The best constant of Sobolev inequality on a bounded interval

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Abstract
The best constants $C_{m,j}$ of Sobolev embedding of $H^m(0,a)$ into $C^j[0,a]$ ($0 \leq j \leq m - 1$) are obtained. Especially, when $a = \infty$, these constants can be represented in a closed form.

Keywords: Sobolev inequality; Best constant; Reproducing kernel

1. Introduction

Let $H^m(0,a)$ be the Hilbertian Sobolev space of order $m$, associated with the inner product

$$ (u, v) := \int_0^a \sum_{s=0}^m u^{(s)}(x)v^{(s)}(x) \, dx, \tag{1} $$

where $u$ and $v$ are arbitrary elements of $H^m(0,a)$, and $u^{(s)}$ denotes the $s$th derivative of $u$ in a distributional sense. The purpose of this note is to investigate the best constant $C_{m,j}$ of the Sobolev inequality

$$ \left( \sup_{0 \leq y \leq a} |u^{(j)}(y)| \right)^2 \leq C \|u\|^2, \tag{2} $$

for $j$ satisfying $0 \leq j \leq m - 1$. For the case $j = 0$, Richardson [7] showed that

$$ C_{m,0} = \frac{2}{m+1} \sum_{k=1}^m \coth\left( \frac{k\pi}{m+1} \right) \sin^3\left( \frac{k\pi}{m+1} \right). $$

We extend this result as follows:

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Theorem 1. The best constant $C_{m,j}$ for the inequality (2) is

$$C_{m,j} = \frac{2}{m+1} \sum_{k=1}^{m} \coth \left( a \sin \left( \frac{k \pi}{m+1} \right) \right) \sin \left( \frac{k \pi}{m+1} \right) \sin^2 \left( \frac{k(j+1) \pi}{m+1} \right).$$

Especially for the case, $a = \infty$ i.e. the case of a half line, $C_{m,j}$ can be represented in a closed form.

Theorem 2. If $a = \infty$, then

$$C_{m,j} = \frac{1}{m+1} \left\{ \frac{\sin \frac{\pi}{m+1}}{2 \sin \left( \frac{(j+1/2) \pi}{m+1} \right) \sin \left( \frac{(j+3/2) \pi}{m+1} \right)} + \frac{\cos \frac{\pi}{2(m+1)}}{\sin \left( \frac{\pi}{2(m+1)} \right)} \right\}.$$  

We remark that the proof of Theorem 1 gives concise proof of the result of [7] (the result corresponds to the case $j = 0$); see Section 2.2. Other results related to Theorems 1 and 2 are Gabushin [2] and Kalyabin [5,6]. In [5,6], Kalyabin obtained simple representation of the best constant of the Sobolev inequality,

$$(\sup_{0 \le y < \infty} |u^{(j)}(y)|)^2 \le C_m \|u^{(m)}\|_{L^2(0,\infty)}^2 + \|u\|_{L^2(0,\infty)}^2,$$

for $j$ satisfying $0 \le j \le m - 1$, as

$$C_{m,j} = \left( \sin \frac{2(j+1)}{2m} \right)^{-1} \left( \prod_{k=1}^{j} \cot \frac{\pi k}{2m} \right)^{2}.$$  

2. Proofs of theorems

From (2) and Riesz’ representation theorem, we see that there uniquely exists the function $K_j(\cdot, y) \in H^m(0,a)$ such that for all $u \in H^m(0,a)$

$$u^{(j)}(y) = (u(\cdot), K_j(\cdot, y))$$

holds. We call such $K_j(\cdot, y)$, the $j$th reproducing kernel of $H^m(0,a)$ ($K_0(\cdot, y)$ is the reproducing kernel in a usual sense; see [1]). For abbreviation, let us define

$$K_j^{(j)}(y, y) := \partial_x^j K_j(x, y)|_{x=y}.$$  

Then, by the Cauchy–Schwarz inequality, we have

$$\left( \sup_{0 \le y \le a} |u^{(j)}(y)| \right)^2 \le \sup_{0 \le y \le a} \|K_j(\cdot, y)\|_2^2 \|u\|_2^2 = \sup_{0 \le y \le a} K_j^{(j)}(y, y) \|u\|^2.$$  

Hence $C_{m,j} \le \sup_{0 \le y \le a} K_j^{(j)}(y, y)$. Suppose that the supremum of $K_j^{(j)}(y, y)$ is attained at some $y_0 \in [0, a]$. Then, by (2), we have

$$|K_j^{(j)}(y_0, y_0)|^2 \le \sup_{0 \le y \le a} |K_j^{(j)}(y, y_0)|^2 \le C_{m,j} \|K_j(\cdot, y_0)\|_2^2 = C_{m,j} K_j^{(j)}(y_0, y_0).$$

Since $0 < C_{m,j} \le K_j^{(j)}(y_0, y_0)$, we have from (8), $K_j^{(j)}(y_0, y_0) \le C_{m,j}$, and hence

$$C_{m,j} = K_j^{(j)}(y_0, y_0).$$  

In fact, we have $y_0 = 0$. We prove this fact separately for the cases $a = \infty$ and $a < \infty$. 

