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# A sufficient condition for pancyclability of graphs

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**Abstract**

Let $G$ be a graph of order $n$ and $S$ be a vertex set of $q$ vertices. We call $G$, $S$-pancyclable, if for every integer $i$ with $3 \leq i \leq q$ there exists a cycle $C$ in $G$ such that $|V(C) \cap S| = i$. For any two nonadjacent vertices $u, v$ of $S$, we say that $u, v$ are of distance two in $S$, denoted by $d_S(u, v) = 2$, if there is a path $P$ in $G$ connecting $u$ and $v$ such that $|V(P) \cap S| \leq 3$. In this paper, we will prove that if $G$ is 2-connected and for all pairs of vertices $u, v$ of $S$ with $d_S(u, v) = 2$, $\max\{d(u), d(v)\} \geq \frac{n}{2}$, then there is a cycle in $G$ containing all the vertices of $S$. Furthermore, if for all pairs of vertices $u, v$ of $S$ with $d_S(u, v) = 2$, $\max\{d(u), d(v)\} \geq \frac{n+1}{2}$, then $G$ is $S$-pancyclable unless the subgraph induced by $S$ is in a class of special graphs. This generalizes a result of Fan [G. Fan, New sufficient conditions for cycles in graphs, J. Combin. Theory B 37 (1984) 221–227] for the case when $S = V(G)$.
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## 1. Preliminaries and main results

We consider only finite undirected graphs without loops or multiple edges. The set of vertices of a graph $G$ is denoted by $V(G)$ or just by $V$; the set of edges by $E(G)$ or just by $E$. We use $|G|$ (the order of $G$) as a symbol for the cardinality of $V(G)$. If $H$ and $S$ are subsets of $V(G)$ or subgraphs of $G$, we denote by $N_H(S)$ the set of vertices in $H$ which are adjacent to some vertex in $S$, and set $d_H(S) = |N_H(S)|$. In particular, when $H = G$, $S = \{u\}$, then let $N(u) = N_G(u)$ and set $d(u) = d_G(u)$. Paths and cycles in a graph $G$ are considered as subgraphs of $G$. We use $G[S]$ to denote the subgraph induced by $S$.

Let $S$ be a vertex set of $G$; $v$ is called an $S$-*vertex* if $v \in S$. Following [3,5], the set $S$ is called *cyclable* in $G$ if all vertices of $S$ belong to a common cycle in $G$. Following [4], the $S$-*length* of a cycle in $G$ is defined as the number of the $S$-vertices that it contains and the graph $G$ is said to be $S$-*pancyclable*, if it contains cycles of all $S$-lengths from 3 to $|S|$. Other notations not defined in this paper can be found in [1].

From the definitions, we see that cyclability and $S$-pancyclability are generalizations of hamiltonicity and pancyclability of the whole graph (set $S = V(G)$), respectively. In recent years, people have given different definitions

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and results on cycles containing certain subsets of vertices, and some related papers can be found in [3–7]. In 1984, Fan [2] proved the following result:

**Theorem 0.** *Let $G$ be a 2-connected graph of order $n$. If $\max\{d(u), d(v)\} \geq \frac{n}{2}$ holds for all pairs $u, v$ of distance two in $G$, then $G$ is hamiltonian.*

Motivated by the above result, we will give sufficient conditions to generalize the hamiltonicity of Theorem 0 to cyclability and $S$-pancyclability. To this end, we first give the following definitions:

For any two nonadjacent vertices $x, y$ of $S$, we say that $x, y$ are of distance two in $S$, denoted by $d_S(x, y) = 2$, if there is a path $P$ in $G$ connecting $x$ and $y$ such that $|V(P) \cap S| \leq 3$.

Given an integer $r \geq 1$, $F_{4r}$ is the graph with $4r$ vertices containing a complete graph $K_{2r}$, a set of $r$ independent edges, denoted by $E_r$ and a matching between the sets of vertices of $K_{2r}$ and $E_r$ (cf. [2]).

The main results of the paper are as follows:

**Theorem 1.** *Let $G$ be a 2-connected graph of order $n$ and $S$ be a vertex set of $G$ with $|S| \geq 3$. If $\max\{d(u), d(v)\} \geq \frac{n}{2}$ holds for all pairs $u, v$ of $S$ with $d_S(u, v) = 2$, then $S$ is cyclable in $G$.*

**Theorem 2.** *Let $G$ be a 2-connected graph of order $n$ and $S$ be a vertex set of $G$ with $|S| \geq 3$. If $\max\{d(u), d(v)\} \geq \frac{n+1}{2}$ holds for all pairs $u, v$ of $S$ with $d_S(u, v) = 2$, then $G$ is $S$-pancyclable unless $|S| = 4r$ and $G[S]$ is a spanning subgraph of $F_{4r}$.*

Theorem 1 generalizes Theorem 0 if we set $S = V(G)$. Notice that $\max\{d(v) : v \in V(F_{4r})\} = 2r$ in $F_{4r}$. By Theorem 2, we have

**Corollary 3.** *Let $G$ be a 2-connected graph of order $n$. If $\max\{d(u), d(v)\} \geq \frac{n+1}{2}$ holds for all pairs $u, v$ of distance two in $G$, then $G$ is pancyclic.*

The proof of Theorem 1 will be given in Section 2 and the proof of Theorem 2 will be given in Section 3. From the proofs provided in Section 3, we believe that the following conjecture might be true.

