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Toughness and the existence of fractional k-factors of graphs☆

Guizhen Liu[a,*], Lanju Zhang[b]

[a] School of Mathematics and System Science, Shandong University, Jinan 250100, PR China
[b] Biostatistics and Data Management, Medimmune Inc., Gaithersburg, MD 20878, USA

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Abstract

The toughness of a graph G, $t(G)$, is defined as $t(G) = \min\{|S|/\omega(G-S) | S \subseteq V(G), \omega(G-S) > 1\}$ where $\omega(G-S)$ denotes the number of components of $G-S$ or $t(G) = +\infty$ if G is a complete graph. Much work has been contributed to the relations between toughness and the existence of factors of a graph. In this paper, we consider the relationship between the toughness and the existence of fractional k-factors. It is proved that a graph G has a fractional 1-factor if $t(G) \geqslant 1$ and has a fractional k-factor if $t(G) \geqslant k - 1/k$ where $k \geqslant 2$. Furthermore, we show that both results are best possible in some sense.

MSC: 05C70

Keywords: Toughness; Fractional k-factor; Fractional matching

1. Introduction

The graphs considered here will be finite undirected graphs which may have multiple edges but no loops. We refer the readers to [2] for the terminologies not defined here. Let G be a graph. We use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively. We use $G[S]$ and $G-S$ to denote the subgraph of G induced by S and $V(G)-S$, respectively, for $S \subseteq V(G)$ and $N_G(x)$ to denote the set of vertices adjacent to x in G. A subset S of $V(G)$ is called a covering set (an independent set) of G if every edge of G is incident with at least (at most) one vertex of S. Let S and T be two disjoint subsets of $V(G)$, we use $E(S,T)$ to denote the set of edges with one end in S and the other end in T and set $S - S' = S \backslash S'$.

Let g and f be two integer-valued functions defined on $V(G)$ with $g(x) \leqslant f(x)$ for any $x \in V(G)$. A subgraph F of G is called a (g,f)-factor if $g(x) \leqslant d_F(x) \leqslant f(x)$ holds for any vertex $x \in V(G)$. A (g,f)-factor is called an $[a,b]$-factor if $g(x) \equiv a$ and $f(x) \equiv b$. An $[a,b]$-factor is called a k-factor if $a = b = k$. Let $h : E(G) \to [0,1]$ be a function. Let $k \geqslant 1$ be an integer. If $\sum_{e \ni x} h(e) = k$ holds for any vertex $x \in V(G)$, we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) | h(e) > 0\}$. A fractional 1-factor is also called a fractional perfect matching [7].

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* Corresponding author. Tel.: +86 531 88363949; fax: +86 531 88364650.
E-mail address: gzliu@sdu.edu.cn (G. Liu).

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A graph is t-tough if for any $S \subseteq V(G)$ and $\omega(G-S)>1$, we have

$$|S| \geqslant t\omega(G-S)$$

holds where $\omega(G-S)$ denotes the number of components of $(G-S)$. A complete graph is t-tough for any positive real number t. If G is not complete, there exists the largest t such that G is t-tough. This number is denoted by $t(G)$ and is called the toughness of G. We define $t(K_n)=+\infty$. If G is not complete,

$$t(G)=\min\left\{\frac{|S|}{\omega(G-S)}\,\middle|\, S\subseteq V(G) \text{ and } \omega(G-S)\geqslant 2\right\}.$$

The toughness of a graph was first introduced by Chvátal in [3]. Since then, much work has been contributed to the relations between toughness and the existence of factors of a graph. The most famous result is that of [4] which confirms a conjecture stated by Chvátal. Its main result is the following Lemma.

Lemma 1.1. *Let G be a graph. If G is k-tough, $|V(G)|\geqslant k+1$ and $k|V(G)|$ is even, then G has a k-factor.*

The result is sharp since for any positive real number ε, there exists a graph G that has no k-factor with $t(G)\geqslant k-\varepsilon$ [4]. Katerinis considered toughness and the existence of $[a,b]$-factors in [5]. In this paper we discuss the relationship between toughness and the existence of fractional k-factors. In [1] Anstee gave a necessary and sufficient condition for a graph to have a fractional (g, f)-factor for which we gave a new proof. The following result can be found in [6].

