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Boundary manifolds of projective hypersurfaces

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Abstract

We study the topology of the boundary manifold of a regular neighborhood of a complex projective hypersurface. We show that, under certain Hodge-theoretic conditions, the cohomology ring of the complement of the hypersurface functorially determines that of the boundary. When the hypersurface defines a hyperplane arrangement, the cohomology of the boundary is completely determined by the combinatorics of the underlying arrangement and the ambient dimension. We also study the LS category and topological complexity of the boundary manifold, as well as the resonance varieties of its cohomology ring.

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1. Introduction

1.1. Boundary manifolds

There are many ways to understand the topology of a homogeneous polynomial $f:\mathbb{C}^{\ell+1}\to\mathbb{C}$. The most direct approach is to study the hypersurface V in $\mathbb{CP}^{\ell}$ defined as the zero locus of f. Another approach is to view the complement, $X=\mathbb{CP}^{\ell}\setminus V$, as the primary object of study. And perhaps the most thorough is to study the Milnor fibration $f:\mathbb{C}^{\ell+1}\setminus\{f(\mathbf{x})=0\}\to\mathbb{C}^*$. Of course, the different approaches are interrelated. For example, if the degree of f is n, then the Milnor fiber $F=f^{-1}(1)$ is a cyclic n-fold cover of X. Consequently, knowledge of the cohomology groups of X with coefficients in certain local systems yields the cohomology groups of F.

In this paper, we take a different (yet still related) tack. We consider the *boundary manifold*, M, defined as the boundary of a closed regular neighborhood N of the subvariety $V\subset\mathbb{CP}^{\ell}$, see Durfee [10]. Clearly, $X\simeq\mathbb{CP}^{\ell}\setminus N^{\circ}$, and M is the boundary of $\mathbb{CP}^{\ell}\setminus N^{\circ}$. While the complement X has the homotopy type of a CW-complex of dimension at most ℓ, the boundary manifold M is a smooth, compact manifold of dimension $2\ell-1$.

There are many questions one can ask about the topology of M, for instance, concerning its fundamental group, and how it relates to the fundamental group of X. In the case where V is the union of an arrangement of lines in $\mathbb{CP}^2$, work in this direction was done by Jiang–Yau [18], Westlund [33], and Hironaka [16]. Here, we resolve the asphericity question for the boundary manifold of an arbitrary hyperplane arrangement (see Propositions 2.14 and 4.8), leaving a more detailed study of the fundamental group and related invariants to future work.

For a general hypersurface V, our main goal in this paper is to compute the cohomology ring of the boundary manifold M. We show that, under fairly mild hypotheses, the cohomology ring of the complement X functorially determines the cohomology ring of M, and derive a number of consequences. For instance, when the hypersurface $V=\bigcup_{H\in\mathcal{A}}H$ is determined by an arrangement of hyperplanes $\mathcal{A}$, these (Hodge-theoretic) hypotheses are satisfied, and the cohomology of $X=X(\mathcal{A})$ is thoroughly understood, thanks to classical results of Brieskorn and Orlik–Solomon. Our results then yield an explicit description of the cohomology ring of the boundary manifold $M=M(\mathcal{A})$.

1.2. Cohomology ring of the boundary

Given a finite-dimensional graded algebra A over a ring R, we construct a new algebra, $\mathsf{D}(A)$. This is a particular case of a more general construction, the "principle of idealization" due to Nagata [24], and popularized by Reiten [29], which associates to a ring A and an A-bimodule B the *trivial extension* ring $A\ltimes B:=A\oplus B$, with multiplication $(a,b)(a',b')=(aa',a\cdot b'+b\cdot a')$. Applying this construction to a graded (commutative) algebra $A=\bigoplus_{k=0}^{\ell}A^k$ and the A-bimodule $B=\bar{A}=\bigoplus_{k=\ell-1}^{2\ell-1}\operatorname{Hom}(A^{2\ell-k-1},R)$ yields a graded (commutative) algebra $\mathsf{D}(A)=A\ltimes\bar{A}$, which we refer to as the *double* of A.

If $V\subset\mathbb{CP}^{\ell}$ is a projective hypersurface, then the cohomology groups of V (with complex coefficients), and those of the complement $X=\mathbb{CP}^{\ell}\setminus V$ admit mixed Hodge structures. For each $k\geqslant 0$, there is an increasing weight filtration $\{W_m\}_{m\leqslant 2k}$ of the kth cohomology group, such that each quotient W_m/W_{m-1} has pure Hodge structure of weight m. Our main results, proved in Section 3, may be summarized as follows.

Theorem. *Let V be a hypersurface in $\mathbb{CP}^\ell$, with complement X and boundary manifold M. If either V is irreducible, or the weight filtration on the top cohomology group of X satisfies $W_{\ell+1}(H^\ell(X;\mathbb{C}))=0$, then the cohomology ring of the boundary manifold is isomorphic to the double of the cohomology ring of the complement*:

$$H^*(M;\mathbb{C}) \cong \mathsf{D}\big(H^*(X;\mathbb{C})\big). \tag{1.1}$$

If $H^\ell(X;\mathbb{C})$ satisfies the above weight condition and the integral cohomology of X is torsion-free, our results can be used to show that the splitting (1.1) holds over the integers, $H^*(M;\mathbb{Z}) \cong \mathsf{D}(H^*(X;\mathbb{Z}))$. This is the case, for example, when X is the complement of a hyperplane arrangement (see Theorem 4.2). On the other hand, this splitting can fail with integral coefficients when V is irreducible (see Example 2.7). With complex coefficients, the splitting (1.1) can fail if neither of the hypotheses stated in the theorem holds (see below).

1.3. Arrangements and curves

When applied to a complex hyperplane arrangement, our result yields an analog for the boundary manifold of a well-known theorem of Orlik and Solomon [25] concerning the cohomology ring of the complement. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{CP}^\ell$, with complement $X(\mathcal{A})$ and boundary manifold $M(\mathcal{A})$. The integral cohomology of the complement, $H^*(X(\mathcal{A});\mathbb{Z})$, is torsion-free, and the ring structure is completely determined by the intersection poset $L(\mathcal{A})$. Moreover, by work of Shapiro [31] and Kim [19], the complex cohomology $H^k(X(\mathcal{A});\mathbb{C})$ is pure of weight $2k$ for each k, $0 \leqslant k \leqslant \ell$. It follows that the integral cohomology ring of the boundary manifold, $H^*(M(\mathcal{A});\mathbb{Z}) \cong \mathsf{D}(H^*(X(\mathcal{A});\mathbb{Z}))$, is determined by the intersection poset $L(\mathcal{A})$ and the ambient dimension ℓ, see Corollary 4.3.

For an algebraic curve $V \subset \mathbb{CP}^2$ (in particular, an arrangement of lines in $\mathbb{CP}^2$), the associated boundary manifold M is a Waldhausen graph manifold. We show in Theorem 3.8 that the “doubling” formula (1.1) holds for a reducible curve V if and only if all its components are rational curves.

Cohomology rings of graph manifolds (with $\mathbb{Z}_2$ coefficients) have been the object of substantial recent study, see Aaslepp et al. [1]. For those graph manifolds which arise as boundary manifolds of arrangements of rational curves in $\mathbb{CP}^2$, our methods, together with Cogolludo's computation of the cohomology ring of the complement of such an arrangement in [4], provide an efficient alternative.

1.4. LS category and topological complexity

Let X^I be the space of continuous paths from the unit interval to X, and let $\pi : X^I \to X \times X$ be the map sending a path to its endpoints. In [13], Farber defines the *topological complexity* of X, denoted by $\mathrm{tc}(X)$, to be the smallest integer k such that $X \times X$ can be covered by k open sets, over each of which π has a section. This numerical invariant, which depends only on the homotopy type of X, is related to the Lusternik–Schnirelmann category by the inequalities $\operatorname{cat}(X) \leqslant \operatorname{tc}(X) \leqslant 2\operatorname{cat}(X) - 1$. Computing the topological complexity of X is crucial to solving the motion planning problem for the space X, see [13].

The topological complexity $\operatorname{tc}(X)$ admits a cohomological lower bound in terms of the zero-divisor length of $H^*(X; \Bbbk)$, similar to the well-known cup-length lower bound for $\operatorname{cat}(X)$. In the case when $X = X(\mathcal{A})$ is the complement of a hyperplane arrangement, explicit computations of $\operatorname{tc}(X)$ were carried out by Farber and Yuzvinsky [14]. In Section 5, we compute the topological complexity of the boundary manifold $M = M(\mathcal{A})$ for various classes of hyperplane arrangements, using our description of the cohomology ring of M and results from [13]. In particular, we show that the difference $\operatorname{tc}(M) - \operatorname{cat}(M)$ can be made arbitrarily large, see Corollary 5.10.

1.5. Resonance

We conclude with a comparison of certain ring-theoretic invariants of the cohomology ring of the complement to those of the cohomology ring of the boundary manifold.

Suppose A is a finite-dimensional, graded, connected algebra over an algebraically closed field $\Bbbk$ of characteristic 0. For each $a \in A^1$, multiplication by a defines a cochain complex (A, a). The *resonance varieties* of A are the jumping loci for the cohomology of these complexes: $\mathcal{R}^k_d(A) = \{a \in A^1 \mid \dim_{\Bbbk} H^k(A, a) \geqslant d\}$.

In Section 6, we study the resonance varieties of the trivial extension, $\mathsf{D}(A) = A \ltimes \bar{A}$. As an application, we obtain information about the structure of the resonance varieties of the boundary manifold of a hyperplane arrangement $\mathcal{A}$. Let $A = H^*(X(\mathcal{A}); \Bbbk)$ be the Orlik–Solomon algebra. It is well known that the components of the resonance varieties $\mathcal{R}^k_d(X(\mathcal{A})) = \mathcal{R}^k_d(A)$ are linear subspaces of $A^1 = \Bbbk^n$. The behavior of the resonance varieties $\mathcal{R}^k_d(M(\mathcal{A})) = \mathcal{R}^k_d(\mathsf{D}(A))$ is dramatically different. Indeed, we produce examples of arrangements for which the resonance varieties of the boundary manifold contain singular, irreducible components of arbitrarily high degree, see Corollary 6.11.

2. The boundary manifold

In this section, we introduce our main character, the boundary manifold of an (algebraic) hypersurface in complex projective space. We then compute its homology groups in terms of those of the complement to the hypersurface, and make a remark on the homotopy groups.

2.1. Thickenings

According to C.T.C. Wall [32], a thickening of a finite, k-dimensional CW-complex Y is a compact, m-dimensional manifold with boundary W^m, which is simply homotopy equivalent to Y. Such a thickening always exists, as soon as $m \geqslant 2k + 1$: embed Y as a sub-polyhedron in $\mathbb{R}^m$, and take W to be a smooth, regular neighborhood of Y.

Let $M = \partial W$ be the boundary of the thickening W. In general, the homotopy type of the boundary manifold M is not determined by the homotopy type of Y. For example, both $\mathbb{CP}^2 \times D^{m-4}$ and the normal disk bundle of $\mathbb{CP}^2 \subset S^m$ are thickenings of $\mathbb{CP}^2$, but their boundary manifolds are not homotopy equivalent, see Lambrechts [21]. Nevertheless, if M is orientable and $m \geqslant 2(k+1)$, then the cohomology ring $H^*(M;\mathbb{Z})$ is completely determined by $H^*(Y;\mathbb{Z})$, by Poincaré duality and degree considerations.

2.2. *Projective hypersurfaces*

Let V be a hypersurface in $\mathbb{CP}^\ell$, given as the zero locus of a homogeneous polynomial $f = f(\mathbf{x})$, where $\mathbf{x} = (x_0, \dots, x_\ell)$ are homogeneous coordinates on $\mathbb{CP}^\ell$. A (closed) regular neighborhood, N, of V in $\mathbb{CP}^\ell$ can be constructed either by triangulation, or by levels sets. In the first approach, triangulate $\mathbb{CP}^\ell$ with V as a subcomplex, and take N to be the closed star of V in the second barycentric subdivision. In the second, define $\phi : \mathbb{CP}^\ell \to \mathbb{R}$ by $\phi(\mathbf{x}) = |f(\mathbf{x})|^2/\|\mathbf{x}\|^{2d}$, where $d = \deg f$, and take $N = \phi^{-1}([0,\delta])$, for sufficiently small $\delta > 0$. As shown by Durfee [10], these constructions yield isotopic neighborhoods, independent of the choices made.

Clearly, N is a thickening of V. Hence, we may define *the boundary manifold* of V to be

$$M = \partial N. \tag{2.1}$$

This is a compact, orientable, smooth manifold of dimension $2\ell - 1$. If $\ell = 1$, then V consists of, say, n points on the sphere, and so M is a disjoint union of n circles. If $\ell > 1$, then M is connected. Here is a simple illustration.