**Conjecture.** *Let $G$ be a 2-connected graph of order $n$ and $S$ be a vertex set of $G$ with $|S| = q \geq 3$. If $\max\{d(u), d(v)\} \geq \frac{n}{2}$ holds for all pairs $u, v$ of $S$ with $d_S(u, v) = 2$, then $G$ is $S$-pancyclable unless $G$ belongs to some exceptional classes of graphs.*

In order to prove the conjecture, more precise discussions are needed and many additional cases must be considered. If the conjecture is true, it will generalize the following result proved independently by Favaron et al. in [4] and Stacho in [6]:

**Theorem 4.** *Let $G$ be a graph of order $n$ and let $S \subseteq V(G)$. If $d(u) + d(v) \geq n$ holds for all nonadjacent pairs $u, v$ of $S$, then $G$ is $S$-pancyclable or $S = V(G)$ and $G = K_{\frac{n}{2},\frac{n}{2}}$ or $|S| = 4$ and $G[S] = K_{2,2}$.*

## 2. Proof of Theorem 1

We first introduce some more notations. For a cycle (or a path) $C$ in $G$ with a given orientation and a vertex $a$ in $C$, $a^+$ and $a^-$ denote the successor and the predecessor of $a$ in $C$, respectively. For two vertices $a$ and $b$ in $C$, we define $C[a, b]$ ($C[a, b)$, $C(a, b)$, respectively) to be the subpath of $C$ from $a$ to $b$ (from $a$ to $b^-$, from $a^+$ to $b^-$, respectively). We use $\overline{C}[b, a]$ for the path from $b$ to $a$ in the reversed direction of $C$.

Theorem 1 will be proved by using the following lemmas:

**Lemma 1.** *Let $P$ be a path connecting $u$ and $v$ in $G$. If $d_P(u) + d_P(v) \geq |P|$, then there exists a cycle $C$ in $G$ such that $V(C) = V(P)$.*

**Proof.** If $uv \in E$, then Lemma 1 holds. If $uv \notin E$, then there exist two consecutive vertices $a, a^+$ in $P$ such that $av \in E$ and $a^+u \in E$. Hence there exists a cycle $C = P[u, a]v\overline{P}(v, a^+]u$ in $G$ such that $V(P) = V(C)$. □

**Lemma 2.** *Let $u, v$ be nonadjacent vertices with $d(u) + d(v) \geq n$ and $G'$ be a graph obtained by adding $uv$ to $G$. Then for any cycle $C'$ in $G'$, there exists a cycle $C$ in $G$ such that $V(C') \subseteq V(C)$.*

**Proof.** Let $C'$ be the cycle in $G'$. Then $uv \in E(G'[C'])$, otherwise $C' = C$ is the required cycle in $G$. Thus there exists a path $P$ starting from $u$ and ending at $v$ in $G$. If $N_{G-P}(u) \cap N_{G-P}(v) \neq \emptyset$, then Lemma 2 holds. If $N_{G-P}(u) \cap N_{G-P}(v) = \emptyset$, then $d_P(u) + d_P(v) \geq |P|$ as $d(u) + d(v) \geq n$. Hence Lemma 2 holds by Lemma 1. □

Now, we turn to prove Theorem 1. Let $T_1 = \{v \in S : d(v) \geq \frac{n}{2}\}$. By repeatedly applying Lemma 2, we can get that $G[T_1]$ is a clique of $G$. Let $C$ be a cycle containing $T_1$ such that $|V(C) \cap S|$ is as large as possible. If $S \subseteq V(C)$, then Theorem 1 holds. If $S \nsubseteq V(C)$, let $u \in S \cap V(G - C)$. Since $G$ is 2-connected, there are two paths $P_1 = P_1[u, w_1]$ and $P_2 = P_2[u, w_2]$ for two distinct vertices $w_1$ and $w_2$ of $C$ with all internal vertices (if any) in $G - C$ and $V(P_1) \cap V(P_2) = \{u\}$. Thus $V(C(w_1, w_2)) \cap S \neq \emptyset$ and $V(C(w_2, w_1)) \cap S \neq \emptyset$, since otherwise we can get a cycle containing all vertices of $V(C) \cap S$ and $u$, contrary to the choice of $C$. Let $x_1$ be the first vertex of $V(C(w_1, w_2)) \cap S$ from $w_1$ to $w_2$ and $x_2$ be the first vertex of $V(C(w_2, w_1)) \cap S$ from $w_2$ to $w_1$. As $T_1 \subseteq V(C)$, we have $u \in S - T_1$. If $x_i \notin T_1$ for some $1 \leq i \leq 2$, then $ux_i \in E$ and by replacing $C[w_i, x_i]P_i[w_i, u]x_i$ we can get a cycle containing all vertices of $V(C) \cap S$ and $u$, contrary to the choice of $C$. Therefore $x_i \in T_1$ for both $1 \leq i \leq 2$. Since $G[T_1]$ is a clique, $x_1x_2 \in E$ and we can get a cycle $C' = C[x_2, w_1]\overline{P}_1(w_1, u]P_2(u, w_2]\overline{C}(w_2, x_1]x_2$ in $G$ such that $|V(C') \cap S| > |V(C) \cap S|$, contrary to the choice of $C$. Hence Theorem 1 is true.

## 3. Proof of Theorem 2

By Theorem 1, there exists a cycle in $G$ containing all the vertices of $S$. Choose such a cycle $C$ with $|C|$ as small as possible and give $C$ an arbitrary orientation. If $|S| = 3$, then Theorem 2 holds. Thus we may assume that $|S| \geq 4$. Put $R = G - C$ and $|S| = q$. Let $x_1, x_2, \ldots, x_q$ be the vertices of $V(C) \cap S$, the order $1, 2, \ldots, q$ following the orientation of $C$, and consider the subscripts modulo $q$ (we use $q$ for 0 when the remainder is 0). Two $S$-vertices $x_i$ and $x_{i+1}$ are said to be *$S$-consecutive*. We use $C_l$ for a cycle of $S$-length $l$ in $G$.