Lemma 1.2. *Let $k\geqslant 1$ be an integer. A graph G has a fractional k-factor if and only if for any subset S of $V(G)$,*

$$k|T|-d_{G-S}(T)\leqslant k|S|, \tag{1}$$

where $T=\{x\in V(G)-S|d_{G-S}(x)\leqslant k-1\}$ and $d_{G-S}(T)=\sum_{x\in T} d_{G-S}(x)$.

In particular, for $k=1$, we have the following result.

Lemma 1.3 (*Scheinerman and Ullman [7]*). *A graph G has a fractional perfect matching if and only if for any $S\subseteq V(G)$,*

$$i(G-S)\leqslant |S|, \tag{2}$$

where $i(G-S)=|\{x\in V(G)-S|d_{G-S}(x)=0\}|$.

Our main results are the following two theorems.

Theorem 1.1. *Let G be a connected graph with $|V(G)|\geqslant 2$. Then G has a fractional perfect matching if $t(G)\geqslant 1$.*

Theorem 1.2. *Let $k\geqslant 2$ be an integer. A graph G with $|V(G)|\geqslant (k+1)$ has a fractional k-factor if $t(G)\geqslant k-1/k$.*

The result in Theorem 1.1 is sharp. To see this, consider the graph $G_1=K_m\vee(m+1)K_1$ where $\vee$ means "join" and m is an arbitrary positive integer. It is easy to find out that $t(G_1)=m/(m+1)<1$ and (1) does not hold if we let $S=V(K_m)$. By Lemma 1.3 G_1 has no fractional perfect matching. But $t(G_1)\to 1$ when $m\to+\infty$.

To see Theorem 1.2 is also sharp, we construct the following graph G_k: $V(G_k)=A\cup B\cup C$ where A, B and C are disjoint with $|A|=|B|=(nk+1)(k-1)$, and $|C|=n(k-1)$. Both A and C are cliques in G_k, while B is isomorphic to $(nk+1)K_{k-1}$. Other edges in G_k are a perfect matching between A and B and all the pairs between B and C. If $k=2$, let $S=(A-\{u\})\cup C$ where $u\in A$, then $|S|=3n$ and $\omega(G-S)=2n+1$; if $k\geqslant 3$, let $S=(A-\{u\})\cup\{v\}\cup C$ where $u\in A$ and $v\in B$ is matched to u in G_k. Then $|S|=(nk+n+1)(k-1)$ and $\omega(G-S)=nk+2$. This follows that

$$t(G_k)=\begin{cases}\dfrac{3n}{2n+1} & \text{if } k=2,\\[2mm] \dfrac{(nk+n+1)(k-1)}{nk+2} & \text{if } k\geqslant 3.\end{cases}$$

But (1) does not hold if we let $S=C$. Thus by Lemma 1.2 G_k has no fractional k-factor. It is easy to see that $t(G_k)$ can be made arbitrarily close to $k-1/k$ when n is large enough. In this sense, the result in Theorem 1.2 is also sharp.

Remark. A graph G that satisfies the condition of Theorem 1.1 has 1-factors if $|V(G)|$ is even by Lemma 1.1. However, a graph G that satisfies the conditions of Theorem 1.2 does not necessarily have a k-factor even if $k|V(G)|$ is even [4].

2. Proofs of theorems

At first let us prove Theorem 1.1.

