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INTERVAL GRAPHS AND INTERVAL ORDERS

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This paper explores the intimate connection between finite interval graphs and interval orders. Special attention is given to the family of interval orders that agree with, or provide representations of, an interval graph. Two characterizations (one by P. Hanlon) of interval graphs with essentially unique agreeing interval orders are noted, and relationships between interval graphs and interval orders that concern the number of lengths required for interval representations and bounds on lengths of representing intervals are discussed.

Two invariants of the family of interval orders that agree with an interval graph are established, namely magnitude, which affects end-point placements, and the property of having the lengths of all representing intervals between specified bounds. Extremization problems for interval graphs and interval orders are also considered.

1. Introduction

Let V, P, I, and F denote a nonempty finite set, an asymmetric binary relation on V, a symmetric and reflexive binary relation on V, and a mapping from V into the set of positive-length closed real intervals respectively. We shall say that (V, I) is an *interval graph* (with loops) if an F exists such that

$$\forall x, y \in V: \quad xIy \text{ iff } F(x) \cap F(y) \neq \emptyset, \tag{1}$$

and that (V, P) is an *interval order* if an F exists such that

$$\forall x, y \in V: \quad xPy \text{ iff } F(x) > F(y), \tag{2}$$

where $F(x) > F(y)$ means that $\inf F(x) > \sup F(y)$.

The aim of this paper is to explore and exploit the close relationship between interval graphs and interval orders that is partly revealed by the following facts.

Given (1), (2) holds for the same F when P is defined by xPy if $F(x) > F(y)$; given (2), (1) holds for the same F when I equals $\mathrm{sc}(P)$, the symmetric complement of P, defined by

$$xIy \quad \text{if } \neg(xPy) \text{ and } \neg(yPx).$$

What makes the relationship between the two structures interesting is that, although there is only one natural interval graph associated with an interval order, namely its symmetric-complement graph, there may be a number of different interval orders that have the same symmetric complement. In other words, given

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an interval graph (V, I), different F assignments that satisfy (1) may produce different interval orders via the definition of P as xPy if $F(x) > F(y)$.

Because of this, we shall consider first the number of distinct interval orders that have the same interval graph for their symmetric-complement graphs. Interval graphs (V, I) that have essentially unique (up to duality) P for which $I = \mathrm{sc}(P)$ will be characterized in two ways. The first follows from Hanlon's analysis of buried subgraphs in his seminal paper [12] on counting interval graphs. The second is based on an equivalence relation L on ordered pairs $ab, xy, \ldots$ of points in V. This relation, defined solely through I, has the informal interpretation that

$abLxy$ if neither aIb nor xIy, and every interval order (V, P) for which $I = \mathrm{sc}(P)$ that has aPb also has xPy.

Our discussion of uniqueness and number of interval orders that agree with a given interval graph appears in Section 3. Section 2 provides additional background on notation and definitions. Later sections focus on aspects of interval end-point placements and lengths in representations of interval graphs and interval orders.

Section 4 establishes an invariant feature of all interval orders that agree with a given interval graph, namely that they require the same minimum number of left (or right) end-points for their interval representations. Section 5 begins an analysis of lengths of intervals for interval representations with a discussion of (V, I) and (V, P) that can be represented by intervals of only two lengths. Interval graphs and orders that require more than two lengths for their representations are considered in Section 6.

It is well known that an interval graph can be represented by unit-length intervals if and only if no induced subgraph is isomorphic to the bipartite star $K_{1,3}$. Sections 5 and 6 show by contrast that, when $k \geq 2$, there is no finite set $\mathscr{I}_0$ of forbidden graphs such that an interval graph can be represented by k or fewer lengths if, and only if, it has no induced subgraph isomorphic to a graph in $\mathscr{I}_0$.

My proof is based on a similar proof for interval orders representable with k or fewer lengths [4] and illustrates a point that is emphasized throughout the paper. The point is that many facts about interval graphs are similar to facts about interval orders. Moreover, since interval orders have more structure than interval graphs, theorems for interval graphs can sometimes be proved by first proving similar theorems for interval orders and then using the relationship between the two concepts to map the results into theorems about interval graphs.

This principle is followed in Section 7 where, in contrast to the results in Sections 5 and 6, we note that the class of interval graphs that can be represented with intervals whose lengths all lie between 1 and fixed $q \in \{1, 2, \ldots\}$ inclusive can be characterized by a single forbidden subgraph. For example, an interval graph has a representation whose interval lengths all lie in $[1, 2]$ if and only if it has no induced subgraph isomorphic to $K_{1,4}$. A related theorem for interval orders appears in [6].

The final section of the paper comments briefly on extremization problems for interval graphs and interval orders. One problem is to determine the largest n for a given k such that every n-point interval graph is representable with k or fewer lengths. Another asks for the largest k such that every n-point interval graph includes a k-point unit interval induced subgraph. Neither problem has been completely solved, and other open problems will be noted as we proceed.