In [4], it was proved:

**Theorem 5.** *Let $G$ be a graph, $S$ be a subset of $V(G)$ such that $S$ is cyclable in $G$, and let $C$ be a shortest cycle through all the vertices of $S$. If $d_C(x) + d_C(y) \geq |C| + 1$ for some pair of $S$-consecutive vertices $x$ and $y$ in $C$, then $G$ is $S$-pancyclable.*

By using the same method as that used in the proof of Theorem 5 in [4], we can get

**Lemma 3.** *Let $G$ be a graph, $S$ be a subset of $V(G)$ such that $S$ is cyclable in $G$ and let $C$ be a shortest cycle through all the vertices of $S$. If there exists some $1 \leq i \leq q$ such that $x_{i-1}x_{i+1} \in E$ and $d_C(x_i) \geq \frac{|C|+1}{2}$, then $G$ is $S$-pancyclable.*

Now, let $T_2 = \{v \in S : d(v) \geq \frac{n+1}{2}\}$. Notice that for any $1 \leq i \leq q$, $x_ix_{i+2} \in E$ when $\{x_i, x_{i+2}\} \subseteq (S - T_2)$ and $x_ix_j \in E$ for any $j \neq i$ when $N(x_i) \cap N(x_j) \neq \emptyset$ and $\{x_i, x_j\} \subseteq (S - T_2)$. It is easy to see the following:

**Remark 1.** If there is no pair of $S$-consecutive vertices $x, y$ in $C[x_i, x_j]$ $(i \neq j)$ such that $\{x, y\} \subseteq T_2$, then $G[V(C[x_i, x_j]) \cap (S - T_2)]$ is a clique of $G$.

**Lemma 4.** *If there exists at most one pair of $S$-consecutive vertices which are both in $T_2$, then Theorem 2 holds.*

**Proof.** If $|S| = 4$, since $G$ is not $S$-pancyclable, $G[S]$ must be a spanning subgraph of $F_4$. Thus Lemma 4 is true. Thus, $|S| \geq 5$. When there is one pair of $S$-consecutive vertices, say $x_q, x_1$ in $T_2$, then by Remark 1, $G[V(C[x_2, x_{q-1}]) \cap (S - T_2)]$ is a clique, especially, $x_2x_{q-1} \in E$. Thus we can easily check that $G$ is $S$-pancyclable. Hence, $\{x_i, x_{i+1}\} \cap (S - T_2) \neq \emptyset$ for any $1 \leq i \leq q$ and it is easy to check that $G$ is $S$-pancyclable as $G[V(C) \cap (S - T_2)]$ is a clique by Remark 1. □

Next, we will show three structural lemmas for some special paths containing vertices of $S$. These three lemmas will play very important roles in the proof of Theorem 2.

**Lemma 5.** *If there is a path $P = u_1 \cdots u_2 \cdots u_{p-1} \cdots u_p$ in $G[V(C)]$ such that $|V(P) \cap S| = l + 1 \geq 4$, $\{u_1, u_2, u_{p-1}, u_p\} \subseteq T_2$, $\{u_1, u_2\}$, $\{u_{p-1}, u_p\}$ are two pairs of $S$-consecutive vertices on $C$ and $(V(P(u_1, u_2)) \cup V(P(u_{p-1}, u_p))) \cap S = \emptyset$, then there exists a $C_l$ in $G$.*

**Proof.** Recall that $R = G - C$. If $N_R(u_1) \cap N_R(u_{p-1}) \neq \emptyset$ or $N_R(u_2) \cap N_R(u_p) \neq \emptyset$, then Lemma 5 holds. If $N_R(u_1) \cap N_R(u_{p-1}) = \emptyset$ and $N_R(u_2) \cap N_R(u_p) = \emptyset$, noting that $\{u_1, u_2, u_{p-1}, u_p\} \subseteq T_2$, we have

$$d_C(u_1) + d_C(u_2) + d_C(u_{p-1}) + d_C(u_p) \geq 2(|C| + 1).$$

Thus either $d_C(u_1) + d_C(u_2) \geq |C| + 1$ or $d_C(u_{p-1}) + d_C(u_p) \geq |C| + 1$. By Theorem 5, $G$ is $S$-pancyclable. Hence Lemma 5 holds. □

**Lemma 6.** *Let $P = u_1 \cdots u_p$ in $G$ such that $|V(P) \cap S| = l \geq 3$. If $\{u_1, u_p\} \subseteq T_2$ and there is no $C_l$ in $G$, then we have*

(i) $|(N(u_1) \cap N(u_p) - V(P)) \cap (V(G) - S)| = \emptyset$;
(ii) $|N(u_1) \cap N(u_p) \cap S \cap (V(G) - V(P))| \geq 2$; *and there exist a $C_4$ and a $C_{l+1}$ which contains $P$ as its subpath;*
(iii) *when $P = C[x_i, x_j]$ for some $j = l + i - 1$ $(3 \leq l \leq q - 2)$ with $\{x_i, x_j\} \subseteq T_2$, then there exists a pair of $S$-consecutive vertices $y$ and $z$ in $V(C(x_j, x_i))$ such that $y \in N(x_i)$ (or $y \in N(x_j)$) and $z \in N(x_j)$ (or $z \in N(x_i)$), and there exists a $C_{l+2}$ which contains $C[x_i, x_j]$ as its subpath.*

**Proof.** Since there is no $C_l$ in $G$, (i) is obvious and $|N(u_1) \cap V(P)| + |N(u_p) \cap V(P)| \leq |V(P)| - 1$ by Lemma 1. As $d(u_1) + d(u_p) \geq n + 1$, by (i), it is easy to check that (ii) holds.