Example 2.3. Let V be a pencil of $n+1$ hyperplanes in $\mathbb{CP}^\ell$, $\ell \geqslant 2$, defined by the polynomial $f = x_0^{n+1} - x_1^{n+1}$. In this case, X may be realized as the complement of n parallel hyperplanes in $\mathbb{C}^\ell$, and so it is homotopy equivalent to the n-fold wedge $\bigvee^n S^1$. On the other hand, $\mathbb{CP}^\ell \setminus N^\circ = (D^2 \setminus \{n \text{ disks}\}) \times D^{2(\ell-1)}$; hence M is diffeomorphic to the n-fold connected sum $\#^n S^1 \times S^{2(\ell-1)}$.

Note that the complement $X = \mathbb{CP}^\ell \setminus V$ is homotopy equivalent to the interior of the manifold with boundary $\mathbb{CP}^\ell \setminus N^\circ$, and that $M = \partial(\mathbb{CP}^\ell \setminus N^\circ)$. Also observe that, while N is a thickening of V, the cohomology ring of $M = \partial N$ is not a priori determined by that of V.

2.4. *Cohomology groups*

We now analyze in detail the cohomology groups of M. We start by relating these cohomology groups to those of X. Throughout this section, we use integral coefficients, unless otherwise noted.

Proposition 2.5. *Let V be a hypersurface in $\mathbb{CP}^\ell$, with complement X and boundary manifold M. For each $0 \leqslant k \leqslant 2\ell - 1$, there is an exact sequence*

$$0 \longrightarrow H^k(X) \longrightarrow H^k(M) \longrightarrow H^{k+1}(X, M) \longrightarrow 0. \tag{2.2}$$

Moreover, $H^{k+1}(X, M) \cong H_{2\ell-k-1}(X)$, and the sequence splits, except possibly when $k = \ell$.

Proof. Let $i : M \to X$ and $j : V \to \mathbb{CP}^\ell$ be the inclusion maps. Consider the following commuting diagram, with rows long exact sequences of pairs, and vertical isomorphisms given by the homotopy equivalence $V \hookrightarrow N$ and excision, respectively:

$$\begin{tikzcd}
\arrow[r] & H^k(\mathbb{CP}^\ell, V) \arrow[r] & H^k(\mathbb{CP}^\ell) \arrow[r, "j^*"] & H^k(V) \arrow[r] & H^{k+1}(\mathbb{CP}^\ell, V) \arrow[r] & {} \\
\arrow[r] & H^k(\mathbb{CP}^\ell, N) \arrow[r] \arrow[u, "\cong"'] \arrow[d, "\cong"] & H^k(\mathbb{CP}^\ell) \arrow[r] \arrow[u, "="'] \arrow[d] & H^k(N) \arrow[r] \arrow[u, "\cong"'] \arrow[d] & H^{k+1}(\mathbb{CP}^\ell, N) \arrow[r] \arrow[u, "\cong"'] \arrow[d, "\cong"] & {} \\
\arrow[r] & H^k(X, M) \arrow[r] & H^k(X) \arrow[r, "i^*"] & H^k(M) \arrow[r] & H^{k+1}(X, M) \arrow[r] & {}
\end{tikzcd} \tag{2.3}$$

By Lefschetz duality, $H^k(\mathbb{CP}^\ell \setminus N^\circ, M) \cong H_{2\ell-k}(\mathbb{CP}^\ell \setminus N^\circ)$ for each $k \geqslant 0$. Since $X \simeq \mathbb{CP}^\ell \setminus N^\circ$, we obtain $H^k(X, M) \cong H_{2\ell-k}(X)$.

By the Lefschetz theorem (see [8, Chapter 5, (2.6)]), the map $j^* : H^k(\mathbb{CP}^\ell) \to H^k(V)$ is an isomorphism for $k \leqslant \ell - 2$ and a monomorphism for $k = \ell - 1$. Chasing the diagram, we find that sequence (2.2) is exact, for each $k \leqslant \ell - 2$.

Now, it is well known that X is a Stein space, and thus has the homotopy type of a CW-complex of dimension at most ℓ. In particular, $H^k(X) = 0$ for $k > \ell$, and $H_\ell(X)$ is finitely generated and torsion-free. Furthermore, the boundary map $H^k(\mathbb{CP}^\ell, V; \mathbb{Q}) \to H^k(\mathbb{CP}^\ell; \mathbb{Q})$ is the zero map; see [8, p. 146]. By Lefschetz duality, $H^\ell(X, M) \cong H_\ell(X)$. Hence the map $H^\ell(X, M) \to H^\ell(X)$ is the zero map. We conclude that sequence (2.2) is exact for $k \geqslant \ell - 1$, as well.

For $k < \ell - 1$ or $k > \ell$, one of the side terms in (2.2) vanishes, so obviously the sequence splits. For $k = \ell - 1$, we know $H_\ell(X)$ is torsion-free, so (2.2) splits again. □

Corollary 2.6. *The Betti numbers of the boundary manifold M are given by $b_k(M) = b_k(X) + b_{2\ell-k-1}(X)$. Hence, the Poincaré polynomials of M and X are related by*:

$$P(M, t) = P(X, t) + t^{2\ell-1} \cdot P\bigl(X, t^{-1}\bigr). \tag{2.4}$$

Proposition 2.5 determines the cohomology groups of M in terms of the (co)homology groups of X, except possibly the torsion in $H^\ell(M)$. By the Universal Coefficient Theorem, this torsion fits into the short exact sequence

$$0 \longrightarrow \operatorname{Tors}\bigl(H_{\ell-1}(X)\bigr) \longrightarrow \operatorname{Tors}\bigl(H^\ell(M)\bigr) \longrightarrow \operatorname{Tors}\bigl(H_{\ell-1}(X)\bigr) \longrightarrow 0. \tag{2.5}$$

This sequence may or may not split. As we shall see in examples below, both possibilities can occur.

Example 2.7. Let V be a smooth algebraic hypersurface in $\mathbb{CP}^\ell$ of degree d. In this case, N can be taken to be a tubular neighborhood of V, diffeomorphic to the unit normal disk bundle ν. Hence M is the total space of the S^1-bundle over V with Euler number $e = c_1(\nu)[V]$.

In particular, if $\ell = 2$, then V is a curve of genus $g = \binom{d-1}{2}$, with $e = d^2$. Hence, by the Gysin sequence, $H^2(M) = \mathbb{Z}_{d^2}$. On the other hand, $H_1(X) = \mathbb{Z}_d$. Thus, in this instance, (2.5) is a nonsplit exact sequence, of the form $0 \to \mathbb{Z}_d \to \mathbb{Z}_{d^2} \to \mathbb{Z}_d \to 0$.

2.8. *Affine hypersurfaces and Milnor fibrations*

Let $V_0 \subset \mathbb{C}^\ell$ be an affine hypersurface, defined by the vanishing of a polynomial $f_0 = f_0(x_1, \dots, x_\ell)$ of degree n. Let V be the projective closure of V_0, defined by the vanishing of the homogeneous polynomial $f(x_0, x_1, \dots, x_\ell) = x_0^{n+1} \cdot f_0(x_1/x_0, \dots, x_\ell/x_0)$. Clearly, $\mathbb{CP}^\ell \setminus V = \mathbb{C}^\ell \setminus V_0$.

If f_0 itself is homogeneous, then $f(x_0, x_1, \dots, x_\ell) = x_0 \cdot f_0(x_1, \dots, x_\ell)$. Moreover, we can take the regular neighborhood N of V to be the union of a regular neighborhood of V_0, say N_0, with a tubular neighborhood of the hyperplane at infinity (after rounding corners). Thus, $\mathbb{CP}^\ell \setminus N^\circ$ is diffeomorphic to $D^{2\ell} \setminus (D^{2\ell} \cap N_0^\circ)$, and so

$$M = \bigl(S^{2\ell-1} \setminus \bigl(S^{2\ell-1} \cap N_0^\circ\bigr)\bigr) \cup \bigl(D^{2\ell} \cap \partial N_0\bigr).$$

As shown in [23], each of the two sides of the above decomposition is diffeomorphic to the total space of the Milnor fibration, $F \to Y \to S^1$, determined by the homogeneous polynomial f_0. Thus, M is the double of the manifold with boundary Y:

$$M = \partial(Y \times I) = Y \cup_{\partial Y} Y. \tag{2.6}$$

Furthermore, M fibers over the circle, with fiber the double of F.

Notice that, in this situation, the exact sequence (2.2) always splits. Indeed, the inclusion $Y \to X$ is a homotopy equivalence, which factors through the inclusions $Y \to M$ and $i : M \to X$. Thus, $i^* : H^*(X) \to H^*(M)$ is a split injection.

Example 2.9. Let $f = x_0 x_1 \cdots x_\ell$ be the polynomial defining the Boolean arrangement in $\mathbb{CP}^\ell$. Then $M = S^{\ell-1} \times T^\ell$, where T^ℓ is the ℓ-torus; see [8, Example 2.29].

Example 2.10. Let $f = x_0(x_1^n - x_2^n)$ be the polynomial defining a near-pencil of $n+1$ lines in $\mathbb{CP}^2$. In this case, Y admits a fibration over the circle (different from the Milnor fibration!), with fiber $D^2 \setminus \{n-1 \text{ disks}\}$, and monodromy a Dehn twist about the boundary D^2. It follows that $M = \Sigma_{n-1} \times S^1$, where Σ_g denotes a surface of genus g.

Example 2.11. More generally, let $f = x_0(x_1^{n_1} - y_1^{n_1}) \cdots (x_k^{n_k} - y_k^{n_k})$, with $n_i \geqslant 2$. Then $M = T^k \times (\#^m T^k \times S^{2k-1})$, where $m = \prod_{i=1}^k (n_i - 1)$.

Example 2.12. Let $f = x_0(x_1^2 + \cdots + x_\ell^2)$. In this case, the Milnor fiber F of $f_0 = x_1^2 + \cdots + x_\ell^2$ is diffeomorphic to the unit disk bundle of $S^{\ell-1}$. Thus, M fibers over S^1 with fiber E, where $S^{\ell-1} \to E \to S^{\ell-1}$ is the bundle with Euler number $1 - (-1)^\ell$.

Now assume ℓ is odd and $\ell > 1$. A computation with the Wang sequence for the bundle $F \to Y \to S^1$ shows that $H_{\ell-1}(X) = \mathbb{Z}_2$; see [9, Example 3.2]. Hence, (2.5) is a split exact sequence, of the form $0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0$.

2.13. On asphericity of the boundary

If V is a hypersurface in $\mathbb{CP}^\ell$, the inclusion map $M \to X$ is an $(\ell-1)$-equivalence, see, for instance, [8, Proposition 2.31]; in particular, $\pi_i(M) \cong \pi_i(X)$, for $i < \ell - 1$. A natural question arises: Is M aspherical? In other words, do all the higher homotopy groups of M vanish?

If $\ell = 2$, the manifold M^3 is a graph manifold in the sense of Waldhausen. With a few exceptions (such as lens spaces), manifolds of this type are aspherical. In higher dimensions, though, this never happens.

Proposition 2.14. *Let M be the boundary manifold of a hypersurface in $\mathbb{CP}^\ell$. If $\ell \geqslant 3$, then M is not aspherical.*

Proof. Let $\pi = \pi_1(M)$ be the fundamental group of M. Since the inclusion $i : M \to X$ is an $(\ell-1)$-equivalence, and since $\ell \geqslant 3$, the induced map $i_* : \pi_1(M) \to \pi_1(X)$ is an isomorphism. Let $g : X \to K(\pi, 1)$ be a classifying map for the universal cover $\widetilde{X} \to X$. By definition, $g_* : \pi_1(X) \to \pi$ is an isomorphism. Hence, the composite map $g \circ i : M \to K(\pi, 1)$ is a classifying map for $\widetilde{M} \to M$.

Now suppose M is aspherical. Then the map $g \circ i : M \to K(\pi, 1)$ must be a homotopy equivalence, since it induces an isomorphism on fundamental groups. Consequently, $(g \circ i)^* : H^{2\ell-1}(\pi) \to H^{2\ell-1}(M) = \mathbb{Z}$ is an isomorphism. On the other hand, $i^* : H^{2\ell-1}(X) \to H^{2\ell-1}(M)$ is the zero map, since the CW-complex X has dimension at most ℓ. This contradiction finishes the proof. □

3. The cohomology ring of the boundary manifold

Let $V \subset \mathbb{CP}^\ell$ be a projective hypersurface, with complement $X = \mathbb{CP}^\ell \setminus V$, and associated boundary manifold M. In this section, we determine the structure of the cohomology ring $H^*(M; \mathbb{C})$ under certain conditions. These conditions are given below in terms of the mixed Hodge structure on $H^*(X;\mathbb{C})$, respectively $H^*(V;\mathbb{C})$. First, we discuss the relevant algebraic structure.

3.1. The double of a graded ring

If A is a ring and B is an A-bimodule, the *trivial extension* of A by B, written $A \ltimes B$, is the additive group $A \oplus B$, with multiplication given by $(a,b)(a',b') = (aa', a\cdot b' + b\cdot a')$, see [24,29]. Note that $A \cong \{(a,0)\}$ is a subring of the trivial extension, and that $B \cong \{(0,b)\}$ is a square-zero ideal.