2. Preliminaries

Throughout, V is a set of n labeled points. I denotes a reflexive (xIx) and symmetric ($xIy \Rightarrow yIx$) binary relation on V, and P denotes an asymmetric ($xPy \Rightarrow \neg(yPx)$) binary relation on V. The *dual* (converse, inverse) of P is $P^* = \{(x, y) \in V \times V: yPx\}$. Since I is symmetric, $I^* = I$, and since P is asymmetric, $P^* = P$ iff $P = \emptyset$. A point x in V is *universal* (for I) if xIy for all $y \in V$. When $U \subseteq V$ and $I' = I \cap (U \times U)$, (U, I') is the subgraph on U *induced* by (V, I). If U is the set of all points in V that are *not* universal, then it is easily seen that the induced subgraph (U, I') is either empty (if $I = K_n$, with loops) or is nonempty and has no universals (for I'). A *component* of (V, I) is a maximal connected induced subgraph of (V, I). For convenience, pictures of (V, I) omit loops, and diagrams of interval representations displace intervals vertically.

Theorems. *(V, I) is an interval graph iff every simple cycle of four or more points has a chord, and any three distinct and nonadjacent points can be ordered so that every path from the first to the third passes through a point adjacent to the second point* [11, 14].

(V, P) is an interval order iff for all $a, b, x, y \in V$, $(aPx$ and $bPy) \Rightarrow (aPy$ or $bPx)$ [2].

If these characterizations are used as definitions, then (1) and (2), prefaced by $\exists F$, emerge as representation theorems.

Every F that satisfies (1) or (2) is a *representation* (interval representation) of the corresponding interval graph or order. We shall denote the interval $F(x)$ assigned to x as

$$F(x) = [f(x), f(x) + \rho(x)]$$

and also refer to the pair of functions (f, ρ) as a representation. In (f, ρ), f is the *location function* and $\rho > 0$ is the *length function* of the representation. Such a representation has $|f(V)|$ different left end-points and uses $|\rho(V)|$ different lengths. *All instances of f and ρ are to be understood in the sense of this paragraph.*

An interval graph (V, I) is a *unit interval graph* if it has a representation with $|\rho(V)| = 1$; an interval order (V, P) is a *semiorder* if it has a representation with $|\rho(V)| = 1$.

Theorems. *An interval graph is a unit interval graph iff it has no induced subgraph isomorphic to* $K_{1,3}$ [15].

An interval order is a semiorder iff its symmetric-complement graph is a unit interval graph (iff, whenever xPyPz and $a \in V$*, either xPa or aPz* [16]*).*

3. Agreeing interval orders

An interval order (V, P) *agrees with* an interval graph (V, I) if $\mathrm{sc}(P) = I$. In this section we shall consider $\theta(V, I)$, the number of interval orders that agree with a given interval graph (V, I). Special attention will be given to interval graphs that have unique agreeing interval orders up to duality, i.e., those with $\theta(V, I) = 2$. Our results draw heavily on Hanlon's paper [12] but differ slightly from his analysis in the treatment of symmetric subgraphs.

Clearly, $\theta(V, I) = 1$ iff (V, I) is the complete graph K_n. If I is not complete then $\theta(V, I)$ is an even integer since P agrees with I iff P^* agrees with I. Moreover, because sets of intervals in representations for the components of (V, I) can be arranged in an arbitrary sequence along the line, $\theta(V, I) = m!\,\theta(V_1, I_1) \cdot \theta(V_2, I_2) \cdot \cdots \cdot \theta(V_m, I_m)$ when the (V_j, I_j) are the components of (V, I).

A connected graph with $\theta(V, I) = 4$ is pictured at the top of Fig. 1, above representations of its four agreeing interval orders. According to the ensuing definition of Hanlon, the graph has one buried subgraph, namely $\{e, g\}$. If either e or g is removed, then there are no buried subgraphs and $\theta = 2$.

For any nonempty subset A in V let

$$K(A) = \{x \in V\colon xIa \text{ for all } a \in A\}.$$

We shall say that $B \subseteq V$ is a *buried subgraph* of an interval graph (V, I) if

(a) there are $x, y \in B$ with $\neg(xIy)$,

(b) $K(B) \neq \emptyset$ and $B \cap K(B) = \emptyset$,

(c) if $x \in B$, $y \in V \setminus B$, and xIy, then $y \in K(B)$.

Since $\{d, e, g\} \cap K(\{d, e, g\}) = \{d\}$, the second part of condition (b) rules out $\{d, e, g\}$ as a buried subgraph in Fig. 1. When $B = \{e, g\}$, $K(B) = \{c, d\}$. Condition (c) implies that every I-path from a point in B to a point not in $B \cup K(B)$, such as a or b in Fig. 1, must involve a point in $K(B)$.