(iii) As $d(x_i) + d(x_j) \geq n + 1$ and $S \cap R = \emptyset$, Lemma 1 and Lemma 6(i) imply $|N(x_i) \cap V(C(x_j, x_i)) \cap S| + |N(x_j) \cap V(C(x_j, x_i)) \cap S| \geq |V(C(x_j, x_i)) \cap S| + 2$. Thus (iii) holds. □

**Lemma 7.** *Let $P = u_1 u_2 \cdots u_p$ be a path in $G[V(C)]$ such that $V(P) \cap S = \{v_1, v_2, \ldots, v_l\}$, where $v_1 = u_1$, $v_l = u_p$ and the order $1, 2, \ldots, l$ follows the orientation of $P$ from $u_1$ to $u_p$. Suppose that $l \geq 5$ and there is no $C_l$ in $G$. If there exist a $C_{l+m}$ and a $C_{l+m+1}$ in $G$ ($m \in \{1, 2\}$), both of which contain $P$ as their subpath and $|V(C_{l+m}) \cap S - V(C_{l+m+1}) \cap S| \leq 1$, then for any $1 \leq i \leq l - m - 2$, we have*

(i) $v_i v_{i+m+1} \notin E$ *and* $v_i v_{i+m+2} \notin E$;
(ii) $\{v_i, v_{i+m+2}\} \cap (S - T_2) \neq \emptyset$.

**Proof.** Let $C' = C_{l+m+1}$ and $C^* = C_{l+m}$. Since $P$ is a subgraph of both $C'$ and $C^*$, we have $C'[v_i, v_{i+m+2}] = C^*[v_i, v_{i+m+2}] = P[v_i, v_{i+m+2}]$.

(i) If $v_i v_{i+m+1} \in E$ or $v_i v_{i+m+2} \in E$ for some $1 \leq i \leq l - m - 2$, then replace $C^*[v_i, v_{i+m+1}]$ or $C'[v_i, v_{i+m+2}]$ with the edge $v_i v_{i+m+1}$ or $v_i v_{i+m+2}$, we can get a $C_l$ in G, a contradiction.

(ii) Since there is no $C_l$ in $G$ and $i \leq l - m - 2$, we obtain $N_R(v_i) \cap N_R(v_{i+m+2}) \cap (V - V(C')) = \emptyset$ and $(N(v_i) \cap V(C[v_{i+2}, v_{i+m+2}])) \cup (N(v_{i+m+2}) \cap V(C(v_i, v_{i+2}))) = \emptyset$ and $v_{i+2} \notin N(v_i) \cap N(v_{i+m+2})$, which imply $|N(v_i) \cap V(C(v_i, v_{i+m+2}))| + |N(v_{i+m+2}) \cap V(C(v_i, v_{i+m+2}))| \leq |V(C(v_i, v_{i+m+2}))|$. Notice that $P' = C'[v_{i+m+2}, v_i]$ is a path with $|V(P') \cap S| = l$. We have $v_i v_{i+m+1} \notin E$ by (i) and $d_{C'}(v_i) + d_{C'}(v_{i+m+2}) < |C'|$ by Lemma 1.

If $\{v_i, v_{i+m+2}\} \subseteq T_2$, then there exist at least two vertices, say $x$ and $y$ in $N(v_i) \cap N(v_{i+m+2}) \cap (V(G) - V(C'))$. When $x \notin S$ or $y \notin S$, then there is a $C_l$ which contains $V(C'[v_{i+m+2}, v_i])$ and $x$ (or $y$), a contradiction. When $\{x, y\} \subseteq S$, then $|\{x, y\} \cap V(C^*)| \leq 1$, as $\{x, y\} \subseteq V(G) - V(C')$ and $|V(C^*) \cap S - V(C') \cap S| \leq 1$. Assume that $x \notin V(C^*)$. Then we can get a $C_l$ containing $V(C^*[v_{i+m+2}, v_i])$ and $x$ in $G$, a contradiction. Hence $\{v_i, v_{i+m+2}\} \cap (S - T_2) \neq \emptyset$ and (ii) holds. □

From Lemma 4, we may assume that $|T_2| \geq 3$ and there exist at least two pairs of $S$-consecutive vertices which are all in $T_2$. Without loss of generality, let $\{x_q, x_1\} \subseteq T_2$ such that

$|N_R(x_1) \cap N_R(x_q)| = \min\{|N_R(x) \cap N_R(y)| : x, y \in T_2$ and $x, y$ are $S$-consecutive$\}$.

If $d_C(x_1) + d_C(x_q) \geq |C| + 1$, then Theorem 2 holds by Theorem 5. Thus in the rest of the proof, we assume that $d_C(x_1) + d_C(x_q) \leq |C|$ and let $M_1 = N_R(x_1) \cap N_R(x_q)$.

**Lemma 8.** *If there exists some $1 < i \leq q - 2$ such that $\{x_i, x_{i+1}\} \subseteq T_2$ and $d_C(x_i) + d_C(x_{i+1}) \leq |C|$, then*

(i) $|(N_R(x_1) \cup N_R(x_q)) \cap N_R(x_i) \cap N_R(x_{i+1})| \geq 1$;
(ii) *there exist a cycle* $C_3$ *and a cycle* $C_4$ *in* $G$.

**Proof.** (i) By the choice of $x_1$ and $x_q$, we have $|M_1| \leq |N_R(x_i) \cap N_R(x_{i+1})|$. Thus $|R| + 1 \leq |N_R(x_1) \cup N_R(x_q)| + |M_1| \leq |N_R(x_1) \cup N_R(x_q)| + |N_R(x_i) \cap N_R(x_{i+1})| = |(N_R(x_1) \cup N_R(x_q)) \cup (N_R(x_i) \cap N_R(x_{i+1}))| + |(N_R(x_1) \cup N_R(x_q)) \cap N_R(x_i) \cap N_R(x_{i+1})| \leq |R| + |(N_R(x_1) \cup N_R(x_q)) \cap N_R(x_i) \cap N_R(x_{i+1})|$.

