,+})_{act} \oplus (0^{+,+})_{I_k \cup \{act\}})$. We wish to show that $cycl(\alpha(Num_k))$ is in Π^{loc}_{k-1}, exploiting that $cycl(Num_k)$ is Π^{loc}_{k-1}.

Intuitively, the pictures in $cycl(\alpha(Num_k))$ are those that result from a picture P in $cycl(Num_k)$ by joining an additional component act that is always 1, and then cyclically inserting an infix from $(0^{+,+})_{I\cup\{act\}}$ between the two columns of P that mark the "end" and the "beginning" of the corresponding Num_k-picture in $cycl(\{P\})$.

Our strategy to prove Proposition 3.39 is to give a sufficient criterion for such a mapping α to respect the local hierarchy. For this purpose we abstract somewhat from the particular situation we have.

Let I be a finite set of attributes, $act\notin I$, and $\Gamma=\{0,1\}^I$, $\Omega=\{0,1\}^{I\cup\{act\}}$. Let $c,c',d,d'\in\Gamma$ and $C=c\ominus c'^{*,1}$ and $D=d\ominus d'^{*,1}$.

Define the alphabet projection $\varphi:\Gamma\to\Omega$, $a\mapsto a\bowtie(1)_{act}$ as in the proofs of Proposition 3.38 and Theorem 3.34. We lift φ to pictures and picture languages as usual. Let $\mathbf{0}\in\Omega$ be defined by $\mathsf{pr}_\mu(\mathbf{0})=0$ for all $\mu\in I\cup\{act\}$, again like in the above proofs.

For every $P\in\Gamma^{+,+}$ let

$$\begin{aligned}\alpha(P)=\{\varphi(P)\}&\\ &\cup\bigcup\{\varphi(P'C)\ \oplus\mathbf{0}^{*,*}\ \oplus\varphi(DP'')\mid P\in P'CDP''\},\\ &\cup\{\mathbf{0}^{*,*}\ \oplus\varphi(DP'C)\ \oplus\mathbf{0}^{*,*}|P\in DP'C\}.\end{aligned}$$

Proposition 3.40. (1) $\alpha(L_1\cup L_2)=\alpha(L_1)\cup\alpha(L_2)$ *for all picture languages* $L_1,L_2\subseteq\Gamma^{+,+}$, *and the same is true for* $\cap$ *instead of* $\cup$.

(2) $\alpha(\Gamma^+\backslash L)=\alpha(\Gamma^+)\backslash\alpha(L)$ *for every word language* $L\subseteq\Gamma^+$.

(3) $\alpha(\Gamma^{+,+}\backslash L)=\alpha(\Gamma^{+,+})\backslash\alpha(L)$ *for every picture language* $L\subseteq\Gamma^{+,+}$.

(4) $\alpha(L_1\dots L_n)=(\mathbf{0}^{*,*}\alpha(L_1)\mathbf{0}^{*,*}\dots\mathbf{0}^{*,*}\alpha(L_n)\mathbf{0}^{*,*})\cap\alpha(\Gamma^{*,*})$ *for all picture languages* $L_1,\dots,L_n\subseteq\Gamma^{+,+}$.

Lemma 3.41. *Let* $L\subseteq\Gamma^+$ *be a word language over* Γ. *If* L *is locally threshold testable, then so is* $\alpha(L)$. *In particular,* $\alpha(top^{-1}(L))$ *is in* Π_0^{loc} *for every locally threshold testable word language* L.

Proof. The second claim follows easily from the first one, using the equation $\alpha(top^{-1}(L))=\alpha(\Gamma^{+,+})\cap top^{-1}(\alpha(L))$. The first one is a consequence of the following two auxiliary claims:

1. If is of the form $L=\{w\in\Gamma^+\mid x$ has t occurrences of the subblock $x\}$ for some $x\in\Gamma^+$ and some $t\geqslant 1$, then $\alpha(L)$ is locally threshold testable.
2. $\alpha(x\Gamma^*)$, $\alpha(\Gamma^*x)$ are locally threshold testable for every $x\in\Gamma^+$.

For the first of these auxiliary claims, let $x\in\Gamma^+$, $t\geqslant 1$, and L as above. Then $\alpha(L)$ is the set of all words $w\in\Omega^+$ for which

- $w\in\alpha(\Gamma^+)$ and w has t different occurrences of $\varphi(x)$, or
- there are x',x'' with $x=x'c\,dx''$ and $w\in\varphi(\Gamma^*x'c)\mathbf{0}^+\varphi(dx''\Gamma^*)$ and w has $t-1$ different occurrences of $\varphi(x)$, or
- there is x' with $dx'c$ and $w\in\mathbf{0}^*\varphi(dx'c)\mathbf{0}^*$ and w has $t-1$ different occurrences of $\varphi(x)$.

It follows that $\alpha(L)$ is locally threshold testable.

The second auxiliary claim can be shown similarly but easier. $\square$

Lemma 3.42. *Let $L \subseteq \Gamma^{+,+}$. If L is local or cyclically local, then $\alpha(L)$ is in Π_0^{loc}.*

Proof. We only consider the case that L is local, the other case in analogous. Choose $\Delta \subseteq (\Gamma \cup \{\#\})^{2,2}$ such that $\mathscr{L}(\Delta) = L$. We extend φ to $\Gamma \cup \{\#\}$ by letting $\varphi(\#) = \#$. Let Δ' contain firstly $\varphi(\Delta)$, secondly all (2×2)-subblocks of

$$\begin{matrix} \# & \# & \# & \# \\ \varphi(c) & \mathbf{0} & \mathbf{0} & \varphi(d) \\ \varphi(c') & \mathbf{0} & \mathbf{0} & \varphi(d') \\ \varphi(c') & \mathbf{0} & \mathbf{0} & \varphi(d') \\ \# & \# & \# & \# \end{matrix} \quad \text{or} \quad \begin{matrix} \# & \# & \# \\ \# & \mathbf{0} & \# \\ \# & \mathbf{0} & \# \\ \# & \# & \# \end{matrix}$$

and thirdly, in case

$$\begin{matrix} c & d \\ \# & \# \end{matrix} \in \Delta$$

all (2×2)-subblocks of

$$\begin{matrix} \varphi(c) & \mathbf{0} & \varphi(d) \\ \# & \# & \# \end{matrix}.$$

Then $\alpha(L)$ is the set of all pictures in $\mathscr{L}(\Delta')$ that are in $\varphi(\Gamma^{+,+})\mathbf{0}^{*,*}\varphi(\Gamma^{+,+})$ or in $\mathbf{0}^{*,*}\varphi(\Gamma^{+,+})\mathbf{0}^{*,*}$, but not in $\mathbf{0}^{+,+}$, i.e.,

$$\alpha(L) = \mathscr{L}(\Delta') \cap top^{-1}((\varphi(\Gamma)^{+}\mathbf{0}^{*}\varphi(\Gamma)^{+} \cup \mathbf{0}^{*}\varphi(\Gamma)^{+}\mathbf{0}^{*}))\backslash\mathbf{0}^{+}).$$

Thus $\alpha(L)$ is in Π_0^{loc}. $\square$

Lemma 3.43. *For every $k \geqslant 0$, if $L \in \Pi_k^{\text{loc}}(\Gamma)$, then $\alpha(L) \in \Pi_k^{\text{loc}}(\Omega)$.*

Proof. We show by simultaneous induction on the definition on the local hierarchy that
1. if $L \in \Sigma_k^{\text{loc}}$, then $\alpha(L) \in \Sigma_k^{\text{loc}}$,
2. if $L \in B(\Sigma_k^{\text{loc}})$, then $\alpha(L) \in B(\Sigma_k^{\text{loc}})$,
3. if $L \in \oplus\text{-cl}(B(\Sigma_k^{\text{loc}}))$, then there is an $L' \in \oplus\text{-cl}(B(\Sigma_k^{\text{loc}}))$ with $\alpha(L) = L' \cap \alpha(\Gamma^{+,+})$.

