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Some spacetimes with higher rank Killing–Stäckel tensors

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ABSTRACT

By applying the lightlike Eisenhart lift to several known examples of low-dimensional integrable systems admitting integrals of motion of higher-order in momenta, we obtain four- and higher-dimensional Lorentzian spacetimes with irreducible higher-rank Killing tensors. Such metrics, we believe, are first examples of spacetimes admitting higher-rank Killing tensors. Included in our examples is a four-dimensional supersymmetric pp-wave spacetime, whose geodesic flow is superintegrable. The Killing tensors satisfy a non-trivial Poisson–Schouten–Nijenhuis algebra. We discuss the extension to the quantum regime.

1. Introduction

Since Carter's *tour de force* in separating variables for the Hamilton–Jacobi and Klein–Gordon equations in the Kerr metric [1] there has been a great deal of work on spacetimes $\{\mathcal{M}, g_{ab}\}$ admitting a second rank Killing–Stäckel tensor $K^{ab} = K^{ba}$ which is responsible for the additive separability of the Hamilton–Jacobi equation. Almost nothing is known about higher rank totally symmetric tensors $K^{a_1 a_2 \ldots a_p}$ satisfying the condition that

$$\nabla^{(a_1} K^{a_2 a_3 \ldots a_{p+1})} = 0. \tag{1}$$

While it is known that any such tensor gives rise to a homogeneous function on the cotangent bundle $T^\star \mathcal{M}$, $\mathcal{K}_p = K^{a_1 \ldots a_p} \times p_{a_1} \cdots p_{a_p}$ of degree p in momenta, which Poisson commutes with the Hamiltonian $\mathcal{H} = \frac{1}{2} g^{ab} p_a p_b$ generating the geodesic flow, no non-trivial (i.e. irreducible) examples appear to be known.

Given any two such Killing–Stäckel tensors of rank p and q respectively their Schouten–Nijenhuis bracket $[K_p, K_q]^{a_1 a_2 \ldots a_{p+q-1}}$ is defined in terms of the standard Poisson bracket $\{\mathcal{K}_p, \mathcal{K}_q\}$ as follows

$$\{\mathcal{K}_p, \mathcal{K}_q\} = \frac{\partial \mathcal{K}_p}{\partial q^i} \frac{\partial \mathcal{K}_q}{\partial p_i} - \frac{\partial \mathcal{K}_q}{\partial q^i} \frac{\partial \mathcal{K}_p}{\partial p_i}$$
$$\equiv [K_p, K_q]^{a_1 a_2 \ldots a_{p+q-1}} p_{a_1} p_{a_2} \cdots p_{a_{p+q-1}}. \tag{2}$$

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While examples of spacetimes admitting more than one quadratic Killing tensor satisfying a non-trivial Poisson or Schouten–Nijenhuis bracket algebra exist [2], no such higher rank examples appear to be known. This may well be because the quickest route for finding quadratic Killing tensors is to follow Carter's original path [1] and seek to separate variables in the Hamilton–Jacobi and Klein–Gordon equations. This route is not available for higher rank Killing–Stäckel tensors since there is no obvious connection between their existence and separability. By theorems in [3,4] only rank two Killing tensors apply to separability of the Hamilton–Jacobi equation.

In some cases it is possible to go further and "quantize" the system. In the case of quadratic Killing–Stäckel tensors it is known that subject to certain conditions on the K^{ab} and the Ricci tensor R_{ab}, the second order differential operator $-\nabla_a K^{ab} \nabla_b$ commutes with the wave operator $-\nabla_a g^{ab} \nabla_b$ and this is related to the multiplicative separability of the Klein–Gordon equation [5]. A recent survey of quantum integrability of quadratic Killing–Stäckel tensors may be found in [6]. To our knowledge, there are few if any results to date on the higher rank case.

The Letter is organized as follows. In Section 2 we give details of the lightlike Eisenhart lift and in particular how constants of the motion are lifted. In Section 3 we give examples of spacetimes generated from the classical examples of Liouville integrable dynamical systems describing heavy tops. In Section 4 we discuss how to obtain a supersymmetric spacetime by lifting dynamical systems in E^2 and give a superintegrable example. We conclude in Section 5 and include a brief summary of conventions in Appendix A.

2. The Eisenhart lift

Our examples are all obtained by taking the *Eisenhart lift* or oxidation [2,7–9] of a dynamical system with an n-dimensional configuration space $\{Q_n, g_{ij}, V, A_i\}$ with Lagrangian

$$L = \frac{1}{2} g_{ij}(q^k, t) \dot{q}^i \dot{q}^j - V(q^k, t) + A_i(q^k, t) \dot{q}^i, \tag{3}$$

to give a system of geodesics in an $(n+2)$-dimensional *Bargmann spacetime* $\{\mathcal{M}, g_{ab}, \partial_s\}$, which admits a covariantly constant null Killing vector field ∂_s. The original dynamical trajectories are obtained by a null reduction along the orbits of ∂_s. Since all Bargmann metrics admit a covariantly constant null vector field, it follows that the holonomy is contained within $E(2) \subset SO(3, 1)$, the two-dimensional Euclidean group which stabilizes a null vector. Thus the null congruence is geodesic, expansion, shear and vorticity free. Thus it is also contained within the class of Kundt spacetimes.