Now let $A = \bigoplus_{k=0}^{\ell} A^k$ be a finite-dimensional graded algebra over a base ring R. We will assume R is a commutative ring with 1, and all the graded pieces A^k are finitely generated free R-modules. Define the *double* $\mathsf{D}(A)$ of A to be the trivial extension of A by the graded A-bimodule $\bar{A} = \bigoplus_{k=\ell-1}^{2\ell-1} \bar{A}^k$, where $\bar{A}^k = \mathrm{Hom}(A^{2\ell-k-1}, R)$, and the A-bimodule structure is given by $a \cdot b(x) = b(xa)$ and $b \cdot a(x) = b(ax)$ for $a, x \in A$ and $b \in \bar{A}$. If A is a graded commutative ring, it is readily checked that $\mathsf{D}(A) = A \ltimes \bar{A}$ is a graded commutative ring as well.

Let $\mu : A \otimes A \to A$, $\mu(a,a') = aa'$, denote the multiplication map of the ring A. Then the multiplication map $\mathsf{D}(\mu) : \mathsf{D}(A) \otimes \mathsf{D}(A) \to \mathsf{D}(A)$ of the double restricts to μ on $A \otimes A$ and vanishes on $\bar{A} \otimes \bar{A}$, while on $A \otimes \bar{A}$ it vanishes, except for

$$\mathsf{D}(\mu)\big(a_j^k, \bar{a}_p^r\big) = \sum_i \mu_{i,j,p}\, \bar{a}_i^{r-k}, \quad \text{if} \quad \mu\big(a_i^{r-k}, a_j^k\big) = \sum_p \mu_{i,j,p}\, a_p^r, \tag{3.1}$$

where $\{a_j^k\}$ is a (fixed) homogeneous basis for A^k and $\{\bar{a}_j^k\}$ is the dual basis for $\bar{A}^{2\ell-k-1} = \mathrm{Hom}(A^k, R)$. The proof of the next result is straightforward.

Proposition 3.2. *The doubling construction is functorial. In particular, if A_1 and A_2 are isomorphic as graded rings, then $\mathsf{D}(A_1)$ and $\mathsf{D}(A_2)$ are isomorphic as graded rings.*

Denote the Betti numbers of A by $b_k(A) = \mathrm{rank}\, A^k$, and let

$$\mathrm{Hilb}(A,t) = \sum_{k=0}^{\ell} b_k(A)\cdot t^k$$

be the Hilbert series of A. Then:

$$\mathrm{Hilb}\big(\mathsf{D}(A), t\big) = \mathrm{Hilb}(A,t) + t^{2\ell-1}\cdot \mathrm{Hilb}\big(A, t^{-1}\big). \tag{3.2}$$

In particular, if $R = \Bbbk$ is a field and A is a $\Bbbk$-algebra that is connected (i.e., $b_0(A) = 1$), then $\mathsf{D}(A)$ is an Artin–Gorenstein ring.

Let R be a coefficient ring. If $H^*(X;R)$ is a free R-module, then it follows from Proposition 2.5 and the Universal Coefficient Theorem that $H^q(M;R) \cong H^q(X;R) \oplus H^{q+1}(X,M;R)$, for all q, $0 \leqslant q \leqslant 2\ell - 1$.

Theorem 3.3. *Assume that $H^*(X; R)$ is a free R-module. If $H^*(X, M; R)$ is a square-zero subring of $H^*(M; R)$, then $H^*(M; R) \cong \mathsf{D}(H^*(X; R))$ as graded rings.*

Proof. Recall that the inclusion $i : M \to X$ induces a monomorphism $i^* : H^*(X) \to H^*(M)$ in cohomology. Let $A = i^*(H^*(X; R))$, and note that A is a subring of $H^*(M; R)$. Comparing formulas (2.4) and (3.2), and using the R-freeness assumption for $H^*(X; R)$, we see that $H^*(M; R)$ and $\mathsf{D}(A) = A \ltimes \bar{A}$ are additively isomorphic. So it suffices to show that the cup-product structure in $H^*(M; R)$ coincides with the multiplicative structure in $\mathsf{D}(A)$. This is clearly the case for the restriction to the common subring A.

For simplicity, let us suppress the coefficient ring R from the notation. Fix a generator $\omega \in H^{2\ell-1}(M)$, and note that $\omega \notin A$. For each q, $0 \leqslant q \leqslant \ell$, let $\{a_1^q, \dots, a_{b_q}^q\}$ be a basis for $A_q \cong H^q(X)$, where $b_q = b_q(A)$. By Poincaré duality, there are elements $\bar{a}_1^q, \dots, \bar{a}_{b_q}^q$ in $H^{\bar{q}}(M)$ which are linearly independent and satisfy

$$a_i^q \cup \bar{a}_j^q = \delta_{i,j}\omega,$$

where $\bar{q} = 2\ell - q - 1$ and $\delta_{i,j}$ is the Kronecker index. Since A is a subring of $H^*(M)$ and $\omega \notin A$, the dual classes $\bar{a}_i^q$ are also not in A. Identifying $H^q(M) = H^q(X) \oplus H^{q+1}(X, M)$, it follows that $\{\bar{a}_1^q, \dots, \bar{a}_{b_q}^q\}$ forms a basis for $H^{\bar{q}+1}(X, M) \subset H^{\bar{q}}(M)$. Consequently, $H^q(M)$ has basis $\{a_1^q, \dots, a_{b_q}^q, \bar{a}_1^{\bar{q}}, \dots, \bar{a}_{b_{\bar{q}}}^{\bar{q}}\}$.

By hypothesis, we have $\bar{a}_i^p \cup \bar{a}_j^q = 0$ for all p, q and i, j. It remains to consider the cup-product $a_j^p \cup \bar{a}_k^q \in H^{p+\bar{q}}(M)$. If $p = 0$, then $a_j^p \cup \bar{a}_k^q = 1 \cup \bar{a}_k^q = \bar{a}_k^q$. If $p > q$, then $a_j^p \cup \bar{a}_k^q = 0$. So assume that $0 < p \leqslant q \leqslant \ell$, which implies that $p + \bar{q} \geqslant \ell$.

If $p + \bar{q} > \ell$, then

$$a_j^p \cup \bar{a}_k^q = \sum_{i=1}^{b_{q-p}} c_{i,j,k}\bar{a}_i^{q-p}$$

for some constants $c_{i,j,k}$. Write the multiplication in $A \cong H^*(X)$ as

$$a_i^r \cdot a_j^p = \sum_{l=1}^{b_{r+p}} \mu_{i,j,l} a_l^{r+p},$$

and note that $\mu_{j,i,l} = (-1)^{rp}\mu_{i,j,l}$ in this instance. For a fixed i, cupping with a_i^{q-p} yields

$$a_i^{q-p} \cup a_j^p \cup \bar{a}_k^q = c_{i,j,k}\omega.$$

Since

$$a_i^{q-p} \cup a_j^p \cup \bar{a}_k^q = \left(\sum_{l=1}^{b_{q-p}} \mu_{i,j,l} a_l^q\right) \cup \bar{a}_k^q = \mu_{i,j,k}\omega,$$

we must have $c_{i,j,k} = \mu_{i,j,k}$, and so $a_j^p \cup \bar{a}_k^q = \sum_{i=1}^{b_{q-p}} \mu_{i,j,k}\bar{a}_i^{q-p}$.

We are left with the case $p+\bar{q}=\ell$, that is, $p=1$ and $q=\ell$. We then have

$$a_j^1 \cup \bar{a}_k^\ell = \sum_{i=1}^{b_{\ell-1}} c_{i,j,k}\bar{a}_i^{\ell-1} + \sum_{i=1}^{b_\ell} d_{i,j,k} a_i^\ell$$

for some constants $c_{i,j,k}$ and $d_{i,j,k}$. Since $0=\bar{a}_i^\ell \cup a_j^1 \cup \bar{a}_k^\ell = \pm d_{i,j,k}\omega$, we have $d_{i,j,k}=0$. Then, a calculation as above yields $c_{i,j,k}=\mu_{i,j,k}$, where

$$a_i^{\ell-1}\cdot a_j^1 = \sum_{k=1}^{b_\ell} \mu_{i,j,k} a_k^\ell.$$

Thus,

$$a_j^1 \cup \bar{a}_k^\ell = \sum_{i=1}^{b_{\ell-1}} \mu_{i,j,k}\bar{a}_i^{\ell-1}.$$

Notice that these calculations show that the square-zero subring $H^*(X,M)$ is, in fact, an ideal in $H^*(M)$. Using these calculations, and formula (3.1), it is readily checked that the cup-product structure in $H^*(M)$ coincides with the multiplicative structure in $\mathsf{D}(H^*(X))$. $\square$

The freeness assumption from Theorem 3.3 holds, for example, when $R=\mathbb{Z}$ and $H^*(X)$ is torsion-free, or when $R=\Bbbk$ is a field. This assumption is necessary, as illustrated by the smooth plane curve of degree $d>1$ from Example 2.7. Indeed, for such a curve, $H^2(M;\mathbb{Z})=\mathbb{Z}_{d^2}$ does not split as a direct sum, and so $H^*(M;\mathbb{Z})\ncong \mathsf{D}(H^*(X;\mathbb{Z}))$, even though $H^*(X,M;\mathbb{Z})$ is a square-zero subring of $H^*(M;\mathbb{Z})$, by degree considerations.

3.4. Hodge structures

Now we pursue conditions which insure that the hypotheses of Theorem 3.3 hold. These conditions will be given in terms of mixed Hodge structures. For the rest of this section, we shall take coefficients in the ring $R=\mathbb{C}$.

If V is a smooth projective variety, then, by a classical theorem of Hodge, each cohomology group $H^m(V)$ admits a pure Hodge structure of weight m. That is, for $H=H^m(V)$, there is a direct sum decomposition

$$H=\bigoplus_{p+q=m} H^{p,q}, \tag{3.3}$$

where $\overline{H^{p,q}}=H^{q,p}$ (complex conjugation).

If X is a quasi-projective variety, then, by a well-known theorem of Deligne [7], each cohomology group of X admits a mixed Hodge structure. That is, for each k, there is an increasing weight filtration

$$0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2k} = H^k(X), \tag{3.4}$$

such that each quotient W_m/W_{m-1} of the subspaces $W_m = W_m(H^k(X))$ of $H^k(X)$ admits a pure Hodge structure of weight m as in (3.3).

The following properties of the weight filtration will be of use. See [8,11,28] for further details.

(1) If X is projective, then $W_k = H^k(X)$ for each k.
(2) If X is smooth, then $0 = W_{k-1} \subset H^k(X)$ for each k.
(3) For any smooth compactification $\iota : X \to \bar{X}$ of X, $W_k = \iota^*(H^k(\bar{X}))$ for each k.
(4) The weight filtration is functorial. For an algebraic map $f : X \to Y$, the induced homomorphism f^* strictly preserves the filtration: If $x \in W_m(H^k(X))$ is in the image of f^*, there is an element $y \in W_m(H^k(Y))$ with $f^*(y) = x$.

It follows from work of Durfee and Hain [12] that the cohomology of the boundary manifold M of a projective hypersurface V admits a mixed Hodge structure. Furthermore, the cup-product of $H^*(M)$ is a morphism of mixed Hodge structures, and the top cohomology $H^{2\ell-1}(M)$ is of weight 2ℓ (and type (ℓ, ℓ)).

Theorem 3.5. *Let V be a hypersurface in $\mathbb{CP}^\ell$ with complement X and boundary manifold M. If V is irreducible, then $H^*(M; \mathbb{C}) \cong \mathsf{D}(H^*(X); \mathbb{C})$ as graded algebras.*

Proof. If $\ell = 1$, then V is a point in $\mathbb{CP}^1$. In this instance, X is contractible, M is a circle, and it is readily checked that $H^*(M) \cong \mathsf{D}(H^*(X))$.

So we may assume that $\ell \geqslant 2$. By Theorem 3.3, it suffices to show that $H^*(X, M)$ is a square-zero subalgebra of $H^*(M)$. For this, it is enough to show that $u \cup v = 0$ for $u \in H^{r+1}(X, M) \subset H^r(M)$ and $v \in H^{s+1}(X, M) \subset H^s(M)$, where $\ell - 1 \leqslant r, s \leqslant \ell$.

Recall that, for $k \leqslant 2\ell - 2$, the inclusion $j : V \to \mathbb{CP}^\ell$ induces a monomorphism in kth cohomology. From diagram (2.3), we see that $H^{k+1}(X, M)$ is isomorphic to $H_0^k(V)$, the primitive cohomology of V, given by

$$H_0^k(V) = \operatorname{coker}\bigl[j^* : H^k\bigl(\mathbb{CP}^\ell\bigr) \to H^k(V)\bigr].$$

It is known that the connecting homomorphism in the long exact sequence of the pair is weight-preserving, see [8,11,28]. This fact, and the properties recorded above, imply that all cohomology classes in $H^{k+1}(X, M) \cong H_0^k(V)$ (for $k \leqslant 2\ell - 2$) are of weight at most k.

