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# Propagation behavior of Love waves in a functionally graded half-space with initial stress

Zheng-Hua Qian [a,b,*], Feng Jin [a], Kikuo Kishimoto [b], Tianjian Lu [a]

[a] MOE Key Laboratory for Strength and Vibration, Xi'an Jiaotong University, Xi'an 710049, PR China

[b] Department of Mechanical and Control Engineering, Tokyo Institute of Technology, 2-12-1 O-okayama, Meguro-ku, Tokyo 152-8552, Japan

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ABSTRACT

The propagation behavior of Love waves in a functionally graded material layered non-piezoelectric half-space with initial stress is taken into account. The Wentzel–Kramers–Brillouin (WKB) technique is adopted for the theoretical derivations. The analytical solutions are obtained for the dispersion relations and the distributions of the mechanical displacement and stress along the thickness direction in the layered structure. First, these solutions are used to study the effects of the initial stress on the dispersion relations and the group and phase velocities, then the influences of the initial stress on the distributions of the mechanical displacement and shear stresses along the thickness direction are discussed in detail. Numerical results obtained indicate that the phase velocity of the Love waves increases with the increase in the magnitude of the initial tensile stress, while decreases with the increase in the magnitude of the initial compression stress. The effects on the dispersion relations of the Love wave propagation are negligible as the magnitudes of the initial stress are less than 100 MPa. Some other results are obtained for the distributions of field quantities along thickness direction. The results obtained are not only meaningful for the design of functionally graded structures with high performance but also effective for the evaluation of residual stress distribution in the layered structures.



## 1. Introduction

Functionally graded materials have been known to have extensive applications in many fields, such as aerospace, electronics, protection industries and so on. It is very important and necessary to investigate the mechanical performance of functionally graded materials and structures. Much work has been done on the research like thermomechanical response and fracture behavior of functionally graded materials since their appearance. A key feature of such analyses is that the determination of the current state of the material, as characterized by the distributions of stresses, strains, displacements and damage through the thickness, is a strong function of the initial condition of the material. While most of the foregoing analyses envision the initial state as the 'stress-free' condition at which the sintering, diffusion bonding or spray deposition of the material is accomplished, it is widely recognized that essentially all processing methods produce 'intrinsic' or 'quench' stresses, over and above the 'thermal stresses' induced during temperature excursions from the 'stress-free' processing temperature (or relaxation temperature) to service temperature (Suresh and Mortensen, 1998). A considering of 'initial mechanical state' during the research of functionally graded materials is vital to the success of design with high performance.

For functionally graded materials, it is very important to study the wave propagation problems due to their working conditions. At this point, Liu et al. (1991a), Liu and Tani (1991b, 1992, 1994), Han et al. (2001, 2002a) and Han and Liu (2002b, 2003) have done a lot of work on the wave propagation in the functionally graded materials and the functionally gradient piezoelectric materials. They proposed the numerical methods for investigating the characteristics of waves in a functionally gradient piezoelectric material plate and its transient responses and demonstrated these numerical techniques for the SH waves propagating in the functionally graded material plates. And their studies mostly focused on the wave propagation in the functionally graded material plates and the functionally gradient piezoelectric material plates. In practical applications, functionally graded materials often cover an elastic body as a coating. So, it is necessary to investigate the mechanical performance of such layered structures, i.e. an elastic half-space with a functionally graded material layer. Liu et al. (2004) considered the Love wave propagation in a layered structure with graded materials and discussed the influence of gradient coefficients on

* Corresponding author. Tel.: +81 3 5734 2783; fax: +81 3 5734 3917.
*E-mail addresses:* zhenghua_qian@hotmail.com, qian.z.aa@m.titech.ac.jp (Z.-H. Qian).

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the phase velocity. Furthermore, Li et al. (2004), Liu and Wang (2005) and Du et al. (2007) investigated the propagation of the Love waves in a functionally gradient piezoelectric layered structure. They studied detailedly the effects of the gradient coefficients of the elastic, piezoelectric and dielectric constants on the dispersive curves and the electromechanical coupling factors, respectively. In all the above-mentioned papers, the residual stresses easily induced during the processing of functionally graded materials were not taken into account. The magnitudes of the residual stresses in functionally graded materials are considerable compared with the external loads, so it is meaningful to study the effect of the residual stresses on the wave propagation behavior in functionally graded materials and structures.

In present contribution, we study the propagation behavior of Love waves in a functionally graded layered non-piezoelectric half-space with an initial stress and discussed the effects of the initial stress on the dispersion relations, phase velocities and the distributions of field quantities. Some significant results have been obtained, which can provide some help for the design and production of functionally graded materials in practice.

## 2. Statement of the problem

Consider the wave propagation behavior in a functionally graded layered non-piezoelectric structure, as shown in Fig. 1. It involves an isotropic elastic substrate bonded perfectly to a functionally graded material layer with uniform thickness of $h$. It is assumed that there exist constant initial stresses in the layered structure, and the upper surface of the layer is traction free. Usually, the thickness of the elastic substrate is much greater than that of the layer, such that the structure can be treated as a functionally graded layered half-space, and the initial stress in the elastic substrate is negligible. The propagations of the Love waves are considered along the positive direction of $y$-axis.

For the wave motion of small amplitude, field equations with initial stresses can be expressed as follows (Liu et al., 2001; Qian et al., 2004):

$$\sigma_{ij,j} + (u_{i,k}\sigma^0_{kj})_{,j} = \rho \ddot{u}_i, \tag{1}$$

where $i,j,k = 1,2,3$, $\rho$ is the mass density, $u_i$ denotes the mechanical displacements in the $i$th direction, $\sigma_{ij}$ the stress tensor, $\sigma^0_{kj}$ the initial stress tensor. The dot denotes time differentiation, the comma followed by the subscript $i$ indicates space coordinate differentiation with respect to corresponding coordinate $x_i$, and the repeated subscript index implies summation with respect to that index. In the following derivation the indexes 1,2,3 in Eq. (1) are replaced by $x,y,z$, and the mechanical displacement components $u_i$ are replaced by $u$, $v$ and $w$.