Claims 1 and 2 of the induction basis $k = 0$ follow from Lemmas 3.41 and 3.42 using Proposition 3.40 (Items 1 and 3).

Claim 3 (both of the induction basis and the induction step) follows from Proposition 3.40, Item 4. Claims 1 and 2 of the induction step follow from Proposition 3.40, Items 1 and 3. This completes the induction.

Proposition 3.43 follows from Item 1 by Lemma 3.40, Item 3. $\square$

Now, we have collected all lemmas to present the proof of the result of this section.

Proof of Proposition 3.39. Choose I from the beginning of this section to be I_k. Let C and D be the set of columns occurring as the last column or the first column, respectively, of pictures in Num_k, i.e., $C = c \ominus (c')^{*,1}$, where $c, c' \in \{0,1\}^{I_k}$ such that $\mathsf{pr}_\mu(c) = 1$ for all $\mu \in I_k$ and $\mathsf{pr}_\mu(c') = 1$ for both $\mu \in \{num\text{-}1, end\text{-}1\}$ and $\mathsf{pr}_\mu(c')\mu = 0$ for all $\mu \in I_k \backslash \{num\text{-}1, end\text{-}1\}$; and $D = d^{+,1}$, where $d \in \{0,1\}^{I_k}$ such that $\mathsf{pr}_\mu(d) = 0$ for all $\mu \in I_k$.

Proposition 3.39 follows from Lemma 3.43 and Theorem 3.16 and the observation that

$$cycl(Num_k \otimes (1^{+,+})_{act}) \oplus \mathbf{0}^{*,*} = \alpha(cycl(Num_k)).$$

This completes the proof and thus fills the gap in the proof of Theorem 3.34. □

3.5.4. Assembling the formulas

In this paragraph we shall prove the following main result of this subsection.

Theorem 3.44. *For every $k \geqslant 1$, there is a $\Pi^0_1(\Delta^{\mathrm{U}}_1(\Pi^0_{k-1}(\Delta^{\mathrm{U}}_1)))$-sentence π and a $\Pi^0_2(\Delta^{\mathrm{U}}_1(\Pi^0_{k-1}(\Delta^{\mathrm{U}}_1)))$-sentence π' that assert for the grid $\underline{[m] \times [n]}$ that $g_k(m) \mid n$ and that $g_k(m) = n$, respectively.*

Before we begin to develop the proof, we note that the formula classes of this theorem are fragments of two more important (and easier to understand) fragments, which allows to deduce Corollary 3.49.

Remark 3.45. For every $k \geqslant 1$,

- $\underline{\Delta^{\mathrm{U}}_1(\Pi^0_{k=1}(\Delta^{\mathrm{U}}_1))} \subseteq \underline{\Delta_k}$,
- $\underline{\Pi^0_1(\Delta_k)} \subseteq \underline{\Pi_k}$,
- $\Pi^0_1(\Delta_k), \underline{\Pi^0_2(\Delta_k)} \subset \underline{FO(\Delta_k)}$,
- $\Pi^0_2(\Delta_k) \subseteq \underline{\Pi^0_1(\Sigma_k)}$.

Thus the sentence π from Theorem 3.44 is in $\bar{\Pi}_k$ and both the sentences π and π' are in the first-order closure of Δ_k.

Together with the non-definability results Corollary 4.4 and Theorem 4.6, the above implies (among others) the following separation results over the class of grids: $\underline{\Pi_k} \nsubseteq \underline{\Sigma_k}$ and $\underline{\Pi^0_2(\Delta_{k+1})} \nsubseteq FO(\Delta_k)$ for all $k \geqslant 1$. The first non-inclusion solves an open problem of [18, 21].

Again, we fix some $k \geqslant 1$.

Lemma 3.46. *There is a first-order formula $mleq(X, Y)$ that asserts for a grid $\underline{[m] \times [n]}$ and two sets $X, Y \subseteq [m] \times [n]$ that X is the unique set of top row positions $\{0\} \times \{1, \dots, j\}$, where $j \leqslant n-1$ is maximal with $(\{0\} \times \{1, \dots, j-1\}) \cap Y = \emptyset$.*

The proof is easy, cf. proof of Proposition 3.38, formula *less*.

Proposition 3.47. *For every $m, n \geqslant 1$ and every $j \in \{1, \dots, f_k(m)\}$ there is exactly one picture of size (m, n) in $(twin_{k,m,j})^* \oplus ptwin_{k,m,j}$.*

This is immediate by definition of $twin_{k,m,j}$ and $ptwin_{k,m,j}$.

Lemma 3.48. *Let us consider the attribute set* $J = I_k \cup \iota(I_k) \cup \{act, \iota(act), leq, offs\}$. *We write* $\bar{X}$ *to abbreviate the variable tuple* $(\bar{X}_\mu)_{\mu \in J}$.

There is a formula $\varphi(\bar{X}, x)$ *in* $\Pi^0_{k-1}(\Delta^{\mathrm{U}}_1)$ *such that for all* $m, n \geqslant 1$ *and all* $x \in [m] \times [n]$ *there is exactly one picture* P *of size* (m, n) *over* $\{0,1\}^J$ *such that* $P, x \models \varphi$.

Moreover, we have the following for this picture P *and this position* x: *if* $x = (i, j) \in \{0\} \times \{1, \ldots, f_k(m)\}$, *then* $\mathsf{restr}_{I_k \cup \iota(I_k) \cup \{act, \iota(act)\}}(P) \in (twin_{k,m,j})^* \oplus ptwin_{k,m,j}$; *and* $x \notin \{0\} \times \{1, \ldots, f_k(m)\}$ *iff* $(0,0) \in X^P_{offs}$.

Proof. By Theorem 3.34, there exists a $\Pi^0_{k-1}(\Delta^{\mathrm{U}}_1)$-formula $iso\text{-}xtwin_k(\bar{X})$ that asserts for a picture P over $\{0,1\}^J$ that $\mathsf{restr}_{I_k \cup \iota(I_k) \cup \{act, \iota(act)\}}(P) \in iso\text{-}xtwin_k$.

The first-order formula $\psi(X_{offs}, x) = top(x) \wedge X_{offs}(x) \wedge \forall x'(S_2 x x' \to \neg X_{offs}\, x')$ asserts that x is the minimal top row position of $X_{\iota(act)}$, provided that $mleq(X_{offs}, X_{\iota(act)})$ holds.

Let $\varphi_1(\bar{X}, x)$ be the following $\Pi^0_{k-1}(\Delta^{\mathrm{U}}_1)$-formula.

$$\varphi_1 = iso\text{-}xtwin_k \wedge mleq(X_{offs}, X_{\iota(act)}) \wedge mleq(X_{leq}, X_{end\text{-}k}) \wedge \psi.$$

For every $m, n \geqslant 1$ and every $x \in [m] \times [n]$, we have the following: $x \in \{0\} \times \{1, \ldots, f_k(m)\}$ iff there is a tuple $\bar{X} = (\bar{X}_\mu)_{\mu \in J}$ of subsets of $[m] \times [n]$ such that the associated picture fulfils φ_1, and in this case $\bar{X}$ is determined uniquely by x (see Proposition 3.47) in such a way that the associated picture is in $iso\text{-}xtwin_k$.

By Theorems 3.27 and 3.4 there is a $\Pi^0_{k-1}(\Delta^{\mathrm{U}}_1)$-formula $pmarknum_k(\bar{X})$ that asserts for a non-empty picture P that $\mathsf{restr}_{I_k} P \in pref_+(Num^+_k)$. Let $\psi'(\bar{X})$ be a first-order formula that asserts that $X_\mu = \operatorname{dom} P$ for all $\mu \in \iota(I_k) \cup \{act, \iota(act), offs\}$.