Theorem 1. *Suppose (V, I) is an interval graph and $(V, I) \neq K_n$. Let U be the set of nonuniversal points in V, and let (U, I') be the subgraph on U induced by (V, I). Then $\theta(V, I) = 2$ if and only if either*

(a) *(U, I') is the union of two complete graphs, or*

(b) *(U, I') is connected and has no buried subgraph.*

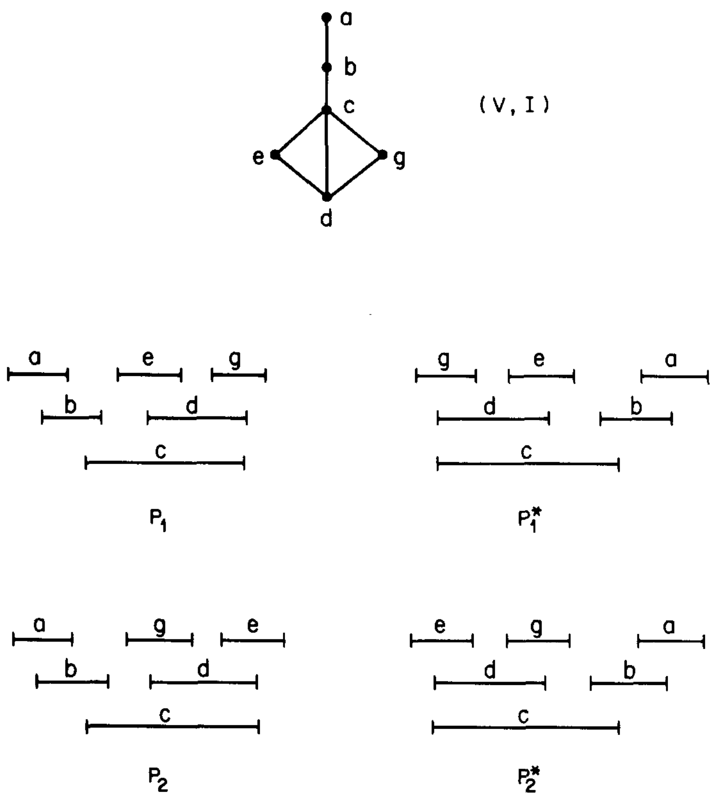

Fig. 1. Interval graph with four agreeing interval orders.

Proof. Assume the initial hypotheses of the theorem. Since universal points have no effect on $\theta(V, I)$, assume also with no loss in generality that (V, I) has no universal points.

Suppose first that (V, I) is not connected. Then it is easily seen that $\theta(V, I) = 2$ iff (V, I) has two components, each of which is complete, i.e., $\theta(V_j, I_j) = 1$ for $j = 1, 2$.

Suppose henceforth that (V, I) is connected. If (V, I) has no buried subgraph, then Theorem 1 in Hanlon [12] and the fact that (V, I) is not complete imply that $\theta(V, I) = 2$. If (V, I) has buried subgraphs, let B be one of them. Also let $(V, P_1 \cup P_2)$ be an agreeing interval order with

$$P_1 \cap (B \times B) = \emptyset, \qquad P_2 \subseteq B \times B, \qquad P_1 \cap P_2 = \emptyset.$$

Since (V, I) has no universal point, $B \cup K(B) \neq V$ and $P_1 \neq \emptyset$. Since $\neg(xIy)$ for some $x, y \in B$, $P_2 \neq \emptyset$. It then follows that $P_1 \cup P_2$, $P_1 \cup P_2^*$, $P_1^* \cup P_2$, and $P_1^* \cup P_2^*$ are distinct orders whose symmetric complements equal I, and therefore $\theta(V, I) \geqslant 4$. $\square$

Theorem 1 and Hanlon's analysis show how to compute $\theta(V, I)$ when (V, I) has buried subgraphs. Suppose (V, I) is connected, has no universal point, and has $m \geq 1$ maximal buried subgraphs $B_1, \ldots, B_m$ whose induced subgraphs are $(B_1, I_1), \ldots, (B_m, I_m)$. The B_i are disjoint and have no I connections between them, so that any (B_i, P_i) that agrees with (B_i, I_i) can be used in an agreeing interval order (V, P) with the requisite connections to the points in $K(B_i)$, see Fig. 2. Since the interval graph that shrinks each B_i to a single point has $\theta = 2$ (Theorem 1), it follows that

$$\theta(V, I) = 2\theta(B_1, I_1) \cdot \theta(B_2, I_2) \cdot \cdots \cdot \theta(B_m, I_m).$$

We now consider a second characterization of $\theta = 2$ which shows more explicitly how pairs of nonadjacent points must be mutually oriented to obtain an agreeing interval order. Given an interval graph (V, I), let

$$c(I) = \{xy \in V \times V: \quad \neg(xIy)\},$$

the complement of I. If (V, P) agrees with (V, I), then either xPy or yPx (but not both) for every $xy \in c(I)$. Define a binary relation L_0 on $c(I)$ by

$$abL_0xy \quad \text{if } ab, xy \in c(I),\ aIx, \text{ and } bIy,$$

and let L be the transitive closure of L_0 on $c(I)$, so that $abLxy$ if $ab = u_0v_0, u_1v_1, \ldots, u_mv_m = xy$ for some m with $u_iv_iL_0u_{i+1}v_{i+1}$ for $i = 0, \ldots, m-1$. Since it is clear that aPb in an agreeing interval order forces xPy when abL_0xy, aPb also forces xPy when $abLxy$. (See G^* in [1, p. 156] for a close correspondent of $(c(I), L_0)$.)