It is simplest to work with the Hamiltonian formulation in order to see how the lift affects constants of the motion. We consider dynamics on the cotangent bundle, T^*M, of some manifold M which is equipped with a natural symplectic form given in local coordinates by $\omega = dq^i \wedge dp_i$, with associated Poisson bracket $\{,\}$. We assume that the Hamiltonian is a polynomial of degree two in momenta:

$$H = H^{(2)} + H^{(1)} + H^{(0)}, \tag{4}$$

where $H^{(i)}$ has degree i in momenta. We do not need to assume that H is independent of t. We lift H to a Hamiltonian on $T^*(M \times \mathbb{R}^2)$ by promoting t to a configuration space coordinate and introducing a new coordinate s. The conjugate momenta are denoted p_t, p_s and the new symplectic form is $\omega' = \omega + dt \wedge dp_t + ds \wedge dp_s$, with associated Poisson bracket $\{,\}'$. The Hamiltonian on this enlarged phase space is

$$\mathcal{H} = H^{(2)} + p_s H^{(1)} + p_s^2 H^{(0)} + p_s p_t. \tag{5}$$

Projecting the integral curves of this system onto the $T^*M \times \mathbb{R}_t$ factor of the phase space gives integral curves of the original Hamiltonian.

Suppose now that the system (H, T^*M) has a constant of the motion which is a polynomial in momenta:

$$K = \sum_{i=0}^{k} K^{(i)}. \tag{6}$$

We calculate the variation of K along an integral curve of (H, T^*M) and find after collecting terms according to their degree in momenta that

$$0 = \frac{dK}{dt} = \{K, H\} + \frac{\partial K}{\partial t} = \sum_{i=0}^{k} \Big[\{K^{(i-1)}, H^{(2)}\} + \{K^{(i)}, H^{(1)}\} \\ + \{K^{(i+1)}, H^{(0)}\} + \frac{\partial K^{(i)}}{\partial t} \Big], \tag{7}$$

Since K should be constant along *any* integral curve, the terms in the sum should vanish independently for each i. We lift K to the extended phase space as

$$\mathcal{K} = \sum_{i=0}^{k} p_s^{k-i} K^{(i)}. \tag{8}$$

Now, along an integral curve of $(\mathcal{H}, T^*(M \times \mathbb{R}^2))$ we have

$$\frac{d\mathcal{K}}{d\lambda} = \{\mathcal{K}, \mathcal{H}\}' = \sum_{i=0}^{k} p_s^{k-i+1} \Big[\{K^{(i-1)}, H^{(2)}\} + \{K^{(i)}, H^{(1)}\} \\ + \{K^{(i+1)}, H^{(0)}\} + \frac{\partial K^{(i)}}{\partial t} \Big]. \tag{9}$$

Clearly this vanishes iff K is a constant of the motion for the original system. Furthermore, since $\mathcal{H}$ is a homogeneous polynomial of degree two in momenta we may interpret it as generating the geodesic flow of a (pseudo-)Riemannian metric. $\mathcal{K}$ is a constant along geodesics which is a homogeneous polynomial in momenta and so corresponds to a Killing tensor of this metric. A similar calculation shows that for constants of the motion for the original system K_1, K_2, K_3 which lift to $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$ we have

$$\{K_1, K_2\} = K_3 \quad \Leftrightarrow \quad \{\mathcal{K}_1, \mathcal{K}_2\}' = \mathcal{K}_3. \tag{10}$$

As a result, the Shouten–Nijenhuis algebra of the Killing tensors in the lifted spacetime will be the same as the Poisson algebra of the constants of the motion for the original dynamical system. We also note that whilst we have increased the dimension of the configuration space by two, we have also gained[1] two new constants of the motion: p_s and p_t. Thus the *degree of integrability* of the system is unchanged by the lift—if the original system is Liouville integrable (i.e. admits n functionally independent constants of the motion in involution) or super-integrable (admits further constants of the motion) then so will the lifted system be.

Applying this method to the system $\{Q_n, g_{ij}, V, A_i\}$ defined above, we find that the lifted system is equivalent to geodesic motion on the spacetime with metric

$$ds^2 = g_{ij}(q^k, t) dq^i dq^j - 2V(q^k, t) dt^2 \\ + 2A_i(q^k, t) dq^i dt + 2 dt\, ds. \tag{11}$$

3. Eisenhart lift of Goryachev–Chaplygin and Kovalevskaya's tops

3.1. Eisenhart lift of the Goryachev–Chaplygin top

In this section we shall illustrate our general procedure by starting with the well-known Liouville integrable system known as the Goryachev–Chaplygin top [10,11]. After introducing the Goryachev–Chaplygin Hamiltonian and the corresponding constant of motion, we proceed to their Eisenhart lift. We demonstrate that the obtained four-dimensional Lorentzian spacetime, which we call the Goryachev–Chaplygin spacetime, admits a rank-3 irreducible Killing tensor. We conclude by making several comments on the quantization of the Goryachev–Chaplygin top and the corresponding results in the Goryachev–Chaplygin spacetime.