Let $\pi(X_{leq}, x) = \neg X_{leq} x \wedge \forall x'(S_2 x' x \to \neg X_{leq} x')$. The first-order formula π asserts for a set $X_{leq} = \{0\} \times \{1, \ldots, j-1\}$ that $x \notin \{0\} \times \{1, \ldots, j\}$.

Let

$$\varphi_2(\bar{X}, x) = \ pmarknum_k \wedge mleq(X_{leq}, X_{end\text{-}k}) \wedge \pi \wedge \psi'.$$

For every $m, n \geqslant 1$ and every $x \in [m] \times [n]$, we have the following: $x \notin \{0\} \times \{1, \ldots, f_k(m)\}$ iff there is a tuple $\bar{X} = (\bar{X}_\mu)_{\mu \in J}$ of subsets of $[m] \times [n]$ such that the associated picture fulfils φ_2, and in this case $\bar{X}$ is determined uniquely (independently from x). The latter is because the conjunct $pmarknum_k$ determines the choice of X_μ for every attribute $\mu \in I_k$, the conjunct $mleq(X_{leq}, X_{end\text{-}k})$ determines the choice of X_{leq} (namely to be $\{0\} \times \{1, \ldots, \max\{n-1, f_k(m)-1\}\}$), and ψ' determines the choice of X_μ for the remaining attributes $\mu \in \iota(I_k) \cup \{act, \iota(act), offs\}$ (namely to $\operatorname{dom} P$).

Let $\varphi = \varphi_1 \vee \varphi_2$. Then φ has the desired property. $\square$

In the following proof we use the mapping $col : \mathbb{N}^2 \to \mathbb{N}$, defined by $col(i, j) = i$. For the position x of some grid, $col(x)$ is the column number of x.

Proof of Theorem 3.44. Let $\psi(\bar{X}, x, y) = \exists z(top(z) \wedge left(z) \wedge X_{offs} z) \vee X_{\iota(end\text{-}k)} y$. Let $\varphi(\bar{X}, x, y)$ be as in the above lemma.[6] Then $\varrho(x, y) = \psi(\iota\bar{X}(\varphi(\bar{X}, x, y)), x, y)$ is an allowed $\Delta_1^{\mathrm{U}}(\Pi_{k-1}^0(\Delta_1^{\mathrm{U}}))$-formula that asserts the following for a picture P of size (m, n) and $y \in \{0\} \times [n]$:

$$x \notin \{0\} \times \{1, \ldots, f_k(m)\} \vee f_k(m) + col(x) \mid col(y) + 1.$$

The latter is true because in a picture of size (m, n) in $(twin_{k,m,j})^* \oplus ptwin_{k,m,j}$ (where $1 \leqslant j \leqslant f_k(m)$), the component corresponding to the attribute $\iota(end\text{-}k)$ is 1 exactly in those positions y that are in the top row and fulfil $f_k(m) + j \mid col(y) + 1$.

Let $\sigma(y) = \forall x(\varrho(x, y))$. Then $\sigma(y)$ is a $\Pi_1^0(\Delta_1^{\mathrm{U}}(\Pi_{k-1}^0(\Delta_1^0)))$-formula that asserts for a position y in the top row that

$$\forall j \in \{1, \ldots, f_k(m)\}: f_k(m) + j \mid col(y) + 1,$$

which is equivalent to

$$g_k(m) = \mathrm{lcm}\{f_k(m) + 1, \ldots, 2f_k(m)\} \mid col(y) + 1.$$

Choose

$$\pi = \forall y(top(y) \wedge right(y) \rightarrow \sigma(y)),$$
$$\pi' = \forall y(top(y) \rightarrow (right(y) \leftrightarrow \sigma(y))).$$

Then π and π' assert for the grid $\underline{[m] \times [n]}$ that n is a (or the least, respectively) number with $g_k(m) \mid n$.

The sentence π is indeed in $\Pi_1^0(\Delta_1^{\mathrm{U}}(\Pi_{k-1}^0(\Delta_1^{\mathrm{U}})))$, whereas the sentence π' is in $\Pi_1^0(B(\Pi_1^0(\Delta_1^{\mathrm{U}}(\Pi_{k-1}^0(\Delta_1^{\mathrm{U}}))))) \subseteq \Pi_2^0(\Delta_1^{\mathrm{U}}(\Pi_{k-1}^0(\Delta_1^{\mathrm{U}})))$, as required. □

The sentence π from Theorem 3.44 shows (with Proposition 3.50, Item 4) the definability result of Theorem 2.26. The corresponding non-definability part is Corollary 4.4.

From Theorem 3.44 we conclude the following.

Corollary 3.49. *For every $k \geqslant 1$, the formula class $\Pi_2^0(\Delta_1^{\mathrm{U}}(\Pi_{k-1}^0(\Delta_1^{\mathrm{U}})))$ is at least $(k+1)$-fold exponential, and thus so are $\Pi_2^0(\Delta_k)$, $\Pi_1^0(\Sigma_k)$, $FO(\Sigma_k)$.*

Proof. Take π' from Theorem 3.44. Membership in the other classes follows from Remark 3.45. □

This corollary provides the definability result for Theorem 2.24. The corresponding non-definability result (i.e., the upper bound for the asymptotic growth rate) is stated in Theorem 4.6.

[6] Note that we add x and y in the list of potentially free variables of ψ and φ, respectively, for formal reasons.

3.5.5. A number-theoretic proposition

Now we present the missing proof of the fact that for every $k \geqslant 1$, the function g_k from Theorem 3.44 is indeed $(k+1)$-fold exponential.

For every $m \geqslant 1$, let

$$F(m) = \max\left\{\mathrm{lcm}\{x_1,\ldots,x_k\} \mid k \geqslant 1, \sum_i x_i = m\right\},$$

$$G(m) = \mathrm{lcm}\{m+1, m+2, \ldots, 2m\},$$

$$H(m) = 2^{\sqrt{m \log m}}.$$

Recall that $s_0(m) = m$ and $s_{k+1}(m) = 2^{s_k(m)}$ for every $m \geqslant 1$, $k \geqslant 0$; and $g_k(m) = G(f_k(m))$ for every $k, m \geqslant 1$.

We have $f_k(m)$ is $s_k(\Theta(m))$. It is easy to show that $F(m) \leqslant 2^m$ for every $m \geqslant 1$, but this is a very rough bound. Landau [10] and Szaloy [23] (see also [4]) show very precise approximations for F. We will only need the fact that $F(m)$ is $\Theta(H(m))$.

Proposition 3.50. *Let* $k \geqslant 1$.
1. $G(m)$ *is* $s_1(\Omega(m))$.
2. $G(m)$ *is* $s_1(m^{\mathcal{O}(1)})$.
3. $F \circ f_k$ *is* $(k+1)$*-fold exponential.*
4. g_k *is* $(k+1)$*-fold exponential.*

Proof. For the first claim, let $m \geqslant 1$. Let $p_1, \ldots, p_k$ be the distinct prime numbers $< m$, and $p = \prod_i p_i$, and $l_i = \max\{l \mid p_i^l < m\}$ for every $i \leqslant k$. Then $\mathrm{lcm}\{1, \ldots, m-1\} = \prod_i p_i^{l_i} = (1/p) \prod_i p_i^{l_i+1} \geqslant m^k/p$. On the other hand, $\mathrm{lcm}\{1, \ldots, m-1\} \geqslant \prod_i p_i = p$. Thus $(\mathrm{lcm}\{1, \ldots, m-1\})^2 \geqslant p \cdot m^k/p = m^k$. k is $\Omega(m/\log m)$, thus $G(m) = \mathrm{lcm}\{m+1, \ldots, 2m\} \geqslant \mathrm{lcm}\{1, \ldots, m-1\} \geqslant m^{k/2}$ is $m^{\Omega(m/\log m)} = 2^{\Omega(m)}$, showing the first claim.