2.1. The case $a = \infty$

**Lemma 3.** Let $a = \infty$, then $C_{m,j} = K^{(j)}_j(0,0)$.

**Proof.** Let $y > 0$ and $u$ be an arbitrary element of $H^m(0,\infty)$. Moreover, let $\tilde{u}(\cdot) = u(\cdot + y)$. Since $\tilde{u} \in H^m(0,\infty)$, we have

$$|u^{(j)}(y)| = |(\tilde{u}(\cdot), K_j(\cdot, 0))| \leq K^{(j)}_j(0,0)\|\tilde{u}\| \leq K^{(j)}_j(0,0)\|u\|.$$ 

Thus, $C_{m,j} \leq K^{(j)}_j(0,0)$. Again, by the same argument as above, we have $C_{m,j} = K^{(j)}_j(0,0)$ and the function which attains the best constant is $K_j(\cdot, 0)$. □

So, let us construct $K_j(\cdot, 0)$ concretely. It is easy to see, with integration by parts, that $K_j(\cdot, 0)$ is obtained as a function satisfying

$$\sum_{k=0}^{m} (-1)^k u^{(2k)} = 0,$$  

with boundary conditions,

$$\sum_{p=s}^{m-1} (-1)^{p-s} u^{(2p-s+1)}(0) = -\delta_{s,j} \quad (0 \leq s \leq m-1).$$

Therefore, $K_j(\cdot, 0)$ is given by

$$K_j(x,0) = \sum_{k=1}^{m} c_ke^{\lambda_k x},$$

where $\lambda_k$ and $c_k$ satisfy

$$\begin{cases}
\lambda_k = ie^{\frac{-\pi i}{m+1}}, \\
\sum_{k=1}^{m} \frac{\lambda_k^{s+1} + (-1)^s \lambda_k^{-(s+1)}}{1 + \lambda_k^2} = -\delta_{s,j} \quad (0 \leq s \leq m-1)
\end{cases}$$

$(i$ is the imaginary unit $\sqrt{-1})$. Here, we prepare the following lemma.

**Lemma 4.**

$$\sum_{k=1}^{m} (\lambda_k^{s+1} + (-1)^s \lambda_k^{-(s+1)}) (\lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)}) = (-1)^j \cdot 2(m+1)\delta_{s,j}.$$  

**Proof.** Let $\theta := \pi/(m+1)$, Then, the left-hand side of (14) becomes

$$= \sum_{k=1}^{m} (\lambda_k^{s+j+2} + (-1)^{s+j} \lambda_k^{-(s+j+2)}) + \sum_{k=1}^{m} ((-1)^j \lambda_k^{s-j} + (-1)^j \lambda_k^{-(s-j)})$$

$$= (-1)^{s+j} \sum_{k=1}^{m} e^{i(s+j+2)k\theta} + e^{-i(s+j+2)k\theta} + \sum_{k=1}^{m} e^{i(s-j)k\theta} + e^{-i(s-j)k\theta}.$$
Proof of Theorem 2. Using (16), we can prove Theorem 2.

Similarly, the second term of (15) is 

\(- \frac{e^{i(s+j+2)\theta}}{1 - e^{i(s+j+2)\theta}} \left(1 + (-1)^{s-j} \right)\), if \( s \neq j \). Therefore, (15) equals

\[-(1)^{s+j} \left(1 + (-1)^{s-j} \right) = 0, \quad s \neq j, \]

\[-(1)^j \cdot 2 + (-1)^j \cdot 2m = (-1)^j \cdot 2(m+1), \quad s = j. \]

Hence, \( c_k \) in (13) is

\[c_k = \frac{(-1)^j+1(1 + \lambda_k^2)(\lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)})}{2(m+1)}.
\]

Using (16), we can prove Theorem 2.