Proof of Theorem 1.1. If G is complete, obviously G has a fractional perfect matching as $|V(G)|\geqslant 2$. In the following we assume that G is not complete. Suppose that G satisfies the conditions in Theorem 1.1, but G has no fractional perfect matching. From Lemma 1.3, there exists a subset S of $V(G)$ such that

$$i(G-S)>|S|.$$

Since G is connected, $S\neq\emptyset$. Thus $i(G-S)\geqslant 2$. Then

$$t(G)\leqslant\frac{|S|}{\omega(G-S)}\leqslant\frac{|S|}{i(G-S)}<1$$

contradicting to $t(G)\geqslant 1$. $\square$

To prove Theorem 1.2, we need the following Lemmas.

Lemma 2.1 (*Chvátal [3]*). *If a graph G is not complete, then* $t(G)\leqslant\frac{1}{2}\delta(G)$.

Lemma 2.2. *Let G be a graph and let $H=G[T]$ such that $d_G(x)=k-1$ for every $x\in V(H)$ and no component of H is isomorphic to K_k where $T\subseteq V(G)$ and $k\geqslant 2$. Then there exists an independent set I and the covering set $C=V(H)-I$ of H satisfying*

$$|V(H)|\leqslant\left(k-\frac{1}{k+1}\right)|I| \tag{3}$$

and

$$|C|\leqslant\left(k-1-\frac{1}{k+1}\right)|I|. \tag{4}$$

Proof. Suppose that H has m components. For each component H_n, let I_n be a maximum independent set of H_n. First we claim that for each vertex $x\in I_n$ and $d_{H_n}(x)=k-1$, there exists a vertex $y\in I_n-\{x\}$ such that $N_{H_n}(x)\cap N_{H_n}(y)\neq\emptyset$. For this, we show that $H_n[N_{H_n}(x)]$ is not complete. Otherwise, $H_n'=H_n[\{x\}\cup N_{H_n}(x)]$ is isomorphic to K_k. Since H_n is connected and for every vertex $x\in V(H_n)$, $d_{H_n}(x)\leqslant k-1$, it follows that $H_n=H_n'$, which contradicts to that H_n is not isomorphic to K_k. Now if for any $y\in I_n-\{x\}$, $N_{H_n}(x)\cap N_{H_n}(y)=\emptyset$, then $E(\{x\}\cup N_{H_n}(x),\ I_n-\{x\})=\emptyset$. Let x' and y' be two vertices in $H_n[N_{H_n}(x)]$ that are not adjacent. Then $(I_n-\{x\})\cup\{x',\ y'\}$will be an independent set of H_n, contradicting to that I_n is a maximum independent set of H_n. So what we desire follows. Let $I_n'=\{x|x\in I_n$ and $d_{H_n}(x)=k-1\}$ and $I_n''=I_n-I_n'$. Then for every $x\in I_n''$, $d_{H_n}(x)\leqslant k-2$. Note that both of I_n' and I_n'' are independent sets of H_n. Since for every vertex $x\in I_n'$, $d_{H_n}(x)=k-1$, and for every $x\in I_n''$, $d_{H_n}(x)\leqslant k-2$, where $k\geqslant 2$, by the above claim we have the following inequality:

$$|V(H_n)|\leqslant k|I_n'|-\left\lceil\frac{|I_n'|}{2}\right\rceil+(k-1)|I_n''|\leqslant k|I_n|-\left\lceil\frac{|I_n|}{2}\right\rceil\leqslant\left(k-\frac{1}{k+1}\right)|I_n|$$

for each $n=1,\ldots,m$. Let $I=\bigcup_{n=1}^m I_n$, Then $|I|=\sum_{n=1}^m|I_n|$ and I is a maximum independent set of H. Thus

$$|V(H)|=\sum_{n=1}^m|V(H_n)|\leqslant\sum_{n=1}^m\left(k-\frac{1}{k+1}\right)|I_n|=\left(k-\frac{1}{k+1}\right)|I|$$

which is inequality (3). Let $C = V(H) - I$. Then $|C| = |V(H)| - |I|$ and the result (4) follows easily from (3). The proof is completed. □

The following Lemma 2.3 is similar to Lemma 5 of [5]. However, it has been strengthened not only in its conditions but also in its result.