Now we need that $F(m)$ is $\Theta(H(m))$. Besides, $s_1(m^{1/2}) = 2^{\sqrt{m}} \leqslant H(m) \leqslant s_1(m)$ for all m, thus $H(m)$ and $F(m)$ are $s_1(m^{\Theta(1)})$. This implies the second and third claim: $G(m) = \mathrm{lcm}\{m+1, \ldots, 2m\} \leqslant F(\sum_{i=m+1}^{2m} i)$ is $F(\mathcal{O}(m^2)) = H(\mathcal{O}(m^2)) \leqslant s_1(m^{\mathcal{O}(1)})$; and $F(f_k(m))$ is $F(s_k(\Theta(m))) = s_1((s_k(\Theta(m)))^{\Theta(1)}) = s_{k+1}(\Theta(m))$.

The first and second claim imply that $G(m)$ is $s_1(m^{\Theta(1)})$. Thus $g_k(m) = G(f_k(m))$ is $G(s_k(\Theta(m)) = s_{k+1}(\Theta(m))$, i.e. the fourth claim. $\square$

The proof of the first claim is by Thomasz Schoen (private communication).

4. Non-definability results

In the previous section we have developed definability results, i.e., we have described particular picture languages like Num_k that are definable in certain classes of formulas, e.g. level k of the monadic alternation hierarchy.

Our aim was to develop separation results, so it remains to present the corresponding non-definability results. This will be done in this section. For example, in Section 4.1 we will show that Num_k is not definable in level $k-1$ of the monadic hierarchy. This part of a proof is often referred to as "lower bound" proof, but we avoid this phrase because in our case, the non-definability result (at least for the class of non-coloured grids) establishes an *upper bound* on the growth rates of functions that are definable in, say, Σ_k.

In Section 4.1 we shall give some more definitions relevant for this section.

4.1. The automaton method

The main idea for our non-definability proofs is to invoke standard pumping techniques for finite automata on words. In [17, 18] we have shown how to pass from a formula φ of monadic second-order logic over pictures to a family $(\mathfrak{A}_m)_{m\geqslant 1}$ of non-deterministic finite automata (NFAs). The NFA $\mathfrak{A}_m$ recognizes the set of those words of columns of height m that satisfy (when considered as pictures of height m) the formula φ. The important point is that if φ is in Σ_k, then $(\mathfrak{A}_m)_m$ can be chosen such that the number of states in $\mathfrak{A}_m$ is k-fold exponential in m.

Finite automata—some more notation: We use standard notations for nondeterministic finite automata on words. An NFA over alphabet Γ is a quintuple (Q,Γ,q_0,Δ,F), where Q is a finite set of states, $q_0\in Q$ is an initial state, $F\subseteq Q$ is a set of final states, and $\Delta\subseteq Q\times\Gamma\times Q$ is a transition relation. If $\mathfrak{A}$ is such an NFA, q,p are states of $\mathfrak{A}$, and w, is a word over Γ, we write $\mathfrak{A}: q\xrightarrow{w} p$ iff there is a path from q to p labelled w. If $P\subseteq Q$ we write $\mathfrak{A}$: $q\xrightarrow{w} P$ iff there is a $p\in P$ with $\mathfrak{A}: q\xrightarrow{w} p$. The set of (non-empty) words *recognized by* $\mathfrak{A}$ is $\{w\in\Gamma^+ \mid q_0\xrightarrow{w} F\}$. (The reference to non-empty words has technical reasons.)

$\mathfrak{A}$ is called *deterministic* (*DFA* for short) iff for every $(q,a)\in Q\times\Gamma$ there is at most one $p\in Q$ with $(q,a,p)\in\Delta$.

If $R\subseteq\mathbb{N}\times\mathbb{N}$ and $m\geqslant 1$, we write $R(m)$ for $\{n \mid (m,n)\in R\}$.

From pictures to words: For our non-definability results for picture languages we will use standard combinatorial methods for NFAs on words. Here we introduce notations that perform the link.

Definition 4.1. The *column word* of a non-empty picture, say of size (m,n), over alphabet Γ is a word over $\Gamma^{m,1}$, namely

$$\textit{column-word}\begin{pmatrix} a_{00} & \cdots & a_{0,n-1} \\ \vdots & & \vdots \\ a_{m-1,0} & \cdots & a_{m-1,n-1}\end{pmatrix} = \begin{pmatrix} a_{00} \\ \vdots \\ a_{m-1,0}\end{pmatrix}\cdots\begin{pmatrix} a_{0,n-1} \\ \vdots \\ a_{m-1,n-1}\end{pmatrix}.$$

For a picture language $L\subseteq\Gamma^{+,+}$ and an integer $m\geqslant 1$, the *height-m fragment of L*, *fragment* denoted by $L[m]$, is the set of all column words of pictures of height m in L.

Let $t \geqslant 1$. The column word of a t-bit grid R is the column word of the picture over $\{0,1\}^t$ associated to R. For a class M of t-bit grids, the *height-m fragment of M* (denoted $M[m]$) is the height-m fragment of the set of pictures over $\{0,1\}^t$ associated to grids from M.

In Subsection 4.2.2, Definition 4.16, we will slightly generalize this definition.

Recall the definition of $Mod_t(\varphi)$ from Section 13. The following has been shown in [17, 18].

Theorem 4.2. *Let $k \geqslant 1$, $t \geqslant 0$, and $\varphi(X_1,\dots,X_t)$ be a Σ_k-formula over τ_{Grids}. Let $\Gamma = \{0,1\}^t$. There exists $c \geqslant 1$ such that for every $m \geqslant 1$ there is an NFA with $s_{k-1}(c^m)$ states that recognizes the word language $Mod_t(\varphi)[m]$ over alphabet $\Gamma^{m,1}$.*

Corollary 4.3. *Let $k \geqslant 1$, $t \geqslant 0$. Let L be a Σ_k-definable picture language over alphabet $\{0,1\}^t$. The partial function $f : \mathbb{N} - \rightarrow \mathbb{N}$, $m \mapsto \min(\{n \mid \exists P \in L : \text{size } P = (m,n)\}$is at most k-fold exponential.*

Proof. Let $n \geqslant 1$ and $\varphi(X_1,\dots,X_t)$ be a Σ_n-formula that defines L. By Theorem 4.2 there is a c such that for all $m \geqslant 1$ there is an NFA with $s_{n-1}(c^m)$ states that accepts $L[m]$. Since the shortest word accepted by an NFA (if it exists) cannot be longer than number of states of this NFA, this implies that $f(m) = \min\{|w| \mid w \in L[m]\} \leqslant s_{n-1}(c^m) = s_n(m \log_2 c)$ for all $m \in \operatorname{dom} f$. $\square$

The following can be inferred from the above in the case[7] $t = 0$.

Corollary 4.4. *Let R be Σ_k-definable relation, $k \geqslant 1$. Then the partial function $\mathbb{N} - \rightarrow \mathbb{N}$, $m \mapsto \min(R(m))$ is at most k-fold exponential.*

Theorem 3.44, Remark 3.45, and the above imply Theorem 2.26. In particular, one may show the separation $\underline{\Sigma}_k \neq \underline{\Pi}_k$ for the class of grids and every $k \geqslant 1$ by considering the relation $\{(m,n) \mid g_k(m) \text{ divides } n\}$.

The following has been shown as a consequence of 4.2 in [17, 18].

Theorem 4.5. *For every k, the formula class $B(\Sigma_k)$ is k-fold exponential.*

4.2. Arguing against the first-order closure

The results of this subsection can be found as preliminary communications in [13]. The following is the main result of this subsection.

Theorem 4.6. *For all $k \geqslant 1$, every $FO(\Sigma_k)$-definable function is at most $(k+1)$-fold exponential.*

[7] We adopt the view that $\{0,1\}^0$ is some singleton.

The concept to show this bound is to exhibit a particular periodicity property of Σ_k-definable picture languages and to show that it is not destroyed by the first-order closure. For this aim, we need to consider properties of classes of "pebbled" grids. These are typical models of grid formulas with free first-order variables but without free set variables. This concept is now explained in an example.