3.1.1. Goryachev–Chaplygin top

Following Whittaker [10] we consider the motion of Goryachev–Chaplygin top as a constrained motion of a heavy top with principle moments of inertia $A = B = 4C$ and whose centre of gravity lies in the plane determined by the two equal moments of inertia, so we start with:

$$L_{top} = \frac{1}{2}(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + \frac{1}{8}(\dot{\psi} + \cos\theta\, \dot{\phi})^2 \\ - \alpha^2 \sin\theta \sin\psi. \tag{12}$$

Proceeding to the Hamiltonian formulation, we find

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[1] The equations of motion derived from $\mathcal{H}$ imply that $p_t = \text{const} - E(t)/p_s$, where $E(t)$ is the energy of the original system, thus when E is constant, we do not lose this constant of the motion by lifting.

$$p_\phi = \sin^2\theta\,\dot\phi + \frac{1}{4}\cos\theta(\dot\psi + \cos\theta\,\dot\phi),$$

$$p_\theta = \dot\theta, \qquad p_\psi = \frac{1}{4}(\dot\psi + \cos\theta\,\dot\phi), \tag{13}$$

and hence the Hamiltonian is

$$H_{top} = \frac{1}{2}p_\theta^2 + 2p_\psi^2 + \frac{1}{2}\left(\frac{p_\phi}{\sin\theta} - \cot\theta\,p_\psi\right)^2 + \alpha^2\sin\theta\sin\psi$$

$$= \frac{1}{2}\left(M_1^2 + M_2^2 + 4M_3^2\right) + \alpha^2 x_2, \tag{14}$$

which, in notations of Appendix A, is the Hamiltonian (1) considered by Komarov [11]. It is obvious that coordinate ϕ is cyclic and hence p_ϕ equals constant. The Hamiltonian of Goryachev–Chaplygin top is obtained if one sets $p_\phi = 0$,

$$H_{GC} = \frac{1}{2}\left(\cot^2\theta + 4\right)p_\psi^2 + \frac{1}{2}p_\theta^2 + \alpha^2\sin\theta\sin\psi. \tag{15}$$

The Hamiltonian (14) has a remarkable property such that the function

$$K_{top} = M_3\left(M_1^2 + M_2^2\right) - \alpha^2 M_2 x_3 \tag{16}$$

obeys

$$\{H_{top}, K_{top}\} = \alpha^2 p_\phi M_1. \tag{17}$$

Hence, for $p_\phi = 0$, i.e. for Goryachev–Chaplygin top, (16) is a constant of motion and reads

$$K_{GC} = p_\psi p_\theta^2 + \cot^2\theta\,p_\psi^3$$

$$+ \alpha^2\cos\theta(\sin\psi\cot\theta\,p_\psi - \cos\psi\,p_\theta). \tag{18}$$

Introducing the following functions (projections of standard functions M_i):

$$m_1 = -\sin\psi\,p_\theta - \cos\psi\cot\theta\,p_\psi,$$

$$m_2 = \cos\psi\,p_\theta - \sin\psi\cot\theta\,p_\psi, \qquad m_3 = p_\psi, \tag{19}$$

we may write the Goryachev–Chaplygin top Hamiltonian and the corresponding constant of motion as

$$H_{GC} = \frac{1}{2}\left(m_1^2 + m_2^2 + 4m_3^2\right) + \alpha^2 x_2,$$

$$K_{GC} = m_3\left(m_1^2 + m_2^2\right) - \alpha^2 m_2 x_3. \tag{20}$$

3.1.2. Eisenhart lift: Goryachev–Chaplygin spacetime

Using the results of Section 2 the Hamiltonian (20) lifts to the four-dimensional Hamiltonian

$$\mathcal{H} = m_1^2 + m_2^2 + 4m_3^2 + 2\alpha^2 p_s^2 x_2 + 2p_s p_t. \tag{21}$$

This generates the geodesic flow of the four-dimensional Lorentzian 4-metric with Killing vector fields $k = \partial_t$ and $l = \partial_s$, the latter of which is lightlike and covariantly constant,

$$g = -2\alpha^2\sin\theta\sin\psi\,dt^2 + 2\,dt\,ds + d\theta^2 + \frac{d\psi^2}{\cot^2\theta + 4}. \tag{22}$$

The constant of motion (20) now reads

$$\mathcal{K} = m_3\left(m_1^2 + m_2^2\right) - \alpha^2 p_s^2 m_2 x_3 \tag{23}$$

and defines a rank-3 Killing tensor K, $\mathcal{K} = K^{abc}p_a p_b p_c$, with non-zero contravariant components

$$K^{\theta\theta\psi} = \frac{1}{3}, \qquad K^{\theta ss} = -\frac{\alpha^2}{3}\cos\psi\cos\theta,$$

$$K^{\psi\psi\psi} = \cot^2\theta, \qquad K^{\psi ss} = \frac{\alpha^2}{3}\frac{\cos^2\theta\sin\psi}{\sin\theta}, \tag{24}$$

together with the other components related by symmetry. One may verify directly that K satisfies the Killing equation, $\nabla^{(a}K^{bcd)} = 0$, however, it is *not* covariantly constant.