The definition of L_0 shows that it is reflexive and symmetric on $c(I)$, and therefore L is an equivalence relation on $c(I)$. The equivalence classes in $c(I)$ determined by L are the members of $c(I)/L$, and $|c(I)/L|$ is the number of these classes. The following theorem says in effect that $\theta(V, I) = 2$ if and only if I is not complete and, given a P orientation on any pair of points $\{x, y\}$ with (xIy), this orientation forces an orientation on every such pair.

Theorem 2. *Suppose (V, I) is an interval graph and $(V, I) \neq K_n$. Then $\theta(V, I) = 2$ iff $|c(I)/L| = 2$.*

A proof of the theorem is given in [10].

4. Magnitudes

Although an interval graph can have many agreeing interval orders, all of these have the same value of an important parameter, referred to as magnitude. We

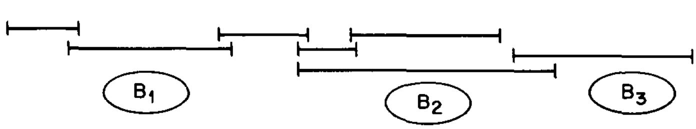

Fig. 2. Three maximal buried subgraphs in a connected interval graph.

define the magnitude of an interval order in a way that reveals its detailed structure.

Given an interval order (V, P) with $I = \mathrm{sc}(P)$, define binary relations $<^-$ and $<^+$ on V by

(a) $a <^- b$ if bPx and xIa for some $x \in V$,

(b) $a <^+ b$ if bIy and yPa for some $y \in V$.

Intuitively, a representation (f, ρ) of (V, P) *must* have $f(a) < f(b)$ (left end-points) iff $a <^- b$, and $f(a)+\rho(a) < f(b)+\rho(b)$ (right end-points) iff $a <^+ b$. It is not hard to prove [2] that the symmetric complements of $<^-$ and $<^+$ are equivalence relations and that the corresponding sets of equivalence classes, $V/\mathrm{sc}(<^-)$ and $V/\mathrm{sc}(<^+)$, are totally ordered in the natural ways. Thus, if we define $A <^- B$ for A, B in $V/\mathrm{sc}(<^-)$ by $a <^- b$ for some (hence for all) $a \in A$ and $b \in B$, then $<^-$ on $V/\mathrm{sc}(<^-)$ is a total order.

It is also easily seen that $|V/\mathrm{sc}(<^-)| = |V/\mathrm{sc}(<^+)|$, and we define the *magnitude* of (V, P) as the number of equivalence classes in $V/\mathrm{sc}(<^-)$, or in $V/\mathrm{sc}(<^+)$. If (f, ρ) is a representation of a magnitude-m interval order (V, P) in which all $2n$ end-points are different, and if

$$V/\mathrm{sc}(<^-) = \{A_1 <^- A_2 <^- \cdots <^- A_m\},$$
$$V/\mathrm{sc}(<^+) = \{B_1 <^+ B_2 <^+ \cdots <^+ B_m\},$$

then end-point placements along the line are ordered as $A_1 < B_1 < A_2 < B_2 < \cdots < A_m < B_m$. Any permutation of left ends within an A_j or of right ends within a B_j yields a representation of (V, P).

We need only use one point on the line for each A_j, and one for each B_j. Consequently, a magnitude-m interval order can use as few as (but no fewer than) m distinct real points for left ends. The minimum needed for both left and right ends might be less than $2m$ since, if $A_j \cap B_j = \emptyset$ $(2 \leqslant j \leqslant m-1)$, the same point might be used for A_j and B_j.

Theorem 3. *If (V, P) and (V, P') are interval orders that agree with interval graph (V, I), then their magnitudes are equal.*

Proof. We use induction on $|V|$. The theorem is obvious if $|V| = 1$. Suppose $|V| = n > 1$, and assume that the theorem holds for interval graphs with fewer than n points. Let (V, P) and (V, P') agree with (V, I), and let x be a point for a representation of (V, P) whose interval has the smallest right end $(x \in A_1 \cap B_1)$. If x is equivalent to $y \neq x$ (for all v, xIv iff yIv), then x's removal has no effect on magnitude, and the induction hypothesis implies that (V, P) and (V, P') have equal magnitudes. Assume henceforth that x is not equivalent to another point.

If xIy for no $y \neq x$ in V, then $\{x\}$ is a component of (V, I), its removal decreases the magnitudes of (V, P) and (V, P') by 1, and the induction hypothesis yields the desired result.