**Proof of Theorem 2.** From (16), we have

\[C_{m,j} = K_j^{(j)}(0,0) = \sum_{k=1}^{m} c_k \lambda_k^j\]

\[= \frac{(-1)^j+1}{2(m+1)} \sum_{k=1}^{m} \left(\lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)}\right)\]

\[= \frac{1}{2(m+1)} \sum_{k=1}^{m} \left((-1)^j+1\left(\lambda_k^{j+1} + \lambda_k^{j+3}\right) - (\lambda_k^{-1} + \lambda_k)\right)\]

\[= \frac{1}{2(m+1)} \sum_{k=1}^{m} \left((-1)^j+1\left(1 + \lambda_k^2\right)\left(\lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)}\right)\right)\]

\[= \frac{(-1)^j}{m+1} \left\{ -\frac{e^{i(2j+1)\theta}}{1 - e^{i(2j+1)\theta}} + \frac{e^{i(2j+3)\theta}}{1 - e^{i(2j+3)\theta}} \right\} - \frac{1 + e^{i\theta}}{1 - e^{i\theta}}\]

\[= \frac{(-1)^j}{m+1} \left\{ -e^{-i\theta} + e^{i\theta} \right\} \frac{1}{(e^{-i(j+\frac{1}{2})\theta} - e^{-i(j+\frac{1}{2})\theta})(e^{-i(j+\frac{1}{2})\theta} - e^{i(j+\frac{1}{2})\theta})} - \frac{e^{-i\theta} + e^{i\theta}}{e^{-i\theta} - e^{i\theta}}\]

\[= \frac{1}{m+1} \left\{ -\frac{\sin \frac{\pi}{m+1}}{2\sin \frac{(j+1/2)\pi}{m+1}} + \frac{\cos \frac{\pi}{2(m+1)}}{\sin \frac{\pi}{2(m+1)}} \right\}. \]

2.2. The case \( a < \infty \)

In this case, it is not so easy to show

\[\sup_{0 \leq y \leq a} K_j^{(j)}(y, y) = K_j^{(j)}(0,0)\]

(17) as in the case \( a = \infty \). Although, Hegland and Marti [3, Theorem 5] show this fact for \( j = 0 \), the proof (essentially) does not seem applicable to our case \( j \geq 1 \). Nevertheless, the method developed in Marti [4] which considers the simplest case \( m = 1 \) and \( j = 0 \) applies to our case. To follow the argument of [4], we first compute \( K_j^{(j)}(0,0) \). As in the case of \( a = \infty \), \( K_j(-,0) \) is given as a function satisfying (10) and boundary conditions.
\[
\begin{align*}
\sum_{p=s}^{m-1} (-1)^p s u^{2p-s+1}(0) &= -\delta_{s,j}, \\
\sum_{p=s}^{m-1} (-1)^p s u^{2p-s+1}(a) &= 0
\end{align*}
\] (0 \leq s \leq m - 1).

Therefore
\[
K_j(x, 0) = \sum_{k=1}^{2m} c_k e^{\lambda_k x},
\]
where \(\lambda_k\) and \(c_k\) satisfy
\[
\begin{align*}
\lambda_k &= \begin{cases} 
    i e^{\frac{k\pi i}{m+1}} & (1 \leq k \leq m), \\
    i e^{-\frac{k\pi i}{m+1}} & (m + 1 \leq k \leq 2m), 
\end{cases} \\
\left\{ \begin{aligned}
    \sum_{k=1}^{2m} c_k e^{\lambda_k} \frac{\lambda_k^{s+1} + (-1)^s \lambda_k^{-(s+1)}}{1 + \lambda_k^2} &= -\delta_{s,j}, \\
    \sum_{k=1}^{2m} c_k e^{\lambda_k} \frac{\lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)}}{1 + \lambda_k^2} &= 0
\end{aligned} \right. 
\] (0 \leq s \leq m - 1).

We can solve Eq. (21) explicitly.