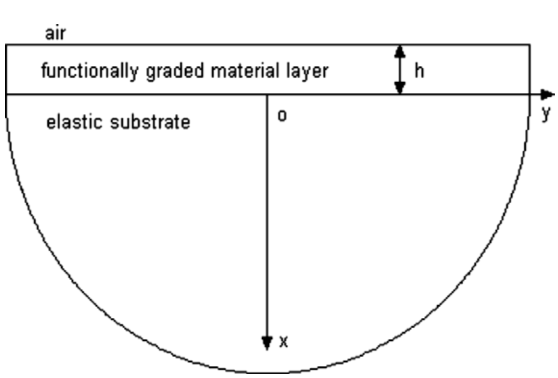


**Fig. 1.** Geometry of a functionally graded layered half-space and the coordinate system.

For a transversely isotropic medium with $z$-axis being the symmetric axis of the material (functionally graded materials can be treated in this way), the constitutive equations can be written in the following forms in term of components:

$$\begin{aligned} \sigma_x &= c_{11}s_x + c_{12}s_y + c_{13}s_z, \\ \sigma_y &= c_{12}s_x + c_{11}s_y + c_{13}s_z, \\ \sigma_z &= c_{13}s_x + c_{13}s_y + c_{33}s_z, \\ \sigma_{yz} &= c_{44}s_{yz}, \\ \sigma_{zx} &= c_{44}s_{zx}, \\ \sigma_{xy} &= \frac{1}{2}(c_{11} - c_{12})s_{xy}, \end{aligned} \tag{2}$$

where $c_{11}$, $c_{12}$, $c_{13}$ and $c_{44}$ are the elastic constants. The strain components $s_{ij}$ in Eq. (2) can be calculated through the following formulae:

$$\begin{aligned} &s_x = \frac{\partial u}{\partial x}, \quad s_y = \frac{\partial v}{\partial y}, \quad s_z = \frac{\partial w}{\partial z}, \\ &s_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \quad s_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad s_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \end{aligned} \tag{3}$$

Here, we assume that the Love waves propagate in the positive direction of $y$-axis without loss of generality. And there exists only one constant initial stress component $\sigma^0_y$ in the layer, such that the mechanical displacement components are as following:

$$u = v = 0, \quad w = w(x, y, t). \tag{4}$$

Substituting Eq. (4) into Eqs. (1) and (3), we have

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \sigma^0_y \frac{\partial^2 w}{\partial y^2} = \rho \frac{\partial^2 w}{\partial t^2} \tag{5}$$

and

$$\begin{aligned} &s_x = s_y = s_z = s_{xy} = 0, \\ &s_{yz} = \frac{\partial w}{\partial y}, \quad s_{zx} = \frac{\partial w}{\partial x}. \end{aligned} \tag{6}$$

Substitution of Eq. (6) into (2) yields

$$\begin{aligned} &\sigma_x = \sigma_y = \sigma_z = \sigma_{xy} = 0, \\ &\sigma_{yz} = c_{44}\frac{\partial w}{\partial y}, \quad \sigma_{zx} = c_{44}\frac{\partial w}{\partial x}. \end{aligned} \tag{7}$$

Let $w_1$, $\mu_1$ and $\rho_1$ denote the mechanical displacement, the shear modulus and the mass density in the region $-h < x < 0$, assume the shear modulus to be gradient in the thickness direction and the mass density to be constant, i.e. $\mu_1(x) = \mu_0 f(x)$ with $\mu_0 = \mu_1(0)$ and $\rho_1$ = constant. Then from Eqs. (5) and (7), we have the following field equations for the functionally graded layer:

$$f'\frac{\partial w_1}{\partial x} + f\frac{\partial^2 w_1}{\partial x^2} + \left(\frac{\sigma^0_y}{\mu_0} + f\right)\frac{\partial^2 w_1}{\partial y^2} = \frac{1}{c_0^2}\frac{\partial^2 w_1}{\partial t^2}, \tag{8}$$

where $c_0 = (\mu_0/\rho_1)^{1/2}$ is the shear wave velocity at the interface $x = 0$.

Let $w_2$, $\mu_2$ and $\rho_2$ denote the mechanical displacement, the shear modulus and the mass density in the region $x > 0$, respectively, then the field equations for the elastic substrate can be expressed as follows from the general elasticity:

$$\frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2} = \frac{1}{c_s^2}\frac{\partial^2 w_2}{\partial t^2}, \tag{9}$$

where $c_s = (\mu_2/\rho_2)^{1/2}$ is the shear wave velocity in the substrate.

When the Love waves propagate in the layered structure as shown in Fig. 1, the mechanical displacement components must satisfy Eqs. (8) and (9). Moreover, the related mechanical variables must satisfy the boundary conditions and the continuity conditions at the interface, which are described as follows:

(1) The traction-free condition at $x = -h$:

$$\sigma_{zx}^{(1)}(-h, y) = 0.$$

(2) The continuity conditions at $x = 0$: the normal components of the stress and the mechanical displacement are continuous:

$$w_1(0, y) = w_2(0, y),$$
$$\sigma_{zx}^{(1)}(0, y) = \sigma_{zx}^{(2)}(0, y).$$

(3) For $x \to +\infty$, $w_2 \to 0$.where the superscripts 1 and 2 denote the quantities in the layer and the substrate, respectively.