Lemma 2.3. *Let G be a graph and let $H = G[T]$ such that $\delta(H) \geqslant 1$ and $1 \leqslant d_G(x) \leqslant k-1$ for every $x \in V(H)$ where $T \subseteq V(G)$ and $k \geqslant 2$. Let $T_1, \ldots, T_{k-1}$ be a partition of the vertices of H satisfying $d_G(x) = j$ for each $x \in T_j$ where we allow some T_j to be empty. If each component of H has a vertex of degree at most $k-2$ in G, then H has a maximal independent set I and a covering set $C = V(H) - I$ such that*

$$\sum_{j=1}^{k-1} (k-j)c_j \leqslant \sum_{j=1}^{k-1} (k-2)(k-j)i_j,$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for every $j = 1, \ldots, k-1$.

Proof. Since $\delta(H) \geqslant 1$ and each component of H has a vertex of degree at most $k-2$ in G , we have $k \geqslant 3$. We prove the lemma by induction on $|V(H)|$. If $|V(H)| = 2$, then H is isomorphic to K_2. Let $V(H) = \{x, y\}$ and suppose $x \in T_{i_0}$ and $y \in T_{j_0}$. We may assume $i_0 \leqslant j_0$. Let $I = \{x\}$ and $C = V(H) - \{x\} = \{y\}$. Then

$$\sum_{j=1}^{k-1} (k-j)c_j = k - j_0 \leqslant k - i_0 \leqslant (k-2)(k-i_0) = \sum_{j=1}^{k-1} (k-2)(k-j)i_j$$

and the result follows. Now we assume that the result holds when $|V(H)| < L$. Now we consider $|V(H)| = L \geqslant 3$. Let $m = \min\{j | T_j \neq \emptyset\}$. Then $1 \leqslant m \leqslant k-2$. Take any $y \in T_m$. Then $H - (\{y\} \cup N_H(y))$ may have some isolated vertices in H. Let I'' be the set of y and these isolated vertices. Now let $H' = H - (I'' \cup N_H(y))$. If $x \in I'' - \{y\}$, then we can see that $d_H(x) \leqslant d_H(y)$ and $d_G(x) \geqslant d_G(y)$ by the definition of I'' and m.

If $|V(H')| = 0$, put $I = I''$ and $C = V(H) - I = N_H(y)$. Note that $T_j = \emptyset$ and $i_j = 0$ for $j < m$. Since $|V(H')| = 0$, we have

$$\sum_{j=m}^{k-1} c_j \leqslant m.$$

Thus

$$\begin{aligned}\sum_{j=1}^{k-1} (k-j)c_j &\leqslant \sum_{j=m}^{k-1} (k-m)c_j \\ &= (k-m)\sum_{j=m}^{k-1} c_j \leqslant (k-m)m \\ &\leqslant (k-2)(k-m) \leqslant \sum_{j=1}^{k-1} (k-2)(k-j)i_j.\end{aligned}$$

Obviously I is a maximal independent set of H.

So we suppose that $|V(H')| \neq 0$ or $V(H) \neq I'' \cup N_H(y)$. Note that if a vertex v is only adjacent to $N_H(y)$, then v is in $(I'' - \{y\})$. If vertex v is adjacent to a vertex $u \in (I'' - \{y\})$, then u is not an isolated vertex of $H - (\{y\} \cup N_H(y))$ in H. This contradicts to that $u \in (I'' - \{y\})$. Thus it follows that $\delta(H') \geqslant 1$. Clearly $\Delta(H') \leqslant k-1$. It is obvious that $|V(H')| \geqslant 2$. From the definition of H' and $\Delta(H) \leqslant k-1$ we can also see that each component of H' has a vertex of degree at most $k-2$ in G as follows. If a component H_0 of H' is also a component of H, clearly, H_0 has a vertex of degree at most $k-2$ by the hypothesis. Otherwise, a component H_0 of H' is not a component of H. Then H_0 is a component of $H_1 - (I'' \cup N_H(y))$ where H_1 is a component of H. Note that there are at least one edge $e = uv$