**Lemma 5.** The solution of (21) is
\[
c_k = \begin{cases} 
    (-1)^{j+1} e^{\lambda_k + a} \left( 1 + \lambda_k^2 \right) \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right) & (1 \leq k \leq m), \\
    (-1)^{j+1} e^{\lambda_k - a} \left( 1 + \lambda_k^2 \right) \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right) & (m + 1 \leq k \leq 2m).
\end{cases}
\]

**Proof.** Noting the relation \((\lambda_k^{s+1} + (-1)^s \lambda_k^{-(s+1)}) = -(\lambda_k^{s+1} + (-1)^s \lambda_k^{-(s+1)})\), we have
\[
\begin{align*}
    c_k e^{\lambda_k} \frac{\lambda_k^{s+1} + (-1)^s \lambda_k^{-(s+1)}}{1 + \lambda_k^2} + c_{k+m} e^{\lambda_{k+m}} \frac{\lambda_{k+m}^{s+1} + (-1)^s \lambda_{k+m}^{-(s+1)}}{1 + \lambda_{k+m}^2} &= 0 \\
    &= (-1)^{j+1} e^{\lambda_k + a} \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right) \left( 1 + \lambda_k^2 \right) \\
    &= (-1)^{j+1} e^{\lambda_k - a} \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right) \left( 1 + \lambda_k^2 \right) \\
    &= \frac{(-1)^{j+1}}{2(m+1)} \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right) \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right) \\
    &= \frac{(-1)^{j+1}}{2(m+1)} \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right) \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right).
\end{align*}
\]

Thus, from Lemma 2, we obtain (21)(a). Next, for (21)(b), we have
\[
\begin{align*}
    c_k e^{\lambda_k} \frac{\lambda_k^{s+1} + (-1)^s \lambda_k^{-(s+1)}}{1 + \lambda_k^2} + c_{k+m} e^{\lambda_{k+m}} \frac{\lambda_{k+m}^{s+1} + (-1)^s \lambda_{k+m}^{-(s+1)}}{1 + \lambda_{k+m}^2} &= 0 \\
    &= (-1)^{j+1} e^{\lambda_k + a} \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right) \left( 1 + \lambda_k^2 \right) \\
    &= (-1)^{j+1} e^{\lambda_k - a} \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right) \left( 1 + \lambda_k^2 \right) \\
    &= \frac{(-1)^{j+1}}{2(m+1)} \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right) \left( \lambda_k^{j+1} + (-1)^j \lambda_k^{-(j+1)} \right) \\
    &= 0.
\end{align*}
\]

So, we have proven the lemma. \(\square\)
Using (22), we obtain
\[
K_j^{(j)}(0, 0) = \left\{ \begin{array}{l}
\sum_{k=1}^{m} c_k \lambda^j_k = \sum_{k=1}^{m} (c_k \lambda^j_k + c_{k+m} \lambda^{j+1}_{k+m}) \\
= \frac{(-1)^{j+1}}{2(m+1)} \sum_{k=1}^{m} \left[ \frac{e^{\lambda_{k+m}a}}{e^{\lambda_{k+m}a} - e^{\lambda_ka}} \lambda^j_k (1 + \lambda^2_k (\lambda^j_k + (-1)^{j+1} \lambda^{-(j+1)}_k) \right] \\
+ \frac{e^{\lambda_ka}}{e^{\lambda_{k+m}a} - e^{\lambda_ka}} \lambda^j_{k+m} (1 + \lambda^2_{k+m} (\lambda^j_{k+m} + (-1)^{j+1} \lambda^{-(j+1)}_{k+m})) \right].
\end{array} \right.
\]
(23)

Let \( \alpha = a \sin(k\theta) \), and \( \beta = a \cos(k\theta) \). Then, we have
\[
\lambda_k a = a e^{k\theta i} = ai \left( \cos(k\theta) + i \sin(k\theta) \right) = -\alpha + i\beta,
\]
\[
\lambda_{k+m} a = a e^{-k\theta i} = ai \left( \cos(k\theta) - i \sin(k\theta) \right) = \alpha + i\beta,
\]
and hence \( e^{\lambda_k a} = e^{-\alpha} (\cos(\beta) + i \sin(\beta)) \) and \( e^{\lambda_{k+m} a} = e^\alpha (\cos(\beta) + i \sin(\beta)) \). So, the right-hand-side of (23) is
\[
\sum_{k=1}^{m} \left[ \frac{e^\alpha}{e^{\alpha} - e^{-\alpha}} (-1)^j e^{k\theta i} (2 \sin^2(k\theta) - i \sin(2k\theta)) (-2(-1)^j \sin((j+1)k\theta)) \right]
\]
Taking the real part of (24), we have
\[
K_j^{(j)}(0, 0) = 4(-1)^{j+1} \sum_{k=1}^{m} \frac{e^\alpha + e^{-\alpha}}{e^{\alpha} - e^{-\alpha}} \sin(k\theta) \sin^2((j+1)k\theta)
\]
\[
= \frac{2}{m+1} \sum_{k=1}^{m} \coth \left( \alpha \sin \left( \frac{k\pi}{m+1} \right) \right) \sin \left( \frac{k\pi}{m+1} \right) \sin^2 \left( \frac{k(j+1)\pi}{m+1} \right).
\]
(25)