Now take $u \in H^{r+1}(X, M) \subset H^r(M)$ and $v \in H^{s+1}(X, M) \subset H^s(M)$ as above. If $r = s = \ell$, then clearly $u \cup v = 0$. If, say, $r = \ell - 1$ and $s = \ell$, then u is of weight at most $\ell - 1$ and v is of weight at most ℓ. Hence, $u \cup v$ is of weight at most $2\ell - 1$ in $H^{2\ell-1}(M)$. But $W_{2\ell-1}(H^{2\ell-1}(M)) = 0$ by the results of Durfee and Hain noted above. So we must have $u \cup v = 0$.

Finally, if $r = s = \ell - 1$, then $u \cup v$ is of weight at most $2\ell - 2$ in $H^{2\ell-2}(M)$. Since V is irreducible, $H^1(X) = 0$, the map $j^* : H^{2\ell-2}(\mathbb{CP}^\ell) \to H^{2\ell-2}(V)$ is an isomorphism, and $H^{2\ell-1}(X, M) \cong H_0^{2\ell-2}(V) = 0$. If $\ell = 2$, then all nontrivial classes in $H^2(M) = H^2(X)$ are of weight at least 3 by Poincaré duality, since all classes in $H^1(M) \cong H_0^1(V) = H^1(V)$ are of weight at most 1. If $\ell \geqslant 3$, then $H^{2\ell-2}(M) = H^{2\ell-2}(X) = 0$ since X has the homotopy type of an ℓ-dimensional complex. It follows that $u \cup v = 0$ in either case. □

Theorem 3.6. *Let V be a hypersurface in $\mathbb{CP}^\ell$ with complement X and boundary manifold M. If $W_{\ell+1}(H^\ell(X;\mathbb{C})) = 0$, then $H^*(M;\mathbb{C}) \cong \mathsf{D}(H^*(X;\mathbb{C}))$ as graded algebras.*

Proof. If $\ell = 1$, then V is a union of, say, $n + 1$ points in $\mathbb{CP}^1$. In this instance, X is homotopic to a bouquet of n circles, M is a disjoint union of $n + 1$ circles, and it is readily checked that $H^*(M) \cong \mathsf{D}(H^*(X))$.

If $\ell \geqslant 2$, by Theorem 3.3, it suffices to show that $H^*(X, M)$ is a square-zero subalgebra of $H^*(M)$. For this, as above, it is enough to show that $u \cup v = 0$ for $u \in H^{r+1}(X, M) \subset H^r(M)$ and $v \in H^{s+1}(X, M) \subset H^s(M)$, where $(r, s) = (\ell - 1, \ell)$ or $(r, s) = (\ell - 1, \ell - 1)$. By Poincaré duality, there are elements $a, b \in H^*(X) \subset H^*(M)$ so that $a \cup u = b \cup v = \omega \in H^{2\ell-1}(M)$.

If $(r, s) = (\ell - 1, \ell)$, then $a \in H^\ell(X)$ and $b \in H^{\ell-1}(X)$. Then, since X is smooth, $W_{\ell-2}(H^{\ell-1}(X)) = 0$, and b is of weight at least $\ell - 1$. Since $W_{\ell+1}(H^\ell(X)) = 0$ by hypothesis, a is of weight at least $\ell + 2$. Since ω is of weight 2ℓ, is follows that u is of weight at most $\ell - 2$ and v is of weight at most $\ell + 1$. Consequently, $u \cup v$ is of weight at most $2\ell - 1$ in $H^{2\ell-1}(M)$, which is pure of weight 2ℓ. Hence $u \cup v = 0$. If $(r, s) = (\ell - 1, \ell - 1)$, then $a, b \in H^\ell(X)$ are both of weight at least $\ell + 2$, and a similar argument shows that $u \cup v = 0$. □

3.7. Plane algebraic curves

For an arbitrary projective hypersurface, the cohomology ring of the boundary manifold (with $\mathbb{C}$ coefficients) need not admit the structure of a double. We illustrate this phenomenon in dimension two.

Theorem 3.8. *Let $V = V_1 \cup \cdots \cup V_k$ be a reducible algebraic curve in $\mathbb{CP}^2$, with complement X and boundary manifold M. Then $H^*(M;\mathbb{C}) \cong \mathsf{D}(H^*(X;\mathbb{C}))$ if and only if all the irreducible components V_j are rational curves.*

Proof. If an irreducible component V_j of V is a rational curve, then the normalization of V_j has genus 0. It follows that all nontrivial cohomology classes in $H_0^1(V_j) = H^1(V_j)$ are of weight 0. Using this, an inductive argument with the Mayer–Vietoris sequence reveals that the same holds for $H_0^1(V) = H^1(V)$. It follows that $H^2(X) \cong H_0^1(V)$ is pure of weight 4, see [8, p. 246]. So $H^*(M) \cong \mathsf{D}(H^*(X))$ by Theorem 3.6.

Conversely, if an irreducible component V_j of V is not a rational curve, then the degree of V_j is necessarily at least three. In this situation, $H^1(V) = H_0^1(V)$ contains nontrivial classes of weights 0 and 1 (see [11]). It follows that $H^2(X)$ contains classes of weights 3 and 4 (see [8]). (Note that the weight condition of Theorem 3.6 fails.) In this instance, it is readily checked that the cup-product $H_0^1(V) \otimes H_0^1(V) \to H_0^2(V)$ is nontrivial. Hence, $H_0^*(V) \subset H^*(M)$ is not a square-zero subalgebra, compare Theorem 3.3, and $H^*(M) \ncong \mathsf{D}(H^*(X))$. $\square$

Suppose V is an arrangement of rational curves in $\mathbb{CP}^2$, with complement X, and boundary manifold M. A presentation for the cohomology ring $H^*(X;\mathbb{C})$ was given in [4, Theorem 0.4]. Our Theorem 3.8 can now be used to compute the cohomology ring $H^*(M;\mathbb{C})$.

4. Hyperplane arrangements

Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{CP}^\ell$. For each hyperplane H of $\mathcal{A}$, let f_H be a linear form with $H = \{f_H = 0\}$. Then $f = Q(\mathcal{A}) = \prod_{H\in\mathcal{A}} f_H$ is a defining polynomial for $\mathcal{A}$, the hypersurface $V = V(\mathcal{A})$ is given by

$$V = f^{-1}(0) = \bigcup_{H\in\mathcal{A}} H,$$

and the complement of the arrangement is $X = X(\mathcal{A}) = \mathbb{CP}^\ell \setminus V$.

4.1. Boundary manifold of an arrangement

Let $M = M(\mathcal{A})$ be the boundary manifold of the hypersurface $V = V(\mathcal{A})$. The next theorem expresses the (integral) cohomology ring of M in terms of the Orlik–Solomon algebra $A = A(\mathcal{A}) = H^*(X(\mathcal{A});\mathbb{Z})$ of the arrangement $\mathcal{A}$.

Theorem 4.2. *Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{CP}^\ell$ with complement X and boundary manifold M. Then $H^*(M;\mathbb{Z}) \cong \mathsf{D}(H^*(X;\mathbb{Z}))$.*

Proof. For any hyperplane arrangement $\mathcal{A}$, the cohomology $H^k(X;\mathbb{C})$ is pure of weight $2k$, that is, the weight filtration takes the form $0 = W_{2k-1} \subset W_{2k} = H^k(X;\mathbb{C})$, for every k, see Shapiro [31], and also Kim [19]. Hence, by Theorem 3.6, we have $H^*(M;\mathbb{C}) \cong \mathsf{D}(H^*(X;\mathbb{C}))$.

Let $A = H^*(X;\mathbb{Z})$ be the integral Orlik–Solomon algebra of $\mathcal{A}$. It is well known that $A = \bigoplus_{k=0}^{\ell} A^k$ is torsion-free. Let $\mathsf{D}(A) = A \ltimes \bar{A}$ be the integral double of A, the trivial extension of A by

$$\bar{A} = \bigoplus_{k=\ell-1}^{2\ell-1} \operatorname{Hom}_{\mathbb{Z}}\bigl(A^{2\ell-k-1}, \mathbb{Z}\bigr),$$

with A-bimodule structure as given in Section 3.1. Since $A = H^*(X;\mathbb{Z})$ is torsion-free, $H^*(M;\mathbb{Z})$ is also torsion-free, see Proposition 2.5. Since $H^*(M;\mathbb{C}) \cong \mathsf{D}(H^*(X;\mathbb{C}))$, it follows that $H^*(M;\mathbb{Z}) \cong \mathsf{D}(A)$. $\square$

Let $L(\mathcal{A})$ be the intersection poset of the arrangement $\mathcal{A}$, the set of all nonempty intersections of elements of $\mathcal{A}$, ordered by reverse inclusion. By the Orlik–Solomon theorem (see [26,34]), the integral cohomology ring of $X(\mathcal{A})$ is determined by $L(\mathcal{A})$. Our next result shows that the cohomology of $M(\mathcal{A})$ is determined by $L(\mathcal{A})$ and the ambient dimension.

Corollary 4.3. *If $\mathcal{A}_1$ and $\mathcal{A}_2$ are hyperplane arrangements in $\mathbb{CP}^\ell$ with isomorphic intersection posets, then $H^*(M(\mathcal{A}_1);\mathbb{Z}) \cong H^*(M(\mathcal{A}_2);\mathbb{Z})$.*

Proof. By the Orlik–Solomon theorem, if $L(\mathcal{A}_1) \cong L(\mathcal{A}_2)$, then $A(\mathcal{A}_1) \cong A(\mathcal{A}_2)$. Proposition 3.2 implies that the (integral) doubles are isomorphic. Thus, by Theorem 4.2, $H^*(M(\mathcal{A}_1);\mathbb{Z}) \cong H^*(M(\mathcal{A}_2);\mathbb{Z})$. $\square$

4.4. Computing cup products

We now exhibit an explicit basis for the cohomology of the boundary manifold of an arrangement, and compute cup products in that basis. Write $\mathcal{A} = \{H_0, H_1, \dots, H_n\}$, and designate H_0 as the hyperplane at infinity in $\mathbb{CP}^\ell$. Let $\mathcal{A}' = \{H_1, \dots, H_n\}$ be the corresponding affine arrangement in $\mathbb{C}^\ell = \mathbb{CP}^\ell \setminus H_0$. Notice that $\mathcal{A}$ is the projective closure of $\mathcal{A}'$.

The rank of the affine arrangement $\mathcal{A}'$ is the maximal number of linearly independent hyperplanes in $\mathcal{A}'$. If $\mathcal{A}' \subset \mathbb{C}^\ell$ has rank ℓ, then $\mathcal{A}'$ is said to be essential. Observe that the projective arrangement $\mathcal{A} \subset \mathbb{CP}^\ell$ is essential if it contains $\ell + 1$ independent hyperplanes. For an arrangement of rank r, it is well known that the Betti numbers, $b_k(X)$, of the complement are nonzero for all k, $0 \leqslant k \leqslant r$. See [26] as a general reference.

Order the hyperplanes of $\mathcal{A}' = \{H_1, \dots, H_n\}$ by their indices. A circuit is an inclusion-minimal dependent set of hyperplanes (in $\mathcal{A}'$), and a broken circuit is a set S for which there exists $j < \min(S)$ so that $\{H_j\} \cup \{H_i \mid i \in S\}$ is a circuit. Let $\mathbf{nbc} = \mathbf{nbc}(\mathcal{A}')$ denote the collection of subsets $I \subset [n]$ for which $\bigcap_{i\in I} H_i \neq \emptyset$ and I contains no broken circuits. If the rank of $\mathcal{A}'$ is r, then all elements of $\mathbf{nbc}$ are of cardinality at most r. Note also that $\emptyset \in \mathbf{nbc}$.

Clearly, the complement of $\mathcal{A}$ in $\mathbb{CP}^\ell$ is diffeomorphic to the complement of $\mathcal{A}'$ in $\mathbb{C}^\ell$. The integral cohomology of $X = X(\mathcal{A}) = X(\mathcal{A}')$ is isomorphic to the Orlik–Solomon algebra $A = A(\mathcal{A}')$, a quotient of an exterior algebra on n generators in degree 1. A basis for A is indexed by the set $\mathbf{nbc}$; denote this basis for A by $\{a_I \mid I \in \mathbf{nbc}\}$. If $|I| = k$, then $a_I \in A^k$. In particular, the unit in A is $1 = a_\emptyset \in A^0$. Express the cup-product in $A = H^*(X)$ by

$$a_I a_J = \sum_{K \in \mathbf{nbc}} \mu_{I,J,K} a_K. \tag{4.1}$$

Denote the images of the generators a_I of $A = H^*(X)$ in $H^*(M)$ by the same symbols. By Poincaré duality, there are elements $\bar{a}_I \in H^*(M)$ so that $a_I \bar{a}_J = \delta_{I,J}\omega$, where ω is a (fixed) generator of $H^{2\ell-1}(M) \cong \mathbb{Z}$. In particular, $\bar{a}_\emptyset = \omega$. Since $H^*(M) = \mathsf{D}(A)$, using (3.1), we obtain the following.

