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Fixed Point Index for Weakly Inward Mappings

SHOUCHUAN HU

Department of Mathematics, Southwest Missouri State University, Springfield, Missouri 65804

AND

YONG SUN

Department of Applied Mathematics, Florida Insitute of Technology, Melbourne, Florida 32901

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The fixed point index for weakly inward mappings, which are not cone mappings, is defined in this paper. Its properties are also investigated. Consequently we study the derivates of weakly inward mappings and nonzero fixed points of such mappings.

1. INTRODUCTION

The Leray–Schauder degree theory plays an important role in the study of fixed points of completely continuous mappings defined on open sets of some Banach space. Consequently, it has been applied to solve various equations. However, if a mapping, which might be derived from practice, is defined only on a relatively open subset of some retract (like a cone of a Banach space), or the interest is to find fixed points in a special region, e.g., positive fixed points, one will have to apply the fixed point index theory instead. Even though the fixed point index theory has been a very effective method, one has to pay attention to its restrictions. A key condition in its application is that the mapping must transfer the relatively open subset of the retract into the retract itself. Otherwise the fixed point index is not applicable.

Suppose that X is a real Banach space, $P \subset X$ is a cone, $\Omega \subset P$ is a relatively open bounded subset of P, and $A: \bar{\Omega} \to X$ is a completely continuous mapping. Let $\partial\Omega(P)$ denote the relative boundary of Ω in P such

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that A has no fixed point on $\partial\Omega(P)$. We can, in general expect to define a fixed point index by

$$i(A, \Omega, P) = \deg(I - Ar, B_R \cap r^{-1}(\Omega), 0), \tag{1}$$

where R is sufficiently large positive number such that $\bar{\Omega} \subset B_R$, $B_R = \{x \in X : \|x\| < R\}$. And hopefully this definition is independent of the positive number R. Indeed, it will be the case if $A : \bar{\Omega} \to P$, since then all the fixed points of Ar in X are actually in P, and therefore in Ω. However, if A does not map $\bar{\Omega}$ into P, the number defined in (1) in general depends on R. Hence, in order for an index to be well defined, it has been necessary that the following condition be fulfilled

$$A : \bar{\Omega} \to P. \tag{2}$$

In some applications, we have to deal with operators that may not satisfy condition (2), nevertheless, we need to know if they have fixed points in Ω. Thus the classical index theory is not applicable to this situation. The aim of this paper is to fill in this gap, namely, to define a generalized index for mappings satisfying the weakly inward condition which is weaker than (2). Therefore the applicability of the index theory will be broadened.

2. Preliminaries

Throughout this paper, X denotes a real Banach space and X^* the dual space of X. For any nonempty convex subset $D \subset X$ and $x \in D$, we define the weakly inward set of D at x by $\bar{I}_D(x)$, where

$$I_D(x) = \{x + t(y - x) : t \geq 0, y \in D\}.$$

A mapping $A : D(A) \subset D \to X$ is said to be weakly inward with respect to D if $Ax \in \bar{I}_D(x)$ for any $x \in D(A)$. The following lemma will be necessary in the sequal, see Deimling [2] for a proof.

Lemma 2.1. *Let $A : D(A) \subset D \to X$, D be closed and convex in X. Then*

(a) *A weakly inward with respect to D if and only if*

$$x \in D(A) \cap \partial D,\ x^* \in X^*, \text{ and } x^*(x) = \sup_D x^*(y)$$
$$\text{imply } x^*(Ax) \leq x^*(x); \tag{3}$$

(b) *If $D=P$ is a cone in X, then A is weakly inward with respect to P if and only if*

$$x \in D(A) \cap \partial D,\ x^* \in P^*,\ \text{and } x^*(x)=0 \text{ imply } x^*(Ax) \geqslant 0,$$

where $P^* = \{x^* \in X^* : x^* \geqslant 0 \text{ for all } x \in P\}$.

Recall the $D \subset X$ is said to be a retract of X if there is a retraction $r: X \to D$, i.e., r is continuous and $r|_D = i_D$, where i_D is the identity mapping on D. In order to establish a fixed point index for weakly inward mappings, we may follow the usual procedure. But, as explained earlier, an arbitrary retraction mapping may not work. Therefore we need to select certain retractions.