Suppose we have a $FO(\bar{\Sigma}_k)$-formula $\exists x_1 \forall x_2 \varphi(x_1, x_2)$, where φ is in $\bar{\Sigma}_k$. Then we may associate to φ the set $Mod(\varphi)$ of "pebbled grids", i.e., grids with two differently marked positions. If m is, again, a fixed height, we may pass to the "height-m-fragment" $Mod(\varphi)[m]$ as before. But this time, the word language $Mod(\varphi)[m]$ will not contain words over a *singleton* alphabet, but words in which all but one or two positions (corresponding to the columns of two marked grid positions) carry the same letter. Again, $Mod(\varphi)[m]$ is recognizable by an NFA whose state set size is at most k-fold exponential in m. Since in input words for this NFA all but two positions carry the same letter, there are similar periodicity properties as for NFAs with a singleton input alphabet. We will exhibit what happens to these periodicity properties when taking the first-order quantifications into account.

Theorem 4.6 was shown for the special case $k = 1$ in [1, 2] using Ehrenfeucht–Fraïssé games that are specifically tailored for $FO(\Sigma_1)$. Our method to argue against the first-order closure results from the one in [1] by extracting the periodicity technique from the description of Duplicator's winning strategy (thus "de-gamifying" this proof). This is combined with the automata-theoretic argument Theorem 4.2 against definability in Σ_k.

Some More Notation. The following definitions will be needed later. Let M, N be sets, $R \subseteq M \times N$ a relation, and $L \subseteq N$. Then

$$\langle R \rangle L = \{m \in M \mid \exists n \in L\colon (m, n) \in R\}.$$

Let $t, r \geqslant 0$. An *r-pebbled t-bit grid* is a structure in the signature $\tau_{Grids_t} \cup \{x_1, \ldots, x_r\} = \{S_1, S_2, X_1, \ldots, X_t, x_1, \ldots, x_r\}$, where $X_1, \ldots, X_t$ are unary predicate symbols and $x_1, \ldots, x_r$ are constant symbols, such that the restriction to $\tau_{Grids} = \{S_1, S_2\}$ is a grid. The set of all r-pebbled t-bit grids is denoted $Grids_{t,r}$.

An r-pebbled t-bit grid is a *model* of a formula $\varphi(X_1, \ldots, X_t, x_1, \ldots, x_r)$ if it makes φ true with the implicitly given assignment. The class of r-pebbled t-bit grids that are models of φ is denoted $Mod_{t,r}(\varphi)$. □

4.2.1. Automata and periodicity wrt a certain letter

Let Γ be an alphabet and $a \in \Gamma$ be some fixed letter. Let $n \geqslant 0$ and $p \geqslant 1$. A word language L over alphabet Γ is *periodic* (*with threshold n*, *period p*, *wrt a*) iff for every $u, v \in \Gamma^*$: $ua^n v \in L \Leftrightarrow ua^{n+p} v \in L$. By (p, n)-*periodic* we mean "periodic with threshold n and period p". The explicit mentioning of a will be dropped if it is clear from the context. A set $N \subseteq \mathbb{N} \cup \{0\}$ will be called (p, n)-periodic iff $\{a^n \mid n \in N\}$ is (p, n)-periodic, i.e., if $n' \in N \leftrightarrow n' + p \in N$ for every $n' \geqslant n$.

The following definition will prove to be useful.

Definition 4.7. Let Γ, Ω be alphabets, $\alpha_0, \alpha_1 \subseteq \Gamma \times \Omega$.

The relation $R_{(\alpha_0,\alpha_1)} \subseteq \Gamma^+ \times \Omega^+$associated to (α_0, α_1) is given by

$$\{((a_1 \dots a_n), (a'_1 \dots a'_n)) \mid n \geqslant 0 \wedge \exists i((a_i, a'_i) \in \alpha_1 \wedge \forall j \neq i((a_j, a'_j) \in \alpha_0))\}.$$

In other words, if $(u, v) \in R_{(\alpha_0,\alpha_1)}$, then v results from u by replacing one letter according to α_1 and all the others according to α_0. In our applications, α_0 will be total, functional, and injective, so that this replacement amounts to marking a position and renaming of the other letters.

With this definition we can mimic the effect of first-order quantifications on the level of height-m fragments of grid languages. The reader who wants a better motivation for this definition *now* may read Sections 4.2.2 and 4.2.3 up to Lemma 4.21 and then continue here.

Remark 4.8. Let $p \geqslant 1$, $n \geqslant 0$.

1. Every (p, n)-periodic word language is (p', n')-periodic for every multiple p' of p and every $n' \geqslant n$.
2. Every boolean combination of (p, n)-periodic word languages is (p, n)-periodic.
3. If $N \subseteq \mathbb{N}$ is (p, n)-periodic, and $n' \geqslant n$, then $\{k \mid k + n' \in N\}$ is $(p, 0)$-periodic.

We are interested in an asymptotic bound for the "threshold" n of the periodicity of fixed height fragments definable by nested first-order quantifications. (Again, we refer to Lemma 4.21 for a precise exposition of the relation to first-order quantification.) The next lemma shows why the length p of the period is essential for this.

Lemma 4.9. *Let Γ, Ω be alphabets, $\alpha_0, \alpha_1 \subseteq \Gamma \times \Omega$ and α_0 total, functional and injective, $(a, a') \in \alpha_0$. Let $R = R_{(\alpha_0,\alpha_1)}$. Let $L \subseteq \Omega^*$ be (p, n)-periodic wrt a'. Then $\langle R \rangle L$ is $(p, 2n + p)$-periodic wrt a.*

Proof. We have to show that $ua^{2n+p}v \in \langle R \rangle L \Leftrightarrow ua^{2n+2p}v \in \langle R \rangle L$ for every $u, v \in \Sigma^*$.

So let $ua^{2n+p}v \in \langle R \rangle L$. There is a word in L that results from $ua^{2n+p}v$ by replacing one letter at a single position according to α_1 and all the others according to α_0. We will make a case distinction whether this position is in the "middle part" or not.

Formally, there are $u', x', v' \in \Omega^*$ such that $|u| = |u'|$, $|v| = |v'|$, and $|x'| = 2n + p$ such that $ua^{2n+p}vRu'x'v' \in L$.

In case $x' = a'^{2n+p}$ we have $ua^{2n+2p}vRu'a'^{2n+2p}v' \in L$ because L is (p, n)-periodic.

In case $x' = a'^i a_1 a'^j$ for some a_1 with $(a, a_1) \in \alpha_1$ and $i + j + 1 = 2n + p$ we have $i \geqslant n$ or $j \geqslant n$. Assume that $i \geqslant n$ (the other case is analogous). Then $ua^{2n+2p}vRu'a'^{i+p}a_1a'^j v' \in L$.

So in every case $ua^{2n+2p}v \in \langle R \rangle L$.

Conversely, let $ua^{2n+2p}v \in \langle R \rangle L$. There are $u', v', x' \in \Omega^*$ with $|u| = |u'|$, $|v| = |v'|$, $|x'| = 2n + 2p$, and $ua^{2n+2p}vRu'x'v' \in L$.

In case $x' = a'^{2n+2p}$ we have $ua^{2n+p}vRu'a'^{2n+p}v' \in L$ as before.

In case $x' = a'^i a_1 a'^j$ for some a_1 with $(a, a_1) \in \alpha_1$ and $i + j + 1 = 2n + 2p$ we have $i \geqslant n + p$ or $j \geqslant n + p$. Assume that $i \geqslant n + p$ (the other case is analogous). Then $ua^{2n+p}vRu'a'^{i-p}a_1a'^jv' \in L$.