Corollary 4.5. *The set* $\{a_I, \bar{a}_I \mid I \in \mathbf{nbc}\}$ *forms a basis for* $H^*(M)$*, and the cup-product in* $H^*(M)$ *is given by*

$$a_I a_J = \sum_{K \in \mathbf{nbc}} \mu_{I,J,K} a_K, \qquad a_J \bar{a}_K = \sum_{I \in \mathbf{nbc}} \mu_{I,J,K} \bar{a}_I, \qquad \bar{a}_I \bar{a}_J = 0.$$

Example 4.6. Let $\mathcal{A}$ be a near-pencil in $\mathbb{CP}^2$, defined by the polynomial

$$Q(\mathcal{A}) = x_0\big(x_1^n - x_2^n\big).$$

As noted in Example 2.10, the boundary manifold M is diffeomorphic to $\Sigma_{n-1} \times S^1$.

The complement X of $\mathcal{A}$ has Poincaré polynomial $P(X,t) = 1 + nt + (n-1)t^2$. The **nbc** basis of the Orlik–Solomon algebra $A = H^*(X)$ is given by $\{1 = a_\emptyset, a_1, \dots, a_n, a_{1,2}, \dots, a_{1,n}\}$. The cup-product in A is given by $a_1 a_j = a_{1,j}$ and $a_i a_j = a_{1,j} - a_{1,i}$ for $i > 1$.

The boundary manifold M has Poincaré polynomial

$$P(M,t) = 1 + (2n-1)t + (2n-1)t^2 + t^3.$$

A basis for the cohomology ring $\mathsf{D}(A) = H^*(M)$ is given by the above basis for the Orlik–Solomon algebra, together with the dual classes $\{\bar{a}_{1,2}, \dots, \bar{a}_{1,n}, \bar{a}_1, \dots, \bar{a}_n, \bar{a}_\emptyset = \omega\}$. By Corollary 4.5, the cup-product in $\mathsf{D}(A)$ is given by the multiplication in A recorded above, $\bar{a}_I \bar{a}_J = 0$ for all I and J, $a_j \bar{a}_k = a_{1,j} \bar{a}_{1,k} = \delta_{j,k}\omega$, and $a_j \bar{a}_{1,k} = -\bar{a}_k + \delta_{j,k}(\bar{a}_1 + \dots + \bar{a}_n)$.

Now, $H^*(\Sigma_{n-1} \times S^1) = H^*(\Sigma_{n-1}) \otimes H^*(S^1)$ is generated by $\alpha_j \otimes 1$, $\beta_j \otimes 1$, $1 \leqslant j \leqslant n-1$, $\Gamma \otimes 1$, and $1 \otimes z$, where α_j, β_j, Γ generate $H^*(\Sigma_{n-1})$ and satisfy $\alpha_j \beta_k = \delta_{j,k}\Gamma$, and z generates $H^*(S^1)$. An explicit isomorphism from $H^*(\Sigma_{n-1} \times S^1)$ to $\mathsf{D}(A)$ is defined by

$$\alpha_j \otimes 1 \mapsto a_{j+1} - a_1, \quad \beta_j \otimes 1 \mapsto \bar{a}_{1,j+1}, \quad 1 \otimes z \mapsto a_1, \quad \Gamma \otimes 1 \mapsto \bar{a}_1 + \dots + \bar{a}_n.$$

4.7. The $K(\pi,1)$ problem

A hyperplane arrangement $\mathcal{A}$ is said to be a $K(\pi,1)$-arrangement if the complement $X = X(\mathcal{A})$ is aspherical, i.e., its universal cover is contractible. Classical examples include the braid arrangement (Fadell–Neuwirth), certain reflection arrangements (Brieskorn) and simplicial arrangements (Deligne).

The boundary manifold of an arrangement in $\mathbb{CP}^1$ is a disjoint union of circles. For $\ell \geqslant 3$, Proposition 2.14 shows that the boundary manifold of an arrangement in $\mathbb{CP}^\ell$ is never aspherical. In the remaining case, $\ell = 2$, we have the following result.

Proposition 4.8. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{CP}^2$. The boundary manifold $M = M(\mathcal{A})$ is aspherical if and only if $\mathcal{A}$ is essential.*

Proof. If $\mathcal{A}$ is not essential, then $\mathcal{A}$ is a pencil of lines in $\mathbb{CP}^2$, and so, by Example 2.3, M is a connected sum of $S^1 \times S^2$'s. Thus, $\pi_2(M) \neq 0$.

If $\mathcal{A}$ is essential, it follows from work of Jiang and Yau [18] that M is an irreducible, sufficiently large Waldhausen graph manifold. Hence, M is aspherical. (In fact, by [30], M admits a metric of nonpositive curvature.) □

5. Topological complexity

In this section, we relate the topological complexity of the boundary manifold of a hyperplane arrangement to that of the complement. We start by relating the zero-divisor length of a graded algebra to that of its double.

5.1. Cup length and zero-divisor length

Let $A = \bigoplus_{k=0}^{\ell} A^k$ be a graded algebra over a field $\Bbbk$ (as usual, we assume all graded pieces are finite-dimensional). Define the *cup length* of A, denoted $\mathrm{cl}(A)$, to be the largest integer q for which there exist homogeneous elements $a_1, \dots, a_q \in A^{>0}$ such that $a_1 \cdots a_q \neq 0$.

The tensor product $A \otimes A$ has a natural graded algebra structure, with multiplication given by $(u_1 \otimes v_1) \cdot (u_2 \otimes v_2) = (-1)^{|v_1|\cdot|u_2|} u_1 u_2 \otimes v_1 v_2$. Multiplication in A defines an algebra homomorphism $\mu : A \otimes A \to A$. Let $J(A)$ be the kernel of this map. The *zero-divisor length* of A, denoted by $\mathrm{zcl}(A)$, is the length of the longest nontrivial product in this ideal.

Lemma 5.2. *The ideal $J(A) = \ker(\mu : A \otimes A \to A)$ is generated by the set $\{\zeta_a := a \otimes 1 - 1 \otimes a \mid a \in A\}$.*

Proof. Let $z = \sum_{i=1}^{k} a_i \otimes b_i$ be an element of $J(A)$. Then $\sum_{i=1}^{k} a_i b_i = 0$ in A, and it is readily checked that $z - \sum_{i=1}^{k} \zeta_{a_i}(1 \otimes b_i) = \sum_{i=1}^{k} 1 \otimes a_i b_i = 1 \otimes (\sum_{i=1}^{k} a_i b_i) = 0$ in $A \otimes A$. □

These two notions of length behave quite nicely with respect to the doubling operation for graded algebras.

Proposition 5.3. *Let A be a connected, finite-dimensional graded algebra, with double $\mathsf{D}(A) = A \ltimes \bar{A}$. Then,* $\mathrm{cl}(\mathsf{D}(A)) = \mathrm{cl}(A) + 1$ *and* $\mathrm{zcl}(\mathsf{D}(A)) = \mathrm{zcl}(A) + 2$.

Proof. Suppose that $\mathrm{cl}(A) = q$, and let $a = a_1 \cdots a_q$ be an element in A of length q. Then $a \cdot \bar{a}$ is a nonzero element in $\mathsf{D}(A)$, of length $q+1$. Thus $\mathrm{cl}(\mathsf{D}(A)) \geqslant \mathrm{cl}(A)+1$. The equality $\mathrm{cl}(\mathsf{D}(A)) = \mathrm{cl}(A) + 1$ then follows from the fact that $\bar{A}$ is a square-zero ideal in $\mathsf{D}(A)$.

Next, suppose that $\mathrm{zcl}(A) = q$, and let $z = z_1 \cdots z_q$ be an element in $J(A)$ of length q. Recall the basis $\{a_j^k\}$ of A from Section 3.1, and write

$$z = \sum c_{j_1,j_2}^{k_1,k_2} a_{j_1}^{k_1} \otimes a_{j_2}^{k_2}.$$

Let m be maximal so that $i_1 + i_2 = m$ and there is a nonzero coefficient $c_{r_1,r_2}^{i_1,i_2}$ in this sum. Then, one can check that

$$z\big(\bar{a}_{r_1}^{i_1} \otimes 1\big)\big(1 \otimes \bar{a}_{r_2}^{i_2}\big) = \pm c_{r_1,r_2}^{i_1,i_2} \omega \otimes \omega + z',$$

where $\omega = \bar{1}$ generates $\mathsf{D}(A)^{2\ell-1}$ and z' is a linear combination of elements

$$a_{j_1}^{k_1} \bar{a}_{r_1}^{i_1} \otimes a_{j_2}^{k_2} \bar{a}_{r_2}^{i_2}$$

in $\mathsf{D}(A)$ of bidegree different from $(2\ell - 1, 2\ell - 1)$. So $\hat{z} = z(\bar{a}_{r_1}^{i_1} \otimes 1)(1 \otimes \bar{a}_{r_2}^{i_2})$ is a nonzero element in $J(\mathsf{D}(A))$, of length at least $q + 2$. Thus $\operatorname{zcl}(\mathsf{D}(A)) \geqslant \operatorname{zcl}(A) + 2$.

To show that $\operatorname{zcl}(\mathsf{D}(A)) = \operatorname{zcl}(A) + 2$, it suffices to check that $\hat{z}\zeta_\alpha = \hat{z}(\alpha \otimes 1 - 1 \otimes \alpha) = 0$ for $\alpha \in \mathsf{D}(A)$. We may assume that α is an element of the basis $\{a_j^k, \bar{a}_j^k\}$ for $\mathsf{D}(A)$. If $\alpha = \bar{a}_j^k \in \bar{A}$, then $\hat{z}\zeta_\alpha = 0$ since $\bar{A}$ is a square-zero ideal in $\mathsf{D}(A)$. If $\alpha = a_j^k \in A$ and $\hat{z}\zeta_\alpha \neq 0$, then $z\zeta_\alpha$ is a nonzero element of length $q + 1$ in $J(A)$, contradicting the assumption that $\operatorname{zcl}(A) = q$. □

5.4. LS category and topological complexity

Let $p : Y \to X$ be a fibration. The *sectional category* of p, denoted $\operatorname{secat}(p)$, is the smallest integer q such that X can be covered by q open subsets, over each of which p has a section. A cohomological lower bound is given by:

$$\operatorname{secat}(p) > \operatorname{cl}\big(\ker\big(p^* : H^*(X; \Bbbk) \to H^*(Y; \Bbbk)\big)\big), \tag{5.1}$$

see James [17] as a classical reference. If $p : PX \to X$ is the path-fibration of a pointed space X, then $\operatorname{secat}(p) = \operatorname{cat}(X)$, the Lusternik–Schnirelmann category of X. The category of X depends only on the homotopy type of X. Since PX is contractible, the inequality (5.1) reduces to $\operatorname{cat}(X) > \operatorname{cl}(X) := \operatorname{cl}(H^*(X; \Bbbk))$. If X is a finite simplicial complex, then $\operatorname{cat}(X) \leqslant \dim(X) + 1$. Furthermore, $\operatorname{cat}(X \times Y) \leqslant \operatorname{cat}(X) + \operatorname{cat}(Y) - 1$.

Now let X^I be the space of all continuous paths from $I = [0, 1]$ to X, with the compact-open topology, and let $\pi : X^I \to X \times X$ be the fibration given by $\pi(\gamma) = (\gamma(0), \gamma(1))$. The *topological complexity* of X, introduced by Farber in [13] and denoted by $\operatorname{tc}(X)$, may be realized as the sectional category of π. Again, $\operatorname{tc}(X) = \operatorname{secat}(\pi)$ depends only on the homotopy type of X. Using the fact that $X^I \simeq X$, and the Künneth formula, (5.1) reduces to $\operatorname{tc}(X) > \operatorname{zcl}(X) := \operatorname{zcl}(H^*(X; \Bbbk))$. If X is a finite simplicial complex, then $\operatorname{cat}(X) \leqslant \operatorname{tc}(X) \leqslant 2\operatorname{cat}(X) - 1$; in particular, $\operatorname{tc}(X) \leqslant 2\dim(X) + 1$. Furthermore, $\operatorname{tc}(X \times Y) \leqslant \operatorname{tc}(X) + \operatorname{tc}(Y) - 1$.

As noted in [13], topological complexity is *not* determined by the LS category. For example, $\operatorname{cat}(S^n) = 2$ for all $n \geqslant 1$, whereas $\operatorname{tc}(S^n) = 2$ for n odd and $\operatorname{tc}(S^n) = 3$ for n even; also, $\operatorname{cat}(T^n) = \operatorname{tc}(T^n) = n + 1$, but $\operatorname{cat}(\Sigma_g) = 3$ and $\operatorname{tc}(\Sigma_g) = 5$ for $g \geqslant 2$.

In [14], Farber and Yuzvinsky study the invariants $\operatorname{tc}(X)$ and $\operatorname{zcl}(X)$ in the case when X is the complement of a (central, essential) hyperplane arrangement in $\mathbb{C}^\ell$. They show that

$\operatorname{tc}(X) \leqslant 2\ell$, and that this upper bound is attained for some classes of arrangements, including generic arrangements of sufficiently large cardinality and the reflection arrangements of types A, B, and D.