DEFINITION 2.2. $D \subset X$ is said to be a retract of X with property (P) if there is a retraction $r: X \to D$ such that for any $x \in X \setminus D$ there is an $x^* \in X^*$ such that

$$x^*(x) > x^*(rx) = \sup_D x^*(y). \tag{4}$$

Remark. From the definition given above it is clear that if D is a retract with property (P) and r is the corresponding retraction, then D must be closed and convex, and rx must be a supporting point of D for each $x \in X \setminus D$. In view of V. L. Klee's results on non-support points of convex sets [3], it is not likely that every closed and convex subset of an arbitrary Banach space is a retract with property (P). Thus, it makes sense to see a few examples of these kinds of retracts first.

LEMMA 2.3. *Let D be a closed and convex subset of X. D is a retract of X with property (P) if one of the following conditions is satisfied*

(a) *D^0, the interior of D, is not empty,*

(b) *X is a reflexive Banach space,*

(c) *There is a metric projection from X onto D.*

Proof. In case (a), take any $x_0 \in D^0$ and define $\phi_\lambda(x) = \lambda x + (1-\lambda) x_0$ for all $x \in X$ and $0 \leqslant \lambda \leqslant 1$. It is clear that for any $x \in X \setminus D$, there is only one point in the line segment $\phi_\lambda(x): 0 \leqslant \lambda \leqslant 1$ that lies on ∂D, say $\phi_{\lambda_x}(x)$. We now define a retraction of D by

$$r(x) = \begin{cases} x, & x \in D, \\ \phi_{\lambda_x}(x), & x \notin D. \end{cases} \tag{5}$$

It can be easily checked that $r(x)$ is indeed continuous. If $x \notin D$, then

$r(x) \in \partial D$, hence Mazur's theorem of separation implies the existence of $0 \neq x^* \in X^*$ such that

$$x^*(r(x)) > x^*(y) \qquad \text{for all} \quad y \in D^0.$$

Therefore,

$$x^*(x) > x^*(r(x)) = \sup_D x^*(y).$$

In case (b), since X can be renormed to become strictly convex and the relation (4) is topological rather than metric, we can without loss of generality assume that X is strictly convex. Thus there is a unique metric projection r of X onto D, i.e.,

$$r(x) \in D, \qquad \|x - r(x)\| = \min_D \|x - y\| \qquad \text{for all} \quad x \in X,$$

where $r(x)$ is even continuous. We need to verify the relation (4) for this r.

For any $x \notin D$, making a translation if necessary, we may assume $x = 0 \notin D$ and just verify the relation (4) for $x = 0$. Let $\eta = \min_D \|0 - y\| > 0$ and observe $B_\eta \cap D = \phi$. An application again of Mazur's theorem implies the existence of $0 \neq x^* \in X$ such that

$$\inf_{B_\eta} x^*(y) \geqslant \sup_D x^*(y).$$

Since $\inf_{B_\eta} x^*(y) < 0$ and $r(0) \in \bar{B}_\eta$, we have

$$x^*(0) = 0 > x^*(r(0)) = \sup_D x^*(y).$$

The approach used to show case (b) works when there is a metric projection from X onto D, namely case (c). And we omit the proof.

The proof of the lemma is therefore complete.

3. Fixed Point index

We first recall a well known result that if $D \subset X$ is bounded, closed, and convex, $A : C \to X$ is completely continuous and weakly inward with respect to D, then A has a fixed point in D. This result may give some indication that the condition (2) can be replaced by the weakly inward condition with respect to P for the purpose of defining an index $i(A, \Omega, P)$. We show that this is indeed true.

Definition 3.1. Assume that D is a retract of X with property (P). We choose a retraction $r(x)$ satisfying condition (4). Let $\Omega \subset D$ be a bounded and relatively open subset of D, $A: \bar{\Omega} \to X$ a completely continuous mapping that is weakly inward with respect to D. If $Ax \neq x$ for all $x \in \partial\Omega(D)$, then we define the fixed point index $i(A, \Omega, D)$ by

$$i(A, \Omega, D) = \deg(I - Ar, B_R \cap r^{-1}(\Omega), 0) \tag{6}$$

where $R \geq 0$ is such that $\bar{\Omega} \subset B_R$.