So in every case $ua^{2n+p}v \in \langle R\rangle L$. This completes the proof of Lemma 4.9. □

The essential point of the above lemma is that when passing from L to $\langle R\rangle L$, the period p remains the same whereas the threshold where this period starts increases by a summand p as well as by a constant factor. We will need this lemma in a situation where the period p is exponentially larger than the threshold n, so that a (single or boundedly often repeated) application of this lemma results in a singly exponential increase of that threshold.

The argument of the above proof is extracted from Duplicator's winning strategy in the proof of [1, Lemma 12.6].

Remark 4.10. Let $n \geqslant 0$ and $p \geqslant 1$. A word language $L \subseteq \{a\}^+$ is (p, n)-periodic iff there is a DFA $\mathfrak{A}$ with $S(\mathfrak{A}) = (p, n)$ that recognizes L.

As explained in the introduction of this subsection, we have to investigate periodicity properties of NFAs whose input consists of words most of whose letters are identical. Our aim is to show that when an NFA reads such an input, then in the "gaps" between the designated positions it behaves very much like a unary NFA. This is stated in Lemma 4.13, which is prepared by the following definition.

Definition 4.11. Let $a \in \Gamma$. Let $\mathfrak{A} = (Q, \Gamma, q_0, \Delta, F)$ and $\mathfrak{A}' = (Q', \{a\}, q_0', \Delta', \emptyset)$ be NFAs. We say that $\mathfrak{A}'$ *simulates* $\mathfrak{A}$ *wrt* a iff there is a family $(F_{q,q'})_{q,q' \in Q}$ of subsets of Q' such that

$$\forall k \geqslant 0\colon\ \mathfrak{A} : q \xrightarrow{a^k} q' \ \Leftrightarrow\ \mathfrak{A}' : q_0' \xrightarrow{a^k} F_{q,q'}. \tag{8}$$

The following is shown in [4, Lemma 4.3, Theorem 4.4].

Theorem 4.12. *Let $N \subseteq \mathbb{N}$ be recognizable by some n-state NFA. Then there are some $k \leqslant (n+2)^2$ and an integer p such that N is recognized by a DFA $\mathfrak{B}$ with states $0, \ldots, (k-1)+p$ such that $\mathfrak{B}$ reaches the state $k + ((l-k) \bmod p)$ after reading an input of length $l \geqslant k$.*

We need a slightly stronger formulation of that theorem:

Lemma 4.13. *Let $\mathfrak{A}$ be an NFA with n states and a be a letter from its input alphabet. Then there is $p \leqslant n$ and a unary NFA $\mathfrak{A}'$ with $S(\mathfrak{A}') = (p, n^2 + n)$ such that* (1) *$\mathfrak{A}'$ simulates $\mathfrak{A}$ wrt a, and* (2) *no state of $\mathfrak{A}'$ on a cycle has more than one successor.*

We do not present the proof of this lemma here. It is almost the same as for [4, Lemma 4.3]. The essential observation is that the transition structure of the NFA constructed there does not depend on the initial and final state, so it can be carried out uniformly for all pairs of initial/final state. The original proof, too, gives these accurate bounds on the numbers of states (rather than the asymptotic ones stated in [4, Lemma 4.3]). See [16] for details.

Recall the definition of F from Section 3.5.5, and that $F(m) \leqslant 2^m$ for all m.

Corollary 4.14. *Let $\mathfrak{A}$ be an NFA with n states and a be a letter from its input alphabet. Then there is $p \in \{1, \ldots, F(n)\}$ such that for all states q, q' of $\mathfrak{A}$,*

$$\mathfrak{A} : q \overset{a^{(n+1)^2}}{\longrightarrow} q' \quad \Leftrightarrow \quad \mathfrak{A} : q \overset{a^{(n+1)^2+p}}{\longrightarrow} q'.$$

Proof. By Lemma 4.13 there is an NFA $\mathfrak{A}'$ and $p' \leqslant n$ with $S(\mathfrak{A}') = (p', n^2 + n)$ such that no state on a cycle of $\mathfrak{A}'$ has more than one successor and $\mathfrak{A}'$ simulates $\mathfrak{A}$ wrt a. Choose a family $(F_{q,q'})_{q,q' \in Q}$ of sets of $\mathfrak{A}'$-states according to Definition 4.11, i.e.

$$\forall q, q' \in Q \, \forall k \geqslant 0 : \mathfrak{A} : q \overset{a^k}{\rightarrow} q' \Leftrightarrow \mathfrak{A}' : q'_0 \overset{a^k}{\rightarrow} F_{q,q'}.$$

Let $p \geqslant 1$ be the least common multiple of the length of cycles in $\mathfrak{A}'$. Since the cycles in $\mathfrak{A}'$ have pairwise disjoint state sets, $p \leqslant F(p') \leqslant F(n)$. Every path through $\mathfrak{A}'$ of length $\geqslant (n+1)^2 > n^2 + n$ passes through a state on a cycle, whose length divides p. This implies the claim. $\square$

Now we will state the main result of this Section 4.2.1.

Theorem 4.15. *Let $a \in \Gamma$. For every word language $L \subseteq \Gamma^+$ that is recognized by an NFA with n states, there is a $p \in \{1, \ldots, F(n)\}$ such that L is $(p, (n+1)^2)$-periodic wrt a.*

Proof. Let $\mathfrak{A}$ be an NFA with n states that recognizes L. Choose $p \leqslant F(n)$ according to Corollary 4.14. Then $ua^{(n+1)^2}v \in L$ iff $\exists q_1, q_2 \in Q \, \exists q_3 \in F : q_0 \overset{u}{\rightarrow} q_1 \overset{a^{(n+1)^2}}{\rightarrow} q_2 \overset{v}{\rightarrow} q_3$ iff $\exists q_1, q_2 \in Q \, \exists q_3 \in F : q_0 \overset{u}{\rightarrow} q_1 \overset{a^{(n+1)^2+p}}{\rightarrow} q_2 \overset{v}{\rightarrow} q_3$ iff $ua^{(n+1)^2+p}v \in L$. $\square$

4.2.2. From words to pebbled grids

The following definition is a slight generalization of Definition 4.1.

Definition 4.16. Let R be an r-pebbled t-bit grid of size (m, n). Then the *column word of R* is a word of length n over alphabet $(\{0,1\}^{t+r})^m$ defined as

$$\textit{column-word}(R) = \begin{pmatrix} \bar{a}_{11} \\ \vdots \\ \bar{a}_{m1} \end{pmatrix} \cdots \begin{pmatrix} \bar{a}_{1n} \\ \vdots \\ \bar{a}_{mn} \end{pmatrix},$$

where for every $(i,j) \in [m] \times [n]$, the word $\bar{a}_{ij} = a^1_{ij} \ldots a^t_{ij} a^{t+1}_{ij} \ldots a^{t+r}_{ij}$ over $\{0,1\}$ is such that

- $a_{ij}^s = 1$ iff $(i,j) \in X_s^R$ for all $s \in \{1,\ldots,t\}$,
- $a_{ij}^{t+s} = 1$ iff $(i,j) = x_s^R$ for all $s \in \{1,\ldots,r\}$.

For a set L of r-pebbled t-bit grids and $m \geqslant 1$, the *height-m fragment of L* is given by $L[m] = \{column\text{-}word(R) \mid R \text{ is of height } m \text{ and } R \in L\}$.

Proposition 4.17. *If $L \subseteq Grids_{0,r}[m]$ is (p,n)-periodic, then so is its complement $Grids_{0,r}[m] \backslash L$.*

Proof. Immediate by the closure of (p,n)-periodic sets under boolean combinations (Remark 4.8) and the fact that $Grids_{0,r}[m]$ is $(1,0)$-periodic. $\square$

4.2.3. Periodicity properties of $FO(\Sigma_k)$-definable relations

The following lemma can be proved easily.