joining H_0 and $I'' \cup N_H(y)$ in H_1. We may assume that vertex v is in H_0. Since $d_G(x) \leqslant k-1$ for every vertex x of H_0, $d_G(v) \leqslant k-1$. Thus vertex v is adjacent to at most $k-2$ vertices in H_0. It is easy to see that H_0 must have a vertex of degree at most $k-2$. Let $T'_j = T_j \cap V(H')$. Since $|V(H')| < L$, by induction hypothesis, there exists a maximal independent set I' and a covering set $C' = V(H') - I'$ of H' such that

$$\sum_{j=1}^{k-1} (k-j)c'_j \leqslant \sum_{j=1}^{k-1} (k-2)(k-j)i'_j,$$

where $i'_j = |I' \cap T'_j|$ and $c'_j = |C' \cap T'_j|$. Now let $I = I' \cup I''$ and $C = V(H) - I = C' \cup N_H(y)$. Obviously, I is a maximal independent set of H. Then

$$\begin{aligned}\sum_{j=1}^{k-1} (k-2)(k-j)i_j &\geqslant \sum_{j=1}^{k-1} (k-2)(k-j)i'_j + m(k-m) \\ &\geqslant \sum_{j=1}^{k-1} (k-j)c'_j + m(k-m).\end{aligned}$$

Since $d_G(y) \leqslant m$ and $m = \min\{j | T_j \neq \phi\}$, we have

$$\sum_{j=1}^{k-1} (k-j)c_j \leqslant \sum_{j=1}^{k-1} (k-j)c'_j + m(k-m).$$

Thus

$$\sum_{j=1}^{k-1} (k-j)c_j \leqslant \sum_{j=1}^{k-1} (k-2)(k-j)i_j$$

completing the proof. □

Proof of Theorem 1.2. If G is complete, since $|V(G)| \geqslant k+1$, obviously, G has a fractional k-factor. In the following we assume that G is not complete. Suppose that G satisfies the conditions of Theorem 1.2 , but has no fractional k-factors. From Lemma 1.2 there exists a subset S of $V(G)$ such that

$$k|T| - d_{G-S}(T) > k|S|, \tag{5}$$

where $T = \{x \in V(G) - S | d_{G-S}(x) \leqslant k-1\}$. By Lemma 2.1, we have $\delta(G) \geqslant 2t(G) \geqslant 2k - 2/k \geqslant k+1$. Therefore $S \neq \emptyset$ by (5). Let l be the number of the components of $H' = G[T]$ which are isomorphic to K_k and let $T_0 = \{x \in V(H') | d_{G-S}(x) = 0\}$. Let H be the subgraph obtained from $H' - T_0$ by deleting those l components isomorphic to K_k.

If $|V(H)| = 0$, then from (5) we obtain

$$k|T_0| + lk > k|S|$$

or

$$1 \leqslant |S| < |T_0| + l.$$

Hence $\omega(G-S) \geqslant l + |T_0| > 1$ and

$$t(G) \leqslant \frac{|S|}{\omega(G-S)} = \frac{|S|}{l+|T_0|} < 1.$$

This contradicts that $t(G) \geqslant k - 1/k \geqslant \frac{3}{2}$.

Now we consider that $|V(H)|>0$ and $\delta(H)\geqslant 1$. Let $H=H_1\cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x)=k-1$ for every vertex $x\in V(H_1)$ and $H_2=H-H_1$. By Lemma 2.2, H_1 has a maximum independent set I_1 and the covering set $C_1=V(H_1)-I_1$ such that

$$|V(H_1)|\leqslant\left(k-\frac{1}{k+1}\right)|I_1| \tag{6}$$

and

$$|C_1|\leqslant\left(k-1-\frac{1}{k+1}\right)|I_1|. \tag{7}$$

On the other hand, it is obvious that $\delta(H_2)\geqslant 1$ and $\Delta(H_2)\leqslant k-1$. Let $T_j=\{x\in V(H_2)|d_{G-S}(x)=j\}$ for $1\leqslant j\leqslant k-1$. By the definition of H and H_2 we can also see that each component of H_2 has a vertex of degree at most $k-2$ in $G-S$. According to Lemma 2.3, H_2 has a maximal independent set I_2 and the covering set $C_2=V(H_2)-I_2$ such that