Assume henceforth that xIy for some $y \neq x$. Then, with respect to (V, P), $A_1 \backslash \{x\} \neq \emptyset$, $B_1 = \{x\}$, and $A_2 \neq \emptyset$. Moreover, yIz for all $y \in A_1 \backslash \{x\}$ and all $z \in A_2$.

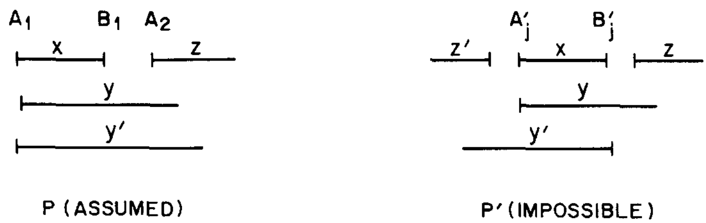

Fig. 3.

This is pictured on the left part of Fig. 3. If x is removed, B_1 disappears and A_1 and A_2 merge, so the magnitude of (V, P) decreases by 1. With respect to end-point equivalence classes for P', say $A'_1, \ldots, B'_r$, $x \in A'_j \cap B'_j$ for some j: otherwise there would be y and y′ with xIy, xIy' and $\neg(yIy')$, which is impossible by the choice of x. Suppose then that $x \in A'_j \cap B'_j$. If $|A'_j| \geqslant 2$ and $|B'_j| \geqslant 2$, we must have the situation shown on the right of Fig. 3 for P'. But this is impossible since y and y′ intersect something besides x (e.g., z on the left of the figure) that does not itself intersect x. Therefore exactly one of $|A'_j|$ and $|B'_j|$ equals 1 (for x), so that x's removal from (V, P') removes either A'_j or B'_j and reduces the magnitude of (V, P') by 1. Since the reduced interval orders have the same magnitude by the induction hypothesis, (V, P) and (V, P') have the same magnitude. □

Because of Theorem 3, the *magnitude of an interval graph* is unambiguously defined as the magnitude of any interval order that agrees with the interval graph. Various counting and extremization questions can then be asked about the families of interval orders of magnitude m and of interval graphs of magnitude m. For example, for $k \leqslant \lfloor (m+1)/2 \rfloor$, what is the fewest number of points in an interval order (interval graph) of magnitude m all of whose representations have intervals with at least k different lengths? This is presently an open problem. We shall return to it briefly in Section 8.

5. Two-length representations

As defined in Section 2, an interval graph can be represented by intervals of one length iff it is a unit interval graph, and an interval order can be represented by intervals of one length iff it is a semiorder. Moreover, each case has a simple characterization in terms of a forbidden four-point subgraph.

The situation for representability by two lengths is very different. Leibowitz [13] identifies three types of interval graphs that have representations with $|\rho(V)| \leqslant 2$. They are interval graphs with induced unit interval subgraphs on $n-1$ points, trees that are interval graphs, and threshold graphs [1, 11]. However, the problems of characterizing interval graphs and interval orders that are representable by two lengths remain open.

One reason for this is shown by the following theorem, based on a similar theorem for interval orders in Fishburn [4]. It says that it is impossible to characterize two-length interval graphs with a *finite* set of forbidden subgraphs.

Theorem 4. *Suppose $\mathscr{I}_0$ is a family of interval graphs such that an interval graph has a representation with $|\rho(V)| \leq 2$ iff it has no induced subgraph that is isomorphic to a graph in $\mathscr{I}_0$. Then $\mathscr{I}_0$ is infinite.*

Proof. The graph on $2(m+1)+5$ points $(m \geq 3)$ shown in Fig. 4 has one maximal buried subgraph, $\{b_1, b_2, \ldots, b_{m+1}\}$. An interval order that agrees with this graph is pictured in [4]. By Theorem 1, an agreeing interval order is unique up to duality and permutations on the order of b_1 through b_{m+1}. It is shown in [4] that the interval order cannot be represented with two lengths, but that every restriction of the order that deletes one or more points has a two-length representation. Consequently, the graph of Fig. 4 has no two-length representation even though every proper induced subgraph does. Since m can be arbitrarily large, it follows that $\mathscr{I}_0$ as hypothesized in the theorem is infinite. □

Two further observations illustrate other differences between the one-length and two-length cases. First, if $|\rho(V)| \leq 2$ for some representation of (V, I), there may be other interval orders that agree with (V, I) whose representations require more than two lengths. The interval graph at the top of Fig. 5 has a two-length representation for agreeing interval order (V, P) but no two-length representation for (V, P'). Thus, unlike magnitude, $\min |\rho(V)|$ for representations of interval orders that agree with an interval graph is not an invariant, except for unit interval graphs.

Second, if an interval order or interval graph can be represented with two lengths, and if the shorter length is fixed at 1, then the set of possible longer lengths need not be an interval. For example, [9] shows that for any $m \geq 2$, there are two-length interval orders whose longer length can lie anywhere in $(2-1/m, 2) \cup (m, \infty)$ but nowhere else. Since the interval orders used there have no buried subgraphs in their associated interval graphs, it follows from Theorem 1 that the same thing is true for two-length interval graphs.