We can see in an elementary way that K is not decomposable into lower rank Killing tensors. This follows from the fact that k and l are the only Killing vectors of the spacetime (22). Suppose K were decomposable, then it would be the sum of terms of the form

$$K_{(1)}^{(a}K_{(2)}^{bc)}, \quad \text{or} \quad K_{(3)}^{(a}K_{(4)}^{b}K_{(5)}^{c)}, \tag{25}$$

where the $K_{(i)}$ are Killing tensors. Since a rank 1 Killing tensor is a Killing vector, by our assumption at least one of the factors in each term must be either k or l. Such terms will only have non-zero components when at least one of a, b, c is either t or s. Since K has a non-zero $\psi\psi\psi$-component, K cannot be decomposed into a sum of lower rank Killing tensors.

One may verify that the following holds:

$$[k, l] = 0, \qquad \mathcal{L}_k K = 0, \qquad \mathcal{L}_l K = 0, \tag{26}$$

which implies that the associated constants of the geodesic motion are in involution; the motion is Liouville integrable.

Let us finally mention some properties of the Goryachev–Chaplygin spacetime. The spacetime is not Ricci flat, nor does the Ricci scalar vanish. This means that it does not admit a Killing spinor, e.g., [12]. We also note that

$$R_{ab}l^b = 0, \tag{27}$$

however R_{ab} clearly has rank 3 (for typical values of the coordinates) and so $R_{ab} \neq Am_a m_b$ for any vector m^a. The Einstein tensor has non-zero components

$$G_{tt} = \frac{-12\alpha^2(3\cos^4\theta - 10\cos^2\theta + 6)\sin\theta\sin\phi}{(3\cos^2\theta - 4)^2},$$

$$G_{ts} = -\frac{2(3\cos^2\theta + 2)}{(3\cos^2\theta - 4)^2}, \tag{28}$$

and obeys $G_{ab}l^a l^b = 0$, which is, of course, obvious from the equivalent result for the Ricci tensor, together with the fact that l is null.

3.1.3. Quantum mechanics of Goryachev–Chaplygin top

The quantum mechanics of the Goryachev–Chaplygin top was studied by Komarov [11]. Specifically, it was shown that (17) admits a quantum analogue

$$[\hat{H}_{top}, \hat{K}_{top}] = -\alpha^2 J_1 \partial_\phi, \tag{29}$$

where operators $\hat{H}_{top}$ and $\hat{K}_{top}$ are given by

$$\hat{H}_{top} = \frac{1}{2}\left(J_1^2 + J_2^2 + 4J_3^2\right) + \alpha^2 x_2,$$

$$\hat{K}_{top} = J_3\left(J_1^2 + J_2^2\right) - \frac{1}{4}J_3 - \frac{1}{2}\alpha^2(J_2 x_3 + x_3 J_2), \tag{30}$$

and J_i are defined in (70). This means that acting on a wave function independent of ϕ, the operators (30) commute.

By employing the Eisenhart lift on these operators one finds that the operators

$$\hat{\mathcal{H}}_{top} = J_1^2 + J_2^2 + 4J_3^2 + 2\alpha^2 x_2 \partial_s^2 + 2\partial_s \partial_t,$$

$$\hat{\mathcal{K}}_{top} = J_3(J_1^2 + J_2^2) - \frac{1}{4}J_3 - \frac{1}{2}\alpha^2(J_2 x_3 + x_3 J_2)\partial_s^2, \quad (31)$$

obey $[\hat{\mathcal{H}}_{top}, \hat{\mathcal{K}}_{top}] = -2\alpha^2 J_1 \partial_s^2 \partial_\phi$, and hence commute on ϕ-independent wave function. The former operator is precisely the standard wave operator on the Lorentzian 5-space with the metric g_{top}, obtained by the Eisenhart lift of H_{top}. So we have, $\square_{top} \equiv g_{top}^{ab} \nabla_a \nabla_b = \hat{\mathcal{H}}_{top}$, where

$$g_{top} = 2\,ds\,dt - 2\alpha^2 x_2\,dt^2 + (\sigma^1)^2 + (\sigma^2)^2 + \frac{1}{4}(\sigma^3)^2, \quad (32)$$

and σ^i are the left-invariant forms on $SU(2)$ defined in (68). Moreover, the latter operator can be written as

$$\hat{\mathcal{K}}_{top} = K_{(top)}^{abc} \nabla_a \nabla_b \nabla_c + \frac{3}{2}(\nabla_a K_{(top)}^{abc})\nabla_b \nabla_c - \frac{1}{2}K_{(top)a}{}^{ab}\nabla_b, \quad (33)$$