From the inequalities above, we can easily check that (i) holds.

(ii) Since $(N_R(x_1) \cup N_R(x_q)) \cap N_R(x_i) \cap N_R(x_{i+1}) \neq \emptyset$, without loss of generality, we may choose a vertex, say $v$, in $N_R(x_q) \cap N_R(x_i) \cap N_R(x_{i+1})$. Notice that $\{x_q, x_1, x_i, x_{i+1}\} \subseteq T_2$. Assume that there is no $C_3$ in $G$. Applying Lemma 6(ii) to the path $P = C[x_i, x_{i+1}]vx_q$, we can get $N(x_i) \cap N(x_q) \cap S - V(P) \neq \emptyset$ which implies there is a $C_3$ as $v \notin S$, a contradiction. Thus there is a $C_3$ in $G$. Now assume that there is no $C_4$ in $G$. Applying Lemma 6(ii) to the path $P' = C[x_i, x_{i+1}]vC[x_q, x_1]$, we can get a $C_4$ in $G$, a contradiction. Hence (ii) holds. □

**Lemma 9.** *If there is no* $C_l$ *in* $G$ *for some integer* $l \geq 3$, *then* $l = q - 1$.

**Proof.** By contradiction, assume that $3 \leq l \leq q - 2$. Then by Theorem 5, for any pair of $S$-consecutive vertices $x$ and $y$ in $C$, we have $d_C(x) + d_C(y) \leq |C|$.

Thus by the assumption, $M_1 \neq \emptyset$ as $d_C(x_1) + d_C(x_q) \leq |C|$ and $|\{x_{l-1}, x_l\} \cap T_2| \leq 1$ by applying Lemma 5 to $C[x_q, x_l]$.

**Case 1.** $x_l \in T_2$.

Then $x_{l-1} \notin T_2$. If $x_{l+1} \notin T_2$, then $x_{l-1}x_{l+1} \in E$ and there exists a $C_3$ in $G$. By Lemma 3, $d_C(x_l) \leq \frac{|C|}{2}$ implying $d_R(x_l) \geq \frac{|R|+1}{2}$. Since $N_R(x_l) \cap N_R(x_1) = \emptyset$ by Lemma 6(i) and $d_R(x_1) + d_R(x_q) \geq |R| + 1$, we have $2|R| + |N_R(x_q) \cap N_R(x_l)| \geq |N_R(x_1) \cup N_R(x_l)| + |N_R(x_q) \cup N_R(x_l)| + |N_R(x_q) \cap N_R(x_l)| \geq d_R(x_1) + d_R(x_q) + 2d_R(x_l) \geq 2|R| + 2$, which implies $|N_R(x_q) \cap N_R(x_l)| \geq 2$ and there exist a $C_{l+1} = vC[x_q, x_l]v$ and a $C_{l+2} = vC[x_q, x_{l-1}]x_{l+1}\overline{C}(x_{l+1}, x_l]v$ for some $v \in N_R(x_q) \cap N_R(x_l)$, both of which contain $C[x_q, x_{l-1}]$ as their subpath and $V(C_{l+1}) \cap S \subseteq V(C_{l+2})$. As $\{x_1, x_l\} \subseteq T_2$, by Lemma 6(ii), we have $l \geq 5$. Since $\{x_q, x_1\} \subseteq T_2$, by applying Lemma 7 with $m = 1$, we have $x_2x_4 \notin E$ and $\{x_3, x_4\} \subseteq S - T_2$ which implies $x_2 \in T_2$. When $l \geq 6$, then $x_5 \in S - T_2$ by Lemma 7(ii) which implies $x_3x_5 \in E$ contrary to Lemma 7(i). When $l = 5$, that is, $x_5 \in T_2$, since there is no $C_5$ in $G$, we obtain $N(x_2) \cap V(C(x_3, x_5]) = \emptyset$ and $x_3x_5 \notin E$. Also by the minimality of $|C|$, we have $|N(x_2) \cap V(C(x_2, x_3])| = 1$ and $|N(x_5) \cap V(C(x_5, x_6])| = 1$. As $d(x_2) + d(x_5) \geq n + 1$ and $x_2x_5 \notin E$ by $x_4x_6 \in E$, we have $|N(x_2) \cap N(x_5)| \geq 3$ and hence there exists some vertex, say $v$ in $N(x_2) \cap N(x_5) - V(C[x_2, x_6])$. Noticing that $x_4x_6 \in E$, we can get a $C_5$ which contains $V(C[x_2, x_6] - C(x_4, x_5)) \cup \{v\}$ whenever $v \notin S$ or $V(C[x_2, x_5]) \cup \{v\}$ whenever $v \in S$, a contradiction. Hence we have $x_{l+1} \in T_2$.

Since there is no $C_l$ in $G$, we have $N_R(x_1) \cap N_R(x_l) = \emptyset$ by Lemma 6(i) and $d_C(x_l) + d_C(x_{l+1}) \leq |C|$ by Theorem 5. Thus there is a vertex, say $w$, in $N_R(x_q) \cap N_R(x_l) \cap N_R(x_{l+1})$ and $l \geq 5$ by Lemma 8. Hence there exist a $C_{l+1}$ and a $C_{l+2}$, which contain $w$ and $C[x_q, x_l]$ as their subpath.