## 3. Solutions of the field equations

The solution of the mechanical displacement in Eq. (8) can be assumed to possess the following form:

$$w_1(x, y, t) = W_1(x)\exp[\mathrm{i}k(y - ct)], \tag{10}$$

where $k$ (=$2\pi/\lambda$) is the wave number with $\lambda$ being the wavelength, $\mathrm{i} = (-1)^{1/2}$, $c$ is the phase velocity of wave propagation, $W_1(x)$ is the undetermined function. Substitution of Eq. (10) into (8) yields

$$W_1''(x) + \frac{f'}{f}W_1'(x) + k^2\left(\frac{c^2}{fc_0^2} - \frac{\sigma_y^0}{f\mu_0} - 1\right)W_1(x) = 0. \tag{11}$$

Usually, it is difficult to obtain the exact solution of Eq. (11). But for high-frequency short waves, i.e., $k \gg 1$, the WKB technique (Ishimaru, 1991) can be applied to obtain the asymptotic approximation of Eq. (11) by the following procedure. The discussion on the validity of the WKB solutions obtained in this paper is listed in Appendix.

First, the following transformation is introduced:

$$W_1(x) = \mathrm{e}^{\int \phi(x)\mathrm{d}x}. \tag{12}$$

Therefore, Eq. (11) can be transformed into the following form:

$$\phi^2 + \frac{f'}{f}\phi + \phi' + k^2\left(\frac{c^2}{fc_0^2} - \frac{\sigma_y^0}{f\mu_0} - 1\right) = 0. \tag{13}$$

Expression (13) is a non-linear differential equation regarding the variable $\phi$. In order to solve Eq. (13), we seek an expansion of $\phi$ in inverse powers of $k$, that is to say, we can write

$$\phi = \phi_0 k + \phi_1 + \phi_2 k^{-1} + \phi_3 k^{-2} + \cdots \tag{14}$$

Substitution of Eq. (14) into (13) and expanding Eq. (13) in inverse powers of $k$ yield

$$\left[\phi_0^2 + \left(\frac{c^2}{fc_0^2} - \frac{\sigma_y^0}{f\mu_0} - 1\right)^2\right]k^2 + \left(\phi_0' + 2\phi_0\phi_1 + \frac{f'}{f}\phi_0\right)k + \left(\phi_1' + 2\phi_0\phi_2 + \phi_1^2 + \frac{f'}{f}\phi_1\right) + \left(\phi_2' + 2\phi_0\phi_3 + 2\phi_1\phi_2 + \frac{f'}{f}\phi_2\right)k^{-1} + \cdots = 0. \tag{15}$$

Equating the coefficients of each power of $k$ to be zero, we get an infinite number of equations as follows:

$$\begin{aligned}
&\phi_0^2 + \left(\frac{c^2}{fc_0^2} - \frac{\sigma_y^0}{f\mu_0} - 1\right)^2 = 0,\\
&\phi_0' + 2\phi_0\phi_1 + \frac{f'}{f}\phi_0 = 0,\\
&\phi_1' + 2\phi_0\phi_2 + \phi_1^2 + \frac{f'}{f}\phi_1 = 0,\\
&\phi_2' + 2\phi_0\phi_3 + 2\phi_1\phi_2 + \frac{f'}{f}\phi_2 = 0,\\
&\vdots
\end{aligned} \tag{16}$$

Here, we take $\mu = \mu_0\exp(\gamma x)$ from Dlale and Erdogan (1988), i.e. $f(x) = \exp(\gamma x)$. Then the solutions of $\phi_0$, $\phi_1$, $\phi_2\ldots$ can be obtained from Eq. (16) as

$$\begin{aligned}
&\phi_0 = \pm q\mathrm{i},\\
&\phi_1 = -\frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right),\\
&\phi_2 = \gamma^2\frac{3q^4 + 6q^2 - 5}{32q^5},\\
&\vdots
\end{aligned} \tag{17}$$

where $q = \sqrt{\left(\frac{c^2}{c_0^2} - \frac{\sigma_y^0}{\mu_0}\right)\mathrm{e}^{-\gamma x} - 1}$.

If we only keep the first two solutions, i.e. solutions of $\phi_0$, $\phi_1$ and substitute them into Eq. (14), then the asymptotic solution of Eq. (13) can be easily obtained. Now considering the transformation (12), we can obtain the solution of Eq. (11) with the following form:

$$W_1(x) = \frac{\exp\left(-\frac{1}{2}\gamma x\right)}{\sqrt{q}}\left\{A_1\exp\left[\mathrm{i}\frac{2k}{\gamma}(\arctan q - q)\right] + B_1\exp\left[-\mathrm{i}\frac{2k}{\gamma}(\arctan q - q)\right]\right\}, \tag{18}$$

where $A_1, B_1$ are unknown constants. Moreover, substitution of Eq. (18) into Eq. (10) yields the solution of the mechanical displacement in the functionally graded layer:

$$w_1(x, y, t) = \frac{\exp\left(-\frac{1}{2}\gamma x\right)}{\sqrt{q}}\left\{A_1\exp\left[\mathrm{i}\frac{2k}{\gamma}(\arctan q - q)\right] + B_1\exp\left[-\mathrm{i}\frac{2k}{\gamma}(\arctan q - q)\right]\right\}\exp[\mathrm{i}k(y - ct)]. \tag{19}$$

Considering condition (3) in Section 2, the solution of the mechanical displacement in the elastic substrate can be easily obtained from Eq. (9), i.e.

$$w_2(x, y, t) = B_2\exp(-kpx)\exp[\mathrm{i}k(y - ct)], \tag{20}$$

where $B_2$ is an unknown constant and $p = (1 - c^2/c_s^2)^{1/2}$.