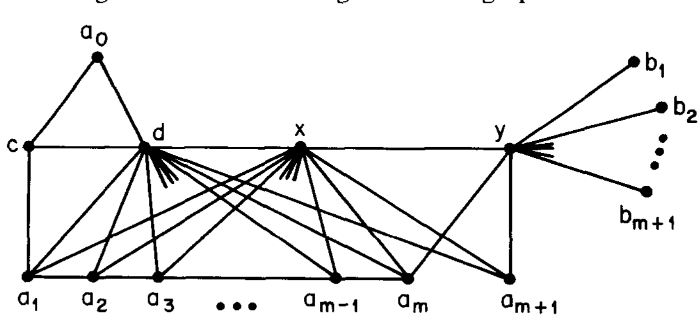

Fig. 4. A three-length interval graph.

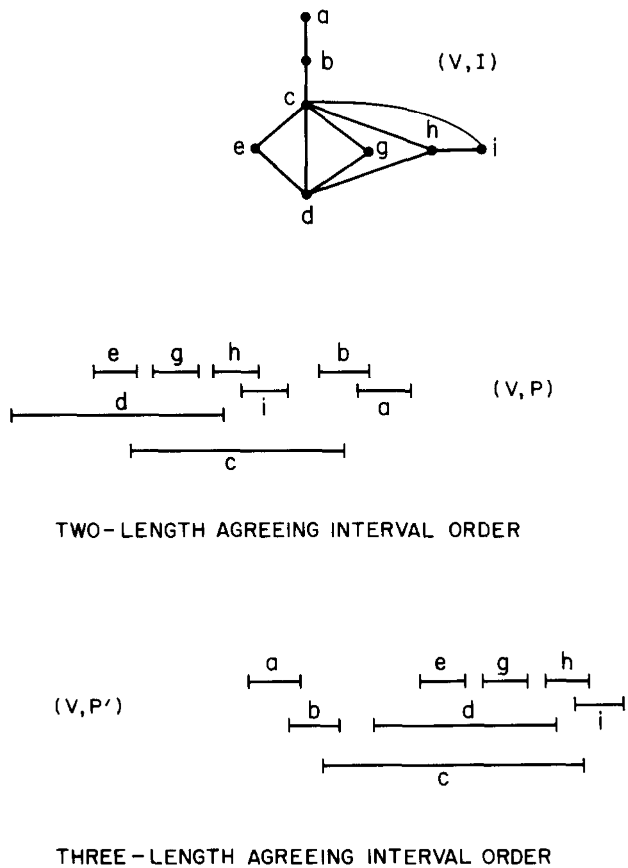

Fig. 5. Different minimum lengths.

6. Multiple-length representations

As might be expected, Theorem 4 generalizes to any finite number of lengths. That is, if $\mathcal{I}_0$ is a family of interval graphs such that an interval graph has a representation with $|\rho(V)| \leqslant k$ (k fixed, $k \geqslant 2$) iff it has no induced subgraph that is isomorphic to a graph in $\mathcal{I}_0$, then $\mathcal{I}_0$ is infinite. This follows from Theorem 4 and the ensuing lemma, which is similar to Lemma 2 in [4] for interval orders. Since the proof of the lemma differs significantly from the corresponding proof for interval orders, it is given here.

Lemma 1. *If $k \geqslant 1$ and there is no finite set $\mathcal{I}_0$ of interval graphs such that an interval graph has a representation (f, ρ) for which $|\rho(V)| \leqslant k$ iff it has no subgraph isomorphic to a graph in $\mathcal{I}_0$, then the same thing is true when k is replaced by $k+1$.*

Proof. Suppose the hypotheses are true for k. Then for every positive integer m there is an interval graph on more than m points that is not representable with k lengths but which has every proper induced subgraph representable by k or fewer lengths. Let G be such a graph for k with $N > m$ points. It is easily seen that G can be represented with $k+1$ lengths. Let G' consist of G and two disjoint copies of G, plus one more point that is universal, i.e. adjacent to every point in G and its disjoint copies. Whether or not G is connected, it is easily seen that G' can be represented with $k+2$ lengths, but no fewer, since, in any $(k+1)$-length representation of G' without its universal point, the same longest length must be involved in at least three components of G' without its universal point and, when the universal point is added, one of the longest prior intervals must be properly included in the universal's interval.

Let G'' be a minimal induced subgraph of G' that requires $k+2$ lengths in a minimum-lengths representation, so that every proper induced subgraph of G'' can be represented with no more than $k+1$ lengths. Then G'' must contain the universal point and at least one of the three copies of every point in G since otherwise G'' without its universal point would be representable with k lengths. Therefore G'' has at least $\frac{1}{3}(3N)+1 = N+1$ points.