where $K_{(top)}$ is a symmetric rank-3 tensor. Introducing the basis

$$L_s = \partial_s, \qquad L_t = \partial_t, \qquad L_i = J_i, \quad (34)$$

one finds that non-vanishing contravariant components of $K_{(top)}$ are

$$K_{(top)}^{ss2} = -2\alpha^2 x_3/3, \qquad K_{(top)}^{113} = K_{(top)}^{223} = 2/3, \quad (35)$$

and that the tensor satisfies $\nabla^{(a} K_{(top)}{}^{bcd)} = -\alpha^2 L_s^{(a} L_s^b (\partial_\phi)^c L_1^{d)}$. Hence, if we restrict to geodesic motion on 5-space with metric g_{top} such that p_ϕ vanishes, $K_{(top)}^{abc} p_a p_b p_c$ defines a constant of motion.

One might wonder whether it is possible to directly carry over the quantization to the Goryachev–Chaplygin four-dimensional spacetime discussed in the previous subsection. The 'naive quantization' of (20) gives

$$\hat{H}_{GC} = \frac{1}{2}(j_1^2 + j_2^2 + 4j_3^2) + \alpha^2 x_2,$$

$$\hat{K}_{GC} = j_3(j_1^2 + j_2^2) - \frac{1}{4}j_3 - \frac{1}{2}\alpha^2(j_2 x_3 + x_3 j_2), \quad (36)$$

where we have defined the operators (projections of J_i)

$$j_1 = -\sin\psi\,\partial_\theta - \cos\psi\cot\theta\,\partial_\psi,$$

$$j_2 = \cos\psi\,\partial_\theta - \sin\psi\cot\theta\,\partial_\psi, \qquad j_3 = \partial_\psi. \quad (37)$$

By lifting the operators (36), one finds

$$\hat{\mathcal{H}} = j_1^2 + j_2^2 + 4j_3^2 + 2\alpha^2 x_2 \partial_s^2 + 2\partial_s \partial_t,$$

$$\hat{\mathcal{K}} = j_3(j_1^2 + j_2^2) - \frac{1}{4}j_3 - \frac{1}{2}\alpha^2(j_2 x_3 + x_3 j_2)\partial_s^2. \quad (38)$$

It is easy to verify that $[\hat{\mathcal{H}}, \hat{\mathcal{K}}] = 0$. However, the operator $\hat{\mathcal{H}}$ is not a standard (geometrical) wave operator on the Goryachev–Chaplygin spacetime. In fact, one finds

$$\square \equiv g^{ab}\nabla_a \nabla_b = \hat{\mathcal{H}} - \frac{3\cot\theta}{4 + \cot^2\theta}\partial_\theta. \quad (39)$$

It is an interesting question whether the operators (36) provide the 'correct quantization' of the Goryachev–Chaplygin top, in which case the operators (38) are 'preferred operators' in the Goryachev–Chaplygin spacetime, or whether some alternative quantization is more appropriate. We leave this problem for the future. We also remark that we were not able to find an operator linear in the Killing tensor K, (24), which commutes with the wave operator $\square$ associated with the Goryachev–Chaplygin metric (22).

3.2. Kovalevskaya's spacetime: quartic Killing tensor

In this case one considers a heavy top with principle moments of inertia $A = B = 2C$ whose centre of gravity lies in the plane determined by the two equal moments of inertia. The Lagrangian is

$$L_K = \frac{1}{2}(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + \frac{1}{4}(\dot{\psi} + \cos\theta\,\dot{\phi})^2 - \alpha^2 \sin\theta\cos\psi. \quad (40)$$

Clearly ϕ is ignorable and the Hamiltonian

$$H_K = \frac{1}{2}\left(p_\theta^2 + \left(\frac{p_\phi}{\sin\theta} - \cot\theta\,p_\psi\right)^2 + 2p_\psi^2\right) + \alpha^2 \sin\theta\cos\psi$$

$$= \frac{1}{2}(M_1^2 + M_2^2 + 2M_3^2) + \alpha^2 x_1 \quad (41)$$

is constant. Kovalevskaya found another constant [10,13] which reads

$$K_K = \left(p_\theta^2 + \left(\frac{p_\phi}{\sin\theta} - \cot\theta\,p_\psi\right)^2\right)^2 + 4\alpha^4 \sin^2\theta$$

$$\quad - 2\alpha^2 \sin\theta\left(e^{i\psi}\left(\frac{p_\phi}{\sin\theta} - \cot\theta\,p_\psi + ip_\theta\right)^2 + \text{c.c.}\right)$$

$$= (M_1^2 + M_2^2)^2 + 4\alpha^4(x_1^2 + x_2^2)$$

$$\quad - 4\alpha^2[x_1(M_1^2 - M_2^2) + 2x_2 M_1 M_2]. \quad (42)$$

This will lift to give a quartic Killing tensor.