5.5. Topological complexity of the boundary manifold

Using Theorem 4.2 and Proposition 5.3, we see that the cup and zero-divisor lengths of the boundary manifold of an arrangement are determined in a simple fashion by the respective lengths of the complement.

Corollary 5.6. *Let $\mathcal{A}$ be an arrangement of hyperplanes in $\mathbb{CP}^\ell$, with complement X and boundary manifold M. Then*:

$$\operatorname{cl}(M) = \operatorname{cl}(X) + 1 \quad \textit{and} \quad \operatorname{zcl}(M) = \operatorname{zcl}(X) + 2.$$

Moreover, if $\mathcal{A}$ is essential, then $\operatorname{cl}(M) = \ell + 1$.

The relationship between the LS category and topological complexity of the boundary manifold on one hand, and the complement on the other hand, is more subtle, as the following example indicates.

Example 5.7. Let $\mathcal{A}$ be the Boolean arrangement $\mathbb{CP}^\ell$. Then $X \simeq T^\ell$ and $M = T^\ell \times S^{\ell-1}$. An easy computation shows that $\operatorname{cat}(M) = \operatorname{cat}(X) + 1 = \ell + 2$; on the other hand, $\operatorname{tc}(M) = \operatorname{tc}(X) + 2 = \ell + 3$ if ℓ is even, but $\operatorname{tc}(M) = \operatorname{tc}(X) + 3 = \ell + 4$ if ℓ is odd.

For projective line arrangements, we can narrow down the possible values of the category and topological complexity of the boundary manifold.

Proposition 5.8. *Let $\mathcal{A}$ be a line arrangement in $\mathbb{CP}^2$, with boundary manifold M. If $\mathcal{A}$ is not essential, then* $\operatorname{cat}(M) = 2$ *or* 3 *and* $\operatorname{tc}(M) = 4$*,* 5*, or* 6*. If $\mathcal{A}$ is essential, then* $\operatorname{cat}(M) = 4$ *and* $\operatorname{tc}(M) = 5$*,* 6*, or* 7*.*

Proof. As shown in [15], the LS category of a closed 3-manifold M depends only on $\pi_1(M)$: it is 2, 3, or 4, according to whether $\pi_1(M)$ is trivial, a nontrivial free group, or not a free group.

Suppose $\mathcal{A}$ is a pencil of $n+1$ lines. If $n = 0$, then $M = S^3$, so $\operatorname{cat}(M) = \operatorname{tc}(M) = 2$. If $n = 1$, then $M = S^1 \times S^2$, so $\operatorname{cat}(M) = 3$ and $\operatorname{tc}(M) = 4$. If $n > 1$, then $M = \#^n S^1 \times S^2$, and so $\operatorname{cat}(M) = 3$, and $\operatorname{tc}(M) = 5$.

On the other hand, if $\mathcal{A}$ is essential, then, as noted in Proposition 4.8, M is aspherical. In particular, $\operatorname{cd}(\pi_1(M)) = 3$, and so $\pi_1(M)$ cannot be free. Hence, $\operatorname{cat}(M) = 4$, and the bounds on $\operatorname{tc}(M)$ follow at once. $\square$

All the various possibilities listed in Proposition 5.8 do occur. For example, if $\mathcal{A}$ is a near-pencil of $n+1 \geqslant 4$ lines, then $M = \Sigma_{n-1} \times S^1$, and so $\operatorname{cat}(M) = 4$ and $\operatorname{tc}(M) = 6$. We summarize in Table 1 the possible values for the LS category and topological complexity

Table 1
Possible values of LS category and topological complexity for the complement X and boundary manifold M of a line arrangement in $\mathbb{CP}^2$

$\mathrm{cat}(X)$	1	2	2	3	3	3
$\mathrm{tc}(X)$	2	3	4	4	5	6
$\mathrm{cat}(M)$	2	3	3	4	4	4
$\mathrm{tc}(M)$	2	4	5	5	6	7
f	x_0	x_0x_1	$x_0^3 - x_1^3$	$x_0x_1x_2$	$x_0(x_1^3 - x_2^3)$	$(x_0^2 - x_1^2)(x_0^2 - x_2^2)(x_1^2 - x_2^2)$

of both the complement and the boundary manifold of an arrangement in $\mathbb{CP}^2$, together with sample representatives for the defining polynomials.

In high dimensions, a complete understanding of the possible values for $\mathrm{cat}(M)$ and $\mathrm{tc}(M)$ is not at hand. Nevertheless, we have the following class of arrangements (mentioned in Example 2.11), where precise formulas can be given.

Proposition 5.9. *Let $\mathcal{A}$ be the hyperplane arrangement in $\mathbb{CP}^{2k}$ defined by the polynomial $f = x_0 \prod_{i=1}^k (x_i^{n_i} - y_i^{n_i})$, with $n_i \geqslant 3$. If X is the complement and M is the boundary manifold, then*:

$$\mathrm{cat}(X) = 2k + 1, \qquad \mathrm{tc}(X) = 3k + 1,$$

$$\mathrm{cat}(M) = 2k + 2, \qquad \mathrm{tc}(M) = 3k + 3.$$

Proof. We have

$$X \simeq T^k \times \prod_{i=1}^{k} \bigvee^{n_i - 1} S^1,$$

while $M = T^k \times (\#^m T^k \times S^{2k-1})$, where $m = \prod_{i=1}^k (n_i - 1)$. A computation shows that $\mathrm{cl}(X) = 2k$ and $\mathrm{zcl}(X) = 3k$. Hence, by Corollary 5.6, $\mathrm{cl}(M) = 2k + 1$ and $\mathrm{zcl}(M) = 3k + 2$.

Let $W = T^k \times S^{2k-1}$. Note that $\mathrm{cl}(W) = k + 1$, while $\mathrm{cat}(W) \leqslant \mathrm{cat}(T^k) + \mathrm{cat}(S^{2k-1}) - 1 = k + 2$; hence $\mathrm{cat}(W) = k + 2$. In fact, if we consider W with its standard CW decomposition, we can take $W = \bigcup_{i=0}^{k+1} U_i$, with U_0 a small ball around the 0-cell e^0, U_i the union of the (open) i and $i + 2k - 1$ cells, for $1 \leqslant i \leqslant k$, and U_{k+1} the top cell e^{3k-1}; plainly, each U_i is contractible in W.

Now, $\#^m W$ is obtained by attaching a top cell to the wedge of m copies of $W \setminus U_{k+1}$ at the basepoint e^0; thus, we may find a decomposition $\#^m W = \bigcup_{i=0}^{k+1} V_i$ as before, with V_i contractible in $\#^m W$. It follows that $\mathrm{cat}(\#^m W) = k + 2$, and so $\mathrm{tc}(\#^m W) \leqslant 2k + 3$. Thus, $\mathrm{tc}(M) \leqslant \mathrm{tc}(T^k) + \mathrm{tc}(\#^m W) - 1 \leqslant 3k + 3$, and we are done. $\square$

As a consequence, we see that the difference between the topological complexity and the LS category of the boundary manifold of an arrangement can be arbitrarily large.

Corollary 5.10. *For each $k \geqslant 1$, there is an arrangement $\mathcal{A}$ with boundary manifold $M = M(\mathcal{A})$ for which* $\mathrm{tc}(M) - \mathrm{cat}(M) = k$.

6. Resonance

In this section, we study the resonance varieties of the trivial extension of a graded algebra. As an application, we obtain information about the structure of the resonance varieties of the boundary manifold of a hyperplane arrangement. Throughout, let $\Bbbk$ be an algebraically closed field of characteristic 0.

6.1. Resonance varieties

Let $A = \bigoplus_{k=0}^{\ell} A^k$ be a graded, graded-commutative, connected algebra over $\Bbbk$. Assume each graded piece A^k is finite-dimensional. For each $a \in A^1$, we have $a \cdot a = 0$; thus, multiplication by a defines a cochain complex

$$(A, a)\colon\ 0 \longrightarrow A^0 \xrightarrow{a} A^1 \xrightarrow{a} A^2 \xrightarrow{a} \cdots \xrightarrow{a} A^{\ell} \longrightarrow 0. \tag{6.1}$$

By definition, the *resonance varieties* of A are the jumping loci for the cohomology of these complexes:

$$\mathcal{R}^k_d(A) = \bigl\{a \in A^1 \mid \dim_{\Bbbk} H^k(A, a) \geqslant d\bigr\}, \tag{6.2}$$

for $0 \leqslant k \leqslant \ell$ and $0 \leqslant d \leqslant b_k = b_k(A)$. Notice that

$$A^1 = \mathcal{R}^k_0(A) \supset \mathcal{R}^k_1(A) \supset \cdots \supset \mathcal{R}^k_{b_k}(A) \supset \{0\}.$$

The sets $\mathcal{R}^k_d(A)$ are algebraic subvarieties of the affine space $A^1 = \Bbbk^n$, and are isomorphism-type invariants of the graded algebra A. They have been the subject of considerable recent interest, particularly in the context of hyperplane arrangements, see, for instance, [6,22, 34], and references therein.

An element $a \in A^1$ is said to be *nonresonant* if the dimensions of the cohomology groups $H^*(A, a)$ are minimal. If A is the Orlik–Solomon algebra of an arrangement of rank ℓ, and $a \in A^1$ is nonresonant, then $H^k(A, a) = 0$ for $k \neq \ell$, see, for instance, [34].

6.2. Resonance varieties of a doubled algebra

We now compare the resonance varieties of A to those of the doubled algebra $\mathsf{D}(A) = A \ltimes \bar{A}$, under the assumption that $\ell \geqslant 3$. Notice that for such ℓ, we have $\mathsf{D}(A)^1 = A^1$.

Theorem 6.3. *If A is a graded, connected $\Bbbk$-algebra and $\ell \geqslant 3$, then the resonance varieties of $\mathsf{D}(A)$ are given by*

$$\mathcal{R}_d^k\big(\mathsf{D}(A)\big) = \begin{cases} \mathcal{R}_d^k(A) & \text{if } k \leqslant \ell-2, \\ \bigcup_{p+q=d}\big(\mathcal{R}_p^{\ell-1}(A) \cap \mathcal{R}_q^{\ell}(A)\big) & \text{if } k=\ell-1 \text{ or } k=\ell, \\ \mathcal{R}_d^{2\ell-k-1}(A) & \text{if } k \geqslant \ell+1. \end{cases}$$

Proof. Fix a basis $\{a_i^p\}$ for A, and let $\{a_i^p, \bar{a}_j^q\}$ be the corresponding basis for the double $\mathsf{D}(A)$ as in Section 3.1. Let $m_k = m_k(a)$ and $\mathsf{D}(m_k) = \mathsf{D}(m_k(a))$ denote the matrices of the maps $A^{k-1} \to A^k$ and $\mathsf{D}(A)^{k-1} \to \mathsf{D}(A)^k$ given by multiplication by $a \in A^1 = \mathsf{D}(A)^1$ in the bases specified above. An exercise in linear algebra reveals that

$$\mathcal{R}_d^k(A) = \big\{a \in A^1 \mid \dim_{\Bbbk} A^k - \operatorname{rank} m_k - \operatorname{rank} m_{k+1} \geqslant d\big\}.$$

Similarly,

$$\mathcal{R}_d^k\big(\mathsf{D}(A)\big) = \big\{a \in \mathsf{D}(A)^1 \mid \dim_{\Bbbk} \mathsf{D}(A)^k - \operatorname{rank} \mathsf{D}(m_k) - \operatorname{rank} \mathsf{D}(m_{k+1}) \geqslant d\big\}.$$

The complex $(\mathsf{D}(A), a)$ has terms $\mathsf{D}(A)^k = A^k$ for $k \leqslant \ell-2$, $\mathsf{D}(A)^{\ell-1} = A^{\ell-1} \oplus \bar{A}^{\ell}$, $\mathsf{D}(A)^{\ell} = \bar{A}^{\ell-1} \oplus A^{\ell}$, and $\mathsf{D}(A)^k = \bar{A}^{2\ell-k-1}$ for $k \geqslant \ell+1$. Using the definition of the multiplication in $\mathsf{D}(A)$, one can check that the boundary maps of this complex have matrices $\mathsf{D}(m_k) = m_k$ for $k \leqslant \ell-2$,

$$\mathsf{D}(m_{\ell-1}) = [m_{\ell-1} \quad 0], \qquad \mathsf{D}(m_\ell) = \begin{bmatrix} 0 & m_\ell \\ \pm m_\ell^\top & 0 \end{bmatrix}, \qquad \mathsf{D}(m_{\ell+1}) = \begin{bmatrix} \pm m_{\ell-1}^\top \\ 0 \end{bmatrix},$$

and $\mathsf{D}(m_k) = \pm m_{2\ell-k}^\top$ for $k \geqslant \ell+2$. Calculating ranks of these matrices, and using the above descriptions of the resonance varieties $\mathcal{R}_d^k(A)$ and $\mathcal{R}_d^k(\mathsf{D}(A))$ yields the result. □

If $\ell = 2$, then $\mathsf{D}(A)^1 = A^1 \oplus \bar{A}^2$. If $(a,b) \cdot (a,b) = 0$ for all $(a,b) \in \mathsf{D}(A)^1$, then $(\mathsf{D}(A), (a,b))$ is a cochain complex for each (a,b) as in (6.1), and the resonance varieties of $\mathsf{D}(A)$ are

$$\mathcal{R}_d^k\big(\mathsf{D}(A)\big) = \big\{(a,b) \in \mathsf{D}(A)^1 \mid \dim_{\Bbbk} H^k\big(\mathsf{D}(A), (a,b)\big) \geqslant d\big\}.$$

In this situation, the boundary maps of the complex $(\mathsf{D}(A), (a,b))$ have matrices

$$\mathsf{D}(m_1) = [m_1 \quad \bar{m}_1], \qquad \mathsf{D}(m_2) = \begin{bmatrix} \phi & m_2 \\ -m_2^\top & 0 \end{bmatrix}, \qquad \mathsf{D}(m_3) = \begin{bmatrix} m_1^\top \\ \bar{m}_1^\top \end{bmatrix},$$

where, as above, $m_k = m_k(a)$ is the matrix of multiplication by a, $A^{k-1} \to A^k$, $\bar{m}_1 = \bar{m}_1(b)$ is the matrix of multiplication by b, $\bar{A}^2 \to \bar{A}^1$, and $\phi = \phi(b)$ is the matrix of multiplication by b, $A^1 \to \bar{A}^1$. Since A and $\mathsf{D}(A)$ are graded commutative, the matrix ϕ is skew-symmetric. The structure of these matrices, $\mathsf{D}(m_3) = \mathsf{D}(m_1)^\top$ and $\mathsf{D}(m_2)^\top = -\mathsf{D}(m_2)$, follows from the multiplication in $\mathsf{D}(A)$, see (3.1).