From (25) and [3, Theorem 5], we obtain the concise proof of [7] (the result for \( j = 0 \)). Next, using (25), we prove that \( K_j^{(j)}(y, y) \) takes its maximum at \( y = 0 \) for \( j \geq 1 \). It should be noted that \( C_{m,j} \) is the optimum value of the variational problem:

(VP) Maximize \( \|u\|^{-2}_{H^m(0,a)} \), subject to \( u \in H^m(0,a), \) and \( \|u^{(j)}\|_{L^\infty(0,a)} = 1 \).

Therefore, there exists the sequence \( \{g_n\} \) in \( H^m(0,a) \) such that \( \|g_n\|_{H^m(0,a)} \to C_{m,j} \) (as \( n \to \infty \)) and \( \|g^{(j)}_n\|_{L^\infty(0,a)} = 1 \). Since \( H^m(0,a) \) is compactly embedded in \( C^j[0,a] \) (\( 0 \leq j \leq m - 1 \)), there is a sub-sequence also denoted by \( \{g_n\} \) such that converges strongly to some \( g_0 \) in \( C^j[0,a] \). So, especially, \( \|g^{(j)}_n - g_0^{(j)}\|_{L^\infty(0,a)} \to 0 \).

Let \( t_n \) be the point satisfying \( g^{(j)}_n(t_n) = 1 \). Then, by the compactness of \( [0,a] \) there is a sub-sequence, again denoted by \( \{t_n\} \) such that \( t_n \) converges to some \( t \) in \( [0,a] \).

\[
|g^{(j)}_n(t) - 1| = |g^{(j)}_n(t) - g^{(j)}_n(t_n)| \leq |g^{(j)}_n(t) - g^{(j)}_0(t)| + |g^{(j)}_0(t) - g^{(j)}_0(t_n)| + |g^{(j)}_0(t_n) - g^{(j)}_0(t_n)|,
\]
we have \( g^{(j)}_n(t) \to 1 \), as \( n \to \infty \). Let us define
\[
U(t) := \inf \left\{ \|g\|_{H^m(0,t)}^2 \mid g \in H^m(0,t), \ g^{(j)}(0) = 1 \right\}.
\]
(26)
Then
\[
C_{m,j} = \lim_{n \to \infty} \left( \frac{1}{\|g^{(j)}_n(t)\|_{H^m(0,a)}} \right)^{-1} \|g^{(j)}_n\|_{H^m(0,a)}^{-2} = \lim_{n \to \infty} \left( \|g^{(j)}_n(t)\|_{H^m(0,t)}^2 + \|g^{(j)}_n(t)\|_{H^m(t,a)}^2 \right)^{-1} \|g^{(j)}_n\|_{H^m(t,a)}^{-2}
\]
\leq \left( U(t) + U(a-t) \right)^{-1}.
\]
Let \( R \) be a sufficiently large real constant. From weak compactness of \( S := \{ g \in H^m(0, a) \mid \| g \| \leq R, g^{(j)}(0) = 1 \} \) and weak lower semi-continuity of the norm \( \| \cdot \| \), we see that \( U(t) \) is the unique minimizer of the variational problem (26); the uniqueness follows from the strict convexity of \( \| g \|^2_{H^m(0, t)} \) on \( \{ g \in H^m(0, t) \mid g^{(j)}(0) = 1 \} \). Hence \( U(t) \) satisfies the Euler–Lagrange equation

\[
\sum_{k=0}^{m} (-1)^k g^{(2k)} = 0 \tag{28}
\]

with boundary conditions

\[
\begin{align*}
\sum_{p=s}^{m-1} (-1)^{p-s} g^{(2p-s+1)}(0) &= 0 \quad (0 \leq s \leq m - 1), \quad s \neq j, \\
g^{(j)}(0) &= 1, \\
\sum_{p=s}^{m-1} (-1)^{p-s} g^{(2p-s+1)}(t) &= 0 \quad (0 \leq s \leq m - 1).
\end{align*}
\tag{29}
\]