## 4. Solutions of the phase velocity

Substituting Eqs. (19) and (20) and their corresponding stress components into the boundary condition (1) and the continuity conditions (2), we can obtain the following algebraic equations with respect to the unknown constants $A_1, B_1, B_2$:

$$\begin{aligned}
&\mu_0 Q_h\mathrm{e}^{-\gamma h}[(P_h + \mathrm{i}S_h)A_1\mathrm{e}^{\mathrm{i}T_h} + (P_h - \mathrm{i}S_h)B_1\mathrm{e}^{-\mathrm{i}T_h}] = 0,\\
&Q_0(A_1\mathrm{e}^{\mathrm{i}T_0} + B_1\mathrm{e}^{-\mathrm{i}T_0}) - B_2 = 0,\\
&\mu_0 Q_0[(P_0 + \mathrm{i}S_0)A_1\mathrm{e}^{\mathrm{i}T_0} + (P_0 - \mathrm{i}S_0)B_1\mathrm{e}^{-\mathrm{i}T_0}] + kp\mu_2 B_2 = 0,
\end{aligned} \tag{21}$$

where

$$a = \frac{c^2}{c_0^2} - \frac{\sigma_y^0}{\mu_0},$$

$$P_0 = -\frac{\gamma}{4}\left(1 - \frac{1}{a - 1}\right), \quad P_h = -\frac{\gamma}{4}\left(1 - \frac{1}{a\mathrm{e}^{\gamma h} - 1}\right),$$

$$Q_0 = \frac{1}{(a - 1)^{1/4}}, \quad Q_h = \frac{\mathrm{e}^{\frac{1}{2}\gamma h}}{(a\mathrm{e}^{\gamma h} - 1)^{1/4}},$$

$$S_0 = k\sqrt{a - 1}, \quad S_h = k\sqrt{a\mathrm{e}^{\gamma h} - 1},$$

$$T_0 = \frac{2k}{\gamma}\left(\arctan\sqrt{a - 1} - \sqrt{a - 1}\right),$$

$$T_h = \frac{2k}{\gamma}\left(\arctan\sqrt{a\mathrm{e}^{\gamma h} - 1} - \sqrt{a\mathrm{e}^{\gamma h} - 1}\right),$$

The non-trivial solutions $A_1$, $B_1$ and $B_2$ can only exist when the determinant of the coefficient matrix of Eq. (21) equals zero, which yields

$$\tan(T_0 - T_h) = \frac{P_0 S_h - P_h S_0 + kpS_h\mu_2/\mu_0}{P_0 P_h + S_0 S_h + kpP_h\mu_2/\mu_0}. \quad (22)$$

Eq. (22) is the phase velocity equation of the Love wave propagation in the functionally graded layered non-piezoelectric half-space with a constant initial stress. It can be seen from the dispersion equation that the wave velocity $c$ is related to the wave number $k$, so the Love wave in such kind of structure is frequency dispersive.

## 5. Solutions of the stress and the displacement fields

From Eq. (21), we can obtain if we take $A_1$ as a known quantity

$$\begin{aligned} B_1 &= -\frac{P_h + iS_h}{P_h - iS_h}\exp(i2T_h)A_1, \\ B_2 &= \frac{Q_0 \exp(iT_h)2i}{P_h - iS_h}[P_h \sin(T_0 - T_h) - S_h \cos(T_0 - T_h)]A_1, \end{aligned} \quad (23)$$

then from Eqs. (19) and (20) we can obtain

$$\begin{aligned} w_1(x,y,t) = A_1 \frac{\exp\left(-\frac{1}{2}\gamma x\right)}{\sqrt{q}} \Big\{ \exp\left[i\frac{2k}{\gamma}(\arctan q - q)\right] - \frac{P_h + iS_h}{P_h - iS_h} \\ \times \exp\left[i2T_h - i\frac{2k}{\gamma}(\arctan q - q)\right] \Big\} \exp[ik(y - ct)], \end{aligned} \quad (24)$$

$$\begin{aligned} w_2(x,y,t) = A_1 \frac{Q_0 \exp(iT_h)2i}{P_h - iS_h}[P_h \sin(T_0 - T_h) - S_h \cos(T_0 - T_h)] \\ \times \exp(-kpx)\exp[ik(y - ct)]. \end{aligned} \quad (25)$$

Substitution of Eq. (24) into (7) yields the following solutions of the stress field in the functionally graded layer, i.e.

$$\begin{aligned} \sigma_{zx}^{(1)}(x,y,t) = A_1 \frac{\mu_0 \exp\left(\frac{1}{2}\gamma x\right)}{\sqrt{q}} \Big\{ \left[-\frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right) + iqk\right] \\ \times \exp\left[i\frac{2k}{\gamma}(\arctan q - q)\right] - \frac{P_h + iS_h}{P_h - iS_h}\left[-\frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right) - iqk\right] \\ \times \exp\left[i2T_h - i\frac{2k}{\gamma}(\arctan q - q)\right] \Big\} \exp[ik(y - ct)], \end{aligned} \quad (26a)$$

$$\begin{aligned} \sigma_{yz}^{(1)}(x,y,t) = A_1 \frac{\mu_0 \exp\left(\frac{1}{2}\gamma x\right) ik}{\sqrt{q}} \Big\{ \exp\left[i\frac{2k}{\gamma}(\arctan q - q)\right] - \frac{P_h + iS_h}{P_h - iS_h} \\ \times \exp\left[i2T_h - i\frac{2k}{\gamma}(\arctan q - q)\right] \Big\} \exp[ik(y - ct)]. \end{aligned} \quad (26b)$$

Similarly, we have the following solutions of the stress field in the elastic substrate from Eqs. (25) and (7), i.e.

$$\begin{aligned} \sigma_{zx}^{(2)}(x,y,t) = -A_1 \frac{Q_0 \exp(iT_h)2i}{P_h - iS_h} kp\mu_2 [P_h \sin(T_0 - T_h) \\ - S_h \cos(T_0 - T_h)] \exp(-kpx)\exp[ik(y - ct)], \end{aligned} \quad (27a)$$

$$\begin{aligned} \sigma_{yz}^{(2)}(x,y,t) = -A_1 \frac{Q_0 \exp(iT_h)2}{P_h - iS_h} k\mu_2 [P_h \sin(T_0 - T_h) \\ - S_h \cos(T_0 - T_h)] \exp(-kpx)\exp[ik(y - ct)]. \end{aligned} \quad (27b)$$

## 6. Numerical simulation and discussions

Up to now, the analytical solutions of the phase velocity, stress field and mechanical displacement field for the functionally graded layered non-piezoelectric structure have been obtained. To study the propagation behavior of the Love waves in this kind of structure and to graphically show the effects of the initial stress on the dispersion relation and the phase velocity, one material combination system is taken into account. The computational material parameters are taken from (Liu et al., 2004; Li et al., 2004) and all the material constants used in the computation are summarized in Table 1.