Since $l \geq 5$ and $\{x_q, x_1\} \subseteq T_2$, by Lemma 7 with $m = 1$, we obtain $\{x_3, x_4\} \subseteq S - T_2$. By applying Lemma 5 to $C[x_1, x_{l+1}]$, we have $x_2 \in S - T_2$ which implies $x_2x_4 \in E$, contrary to Lemma 7(i).

**Case 2.** $x_l \notin T_2$, $x_{l-1} \in T_2$.

If $x_{l-2} \notin T_2$, then $x_{l-2}x_l \in E$ and $(N_R(x_1) \cup N_R(x_q)) \cap N_R(x_{l-1}) = \emptyset$ as there is no $C_l$ in $G$. Since $2|N_R(x_1) \cup N_R(x_q)| \geq d_R(x_q) + d_R(x_1) \geq |R| + 1$, we have $d_C(x_{l-1}) \geq \frac{|C|+1}{2}$ and by Lemma 3 $G$ is $S$-pancyclable. Hence we may assume that $x_{l-2} \in T_2$. When $l \neq 3$, noting that $N_R(x_q) \cap N_R(x_{l-1}) = \emptyset$ by Lemma 6(i), $|N_R(x_1) \cap N_R(x_{l-1}) \cap N_R(x_{l-2})| \geq 1$ and $l \geq 5$ by Lemma 8. As $\{x_q, x_{l-1}\} \subseteq T_2$, by Lemma 6(ii) and (iii), there exist a $C_{l+1}$ and a $C_{l+2}$ which contain $C[x_q, x_{l-1}]$ as their subpath and $|V(C_{l+1}) \cap S - V(C_{l+2}) \cap S| \leq 1$. Thus by Lemma 7(ii) with $m = 1$, we can get $\{x_3, x_4\} \subseteq S - T_2$ and hence $l \geq 7$, $\{x_2, x_5\} \cap (S - T_2) \neq \emptyset$ which imply $x_2x_4 \in E$ or $x_3x_5 \in E$, contrary to Lemma 7(i). When $l = 3$, we have $\{x_q, x_1, x_2\} \subseteq T_2$ and $x_qx_2 \notin E$, $N_R(x_q) \cap N_R(x_2) = \emptyset$, since otherwise there exists a $C_3$. By the minimality of $|C|$, $|N(x_q) \cap V(C(x_q, x_1])| = 1$ and $|N(x_2) \cap V(C[x_1, x_2))| = 1$. Thus $|N(x_q) \cap V(C[x_2, x_q])| + |N(x_2) \cap V(C[x_2, x_q])| \geq |V(C[x_2, x_q])| + 1$. When there is some $i$ with $2 \leq i \leq q - 1$ such that either $\{x_i, x_{i+1}\} \subseteq N(x_q)$ or $\{x_i, x_{i+1}\} \subseteq N(x_2)$, then we can easily get a $C_3$ containing $V(C[x_i, x_{i+1}])$ and $x_q$ or $x_2$, a contradiction. Hence there exists some $i$ with $2 \leq i \leq q - 1$ such that $N(x_q) \cap N(x_2) \cap V(C(x_i, x_{i+1})) \neq \emptyset$ and we can find a $C_3$ containing $x_q, x_1, x_2$, a contradiction.

**Case 3.** $x_l \notin T_2$ and $x_{l-1} \notin T_2$, that is, $\{x_l, x_{l-1}\} \cap T_2 = \emptyset$.

**Case 3.1.** There is no pair of $S$-consecutive vertices $x$ and $y$ in $V(C[x_{l+1}, x_{q-1}])$ such that $\{x, y\} \subseteq T_2$.

Then $G[V(C[x_{l-1}, x_{q-1}]) \cap (S - T_2)]$ is a clique by Remark 1. Since $l \le q - 2$, $|V(C[x_{l-1}, x_{q-1}]) \cap S| \ge 3$.

If $x_{q-1} \notin T_2$, then $x_{l-1}x_{q-1} \in E$ and $x_l x_{q-1} \in E$. Thus there exist a $C_3$, and two cycles $C_{l+1}$, $C_{l+2}$ in $G[V(C)]$, which contain $C[x_{q-1}, x_{l-1}]$ as their subpath. Thus $l \ge 4$ and by Lemma 7(i) $\{x_{l-2}, x_{l-3}\} \subseteq T_2$. By applying Lemma 7 to $C[x_q, x_{l-1}]$ with $m = 1$, we have $\{x_3, x_4\} \subseteq S - T_2$ which implies $x_2 \in T_2$ and $l - 1 \ge 7$ or $l - 1 = 3$ as $\{x_{l-2}, x_{l-3}\} \subseteq T_2$. When $l \ge 8$, by Lemma 7(ii) again, $x_5 \in S - T_2$ as $x_2 \in T_2$. Thus $x_3x_5 \in E$, contrary to Lemma 7(i). Hence $l = 4$. Since there is no $C_4$ in $G$ and $\{x_3, x_4\} \subseteq N(x_{q-1})$, we have $N(x_q) \cap V(C(x_1, x_3]) = \emptyset$ and $N(x_2) \cap V(C[x_{q-1}, x_1)) = \emptyset$. Thus by applying Lemma 1 to $C[x_2, x_3]x_{q-1}C(x_{q-1}, x_q]$, we have $|N(x_q) \cap V(C[x_{q-1}, x_3])| + |N(x_2) \cap V(C[x_{q-1}, x_3])| \le |V(C[x_{q-1}, x_3])|$. Since $d(x_q) + d(x_2) \ge n + 1$, we obtain $N(x_2) \cap N(x_q) - V(C[x_{q-1}, x_3]) \ne \emptyset$. Let $w$ in $N(x_2) \cap N(x_q) - V(C[x_{q-1}, x_3])$ and we can get a $C_4$ in $G$, which contains $V(C[x_q, x_2]) \cup \{w\}$ when $w \in S$ or $V(C[x_{q-1}, x_q]) \cup V(C[x_2, x_3]) \cup \{w\}$ when $w \notin S$, a contradiction.