The conclusion of the lemma follows since, for every m, there is an interval graph on $N' > m$ points that is not representable with $k+1$ lengths but has every proper induced subgraph representable by $k+1$ or fewer lengths. $\square$

7. Bounds on interval lengths

In contrast to the fixed numbers of lengths cases in the two preceding sections, we now consider interval graphs and orders with length-bounded representations. Any number of different interval lengths are allowed so long as all lengths are within the specified bounds. Throughout the section it is assumed that $0 < p \leq q$. Our length-bounded classes are

$$\mathscr{I}[p,q] = \{(V,I)\colon (V,I) \text{ is an interval graph that has } \rho(V) \subseteq [p,q] \text{ for some representation}\},$$

$$\mathscr{P}[p,q] = \{(V,P)\colon (V,P) \text{ is an interval order that has } \rho(V) \subseteq [p,q] \text{ for some representation}\}.$$

Since $\mathscr{I}[p,q] = \mathscr{I}[1, q/p]$ by normalization, the shortest length can be taken to be 1, but it is sometimes more convenient to work in the general $[p,q]$ setting.

When p and q are integers, a basic result for interval orders [6] implies that $\mathscr{P}[p,q]$ can be characterized by a finite set of forbidden interval orders. However, this is no longer true if q/p is irrational. An application of the rational case detailed in [6] to interval graphs yields

Theorem 5. *When $q \in \{1, 2, \ldots\}$, interval graph (V, I) is in $\mathcal{I}[1, q]$ iff no induced subgraph of (V, I) is isomorphic to the bipartite star (with loops) $K_{1,q+2}$.*

Proof. It is shown in [6] that interval order (V, P) is in $\mathcal{P}[1, q]$ iff

$$\forall x, a_1, \ldots, a_{q+2} \in V: \quad a_1 P a_2 P \ldots P a_{q+2} \Rightarrow (a_1 P x \text{ or } x P a_{q+2}). \qquad (*)$$

Hence $(V, I) \in \mathcal{I}[1, q]$ iff $(*)$ holds for some interval order that agrees with (V, I). If (V, I) has no induced subgraph $K_{1,q+2}$, then $(*)$ is true for every (V, P) that agrees with (V, I). Conversely, if (V, I) has induced subgraph $K_{1,q+2}$, then $(*)$ fails for every agreeing (V, P) since P totally orders the independent $(q+2)$-set but the center of $K_{1,q+2}$ bears $I = \mathrm{sc}(P)$ to each of its other $q+2$ points. $\square$

It can also be shown that every $\mathcal{I}[p, q]$ for rational q/p can be characterized by a finite set of forbidden subgraphs. The proof of this is based on the analogous result for interval orders [6] and the following theorem, which holds regardless of whether q/p is rational.

Theorem 6. *An interval graph (V, I) is in $\mathcal{I}[p, q]$ iff every interval order that agrees with (V, I) is in $\mathcal{P}[p, q]$.*

Remarks. By the definitions, $(V, I) \in \mathcal{I}[p, q]$ iff some (V, P) that agrees with (V, I) is in $\mathcal{P}[p, q]$. Theorem 6 implies that either all agreeing (V, P) are in $\mathcal{P}[p, q]$, or no agreeing (V, P) is in $\mathcal{P}[p, q]$. Thus membership in $\mathcal{P}[p, q]$, like magnitude (Theorem 3), is an invariant of the set of interval orders that agree with an interval graph.

Proof outline. Following Hanlon's analysis [12], one can construct a rooted tree for interval graph (V, I) as follows. The root is (V, I). The first level (points adjacent to the root) consists of the components of (V, I) after its universal points, if any, are removed. The second-level points adjacent to a first-level component are the maximal buried subgraphs of that component. The third-level points adjacent to a second-level maximal buried subgraph are the components of that subgraph. We then get maximal buried subgraphs within these components, then components of those maximal buried subgraphs, and so on. Each terminal point of the tree is a connected induced subgraph that includes no buried subgraph.

Hanlon's analysis shows that all interval orders that agree with (V, I) can be obtained from one another by two operations: first, by permutations of the order of the main components (level 1) and of the order of components within a buried subgraph; second, by taking duals of (flipping over) entire components or buried subgraphs at any level.

Now suppose (V, P) is an interval order that agrees with (V, I) and is in $\mathcal{P}[p, q]$, and let (f, ρ) be a representation of (V, P) that has $\rho(V) \subseteq [p, q]$. If (V, I) has no universal point but more than one component, each component's interval array can be uniformly shifted and/or flipped to accommodate main permutations and duality at level 1 without changing the symmetric complement of the order. If (V, I) has universal points (all with coincidental intervals) and more than one component at level 1, the same thing is possible with all intervals intersecting the universal, which is left intact, if, within each component, the left ends of intervals in A_1 and the right ends of intervals in B_m (see Section 4) are moved right or left respectively as far as possible without violating the minimum length of p or destroying a needed intersection of overlapping intervals. After these adjustments, which may be needed to prevent a flip of an end component from putting an interval outside of the universal, flips of components are made about the midpoints of their spans.