6.4. Aomoto complexes

The complex (6.1) may be realized as the specialization at a of the Aomoto complex of the algebra A. Let $R_A = \mathrm{Sym}(A_1)$ be the symmetric algebra on the $\Bbbk$-dual of A^1, and let $\mathbf{x} = \{x_1, \dots, x_n\}$ be the basis for A_1 dual to the basis $\{a_1^1, \dots, a_n^1\}$ for A^1. Then R_A becomes identified with the polynomial ring $R = \Bbbk[\mathbf{x}]$. The Aomoto complex of A is the cochain complex

$$A^0 \otimes_{\Bbbk} R \xrightarrow{d^1} A^1 \otimes_{\Bbbk} R \xrightarrow{d^2} A^2 \otimes_{\Bbbk} R \xrightarrow{d^3} \cdots \xrightarrow{d^\ell} A^\ell \otimes_{\Bbbk} R \tag{6.3}$$

where the boundary maps are multiplication by:

$$\sum_{j=1}^{n} a_j^1 \otimes x_j.$$

Notice that the multiplication map $\mu : A^1 \otimes A^{p-1} \to A^p$ can be recovered from the boundary map d^p. Denote the matrix of d^1 by $d_{\mathbf{x}}$, and that of d^2 by $\Delta = \Delta_A$. If the multiplication $A^1 \otimes A^1 \to A^2$ is given by

$$a_i^1 a_j^1 = \sum_{k=1}^{m} \mu_{i,j,k} a_k^2,$$

the latter is an $n \times m$ matrix of linear forms over R, with entries

$$\Delta_{j,k} = \sum_{i=1}^{n} \mu_{i,j,k} x_i. \tag{6.4}$$

The (transpose of the) matrix Δ_A is the (linearized) *Alexander matrix* of the algebra A, which appears in various guises in, for instance, [5,6,22,27].

The Aomoto complex of the double $\mathsf{D}(A)$ may be constructed analogously. In light of Theorem 6.3, we focus on the case $\ell = 2$. Here, $\mathsf{D}(A)^1 = A^1 \oplus \bar{A}^2$, with basis $\{a_i^1, \bar{a}_j^2\}$, where $1 \leqslant i \leqslant n$ and say $1 \leqslant j \leqslant m$. Identify the ring $R_{\mathsf{D}(A)} = \mathrm{Sym}(A_1 \oplus \bar{A}_2)$ with the polynomial ring $S = \Bbbk[\mathbf{x}, \mathbf{y}]$. Then, the Aomoto complex of $\mathsf{D}(A)$ is the complex

$$\mathsf{D}(A)^0 \otimes_{\Bbbk} S \xrightarrow{D^1} \mathsf{D}(A)^1 \otimes_{\Bbbk} S \xrightarrow{D^2} \mathsf{D}(A)^2 \otimes_{\Bbbk} S \xrightarrow{D^3} \mathsf{D}(A)^3 \otimes_{\Bbbk} S \tag{6.5}$$

where the boundary maps are multiplication by

$$\sum_{i=1}^{n} a_i^1 \otimes x_i + \sum_{j=1}^{m} \bar{a}_j^2 \otimes y_j.$$

Denote the matrix of D^1 by $(d_{\mathbf{x}}\ d_{\mathbf{y}})$, and that of D^2 by $\Delta_{\mathsf{D}(A)}$. Then it follows from (3.1) that the matrix of D^3 is $(d_{\mathbf{x}}\ d_{\mathbf{y}})^\top$, and that

$$\Delta_{\mathsf{D}(A)} = \begin{pmatrix} \Phi & \Delta_A \\ -\Delta_A^\top & 0 \end{pmatrix}, \tag{6.6}$$

where Φ is the $n \times n$ matrix with entries $\Phi_{i,j} = \sum_{k=1}^m \mu_{i,j,k} y_k$. Notice that $\Delta_{\mathsf{D}(A)}$ is a skew-symmetric matrix of linear forms, and that $d_{\mathbf{x}}\Phi = d_{\mathbf{y}}\Delta_A^\top$.

If $A = \bigoplus_{k=0}^{\ell} A^k$ and $\ell \geqslant 3$, the relationship between the Aomoto complexes of A and $\mathsf{D}(A)$ is implicit in the proof of Theorem 6.3. We relate these complexes in the case $\ell = 2$. Consider the Aomoto complex of A and its dual,

$$A^0 \otimes_{\Bbbk} S \xrightarrow{d_{\mathbf{x}}} A^1 \otimes_{\Bbbk} S \xrightarrow{\Delta_A} A^2 \otimes_{\Bbbk} S \quad \text{and}$$

$$\bar{A}^2 \otimes_{\Bbbk} S \xrightarrow{-\Delta_A^\top} \bar{A}^1 \otimes_{\Bbbk} S \xrightarrow{-d_{\mathbf{x}}^\top} \bar{A}^0 \otimes_{\Bbbk} S,$$

where we have extended scalars and changed the signs for reasons which will become apparent.

Lemma 6.5. *The maps $\{d_{\mathbf{y}}, -\Phi, d_{\mathbf{y}}^\top\}$ provide a chain map*

$$\begin{tikzcd} A^0 \otimes_{\Bbbk} S \arrow[r, "d_{\mathbf{x}}"] \arrow[d, "d_{\mathbf{y}}"] & A^1 \otimes_{\Bbbk} S \arrow[r, "\Delta_A"] \arrow[d, "-\Phi"] & A^2 \otimes_{\Bbbk} S \arrow[d, "d_{\mathbf{y}}^\top"] \\ \bar{A}^2 \otimes_{\Bbbk} S \arrow[r, "-\Delta_A^\top"] & \bar{A}^1 \otimes_{\Bbbk} S \arrow[r, "-d_{\mathbf{x}}^\top"] & \bar{A}^0 \otimes_{\Bbbk} S. \end{tikzcd}$$

Furthermore, the Aomoto complex of $\mathsf{D}(A)$ *is the mapping cone of this chain map.*

An alternate way to compute the resonance varieties $\mathcal{R}_d^1(A)$ is by taking the zero locus of the determinantal ideals of the linearized Alexander matrix of A. If Ψ is a $p \times q$ matrix ($p \leqslant q$) with polynomial entries, define $\mathcal{R}_d(\Psi) = V(E_{p-d}(\Psi))$ where $E_r(\Psi)$ is the ideal of $r \times r$ minors. Proceeding as in the proof of Theorem 3.9 from [22] (see also [6]), we find that $\mathcal{R}_d^1(A) = \mathcal{R}_d(\Delta_A)$. Similarly, $\mathcal{R}_d^1(\mathsf{D}(A)) = \mathcal{R}_d(\Delta_{\mathsf{D}(A)})$.

6.6. Resonance of arrangements

For a space X with the homotopy type of a finite CW-complex, define the resonance varieties of X by $\mathcal{R}_d^k(X) = \mathcal{R}_d^k(H^*(X;\Bbbk))$.

Let $\mathcal{A}$ be an arrangement of hyperplanes, with complement X, boundary manifold M, and Orlik–Solomon algebra $A = H^*(X;\Bbbk)$. If $\mathcal{A}$ is an arrangement in $\mathbb{CP}^\ell$ with $\ell \geqslant 3$, then it follows from Theorem 6.3 that the resonance varieties of the complement, $\mathcal{R}_d^k(X) =$

$\mathcal{R}^k_d(A)$, completely determine the resonance varieties, $\mathcal{R}^k_d(M) = \mathcal{R}^k_d(\mathsf{D}(A))$, of the boundary manifold. So assume that $\mathcal{A} \subset \mathbb{CP}^2$ is a line arrangement.

The complex (A, a) of (6.1) may be realized as the specialization $A^\bullet \otimes_{\Bbbk} R|_{\mathbf{x} \mapsto a}$ of the Aomoto complex of A. Since $\mathsf{D}(A)^1 = A^1 \oplus \bar{A}^2$, the resonance varieties of the boundary manifold are given by

$$\mathcal{R}^k_d(M) = \mathcal{R}^k_d\big(\mathsf{D}(A)\big) = \big\{(a, b) \in A^1 \oplus \bar{A}^2 \mid \dim H^k\big(\mathsf{D}(A), (a, b)\big) \geqslant d\big\}.$$

Similarly, the complex $(\mathsf{D}(A), (a, b))$ may be realized as the specialization $\mathsf{D}(A)^\bullet \otimes_{\Bbbk} S|_{(\mathbf{x},\mathbf{y}) \mapsto (a,b)}$ of the Aomoto complex of $\mathsf{D}(A)$. The properties of the boundary maps of the complex (6.5) noted above imply that the resonance varieties of M satisfy $\mathcal{R}^k_d(M) = \mathcal{R}^{3-k}_d(M)$.

Recall that, for nonresonant $a \in A^1$, we have $H^k(A, a) = 0$ for $k = 0, 1$. Write $b_k = b_k(A) = \dim_{\Bbbk} A^k$, and $\beta = 1 - b_1 + b_2$. Note that $\beta = \dim_{\Bbbk} H^2(A, a)$.

Proposition 6.7. *Let $\mathcal{A} \subset \mathbb{CP}^2$ be a line arrangement with Orlik–Solomon algebra A and double $\mathsf{D}(A)$.*

(1) *If $a \in A^1$ is nonresonant for A, then for any b, $(a, b) \in \mathsf{D}(A)^1$ is nonresonant for $\mathsf{D}(A)$. Furthermore, $H^0(\mathsf{D}(A), (a, b)) = H^3(\mathsf{D}(A), (a, b)) = 0$ and $H^1(\mathsf{D}(A), (a, b)) = H^2(\mathsf{D}(A), (a, b)) = \Bbbk^\beta$.*
(2) *If $a \in \mathcal{R}^1_d(A)$ is nonzero, then for any b, $(a, b) \in \mathcal{R}^1_{d+\beta}(\mathsf{D}(A))$.*
(3) *If $b \neq 0$, then $(0, b) \in \mathcal{R}^1_d(\mathsf{D}(A))$, where $d = b_2 - 1 + \dim_{\Bbbk}(\ker \Phi|_{\mathbf{y} \mapsto b})$.*

Proof. Given $(a, b) \in \mathsf{D}(A)^1$, by Lemma 6.5, there is a corresponding short exact sequence of complexes $0 \to (\bar{A}^\#, a)^{-1} \to (\mathsf{D}(A), (a, b)) \to (A, a) \to 0$:

$$\begin{tikzcd}
 & \bar{A}^2 \arrow[r, "-\Delta_A^\top"] \arrow[d] & \bar{A}^1 \arrow[r, "-d_{\mathbf{x}}"] \arrow[d] & \bar{A}^0 \arrow[d] \\
\mathsf{D}(A)^0 \arrow[r, "(d_{\mathbf{x}}\ d_{\mathbf{y}})"] \arrow[d] & \mathsf{D}(A)^1 \arrow[r, "\left(\begin{smallmatrix} \Phi & \Delta_A \\ -\Delta_A^\top & 0 \end{smallmatrix}\right)"] \arrow[d] & \mathsf{D}(A)^2 \arrow[r, "\left(\begin{smallmatrix} d_{\mathbf{x}}^\top \\ d_{\mathbf{y}}^\top \end{smallmatrix}\right)"] \arrow[d] & \mathsf{D}(A)^3 \\
A^0 \arrow[r, "d_{\mathbf{x}}"] & A^1 \arrow[r, "\Delta_A"] & A^2 &
\end{tikzcd}$$

where $(\bar{A}^\#, a)$ denotes the specialization at a of the dual of the Aomoto complex of A. Passing to cohomology yields a long exact sequence

$$0 \to H^0(\mathsf{D}(A)) \to H^0(A) \to H^0(\bar{A}^\#) \to H^1(\mathsf{D}(A)) \to H^1(A)$$
$$\to H^1(\bar{A}^\#) \to H^2(\mathsf{D}(A)) \to H^2(A) \to H^2(\bar{A}^\#) \to H^3(\mathsf{D}(A)) \to 0 \qquad (6.7)$$

where, for instance, $H^k(A) = H^k(A, a)$. Using the fact that $H^k(\bar{A}^\#)$ is isomorphic to $H^{2-k}(A)$, calculations with this long exact sequence may be used to establish all three assertions. □

As a consequence of Proposition 6.7, we obtain the following.