THEOREM 3.2. *The index in the above defintion is well defined and independent of $R > 0$.*

Proof. The key is that for any $x \in X$, $Arx = x$ implies $x \in \Omega$, hence $Ax = x$. To prove this, assume there is an $x \in X \backslash D$ such that $Arx = x$. By Definition 2.2, there is an $x^* \in X^*$ such that $x^*(x) > x^*(rx) = \sup_D x^*(y)$. On the other hand, Lemma 2.1(a) implies that $x^*(x) = x^*(Arx) \leq x^*(rx)$, a contradiction. Now the excision property of the Leray–Schauder degree completes the proof.

Evidently, this index coincides with the well-known fixed point index in case mapping A maps $\bar{\Omega}$ into D. Hence, it is a generalization. The following theorem shows that this generalized index possesses most of the properties that the reguler index does.

THEOREM 3.3. *The fixed point index has the following properties:*

(1) (*Normality*) *$i(A, \Omega, D) = 1$ if $Ay = y_0 \in \Omega$ for all $y \in \Omega$.*

(2) (*Additivity*) *$i(A, \Omega, D) = i(A, \Omega_1, D) + i(A, \Omega_2, D)$ provided that Ω_1, Ω_2 are disjoint relatively open subsets of D such that A has no fixed points in $\bar{\Omega} \backslash (\Omega_1 \cup \Omega_2)$.*

(3) (*Solvability*) *A has a fixed point in Ω if $i(A, \Omega, D) \neq 0$.*

(4) (*Excision*) *$i(A, \Omega, D) = i(A, \Omega_0, D)$ if $\Omega_0 \subset \Omega$ is relatively open in D and A has no fixed point in $\bar{\Omega} \backslash \Omega_0$.*

(5) (*Homotopy Invariance*) *Assume that $H(\ ,\ ): [0, 1] \times \bar{\Omega} \to X$ is continuous and the range $R(H)$ is precompact in X, $H(t, x) \neq x$ $\forall t \in [0, 1]$ and $x \in \partial\Omega(D)$, $H(t, .)$ is weakly inward with respect to D for every $t \in [0, 1]$. Then, $i(H(t,\ ), \Omega, D)$ is independent of t. In particular,*

$$i(H(0, .), \Omega, D) = i(H(1, .), \Omega, D).$$

The proof of Theorem 3.3 is merely an imitation of that of the corresponding results for the known fixed point index, hence it is omitted. As a consequence of Homotopy Invariance, we have

COROLLARY 3.4. *Assume that $A, B: \bar{\Omega} \subset D \to X$ are completely con-*

tinuous and weakly inward with respect to D, $Ax = Bx \neq x$ *for all* $x \in \partial\Omega(D)$. *Then*

$$i(A, \Omega, D) = i(B, \Omega, D).$$

Proof. Let $H(t, x) = tAx + (1-t)Bx$, then $H(t, x) \neq x \ \forall x \in \partial\Omega(D)$, $t \in [0, 1]$. For $x \in \bar{\Omega} \cap \partial D$, $Ax \in \bar{I}_D(x)$ and $Bx \in \bar{I}_D(x)$, hence $H(t, x) \in \bar{I}_D(x)$ since $\bar{I}_D(x)$ is a convex subset of X. Thus, $H(t, .)$ is weakly inward with respect to D for each $t \in [0, 1]$. Now Corollary 3.4 follows from the Homotopy Invariance.

If in particular $D = P$ is a cone of X and $A : \Omega \subset P \to X$ is weakly inward with respect to P, we have the following results.