$$\sum_{j=1}^{k-1}(k-j)c_j\leqslant\sum_{j=1}^{k-1}(k-2)(k-j)i_j, \tag{8}$$

where $c_j=|C_2\cap T_j|$ and $i_j=|I_2\cap T_j|$ for every $j=1,\ldots,k-1$. Set $W=V(G)-S-T$ and $U=S\cup C_1\cup C_2\cup(N_G(I_2)\cap W)$. Then since $|C_2|+|(N_G(I_2)\cap W)\leqslant\sum_{j=1}^{k-1}ji_j$ we obtain

$$|U|\leqslant|S|+|C_1|+\sum_{j=1}^{k-1}ji_j \tag{9}$$

and

$$\omega(G-U)\geqslant t_0+l+|I_1|+\sum_{j=1}^{k-1}i_j, \tag{10}$$

where $t_0=|T_0|$. Let $t(G)=t$. Then when $\omega(G-U)>1$, we have

$$|U|\geqslant t\omega(G-U). \tag{11}$$

In addition, (11) also holds when $\omega(G-U)=1$ as by Lemma 2.1 for any $x\in T$,

$$|U|\geqslant d_{G-S}(x)+|S|\geqslant d(x)\geqslant 2t.$$

By (9)–(11),

$$|S|+|C_1|+\sum_{j=1}^{k-1}ji_j\geqslant t(t_0+l)+t|I_1|+t\sum_{j=1}^{k-1}i_j$$

or

$$|S|+|C_1|\geqslant\sum_{j=1}^{k-1}(t-j)i_j+t(t_0+l)+t|I_1|. \tag{12}$$

From (5) we have

$$kt_0+kl+|V(H_1)|+\sum_{j=1}^{k-1}(k-j)i_j+\sum_{j=1}^{k-1}(k-j)c_j>k|S|.$$

Combining with (12) we have

$$kt_0+kl+|V(H_1)|+\sum_{j=1}^{k-1}(k-j)i_j+\sum_{j=1}^{k-1}(k-j)c_j+k|C_1|>\sum_{j=1}^{k-1}(kt-kj)i_j+kt(t_0+l)+kt|I_1|.$$

Thus

$$\sum_{j=1}^{k-1}(k-j)c_j+|V(H_1)|+k|C_1|>\sum_{j=1}^{k-1}(kt-kj-k+j)i_j+k(t-1)(t_0+l)+kt|I_1|$$
$$\geqslant\sum_{j=1}^{k-1}(kt-kj-k+j)i_j+kt|I_1|. \tag{13}$$

By (6) and (7),

$$|V(H_1)|+k|C_1|\leqslant\left[k-\frac{1}{k+1}+k\left(k-1-\frac{1}{k+1}\right)\right]|I_1|$$
$$=(k^2-1)|I_1|. \tag{14}$$

Using (8), (13) and (14), we have

$$\sum_{j=1}^{k-1}(k-2)(k-j)i_j+(k^2-1)|I_1|>\sum_{j=1}^{k-1}(kt-kj-k+j)i_j+kt|I_1|.$$

Thus at least one of the following two cases must hold.

Case 1: There is at least one j such that

$$(k-2)(k-j)>kt-kj-k+j.$$

It follows that

$$t<\frac{k^2-k+j}{k}.$$

But $j\leqslant(k-1)$, we have

$$t<k-\frac{1}{k},$$

contradicting to the toughness condition of Theorem 1.2.

Case 2: $k^2-1>kt$. In this case we have

$$t<k-\frac{1}{k}.$$

This also contradicts to the toughness condition of Theorem 1, completing the proof of the theorem. □

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