Hence we may assume that $x_{q-1} \in T_2$. Then $x_{q-2} \in S - T_2$ as there is no pair of $S$-consecutive vertices in $V(C[x_{l+1}, x_{q-1}]) \cap T_2$, and $x_{l-1}x_{q-2} \in E$, which implies there exists a $C_{l+2}$ in $G$ which contains $C[x_{q-2}, x_{l-1}]$ as a subpath.

If $l \le q - 3$, then $x_l x_{q-2} \in E$ and there exists a $C_3$. When $x_{l-2} \notin T_2$, then $x_{l-2}x_{q-2} \in E$ and there exist a $C_4$ and a $C_{l+1}$ in $G$ which contains $C[x_{q-2}, x_{l-2}]$ as its subpath. When $x_{l-2} \in T_2$, since $x_{q-1} \in T_2$, by Lemma 6(ii), there exist a $C_4$ and a $C_{l+1}$ in $G$ which contains $C[x_{q-1}, x_{l-2}]$ as its subpath. Thus in both subcases, we have $l \ge 5$ and $|V(C_{l+1}) \cap S - V(C_{l+2}) \cap S| \le 1$. By using Lemma 7 with $m = 1$ and the facts that $x_{l-1} \in S - T_2$ and $\{x_{q-1}, x_q, x_1\} \subseteq T_2$, we can get $\{x_2, x_3, x_4\} \subseteq S - T_2$ and $x_2x_4 \in E$ which implies there exists a $C_l$ in $G$, a contradiction.

If $l = q - 2$ and $x_{l-2} \notin T_2$, then $x_{l-2}x_l \in E$ which implies there exist a $C_3$ and a $C_{l+1}$ in $G$ which contains $C[x_{q-1}, x_{l-2}]$ as its subpath. When $l \ge 5$, by Lemma 7(ii) and $\{x_{q-1}, x_q, x_1\} \subseteq T_2$ we can get $\{x_2, x_3, x_4\} \cap T_2 = \emptyset$ and $x_2x_4 \in E$. Thus Lemma 7(i) implies $l \le 5$. Whenever $l = 5$, $q = 7$ as $l = q - 2$ and $x_2x_5 \in E$ as $x_5 \notin T_2$. So we can get a $C_5 = C[x_5, x_2]x_5$, a contradiction. Hence $l = 4$ and $q = 6$. Since there is no $C_4$ in $G$ and $x_2x_4 \in E$, we can derive that $d_C(x_1) = 2$ and symmetrically $d_C(x_5) = 2$ by the minimality of $|C|$. Since $\{x_5, x_1\} \subseteq T_2$, we have $N_R(x_1) \cap N_R(x_5) \ne \emptyset$ and consequently we can get a $C_4$ containing $V(C[x_1, x_2]) \cup V(C[x_4, x_5])$ and $w$ for some $w$ in $N_R(x_1) \cap N_R(x_5)$, a contradiction.

Hence $x_{l-2} \in T_2$ and $\{x_{l-1}, x_l\} \subseteq N(x_{q-1}) \cap N(x_{l-2})$ by applying Lemma 6(ii) to $C[x_{q-1}, x_{l-2}]$, which implies there exist a $C_3$, a $C_4$ and a $C_{l+1}$ in $G$ which contains $C[x_{q-1}, x_{l-2}]$ as its subpath and consequently, we can derive a contradiction as before by Lemma 7.

**Case 3.2.** There exists a pair of $S$-consecutive vertices $x$ and $y$ in $V(C[x_{l+1}, x_{q-1}])$ such that $\{x, y\} \subseteq T_2$.

Choose $q - 1 > t \ge l + 1$ such that $x_t$ and $x_{t+1}$ are a pair of $S$-consecutive vertices with $\{x_t, x_{t+1}\} \subseteq T_2$ and $t$ as small as possible. Then by Remark 1, we have that $G[V(C[x_{l-1}, x_t)) \cap (S - T_2)]$ is a clique of $G$ which implies $x_{t-1}x_{l-1} \in E$. Let $P = C[x_1, x_{l-1}]x_{t-1}$. By Theorem 5 and Lemma 8, $|(N_R(x_1) \cup N_R(x_q)) \cap N_R(x_t) \cap N_R(x_{t+1})| \ge 1$ and $l \ge 5$. We distinguish the following two subcases.

**Case 3.2.1.** $|N_R(x_1) \cap N_R(x_t) \cap N_R(x_{t+1})| \ge 1$.

Then we can get a $C_{l+1}$ and a $C_{l+2}$ in $G$, both of which contain $P$ as their subpath. Notice that $l \ge 5$. By Lemma 7, we have $x_4 \in S - T_2$ which implies $x_2 \in T_2$. Using Lemma 5 for the path $P' = C[x_2, x_{l-1}]C[x_{t-1}, x_{t+1}]$, we have $x_3 \in S - T_2$ as $\{x_2, x_{t+1}, x_t\} \subseteq T_2$. Thus by Lemma 7 and $\{x_q, x_1, x_2\} \subseteq T_2$, we have $x_j \in S - T_2$ which implies $x_3x_j \in E$, where $j = 5$ when $l \ge 6$ or $j = t - 1$ when $l = 5$, contrary to Lemma 7(i).