**Table 1**
Material parameters used in the computation

| Materials | Shear modulus, $\mu$ ($\times 10^{10}$ N/m$^2$) | Mass density, $\rho$ ($\times 10^3$ kg/m$^3$) |
|---|---|---|
| Functionally graded material layer | 3.05 * exp($-x$) | 7.50 |
| Elastic substrate | 4.43 | 5.68 |

### 6.1. Effect of the initial stress on the dispersion relation

First, we consider the effect of the initial stress on the dispersion relations. For the given material system and initial stress, there are only two variables, i.e. phase velocity $c$ and wave number $k$ in Eq. (22). The two variables are complex valued. It requires a complex root searching, for which we search the roots of phase velocity $c$ for every given wave number $k$, taking into account the existence condition of the Love waves in a layered structure. Much attention should be paid to the roots at the asymptote part and the break point part which probably make the modes become leaky, radiating bulk shear wave into the substrate. The dispersion relations for the initial tensile and compression stresses are shown in Fig. 2(a) and (b) compared with the dispersive relations without initial stress.

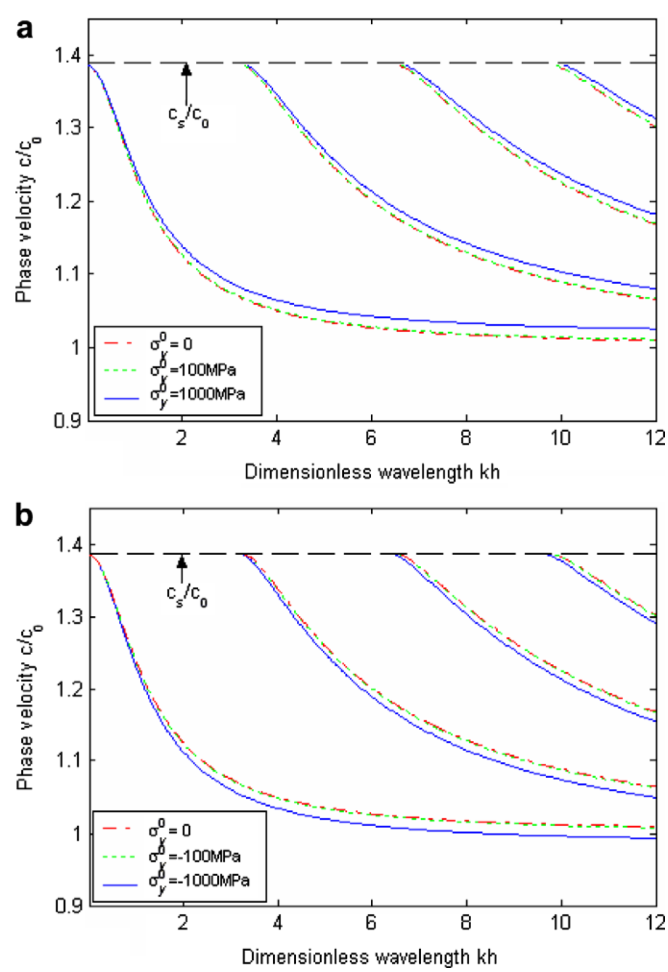


**Fig. 2.** Dispersive relations for the initial stresses compared with that without initial stress: (a) tensile stress case; (b) compression stress case.

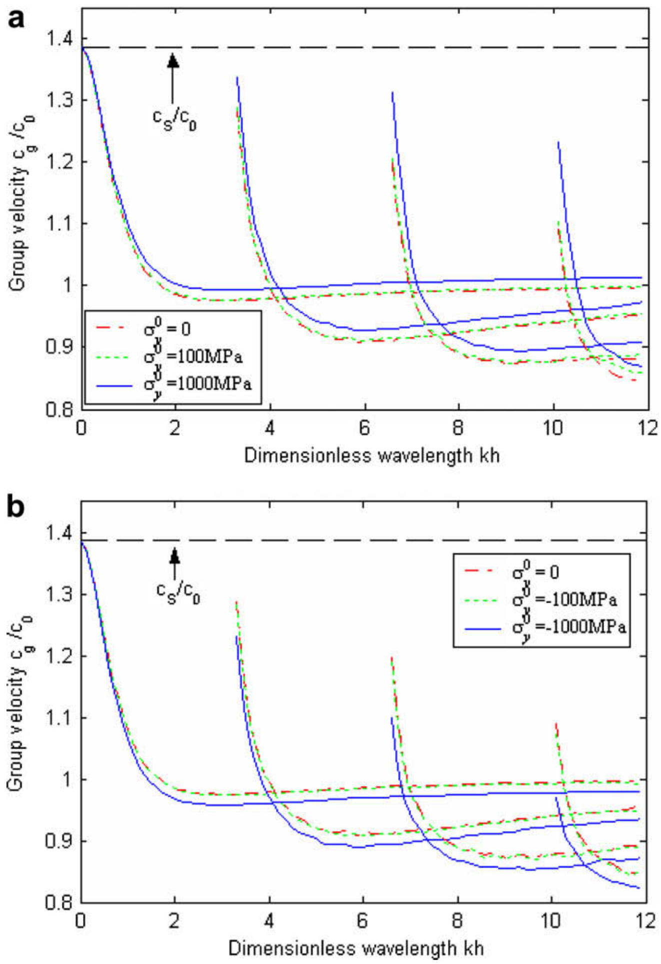


**Fig. 3.** Group velocity for the initial stresses compared with that without initial stress: (a) tensile stress case; (b) compression stress case.