Lemma 4.18. *Let $k \geqslant 0$. For every Σ_k-formula $\varphi(x_1,\ldots,x_r)$ there is a Σ_k-formula $\varphi'(X_1,\ldots,X_r)$ with*

$$Mod_{r,0}(\varphi') = \{(R,\{u_1\},\ldots,\{u_r\}) \mid (R,u_1,\ldots,u_r) \in Mod_{0,r}(\varphi)\}.$$

In particular, $Mod_{r,0}(\varphi')[m] = Mod_{0,r}(\varphi)[m]$ for every m.

By the above lemma, the following corollary a consequence of Theorem 4.2.

Corollary 4.19. *Let $k \geqslant 1$, $r \geqslant 0$, and $\varphi(x_1,\ldots,x_r)$ be a Σ_k-formula over τ_{Grids}. There is a $c \geqslant 1$ such that for every $m \geqslant 1$ there is an $s_{k-1}(c^m)$-state NFA that recognizes the word language $Mod_{0,r}(\varphi)[m]$.*

The following definition mimics the effect of first-order quantifications on the level of formal languages.

Definition 4.20. Let $m \geqslant 1$, $r \geqslant 0$. Let $\Gamma = (\{0,1\}^r)^{m,1}$ and $\Omega = (\{0,1\}^{r+1})^{m,1}$. We define the following subsets of $\Gamma \times \Omega$:

$$\alpha_0 = \left\{ \left(\begin{pmatrix} \overline{a_1} \\ \vdots \\ \overline{a_m} \end{pmatrix}, \begin{pmatrix} \overline{a_1}0 \\ \vdots \\ \overline{a_m}0 \end{pmatrix} \right) \middle| \, \forall i : \overline{a_i} \in \{0,1\}^r \right\},$$

$$\alpha_1 = \left\{ \left(\begin{pmatrix} \overline{a_1} \\ \vdots \\ \overline{a_m} \end{pmatrix}, \begin{pmatrix} \overline{a_1}b_1 \\ \vdots \\ \overline{a_m}b_m \end{pmatrix} \right) \middle| \, b_1 \ldots b_m \in 0^*10^*, \forall i : \overline{a_i} \in \{0,1\}^r \right\}.$$

We define the *position choice relation for height m and index r* as the relation $R_{(\alpha_0,\alpha_1)} \subseteq \Gamma^+ \times \Omega^+$ associated to (α_0,α_1) in the sense of Definition 4.7.

Note that in the above definition, α_0 is total, functional, and injective.

The next lemma states that, roughly speaking, the application of the $\langle R\rangle$-modality for the position choice relation R for some fixed height does the same as an existential first-order quantification.

Lemma 4.21. *Let $m \geqslant 1$. Let $\Gamma = (\{0,1\}^r)^{m,1}$ and $\Omega = (\{0,1\}^{r+1})^{m,1}$. Let $\varphi(x_1,\dots,x_r)$ be a first-order formula. Let R be the position choice relation for height m and index r. Then*

$$Mod_{0,r}(\exists x_{r+1}\varphi)[m] = \langle R\rangle(Mod_{0,r+1}(\varphi)[m]).$$

Proof. Firstly, let $w \in Mod_{0,r}(\exists x_{r+1}\varphi)[m]$, say of length n. For every $(i,j) \in [m] \times [n]$ there is $\overline{a_{ij}} \in \{0,1\}^r$ such that

$$w = \begin{pmatrix} \overline{a_{00}} \\ \vdots \\ \overline{a_{m-1,0}} \end{pmatrix} \cdots \begin{pmatrix} \overline{a_{0,n-1}} \\ \vdots \\ \overline{a_{m-1,n-1}} \end{pmatrix}.$$

There exists a $(0,r)$-grid $A = ([m] \times [n], x_1,\dots,x_r)$ such that $A \models \exists x_{r+1}\varphi$ and w is the column word of A (in the sense of Definition 4.16). Thus there is $x_{r+1} = (\hat{\imath},\hat{\jmath}) \in [m] \times [n]$ with $A,(\hat{\imath},\hat{\jmath}) \models \varphi$. For every $(i,j) \in [m] \times [n]$ we define $\overline{a'_{ij}} \in \{0,1\}^{r+1}$ as follows:

$$\overline{a'_{ij}} = \begin{cases} \overline{a_{ij}}1 & \text{if } (i,j) = (\hat{\imath},\hat{\jmath}), \\ \overline{a_{ij}}0 & \text{else.} \end{cases}$$

Then

$$w = \begin{pmatrix} \overline{a_{00}} \\ \vdots \\ \overline{a_{m-1,0}} \end{pmatrix} \cdots \begin{pmatrix} \overline{a_{0,n-1}} \\ \vdots \\ \overline{a_{m-1,n-1}} \end{pmatrix} R \begin{pmatrix} \overline{a'_{00}} \\ \vdots \\ \overline{a'_{m-1,0}} \end{pmatrix} \cdots \begin{pmatrix} \overline{a'_{0,n-1}} \\ \vdots \\ \overline{a'_{m-1,n-1}} \end{pmatrix} \in$$
$$Mod_{0,r}(\varphi)[m], \text{ thus } w \in \langle R\rangle(Mod_{0,r+1}(\varphi)[m]).$$

The converse direction is similar. □

The above lemma justifies our interest in Definitions 4.7 and 4.20. The reason why I chose to present them and Lemma 4.9 earlier is that the latter is purely word-theoretic and can be understood without any knowledge of grids and logic.

Now we combine the above lemma and Lemma 4.9 to state what a single first-order quantification can do to the periodicity of a fixed height fragment of some set of grids.

Lemma 4.22. *Let $m \geqslant 1$, $r \geqslant 0$, and $\circlearrowright \in \{\exists,\forall\}$, and $\varphi(x_1,\dots,x_{r+1})$ be a formula. If $Mod_{0,r+1}(\varphi)[m]$ is (p,n)-periodic, then $Mod_{0,r}(\circlearrowright x_{r+1}\varphi)[m]$ is $(p,2n+p)$-periodic.*

Proof. In case $\circlearrowright = \exists$, the claim is immediate from Lemmas 4.21 and 4.9.

In case $\circlearrowright = \forall$, we have that $Mod_{0,r}(\neg\varphi)[m]$ is (p,n)-periodic by Proposition 4.17. Thus $Mod_{0,r}(\exists x_{r+1}\neg\varphi)[m]$ is $(p,2n+p)$-periodic, and thus (again by Proposition 4.17) so is $Mod_{0,r}(\forall x_{r+1}\varphi)[m] = Mod_{0,r}(\neg\exists x_{r+1}\neg\varphi)[m]$. □

Since we are interested in a non-definability argument for the first-order closure of a given class of formulas, we have to apply the above lemma in iteration. This will be done now.

Lemma 4.23. *Let $m \geqslant 1$. For every $r \geqslant 0$ and every $\circlearrowright_1, \ldots, \circlearrowright_r \in \{\exists, \forall\}$, if $\varphi(x_1, \ldots, x_r)$ is a formula such that $Mod_{0,r}(\varphi)[m]$ is (p, n)-periodic, then $Mod_{0,0}(\circlearrowright_1 x_1 \ldots \circlearrowright_r x_r \varphi)[m]$ is $(p, 2^r(n+p) - p)$-periodic.*

Proof. The proof is by induction on r. The case $r = 0$ is trivial because $2^0(n+p) - p = n$. So assume the claimed implication is true for some r. Let $\circlearrowright_1, \ldots, \circlearrowright_{r+1} \in \{\exists, \forall\}$ and $\varphi(x_1, \ldots, x_{r+1})$ be a formula such that $Mod_{0,r+1}(\varphi)[m]$ is (p, n)-periodic. By Lemma 4.22, $Mod_{0,r}(\circlearrowright x_{r+1} \varphi)[m]$ is $(p, 2n+p)$-periodic. By induction hypothesis, $Mod_{0,0}(\circlearrowright_1 x_1 \ldots \circlearrowright_{r+1} x_{r+1} \varphi)[m]$ is $(p, 2^r(2n+p) + p) - p))$-periodic, i.e., $(p, 2^{r+1}(n+p) - p)$-periodic. This completes the induction. □

Now we will apply the above in the situation where φ is a boolean combination of Σ_k-formulas. In the introduction to this subsection, I explained our interest in automata running on inputs almost all of whose letters are identical. I mentioned the idea that on such an input an NFA has similar periodicity properties as a unary NFA. However, the truth is a little more complicated, as the proof of the next lemma shows. The point is that we need the fact that if we have several NFAs, then we can find an appropriate periodicity property that is common for these automata by building their cross-product first.