Corollary 6.8. *The resonance varieties of the doubled algebra* $\mathsf{D}(A)$ *satisfy*:

(1) $\mathcal{R}^1_d(\mathsf{D}(A)) = \mathsf{D}^1(A)$ *for* $d \leqslant \beta$.
(2) $\mathcal{R}^1_d(A) \times A^2 \subseteq \mathcal{R}^1_{d+\beta}(\mathsf{D}(A))$.
(3) $\{0\} \times \mathcal{R}_d(\Phi) \subseteq \mathcal{R}^1_{d+b_2(A)}(\mathsf{D}(A))$.

All irreducible components of $\mathcal{R}^1_d(X) = \mathcal{R}^1_d(A)$ are linear, see [6]. From items (1) and (2) in the above corollary, it is clear that the resonance varieties of M contain linear components as well. However, item (3) leaves open the possibility that $\mathcal{R}^1_d(M) = \mathcal{R}^1_d(\mathsf{D}(A))$ contains nonlinear components, for $d \geqslant b_2(A)$. This does indeed occur, as shown next.

6.9. General position arrangements

Let $\mathcal{A}_n$ be a projective line arrangement consisting of $n+1$ lines in general position. We identify the resonance varieties of the boundary manifold $M^3_n = M(\mathcal{A}_n)$.

The Orlik–Solomon algebra, $A = E/\mathfrak{m}^3$, is the rank 2 truncation of the exterior algebra generated by $e_1, \dots, e_n$, where $\mathfrak{m} = (e_1, \dots, e_n)$. Note that A has Betti numbers $b_1 = n$, $b_2 = \binom{n}{2}$, and that $\beta = 1 - b_1 + b_2 = \binom{n-1}{2}$.

For this arrangement, the Alexander matrix Δ_A is the transpose of the matrix of the Koszul differential $\delta_2 : E^2 \otimes S \to E^1 \otimes S$. The submatrix Φ of the Alexander matrix $\Delta_{\mathsf{D}(A)}$ recorded in (6.6) is the generic $n \times n$ skew-symmetric matrix of linear forms Φ_n, with entries $(\Phi_n)_{i,j} = y_{i,j}$ above the diagonal.

Identity $\mathsf{D}(A)^1 = E^1 \times E^2$. Note that $\mathcal{R}^1_d(A) = \mathcal{R}^1_d(E) = \{0\}$ for $d > 0$. An analysis of the long exact sequence (6.7) in light of this observation yields the following sharpening of Corollary 6.8 for general position arrangements.

Proposition 6.10. *The resonance varieties of the boundary manifold* M_n *of a general position arrangement of* $n+1$ *lines in* $\mathbb{CP}^2$ *are given by*:

$$\mathcal{R}^1_d(M_n) = \begin{cases} E^1 \times E^2 & \text{if } d \leqslant \binom{n-1}{2}, \\ \{0\} \times E^2 & \text{if } \binom{n-1}{2} < d < \binom{n}{2}, \\ \{0\} \times \mathcal{R}_{d-\binom{n}{2}}(\Phi_n) & \text{if } \binom{n}{2} \leqslant d < \binom{n}{2} + n, \\ \{0\} \times \{0\} & \text{if } d = \binom{n}{2} + n. \end{cases}$$

If Ψ is a skew-symmetric matrix of size n with polynomial entries, define the *Pfaffian varieties* of Ψ by

$$\mathcal{P}_d(\Psi) = V\bigl(\mathrm{Pf}_{2(\lfloor n/2\rfloor - d)}(\Psi)\bigr), \tag{6.8}$$

where $\mathrm{Pf}_{2r}(\Psi)$ is the ideal of $2r \times 2r$ Pfaffians of Ψ. For n even, the ideal $\mathrm{Pf}_n(\Psi)$ is principal, generated by $\mathrm{Pfaff}(\Psi)$, the maximal Pfaffian of Ψ. Well-known properties of Pfaffians (see, for instance, [3, Corollary 2.6]) may be used to establish the following relationship between the resonance and Pfaffian varieties of Ψ:

$$V\bigl(E_{2r-1}(\Psi)\bigr) = V\bigl(E_{2r}(\Psi)\bigr) = V\bigl(\mathrm{Pf}_{2r}(\Psi)\bigr). \tag{6.9}$$

In other words, for n even, we have $\mathcal{R}^1_{2d+1}(\Psi) = \mathcal{R}^1_{2d}(\Psi) = \mathcal{P}_d(\Psi)$, while for n odd, we have $\mathcal{R}^1_{2d}(\Psi) = \mathcal{R}^1_{2d-1}(\Psi) = \mathcal{P}_{d-1}(\Psi)$.

For $n = 2k$, the Pfaffian of the generic skew-symmetric matrix Φ_n is given by

$$\mathrm{Pfaff}(\Phi_n) = \sum_m \sigma(m)\omega(m), \tag{6.10}$$

with the sum over all perfect matchings $m = \{\{i_1, j_1\}, \{i_2, j_2\}, \ldots, \{i_k, j_k\}\}$ (partitions of $[2k]$ into blocks of size two with $i_p < j_p$), and where $\sigma(m)$ is the sign of the corresponding permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2k-1 & 2k \\ i_1 & j_1 & i_2 & j_2 & \ldots & i_k & j_k \end{pmatrix},$$

and $\omega(m) = y_{i_1,j_1} y_{i_2,j_2} \cdots y_{i_k,j_k}$, see, for instance, [2]. Note that $\mathrm{Pfaff}(\Phi_n)$ is a polynomial of degree $k = n/2$ in the variables $y_{i,j}$.

For arbitrary n, it is known that the Pfaffian variety $\mathcal{P}_d(\Phi_n)$ is irreducible, with singular locus $\mathcal{P}_{d+1}(\Phi_n)$, see [3,20]. These facts, together with Proposition 6.10 and (6.9), yield the following.

Corollary 6.11. *Let M_n be the boundary manifold of the general position arrangement $\mathcal{A}_n$. For $n \geqslant 4$ and $\binom{n}{2} < d < \binom{n}{2} + n - 2$, the resonance variety $\mathcal{R}^1_d(M_n)$ is a singular, irreducible variety.*

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References

[1] K. Aaslepp, M. Drawe, C. Hayat-Legrand, C. Sczesny, H. Zieschang, On the cohomology of Seifert and graph manifolds, Topology Appl. 127 (1–2) (2003) 3–32. MR 2003j:57038.

[2] R. Brualdi, H. Ryser, Combinatorial Matrix Theory, Encyclopedia Math. Appl., vol. 39, Cambridge Univ. Press, Cambridge, 1991. MR 93a:05087.

[3] D. Buchsbaum, D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math. 99 (3) (1977) 447–485. MR 0453723.

[4] J. Cogolludo, Topological invariants of the complement to arrangements of rational plane curves, Mem. Amer. Math. Soc. 159 (756) (2002). MR 2003k:32037.

[5] D. Cohen, A. Suciu, Alexander invariants of complex hyperplane arrangements, Trans. Amer. Math. Soc. 351 (10) (1999) 4043–4067. MR 99m:52019.

[6] D. Cohen, A. Suciu, Characteristic varieties of arrangements, Math. Proc. Cambridge Philos. Soc. 127 (1) (1999) 33–53. MR 2000m:32036.

[7] P. Deligne, Théorie de Hodge II, III, Inst. Hautes Études Sci. Publ. Math. 40 (1972) 5–57, Inst. Hautes Études Sci. Publ. Math. 44 (1974) 5–77. MR 0498551, MR 0498552.

[8] A. Dimca, Singularities and Topology of Hypersurfaces, Universitext, Springer, New York, 1992. MR 94b:32058.

[9] A. Dimca, A. Némethi, Hypersurface complements, Alexander modules and monodromy, in: Real and Complex Singularities, in: Contemp. Math., vol. 354, Amer. Math. Soc., Providence, RI, 2004, pp. 19–43. MR 2005h:32076.

[10] A. Durfee, Neighborhoods of algebraic sets, Trans. Amer. Math. Soc. 276 (2) (1983) 517–530. MR 84m:32013.

[11] A. Durfee, A naive guide to mixed Hodge theory, in: Singularities, Part 1, Arcata, CA, 1981, in: Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 313–320. MR 85b:32044.

[12] A. Durfee, R. Hain, Mixed Hodge structures on the homotopy of links, Math. Ann. 280 (1) (1988) 69–83. MR 89c:14012.

[13] M. Farber, Topological complexity of motion planning, Discrete Comput. Geom. 29 (2) (2003) 211–221. MR 2004c:68132.

[14] M. Farber, S. Yuzvinsky, Topological robotics: Subspace arrangements and collision free motion planning, in: Geometry, Topology, and Mathematical Physics, in: Amer. Math. Soc. Transl. Ser. 2, vol. 212, Amer. Math. Soc., Providence, RI, 2004, pp. 145–156. MR 2005i:55019.

[15] J.C. Gómez-Larrañaga, F. González-Acuña, Lusternik–Schnirelmann category of 3-manifolds, Topology 31 (4) (1992) 791–800. MR 93j:55005.

[16] E. Hironaka, Boundary manifolds of line arrangements, Math. Ann. 319 (1) (2001) 17–32. MR 2001m:32061.

[17] I.M. James, On category, in the sense of Lusternik–Schnirelmann, Topology 17 (4) (1978) 331–348. MR 80i:55001.

[18] T. Jiang, S.S.-T. Yau, Intersection lattices and topological structures of complements of arrangements in $\mathbb{CP}^2$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 26 (2) (1998) 357–381. MR 99g:52020.

[19] M. Kim, Weights in cohomology groups arising from hyperplane arrangements, Proc. Amer. Math. Soc. 120 (3) (1994) 697–703. MR 94e:14027.

[20] H. Kleppe, D. Laksov, The algebraic structure and deformation of Pfaffian schemes, J. Algebra 64 (1) (1980) 167–189. MR 82b:14027.

[21] P. Lambrechts, Cochain model for thickenings and its application to rational LS-category, Manuscripta Math. 103 (2) (2000) 143–160. MR 2001m:55009.

[22] D. Matei, A. Suciu, Cohomology rings and nilpotent quotients of real and complex arrangements, in: Arrangements–Tokyo 1998, in: Adv. Stud. Pure Math., vol. 27, Math. Soc. Japan, Tokyo, 2000, pp. 185–215. MR 2002b:32045.

[23] J.W. Milnor, Singular Points of Complex Hypersurfaces, Ann. of Math. Stud., vol. 61, Princeton Univ. Press, Princeton, NJ, 1968. MR 0239612.

[24] M. Nagata, Local Rings, Interscience Tracts in Pure and Appl. Math., vol. 13, Wiley, New York, 1962. MR 0155856.

[25] P. Orlik, L. Solomon, Combinatorics and topology of complements of hyperplanes, Invent. Math. 56 (2) (1980) 167–189. MR 81e:32015.

[26] P. Orlik, H. Terao, Arrangements of Hyperplanes, Grundlehren Math. Wiss., vol. 300, Springer, Berlin, 1992. MR 94e:52014.

[27] S. Papadima, A. Suciu, Chen Lie algebras, Int. Math. Res. Not. 2004 (21) (2004) 1057–1086. MR 2004m:17043.

[28] C. Peters, J. Steenbrink, Mixed Hodge Theory, book in preparation.

[29] I. Reiten, The converse to a theorem of Sharp on Gorenstein modules, Proc. Amer. Math. Soc. 32 (1972) 417–420. MR 0296067.

[30] V. Schroeder, Rigidity of nonpositively curved graphmanifolds, Math. Ann. 274 (1) (1986) 19–26. MR 87h:53054.

[31] B.Z. Shapiro, The mixed Hodge structure of the complement to an arbitrary arrangement of affine complex hyperplanes is pure, Proc. Amer. Math. Soc. 117 (4) (1993) 931–933. MR 93e:32045.

[32] C.T.C. Wall, Classification problems in differential topology. IV. Thickenings, Topology 5 (1966) 73–94. MR 0192509.

[33] E. Westlund, The boundary manifold of an arrangement, PhD thesis, University of Wisconsin, Madison, WI, 1997.

[34] S. Yuzvinsky, Orlik–Solomon algebras in algebra and topology, Russian Math. Surveys 56 (2) (2001) 293–364. MR 2002i:14047.