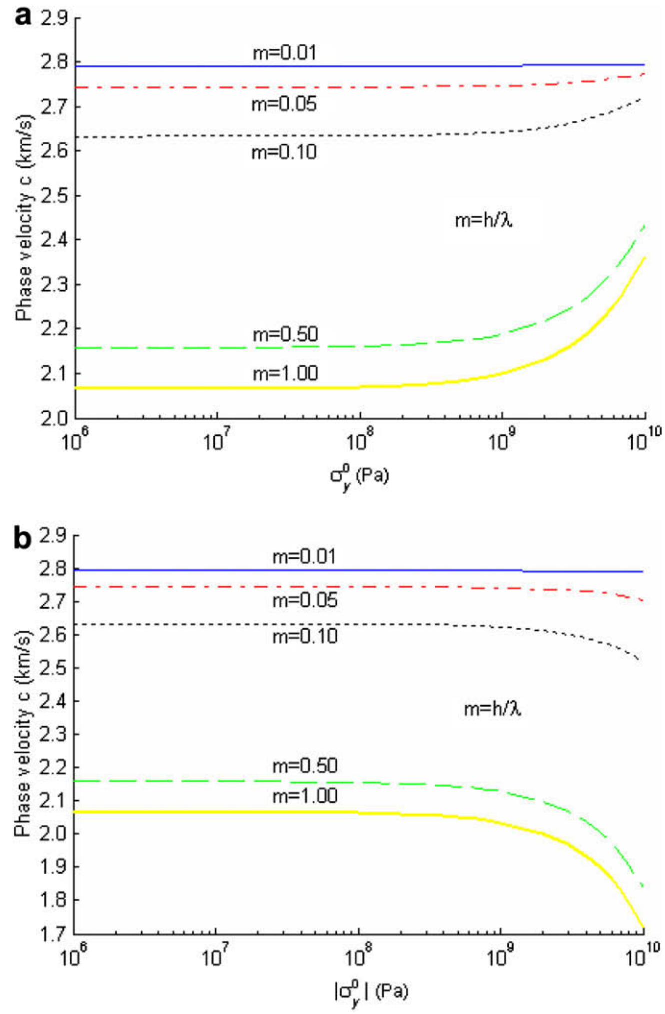


**Fig. 4.** Variations of the phase velocity $c$ vs. the initial stress $\sigma_y^0$ for different values of $m$: (a) tensile stress case; (b) compression stress case.

It can be seen that the effects of the initial stress on the dispersion relations are different for the tensile stress case and the compression stress case. The dispersive curves for the tensile stress shift to the right side of the dispersive curves without stress, while the dispersive curves for the compression stress shift to the left side of the dispersive curves without stress. But the initial stress has no effect on the cutoff frequency for each mode for both tensile case and compression cases.

To take a deep insight into the dispersion behavior of the Love wave propagation, we thus plot the group velocity for initial tensile stress and initial compression stress shown in Fig. 3(a) and (b) compared with that without initial stress. It can be readily seen that the group velocities for all the modes start from a value no more than the corresponding phase velocities, indicating a normal dispersion of the Love waves, decrease first to some values lower than the bulk shear wave velocity in the elastic substrate, and then tend to the asymptote with the increase in the dimensionless wave number. The effects of the initial stress on the group velocities seems similar to that on the dispersion curves.

### 6.2. Effect of the initial stress on the phase velocity

The phase velocity $c$ can be calculated from Eq. (22) for different values of $m$. (Here, $m$ is the ratio of the layer thickness $h$ to the wavelength $\lambda$.) The effect of the initial stress on the phase velocity $c$ is shown in Fig. 4(a) and (b) for tensile case and compression case, respectively. From the discussion on the effect of the initial stress on the dispersion curves in Section 6.1, we know that the effect on the higher modes has no much difference from that on the fundamental mode. The similar phenomenon happens to the effect of the initial stress on the phase velocity. Therefore, only the discussion on the fundamental mode of the Love waves is displayed here.

It is seen that the effect of the initial stress on the phase velocity is negligible as $|\sigma_y^0| < 10^8$ Pa, but the phase velocity increases with the increase in the magnitude of the initial tensile stress while decreases with the increase in the absolute magnitude of the initial compression stress as $|\sigma_y^0| > 10^8$ Pa. It means that we can change the phase velocity of the Love wave in the structure shown in Fig. 1 by adjusting the initial stress state during the processing of the functionally graded material layer. It is readily seen from Fig. 4(a) and (b) that the values of the ratio of the layer thickness $h$ to the wavelength $\lambda$ have a remarkable influence on the phase velocity $c$.

### 6.3. Effects of the initial stress on the stress and the mechanical displacement fields

The stress and mechanical displacement distribution in the functionally graded layered structure will be taken into account for the fundamental mode. Without loss of generality, it is assumed