Lemma 4.24. *Let $k \geqslant 1$ and $r \geqslant 0$ and $\varphi(x_1, \ldots, x_r)$ be a $B(\Sigma_k)$-formula, and $\circlearrowright_1, \ldots, \circlearrowright_r \in \{\exists, \forall\}$. Then there is some $d \geqslant 0$ and such that for every $m \geqslant 1$ there is some $p \in \{1, \ldots, s_{k+1}(dm)\}$ such that*

$$Mod_{0,0}(\circlearrowright_1 x_1 \ldots \circlearrowright_r x_r \varphi)[m]$$

is $(p, s_{k+1}(dm))$-periodic.

Proof. Assume that $\varphi(x_1, \ldots, x_r)$ is a boolean combination of the Σ_k-formulas $\varphi_1, \ldots, \varphi_j$. By Corollary 4.19, for every $i \leqslant j$ there is some constant c_i such that for every $m \geqslant 1$ there is a $s_k(c_i m)$-state NFA that recognizes the word language $Mod_{0,r}(\varphi_i)[m]$.

Choose $c = \max\{c_1, \ldots, c_j\}$ and d with $2^r((s_k(cjm) + 1)^2 + s_{k+1}(cjm)) \leqslant s_{k+1}(dm)$ for every $m \geqslant 1$.

Let $m \geqslant 1$. By choice of c there is, for every i, an NFA $\mathfrak{A}_i = (Q_i, (\{0,1\}^r)^m, q_{0,i}, \Delta_i, F_i)$ with $s_k(cm)$ states that recognizes $Mod_{0,r}(\varphi_i)[m]$. Build the usual "cross-product automaton" $\mathfrak{A} = (Q_1 \times \cdots \times Q_j, (\{0,1\}^r)^m, (q_{01}, \ldots, q_{0j}), \Delta, \emptyset)$ with state set of size $n := (s_k(cm))^j \leqslant s_k(cjm)$. Choose p with $1 \leqslant p \leqslant F(n) \leqslant 2^n \leqslant s_{k+1}(cjm) \leqslant s_{k+1}(dm)$ according to Corollary 4.14, i.e., such that for every $q, q' \in Q$:

$$\mathfrak{A} : q \xrightarrow{a^{(n+1)^2}} q' \Leftrightarrow \mathfrak{A} : q \xrightarrow{a^{(n+1)^2+p}} q'.$$

Then $Mod_{0,r}(\varphi)[m]$ is $(p,(n+1)^2)$-periodic. Thus $Mod_{0,0}(\mho_1 x_1 \cdots \mho_r x_r \varphi)[m]$ is $(p,(2^r((n+1)^2+p))-p)$-periodic by Lemma 4.23. The claim follows by choice of d. □

Now we can prove Theorem 4.6, which states that for every $k \geqslant 1$, the class $FO(\Sigma_k)$ is at most $(k+1)$-fold exponential in the sense of Definition 2.19.

Proof of Theorem 4.6. Let $f : \mathbb{N} - \rightarrow \mathbb{N}$ be definable by a $FO(\Sigma_k)$-sentence ψ. Clearly, we may assume that ψ is of the form $\mho_1 x_1 \cdots \mho_r x_r \varphi$, where $r \geqslant 0$ and φ is a $B(\Sigma_k)$-formula.

For every $m \geqslant 1$, we have $\{a^n \mid f(m) = n\} = L_f[m] = Mod_{0,0}(\psi)[m]$, where a is the element of $(\{0\}^r)^m$.

Choose d as in the above lemma, i.e., such that for all $m \in \operatorname{dom} f$ there is a $p \geqslant 1$ such that the set $\{f(m)\}$ is $(p, s_{k+1}(dm))$-periodic. Then this implies that $f(m) \leqslant s_{k+1}(dm)$ for all $m \in \operatorname{dom} f$. □

As a consequence of Theorem 4.6, all subclasses of $\overline{FO(\Sigma_k)}$ are at most $(k+1)$-fold exponential. Together with Corollary 3.49 this implies that for every $k \geqslant 1$, the formula classes of that corollary are $(k+1)$-fold exponential.

Together with Corollary 4.5, this implies Theorem 2.22. In particular, we now have all separation claims from Corollary 2.25.

5. Conclusion

5.1. Technical review

In the introduction I summarized the separation results that were to be proved in this paper. Let us now review what we have done from a technical point of view. The technical contribution of this paper consists of three parts.

Firstly, we have redone the "definability part" of the proof of [17] showing the strictness of the monadic quantifier alternation hierarchy over pictures. This time, we proceeded via starfree picture languages, i.e., we constructed picture languages (corresponding to sets of coloured grids) beyond a given level of the monadic hierarchy by applying column concatenation and boolean combinations in alternation. This showed that the latter alternation is so powerful that its expressive power cannot be captured by a smaller number of alternations of set quantifier blocks in the monadic hierarchy.

We deduced that "very little" set quantification is necessary to leave any level of the monadic hierarchy. More precisely, we obtained the following, which has been stated already in [17]: In the context of pictures of coloured grids, it is the alternation of *first-order* quantifications followed by a single block of set quantifications that makes our properties hard wrt the monadic hierarchy. This set quantification could even been limited to "unique" ones, where quantification is permitted only to fix a tuple of sets that is uniquely determined by a first-order formula, i.e., we could pass to a formula in the first-order closure of Δ_1^{U} over coloured grids.

The second technical contribution of this paper was to construct a formula in the first-order closure of Σ_k (even of Δ_k) that allows us to define a grid property that represents a $(k+1)$-fold exponential function. As a by-product we obtained a formula in Π_k that allows us to define a grid property that represents a relation R for which $m \mapsto \min R(m)$ is a $(k+1)$-fold exponential function. The latter shows together with known facts that also over non-coloured grids, the levels of the monadic hierarchy are not closed under complement, a problem that had remained open in [17].

The third technical contribution was to show that no formula in the first-order closure of Σ_k allows us to define a grid property representing a function that grows faster than $(k+1)$-fold exponential, a fact that was shown in [1, 2] for $k=1$. Together with the second contribution this gives a tight asymptotic bound for the formulas definable in the first-order closure of the levels of the monadic hierarchy, and this bound is exponentially higher than the tight bound of [17] for the levels of the monadic hierarchy.

These three contributions allow for separation results concerning the connection of the first-order closure to monadic second-order logic: Firstly, the first-order closure is so powerful that when applied to Σ_1, it allows us to define properties beyond arbitrarily high levels of the monadic hierarchy. Secondly, it is so weak that when applied to Σ_k, it results in expressive power that does not subsume that of Δ_{k+2}. Thirdly, when the first-order closure is applied to all levels of the monadic hierarchy, the resulting hierarchy is still strict.

5.2. Connection to polynomial hierarchy

Since part of the interest in the monadic second-order alternation hierarchy stems from the desire to achieve progress in complexity theoretical questions, we should review our proofs with regard to whether they provide methods for attacking at least other fragments of the second-order hierarchy.

As a striking difference to most of the other approaches to separations in monadic second-order logic, our non-definability argument (the “lower bound proof”, Section 4) does not use Ehrenfeucht–Fraïssé games but automata-theoretic arguments. The ultimate core is the standard pumping lemma for finite automata. There is little or no hope that such techniques will carry over to, say, the binary fragment of second-order logic because a run of an automaton is a *colouring* of the input and thus inherently monadic.

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