In order to get a four-dimensional spacetime we perform again the reduction along the ϕ-direction. So we consider

$$H = \frac{1}{2}(m_1^2 + m_2^2 + 2m_3^2) + \alpha^2 x_1,$$

$$K = (m_1^2 + m_2^2)^2 + 4\alpha^4(x_1^2 + x_2^2)$$

$$\quad - 4\alpha^2[x_1(m_1^2 - m_2^2) + 2x_2 m_1 m_2]. \quad (43)$$

The Hamiltonian lifts to

$$\mathcal{H} = m_1^2 + m_2^2 + 2m_3^2 + 2\alpha^2 p_s^2 x_1 + 2p_s p_t, \quad (44)$$

which generates geodesic flow of the Lorenzian 4-metric

$$g = -2\alpha^2 \sin\theta\cos\psi\,dt^2 + 2\,ds\,dt + d\theta^2 + \frac{d\psi^2}{\cot^2\theta + 2}, \quad (45)$$

admitting the rank-4 irreducible tensor K, given by

$$K^{\theta\theta\theta\theta} = 1, \qquad K^{\theta\theta\psi\psi} = \frac{1}{3}\cot^2\theta,$$

$$K^{ss\theta\theta} = \frac{2}{3}\alpha^2 \sin\theta\cos\psi, \qquad K^{\psi\psi\psi\psi} = \cot^4\theta,$$

$$K^{ss\theta\psi} = -\frac{2}{3}\alpha^2 \cos\theta\sin\psi, \qquad K^{ss\psi\psi} = -\frac{2}{3}\alpha^2 \cos\psi\cos\theta\cot\theta,$$

$$K^{ssss} = 4\alpha^4 \sin\theta^2. \quad (46)$$

Properties of the Kovalevskaya spacetime are very similar to properties of the Goryachev–Chaplygin spacetime. In particular, the spacetime admits a covariantly constant null Killing vector $l = \partial_s$, it is not Ricci flat, and does not admit a Killing spinor. We also have that $\square \neq \hat{\mathcal{H}}$, with the latter obtained by a naive quantization described in previous section.

One can again consider a 5D spacetime instead,

$$g_K = -2\alpha^2 \sin\theta \cos\psi \, dt^2 + 2\, ds\, dt + (\sigma^1)^2$$
$$+ (\sigma^2)^2 + \frac{1}{2}(\sigma^3)^2, \tag{47}$$

where one has [14]

$$\Box_K = g_K^{ab} \nabla_a \nabla_b = \hat{\mathcal{H}}_K = J_1^2 + J_2^2 + 2J_3^2 + 2\alpha^2 x_1 \partial_s^2 + 2\partial_s \partial_t,$$
$$\hat{\mathcal{K}}_K = \frac{1}{2}(K_+ K_- + K_- K_+) - 2(J_+ J_- + J_- J_+), \tag{48}$$

where $J_\pm = J_1 \pm iJ_2$, $K_\pm = J_\pm^2 - 2\alpha^2 x_\pm \partial_s^2$ and $x_\pm = x_1 \pm ix_2$. In this case $\hat{\mathcal{K}}_K$ is a real symmetry of the wave operator, $[\Box_K, \hat{\mathcal{K}}_K] = 0$. It is related to the five-dimensional rank-4 irreducible Killing tensor $K_{(K)}$ as

$$\hat{\mathcal{K}}_K = K_{(K)}^{abcd} \nabla_a \nabla_b \nabla_c \nabla_d + 2(\nabla_a K_{(K)}^{abcd})\nabla_b \nabla_c \nabla_d$$
$$+ 3(\nabla_a \nabla_b K_{(K)}^{abcd})\nabla_c \nabla_d - 2K_{(K)c}^{abc}\nabla_a \nabla_b$$
$$- \frac{3}{4} K_{(K)ab}^{ab} L_3^c L_3^d \nabla_c \nabla_d, \tag{49}$$

where in the basis (34) the components of the Killing tensor $K_{(K)}$ are written as

$$K_{(K)}^{ssss} = 4\alpha^4(x_1^2 + x_2^2), \qquad K_{(K)}^{ss11} = -K_{(K)}^{ss22} = -2\alpha^2 x_1/3,$$
$$K_{(K)}^{ss12} = -2\alpha^2 x_2/3, \qquad K_{(K)}^{1111} = 3K_{(K)}^{1122} = K_{(K)}^{2222} = 1. \tag{50}$$

4. Superintegrable systems in E^2: SUSY plane waves

In this section we consider Hamiltonians of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y). \tag{51}$$

For some choices of the potential V this Hamiltonian is superintegrable, e.g., [15] and references therein. The Hamiltonian (51) lifts to

$$\mathcal{H} = p_x^2 + p_y^2 + 2V(x, y)p_s^2 + 2p_s p_t, \tag{52}$$

which generates geodesic flows of Lorentzian 4-metric

$$g = dx^2 + dy^2 - 2V(x, y)\, dt^2 + 2\, dt\, ds. \tag{53}$$

In quantum mechanics, one has the quantized Hamiltonian

$$\hat{\mathcal{H}} = \partial_x^2 + \partial_y^2 + 2V(x, y)\partial_s^2 + 2\partial_s \partial_t \tag{54}$$

and this coincides with the Laplacian of the metric (53), i.e., one has $\Box \equiv \nabla_a g^{ab} \nabla_b = \hat{\mathcal{H}}$.