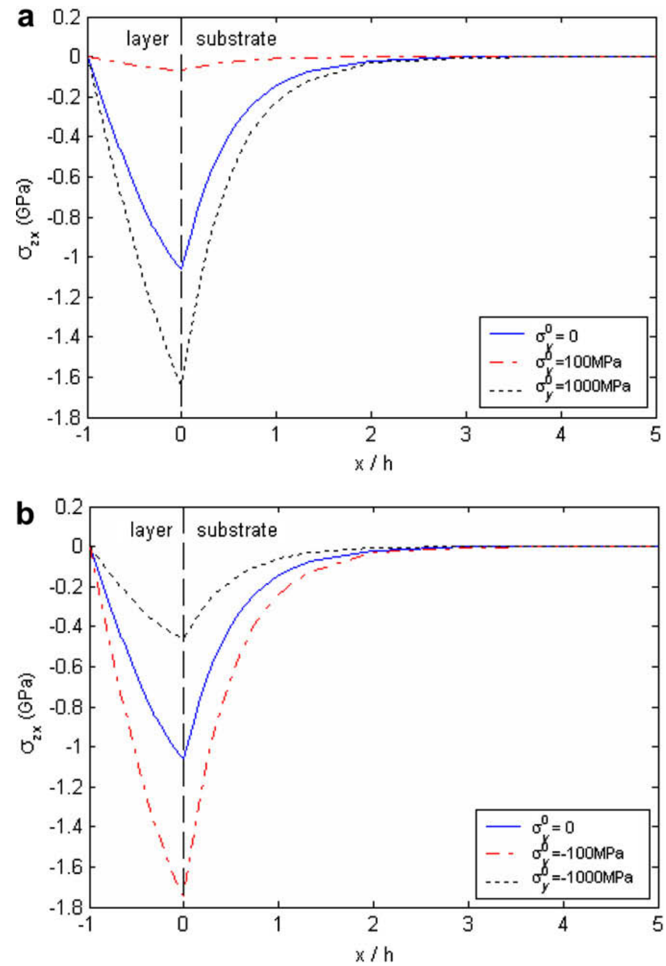


**Fig. 5.** Distribution of the shear stress $\sigma_{zx}$ over the thickness direction for different values of the initial stress: (a) tensile stress case; (b) compression stress case.

that $A_1$ = 0.1 mm and $m$ = 0.5. Variations of the stress components $\sigma_{zx}$ and $\sigma_{yz}$ with $x/h$, at $y = 0$ are shown in Figs. 5(a) and (b) and 6(a) and (b) with different initial stresses for tensile stress case and compression stress case, respectively. Also the variation of the mechanical displacement $w$ with $x/h$, at $y = 0$ is shown in Fig. 7(a) and (b).

It is seen that both the magnitude and the sign of the initial stress have a remarkable influence on the distribution of the stress and the mechanical displacement in the functionally graded layered structure. Each distribution curve of the stress and the mechanical displacement in Figs. 5–7 has not a zero point in the region $-1 < x/h < 0$ due to the fundamental mode taken into account, which also proves that the solutions we obtain in this paper are right. It can be seen that the initial stress changes not only the phase of the stress and mechanical displacement distribution but also the magnitude for both the tensile stress case and the compression stress case.

For the layered structure, small stress near the interface is expected to prevent the layered structure from debonding or fracture. While for surface acoustic wave devices/sensors that often adopt the layered structure, large surface mechanical displacement is needed to increase sensibility. In order to obtain high performance for the surface acoustic wave devices/sensors, small stress near the interface and large surface mechanical displacement can be obtained simultaneously through changing the magnitude and the sign of the initial stress in the layer during the manufacture process.

## 7. Conclusions

Usually, for the design of surface acoustic wave sensors, it is difficult to obtain high performance by using single material. So, it is necessary to seek a combination of materials to fabricate the sensors with a layered structure. However, due to the thermal mismatch of the film and the substrate material and the structural defects in films, there is an unavoidable initial stress in the layered structure during the manufacturing process. Liu et al. (2001) and Qian et al. (2004) investigated the effect of the initial stress on the propagation behavior of the Love waves in a piezoelectric layered structure. In this paper, we have taken into account the propagation behavior of the Love waves in a functionally graded layered non-piezoelectric half-space with an initial stress. Through the theoretical analysis and the numerical simulation, the following meaningful results are provided:

(1) The effects of the initial stress on the dispersion relations are different for the tensile stress case and the compression stress case. In both cases, the effects are obvious as $|\sigma_y^0| > 10^8$ Pa. But the initial stress has no influence on the cutoff frequency of each mode.
(2) The effect of the initial stress on the phase velocity is negligible as $\sigma_y^0 < 10^8$, but the phase velocity increases with the increase in the magnitude of the initial tensile stress while decreases with the increase in the magnitude of the initial compression stress as $|\sigma_y^0| > 10^8$ Pa.
(3) The initial stress changes not only the phase of the stress and the mechanical displacement distributions but also the magnitudes. In order to obtain high performance for the surface acoustic wave devices/sensors, small stress near the interface and large surface mechanical displacement can be obtained simultaneously through changing the initial stress in the functionally graded layer during the manufacture process of functionally graded materials.

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## Appendix. Discussion on the validity of WKB solutions

The WKB solution (12) consists of the wave $W_1$ traveling in the +$x$ direction and the wave $W_2$ traveling in the direction $-x$, which means

$$W_1(x) = \exp\left\{\int\left[iqk - \frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right)\right]dx\right\},$$

$$W_2(x) = \exp\left\{\int\left[-iqk - \frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right)\right]dx\right\}.$$