Similar procedures are followed for permutations of components within buried subgraphs and flips of inner components or buried subgraphs. Figure 6 pictures a buried subgraph B at some node of the tree (B can have several components), and r, s and t identify members of $K(B)$. If we treat the intersection of the intervals for $K(B)$ like the universal interval in the preceding paragraph, and trim the ends of the components of B in the indicated way, then permutations and/or flips can be made without changing the symmetric complement of the order. □

8. Extremization problems

We conclude with a few extremization problems. The first is to determine $\sigma(k)$ ($\hat{\sigma}(k)$), the smallest n such that some n-point interval order (n-point interval graph) requires at least k lengths for its representation. It is known for interval orders [8] that

$$2k \leqslant \sigma(k) \leqslant 3k-2 \quad \text{for all } k \geqslant 2,$$

and that $\sigma(k)=3k-2$ for all $k \leqslant 7$ [7]. The upper bound of $3k-2$ comes from the interval order described in Fig. 7. Since $(V, \mathrm{sc}(P))$ has no buried subgraph when the universal point is removed, Theorem 1 implies that $\hat{\sigma}(k) \leqslant 3k-2$. The lower bound induction proof in [8] for interval orders applies also to interval graphs, and therefore

$$2k \leqslant \hat{\sigma}(k) \leqslant 3k-2 \quad \text{for all } k \geqslant 2.$$

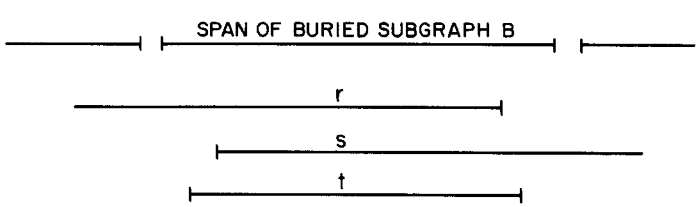

Fig. 6.

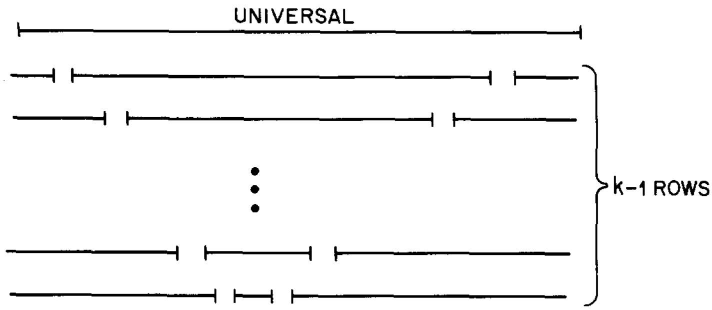

Fig. 7. $\sigma(k) \leqslant 3k-2$.

Moreover, the correspondence between interval graphs and orders implies $\sigma(k) \leqslant \hat{\sigma}(k)$. Little more is presently known about $\hat{\sigma}(k)$.

Attempts to settle the conjecture that $\sigma(k) = 3k-2$ have led to consideration of $\nu(k, m)$, the smallest n for which there is a magnitude-m interval order on n points that is not representable with fewer than k lengths. Given m, [7] shows that the most lengths ever needed to represent magnitude-m interval orders is $\lfloor (m+1)/2 \rfloor$. In addition, for $k \leqslant \lfloor (m+1)/2 \rfloor$,

$$\nu(k, m) \leqslant m + k - 1,$$

and this is an equality when $k \in \{1, 2, 3, 4, \lfloor (m+1)/2 \rfloor\}$. However, strict inequality holds when $k = 5$ and $m \geqslant 29$. The same things are true for the corresponding function $\hat{\nu}(k, m)$ for magnitude-m interval graphs.

The primary open question for ν or $\hat{\nu}$ is whether it is nondecreasing in $m \geqslant 2k-1$ for each fixed k. If this is true, then $\sigma(k) = \hat{\sigma}(k) = 3k-2$ for all k.

Another extremization problem is to determine $s(n)$ ($\hat{s}(n)$), the largest k so that every interval order (interval graph) on n points includes a k-point semiorder (k-point induced unit interval subgraph). The correspondence in Theorem 5 for $q = 1$ shows that $\hat{s} = s$, and $\hat{s}(n)$ is the largest k such that every n-point interval graph has a k-point induced subgraph with no copy of $K_{1,3}$.

It is known from $\hat{s} = s$ and [3, 5] that $\hat{s}(n-1) = \hat{s}(n) = \frac{1}{2}n + 1$ for even n from 4 to 14, but also that $\hat{s}(15) = \hat{s}(16) = \hat{s}(17) = 9$. Moreover,

$$\hat{s}(n) > n/(\log_2 n) \quad \text{for } n \geqslant 3,$$

and $\hat{s}(n)/n \to 0$.

The question of whether $\hat{s}(n)(\log_2 n)/n$ converges is open. For $q \geqslant 2$, I am not aware of any work on the largest k such that every n-point interval graph has a k-point induced subgraph that has no copy of $K_{1,q+2}$. Other open problems related to this section are discussed in [3, 7, 8].

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