Let us mention some basic properties of the spacetime (53). The Ricci curvature has only tt-component,

$$R_{tt} = (\partial_x^2 + \partial_y^2)V, \tag{55}$$

and the scalar curvature vanishes, $R = 0$. Hence $G_{ab} = R_{ab}$ and

$$R_{ab}l^b = 0, \tag{56}$$

where $l \equiv \partial/\partial s$ is a covariantly constant null Killing vector. Since the "transverse" x–y space is flat the metric (53) admits a covariantly constant spinor field ϵ such that

$$\bar{\epsilon}\gamma^a \epsilon = l^a = (\partial_s)^a \tag{57}$$

and hence a covariantly constant null 2-form

$$\ell^{ab} = \bar{\epsilon}\gamma^{[ab]}\epsilon \tag{58}$$

such that $\ell^{ab}l_b = 0$.

There are many examples of interesting (superintegrable) systems of the type (51) which give rise to higher-rank Killing tensors and non-trivial Schouten–Nijenhuis brackets. We refer the reader to recent paper by Kalnins et al. [15] and references therein as well as to Chapter 4.4 in [16]. To illustrate the theory we give the following recent example:

4.1. Post–Winternitz example

In [17], Post and Winternitz give a (Hamilton–Jacobi non-separable) classical super-integrable example of the form (51) with the potential

$$V = \frac{\alpha y}{x^{\frac{2}{3}}}, \tag{59}$$

such that

$$X = 3p_x^2 p_y + 2p_y^3 + 9\alpha x^{\frac{1}{3}} p_x + \frac{6\alpha y p_y}{x^{\frac{2}{3}}}, \tag{60}$$

$$Y = p_x^4 + \frac{4\alpha y p_x^2}{x^{\frac{2}{3}}} - 12\alpha x^{\frac{1}{3}} p_x p_y - \frac{2\alpha^2(9x^2 - 2y^2)}{x^{\frac{4}{3}}}, \tag{61}$$

both Poisson commute with H and satisfy the Heisenberg algebra

$$\{X, Y\} = 108\alpha^3. \tag{62}$$

The spacetime reads

$$g = 2\, ds\, dt - \frac{2y}{x^{\frac{2}{3}}}\, dt^2 + dx^2 + dy^2. \tag{63}$$

The constants X, Y are lifted and give

$$\{\mathcal{X}, \mathcal{Y}\} = 108\alpha^3 p_s^6. \tag{64}$$

Thus, consistent with previous cases ([2] and references therein), the central element in the Heisenberg algebra (62) may be interpreted as the (sixth power of) a null translation.

The spacetime admits rank-3 and rank-4 Killing tensors. Their components X^{abc} and Y^{abcd} can be read of from

$$\mathcal{X} = X^{abc} p_a p_b p_c$$
$$= 3p_x^2 p_y + 2p_y^3 + 9\alpha x^{\frac{1}{3}} p_x p_s^2 + \frac{6\alpha y p_y p_s^2}{x^{\frac{2}{3}}}, \tag{65}$$

$$\mathcal{Y} = Y^{abcd} p_a p_b p_c p_d$$
$$= p_x^4 + \frac{4\alpha y p_x^2 p_s^2}{x^{\frac{2}{3}}} - 12\alpha x^{\frac{1}{3}} p_x p_y p_s^2$$
$$- \frac{2\alpha^2(9x^2 - 2y^2)}{x^{\frac{4}{3}}} p_s^4. \tag{66}$$

Since $l_a dx^a = dt$, we have

$$l_a X^{abc} = 0 = l_a Y^{abcd}. \tag{67}$$

Post and Winternitz have provided a quantization of their model. Thus if $[x, p_x] = i\hbar$ etc., then all products are replaced by half their anti-commutator and in addition one must subtract $\frac{5\hbar^2}{72x^2}$ from the expression for H and add $\frac{25\hbar^4}{1296x^4}$ to the expression for Y.

5. Conclusions

In this Letter we have shown that by applying Eisenhart's light-like lift to dynamical systems admitting constants of the motion of degree greater than two in momenta, one may obtain spacetimes admitting Killing tensors of higher rank than two. Our examples by no means exhaust the possibilities. In [13–15,18–22] more complicated examples are given, but our examples illustrate the point we wish to make.

In some cases we find the Poisson–Schouten–Nijenhuis algebra to be non-trivial. We have also constructed differential operators which realize the classical algebra as $\hbar \to 0$. In some, but not all, cases the Hamiltonian corresponds to the Laplace or wave operator. In general the wave operator must be augmented by quantum corrections which are not always expressible in purely geometric terms. The higher rank conserved quantities also receive quantum corrections not expressible solely in terms of the Killing tensor. In some ways this is one of the most interesting of our findings and is certainly worthy of further study.