Substituting the expression of $W_1$ (or $W_2$) into Eq. (11), we can get

$$\left(\frac{d^2}{dx^2} + \gamma\frac{d}{dx} + k^2q^2\right)W_1 = f \neq 0.$$

Therefore, the range of validity of the WKB solution conditioned by

$$|f| \ll |k^2q^2W_1|,$$

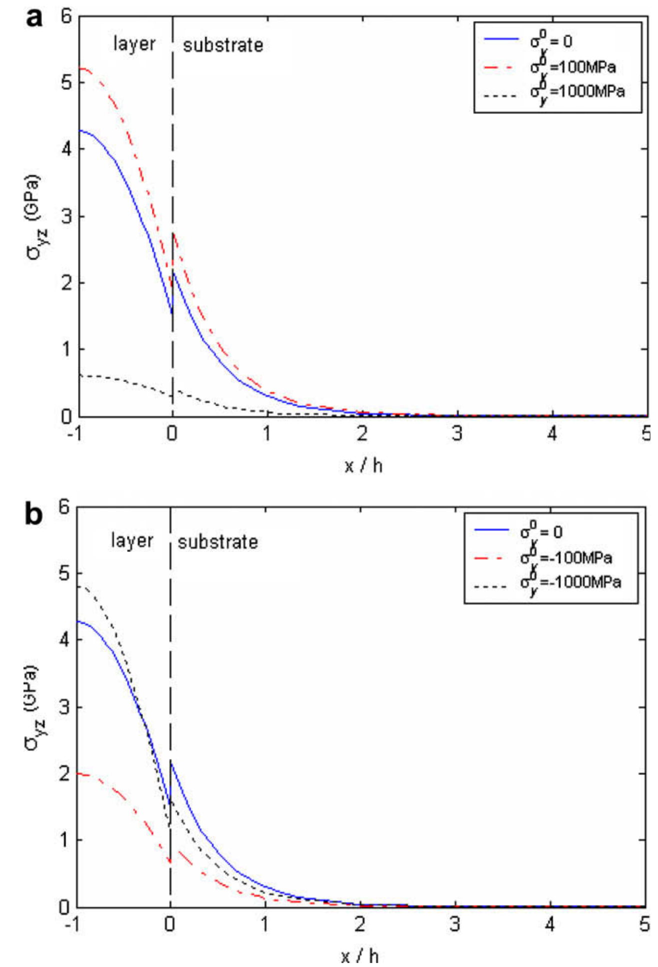


**Fig. 6.** Distribution of the shear stress $\sigma_{yz}$ over the thickness direction for different values of the initial stress: (a) tensile stress case; (b) compression stress case.

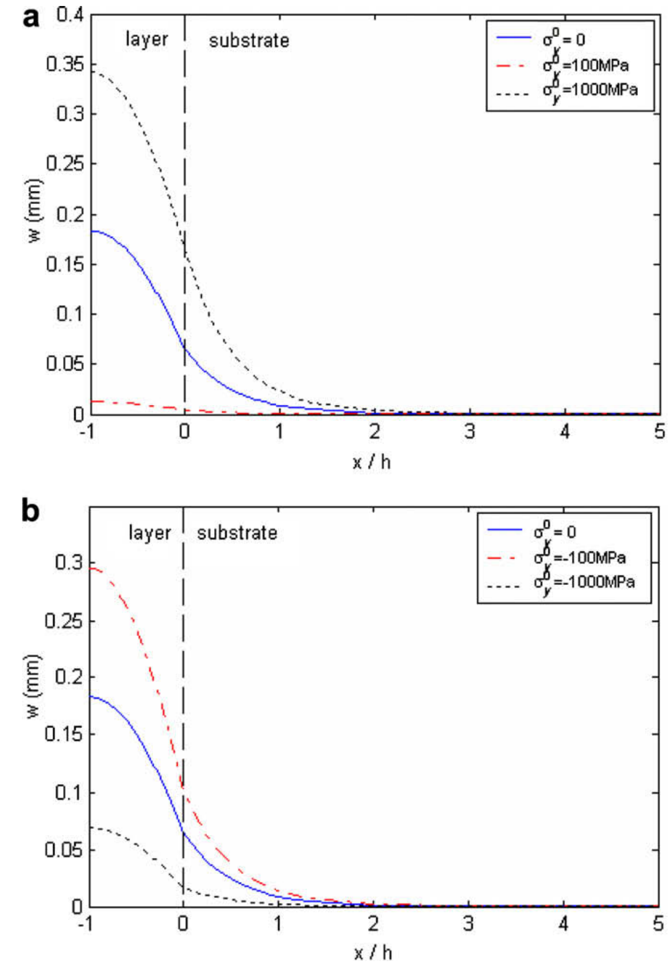


**Fig. 7.** Distribution of the displacement $w$ over the thickness direction for different values of the initial stress: (a) tensile stress case; (b) compression stress case.

that is to say

$$\Delta = \left|\frac{\left(\frac{d^2}{dx^2} + \gamma\frac{d}{dx} + k^2q^2\right)W}{k^2q^2W_1}\right| \ll 1.$$

Calculating

$$\frac{dW_1}{dx} = \left[iqk - \frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right)\right]\exp\left\{\int\left[iqk - \frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right)\right]dx\right\},$$

$$\frac{d^2W_1}{dx^2} = \left[ikq' - \frac{\gamma}{2}\frac{q'}{q^3}\right]\exp\left\{\int\left[iqk - \frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right)\right]dx\right\} + \left[iqk - \frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right)\right]^2\exp\left\{\int\left[iqk - \frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right)\right]dx\right\}$$

leads to

$$\Delta = \left|\frac{\left(\frac{d^2}{dx^2} + \gamma\frac{d}{dx} + k^2q^2\right)W}{k^2q^2W_1}\right| = \left|\frac{\left[ikq' - \frac{\gamma}{2}\frac{q'}{q^3}\right] + \left[iqk - \frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right)\right]^2 + \gamma\left[iqk - \frac{\gamma}{4}\left(1 - \frac{1}{q^2}\right)\right] + k^2q^2}{k^2q^2}\right| = \left|\frac{ikq' - \frac{\gamma}{2}\frac{q'}{q^3} + \frac{\gamma}{2}iqk\left(1 + \frac{1}{q^2}\right) + \frac{\gamma^2}{16}\left(1 + \frac{1}{q^2}\right)^2 - \frac{\gamma^2}{4}}{k^2q^2}\right|.$$

Inserting $q' = -(q^2+1)/2q$, we arrive at

$$\Delta = \left|\frac{ik\frac{\gamma}{2}\left(q + \frac{1}{q} - \frac{q^2+1}{q}\right) + \frac{\gamma^2}{16}\left(1 + \frac{1}{q^2}\right)^2 - \frac{\gamma^2}{4}\left(1 - \frac{q^2+1}{q^4}\right)}{k^2q^2}\right| = \frac{1}{k^2}\frac{\gamma^2}{16}\left|\frac{5}{q^6} + \frac{6}{q^4} - \frac{3}{q^2}\right| \ll 1.$$

In general, $0 < q < 1$, it is thus easily obtained that

$$[q^{-1} + \arctan q]_0^{-h} \ll kh,$$

which is the range of validity of the WKB solutions obtained in the paper.

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