**Case 3.2.2.** $|N_R(x_1) \cap N_R(x_t) \cap N_R(x_{t+1})| = 0$.

By Lemma 8(i), there is a vertex $w$ in $N_R(x_q) \cap N_R(x_t) \cap N_R(x_{t+1})$. Thus there exist a $C_{l+2} = wC[x_q, x_{l-1}]C[x_{t-1}, x_t]w$ and a $C_{l+3} = wC[x_q, x_{l-1}]C[x_{t-1}, x_{t+1}]w$ which contain $P = C[x_1, x_{l-1}]x_{t-1}$ as their subpath. Since $l \ge 5$ and $\{x_q, x_1\} \subseteq T_2$, by applying Lemma 7(ii) with $m = 2$, we have $\{x_4, x_j\} \subseteq S - T_2$ where $j = 5$ when $l > 5$ and $j = t - 1$ when $l = 5$. If $x_2 \notin T_2$, then $x_2x_5 \in E$ or $x_2x_{t-1} \in E$ contrary to Lemma 7(i). Thus $x_2 \in T_2$. For the same reason as above, we have $x_3 \notin T_2$ by Lemma 5 and $x_3x_j \in E$. Since $\{x_2, x_t\} \subseteq T_2$, applying Lemma 6(ii) to the path $P^* = C[x_2, x_{l-1}]C[x_{t-1}, x_t]$, we can get a $C_{l+1}$ containing $P^*$ as a subpath. Noticing that $x_3x_j \in E$, we can get a $C_l$ in $G$, a contradiction. □

Now, we turn to prove Theorem 2. By Lemma 9, there exists a $C_l$ in $G$ for $3 \le l \le q-2$. If there exists a $C_{q-1}$, then Theorem 2 holds. Thus in the rest of the proof we assume that there is no $C_{q-1}$ in $G$, which implies for any $1 \le i \le q$, $x_{i-1}x_{i+1} \notin E$. $N_R(x_{i-1}) \cap N_R(x_{i+1}) = \emptyset$ and consequently, $|\{x_{i-1}, x_{i+1}\} \cap T_2| \ge 1$ as $|V(C[x_{i-1}, x_{i+1}]) \cap S| = 3$.

If there exists some $1 \leq i \leq q$ such that $\{x_{i-1}, x_{i+1}\} \subseteq T_2$, then $d_C(x_{i-1}) + d_C(x_{i+1}) \geq |C| + 1$. Since $N(x_{i-1}) \cap V(C(x_i, x_{i+1}]) = \emptyset$ and $N(x_{i+1}) \cap V(C[x_{i-1}, x_i)) = \emptyset$, we obtain $d_P(x_{i-1}) + d_P(x_{i+1}) \geq |P|$ for $P = C[x_{i+1}, x_{i-1}]$. By Lemma 1 we can get a $C_{q-1}$ in $G$, a contradiction. Thus we may assume that for any $1 \leq i \leq q$, $|\{x_{i-1}, x_{i+1}\} \cap T_2| = 1$. Noting that $\{x_q, x_1\} \subseteq T_2$, we obtain that $q = 4r$, $\{x_2, x_3\} \subseteq S - T_2$, $\{x_{4p}, x_{4p+1}\} \subseteq T_2$ and $\{x_{4p+2}, x_{4p+3}\} \subseteq S - T_2$ implying that $x_{4p+2}x_{4p+3} \in E$ for any $1 \leq p \leq r - 1$ as $C$ is a cycle which contains $S$ with $|C|$ as small as possible.

In order to show that $G[S]$ has the exceptional structure described in the statement of Theorem 2, we need to show that $N(x_{4p+2}) \cap S \subseteq \{x_{4p+1}, x_{4p+3}\}$ and $N(x_{4p+3}) \cap S \subseteq \{x_{4p+2}, x_{4p+4}\}$ for any $0 \leq p \leq r - 1$.

Since there is no $C_{q-1}$, $x_{4p+1}x_{4p+3} \notin E$ and $x_{4p+2}x_{4p+4} \notin E$. Assume that $(N(x_{4p+2}) \cup N(x_{4p+3})) \cap \{x_{4s+1}, x_{4s+2}, x_{4s+3}, x_{4s+4}\} \neq \emptyset$ for some $p$ and $s$ with $1 \leq p \neq s \leq q$, then $G[\{x_{4p+2}, x_{4p+3}, x_{4s+2}, x_{4s+3}\}]$ is a clique since $\{x_{4p+2}, x_{4p+3}, x_{4s+2}, x_{4s+3}\} \subseteq S - T_2$.

Let $P = C[x_{4p+4}, x_{4s+2}]x_{4p+2}x_{4s+3}C(x_{4s+3}, x_{4p+1}]$. Then we have $|V(P) \cap S| = q - 1$. When $d_P(x_{4p+1}) + d_P(x_{4p+4}) \geq |P|$, then we can get a $C_{q-1}$ in $G$ by Lemma 1, a contradiction. Thus $d_P(x_{4p+1}) + d_P(x_{4p+4}) < |P|$. Since $\{x_{4p+1}, x_{4p+4}\} \subseteq T_2$, there is a vertex, say $w$, in $N_{G-P}(x_{4p+1}) \cap N_{G-P}(x_{4p+4}) - \{x_{4p+3}\}$ and we can get a $C_{q-1}$ containing $V(P)$ and $w$ in $G$, a contradiction.

Hence, we have $N(x_{4p+2}) \cap S \subseteq \{x_{4p+1}, x_{4p+3}\}$ and $N(x_{4p+3}) \cap S \subseteq \{x_{4p+2}, x_{4p+4}\}$ for any $0 \leq p \leq r - 1$ and consequently, we can derive that $G[S]$ is a spanning subgraph of $F_{4r}$.

Therefore, the proof of Theorem 2 is complete.

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