Appendix A. Conventions and Euclidean group notation

To fix the conventions for forms on $SU(2)$, we take the following basis for left-invariant forms:

$$\sigma^1 = \sin\theta\cos\psi\, d\phi - \sin\psi\, d\theta,$$
$$\sigma^2 = \sin\theta\sin\psi\, d\phi + \cos\psi\, d\theta,$$
$$\sigma^3 = d\psi + \cos\theta\, d\phi, \tag{68}$$

which obey the relations

$$d\sigma^i = -\frac{1}{2}\epsilon_{ijk}\sigma^j \wedge \sigma^k. \tag{69}$$

The dual vector fields are

$$J_1 = -\sin\psi\, \partial_\theta + \frac{\cos\psi}{\sin\theta}\partial_\phi - \cot\theta\cos\psi\, \partial_\psi,$$
$$J_2 = \cos\psi\, \partial_\theta + \frac{\sin\psi}{\sin\theta}\partial_\phi - \cot\theta\sin\psi\, \partial_\psi, \qquad J_3 = \partial_\psi, \tag{70}$$

and satisfy the algebra:

$$[J_i, J_j] = -\epsilon_{ijk}J_k. \tag{71}$$

Defining the functions

$$x_1 = \sin\theta\cos\psi, \qquad x_2 = \sin\theta\sin\psi, \qquad x_3 = \cos\theta, \tag{72}$$

we have the additional relations

$$[J_i, x_j] = -\epsilon_{ijk}x_k, \tag{73}$$

where we interpret the functions x_i as operators on functions, acting by multiplication.

Both the Goryachev–Chaplygin and the Kovalevskaya tops discussed in the main text are examples of tops whose centre of gravity does not coincide with the pivot point. They admit a description in terms of the Lie algebra of the Euclidean group $E(3)$ and since this is used in some of the literature, e.g. [11,13,14,18,19,23], we give it here.

If $\mathbf{M}$ is the angular momentum of the top one has, in the rotating frame

$$\dot{\mathbf{M}} + \boldsymbol{\omega} \times \mathbf{M} = -mg\mathbf{x}_0 \times \mathbf{x},$$
$$\dot{\mathbf{k}} + \omega\mathbf{x} = 0, \tag{74}$$

where $\mathbf{x}$ is unit vector which is constant in the inertial frame (the constancy of $|\mathbf{x}|$ is a consequence of these equations of motion) and points in the opposition direction to the local direction of gravity and $\mathbf{x}_0$ is a constant vector in the rotating from which gives the centre of gravity. An alternative interpretation, used in analyzing the *Stark effect*, is that $\mathbf{x}_0$ is the electric dipole moment and $mg\mathbf{x}$ is in the direction of the applied electric field. The system of equations admits three constants of the motion

$$\mathbf{x} \cdot \mathbf{x}, \quad \mathbf{x} \cdot \mathbf{M}, \quad \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{M} + mg\mathbf{x}_0 \cdot \mathbf{x}. \tag{75}$$

Choosing coordinates such that the centre of mass relative to the pivot (normalized to unit length) are given by (72), we find that the potential energy of the top is given by

$$V = mg(x_0 \sin\theta\cos\psi + y_0 \sin\theta\sin\psi + z_0 \cos\theta), \tag{76}$$

and one may construct a Lagrangian on $TSO(3)$ and a Hamiltonian on $T^\star SO(3)$ which depend on the principle moments of inertia (A, B, C). For the Goryachev–Chaplygin top we have $A = B = 4C$, and the centre of gravity lies in the plane defined by the two principal axes with equal moments of inertia.

The moment maps for left actions of rotations

$$M_1 = -\sin\psi\, p_\theta + \frac{\cos\psi}{\sin\theta}\, p_\phi - \cos\psi\cot\theta\, p_\psi,$$
$$M_2 = \cos\psi\, p_\theta + \frac{\sin\psi}{\sin\theta}\, p_\phi - \sin\psi\cot\theta\, p_\psi, \qquad M_3 = p_\psi. \tag{77}$$

The Poisson algebra of $\mathbf{M}$ and $\mathbf{x}$ then turns out to be that of the Euclidean group $\mathfrak{e}(3)$. Thus the system of equations (74) may also be interpreted as a Hamiltonian system moving on $\mathfrak{e}^\star(3)$ the dual of the Lie algebra $\mathfrak{e}(3)$. As a consequence one has an isomorphism with the problem of a rigid body moving in a fluid. However it should be noted that the latter has phase space $T^\star E(3)$ which is 12-dimensional while the top has phase space has phase space $T^\star(SO(3))$ which is 6-dimensional. As pointed out in [23] if one imposes the constraints $\mathbf{x} \cdot \mathbf{x} = 1$, $\mathbf{M} \cdot \mathbf{x} = 0$, one gets the standard symplectic structure on $T^\star S^2$.

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