In order to solve (28) with (29), we consider Eq. (29), with (modified) boundary conditions

\[
\begin{align*}
\sum_{p=s}^{m-1} (-1)^{p-s} g^{(2p-s+1)}(0) &= -\alpha \cdot \delta_{s,j} \quad (0 \leq s \leq m - 1), \\
\sum_{p=s}^{m-1} (-1)^{p-s} g^{(2p-s+1)}(t) &= 0 \quad (0 \leq s \leq m - 1),
\end{align*}
\tag{30}
\]

and adjust the value of \( \alpha \) to satisfy (29). Clearly, the solution of (28) with (30) is

\[
g(x) = \sum_{k=1}^{2m} \alpha c_k e^{j \lambda_k x} = \alpha \tilde{K}_j(x, 0), \tag{31}
\]

where \( \tilde{K}_j(\cdot, y) \) is the \( j \)th reproducing kernel of \( H^m(0, t) \). Thus, from the relation

\[
g^{(j)}(0) = \alpha \tilde{K}_j^{(j)}(0, 0) = 1,
\]

we have \( \alpha = 1/\tilde{K}_j^{(j)}(0, 0) \), and hence

\[
U(t) = \| g \|^2_{H^m(0, t)} = \alpha^2 \tilde{K}_j^{(j)}(0, 0) = \frac{1}{\tilde{K}_j^{(j)}(0, 0)} = \frac{1}{\frac{2}{m+1} \sum_{k=1}^{m} \coth(t \sin(\frac{k\pi}{m+1})) \sin(\frac{k\pi}{m+1}) \sin^2(\frac{k(j+1)\pi}{m+1})}.
\]

Here, we prepare the lemma concerning to the behavior of \( U(t) \).

**Lemma 6.** Suppose \( a_k, b_k > 0 \) for \( 1 \leq k \leq m \). Then

\[
f(t) := \frac{1}{\sum_{k=1}^{m} a_k \coth(b_k t)}
\]

is a concave function on \( t \geq 0 \).

**Proof.** Noting that \( \text{csch}(\cdot) = 1/\sinh(\cdot) \), we have the second derivative of \( f \) as

\[
f''(t) = 2 \frac{\left( \sum_{k=1}^{m} a_k b_k \text{csch}^2(b_k t) \right)^2 - \left( \sum_{k=1}^{m} a_k \coth(b_k t) \right) \left( \sum_{k=1}^{m} a_k b_k^2 \coth(b_k t) \text{csch}^2(b_k t) \right)}{\left( \sum_{k=1}^{m} a_k \coth(b_k t) \right)^3}.
\tag{32}
\]

Moreover, by decomposing the summations of the numerator of (32) into diagonal and off-diagonal part, we have
\[
\sum_{k=1}^{m} (a_k b_k)^2 \operatorname{csch}^4(b_k t) \left(1 - \cosh^2(b_k t)\right) \\
- \sum_{1 \leq s < j \leq m} a_s a_j \left(-2 b_s b_j \operatorname{csch}^2(b_s t) \operatorname{csch}^2(b_j t) + b_j^2 \coth(b_j t) \coth(b_j t) \operatorname{csch}^2(b_j t) \right) \\
+ b_s^2 \coth(b_s t) \coth(b_s t) \operatorname{csch}^2(b_s t)) \]  

Clearly, the first term is negative for all \( t > 0 \), and the second term is further deformed to 

\[
\leq - \sum_{1 \leq s < j \leq m} \frac{a_s a_j}{\sinh(b_s t) \sinh(b_j t)} \left(- \frac{2 b_s b_j}{\sinh(b_s t) \sinh(b_j t)} + b_j^2 \frac{\cosh(b_j t) \cosh(b_j t)}{\sinh^2(b_j t)} + b_s^2 \frac{\cosh(b_s t) \cosh(b_s t)}{\sinh^2(b_s t)}\right) \\
= - \sum_{1 \leq s < j \leq m} \frac{a_s a_j}{\sinh(b_s t) \sinh(b_j t)} \left( \frac{b_s}{\sinh(b_s t)} - \frac{b_j}{\sinh(b_j t)} \right)^2 \leq 0, 
\]

so \( f \) is a concave function. \( \square \)

From Lemma 4, we see that \( U(t) \) is concave on \( t > 0 \). Hence \( U(t) + U(a - t) \) is also concave and symmetric with respect to \( t = a/2 \). Thus, \( (U(t) + U(a - t))^{-1} \) takes its maximum at \( t = 0 \) or \( t = a \). Therefore, we have \( C_{m,j} \leq (U(0) + U(a))^{-1} = U(a)^{-1} = K^{(j)}(0,0) \). The reverse inequality \( K^{(j)}(0,0) \leq C_{m,j} \) can be shown as in Lemma 1. This completes the proof of Theorem 1.

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References