THEOREM 3.5. (a) *If* $0 \in \Omega$ *and* $Ax \neq tx$ *for all* $x \in \partial\Omega(P)$ *and* $t \geqslant 1$, *then*

$$i(A, \Omega, P) = 1;$$

(b) *If there is an* $h \in P$, $h \neq 0$ *such that* $x - Ax \neq th$ *for all* $t \geqslant 0$ *and* $x \in \partial\Omega(P)$, *then*

$$i(A, \Omega, P) = 0.$$

Proof. (a) Let $H(t, x) = tAx$, then $H(t, x) \neq x$ for all $x \in \partial\Omega(P)$, $t \in [0, 1]$. For $x \in \bar{\Omega} \cap \partial P$, since $Ax \in \bar{I}_P(x)$ and $0 \in \bar{I}_P(x)$, we have $H(t, x) \in \bar{I}_P(x)$. Therefore, $H(t, .)$ is weakly inward with respect to P for every $t \in [0, 1]$. Hence $i(A, \Omega, P) = i(0, \Omega, P) = 1$.

For part (b), assume $i(A, \Omega, P) \neq 0$. Choose $\eta > 0$ such that

$$\eta > \frac{1}{\|h\|} \sup\{\|x - Ax\| : x \in \bar{\Omega}\} \tag{7}$$

and set $H(t, x) = Ax + t\eta h$. It is obvious that $H(t, \ )$ is weakly inward with respect to P. Also, $H(t, x) \neq x$ for all $x \in \partial\Omega(P)$, $t \in [0, 1]$. Hence, $i(A + \eta h, \Omega, P) = i(A, \Omega, P) \neq 0$. Solvability then implies the existence of an $x \in \Omega$ such that $x = Ax + \eta h$, a contradiction to (7).

Remark. It is not clear to the authors whether the definition of the generalized index actually depends on the choice of the retraction $r: X \to D$.

4. FURTHER REMARKS ON WEAKLY INWARD MAPPINGS

Let P be an cone in X, $A : P \to X$ a weakly inward mapping with respect to P. In this section we want to show that if A is differentiable at $x = 0$ and $x = \infty$ along the cone P, then $A'(0)$ and $A'(\infty)$ are also weakly inward with respect to P. Therefore, weak inwardness is a property of inheritance by the operation of differentiation.

DEFINITION 4.1. (a) *The derivative of A at $x=0$ along P is denoted by $A'(0)$ and is defined to be an operator in $L(\overline{P-P}, X)$ such that*

$$Ax - A(0) = A'(0)\,x + o(\|x\|) \qquad as \qquad x \to \infty \tag{8}$$

for $x \in P$

(b) *The derivative of A at $x=\infty$ along P is denoted by $A'(\infty)$ and is defined to be an operator in $L(\overline{P-P}, X)$ such that*

$$Ax - A'(\infty)\,x = o(\|x\|) \qquad as \qquad x \to \infty \tag{9}$$

for $x \in P$.

LEMMA 4.2. *For every $x \in P$, $\bar{I}_P(x)$ is a wedge.*

Proof. It is easily checked that $\bar{I}_P(x)$ is convex. Hence $t\bar{I}_P(x) \subset \bar{I}_P(x)$ $\forall t \in [0, 1]$ since $0 \in \bar{I}_P(x)$. We need to prove that this inclusion remains valid for $t > 1$. For any $\lambda > 0$, $t > 1$, and $y \in P$,

$$t[x + \lambda(y - x)] = x + \bar{\lambda}(\bar{y} - x) \in \bar{I}_P(x), \tag{10}$$

where $\bar{\lambda} = t\lambda$, $\bar{y} = y + ((t-1)/t\lambda)\,x \in P$.

This yields that $\bar{I}_P(x)$ is a wedge.

THEOREM 4.3. *Assume that $A: P \to X$ is weakly inward with respect to P, and $A'(0)$, $A'(\infty)$ exist.*

(a) *If $A0 = 0$, then $A'(0)$ is weakly inward with respect to P;*

(b) *$A'(\infty)$ is weakly inward with respect to P.*

Proof. (a) Fix an $x \in P$, for $t > 0$ we have by (8)

$$A(tx) = tA'(0)\,x + o(t) \qquad as \qquad t \to 0.$$

Since $1/t\, A(tx) \in \bar{I}_P(tx) = t\bar{I}_P(x) \subset \bar{I}_P(x)$ by Lemma 4.2, $A'(0)\,x \in \bar{I}_P(x)$ follows immediately. Part (b) can be proved similarly.

We now can follow Amann's thought [1] to study the operator A via $A'(0)$ and $A'(\infty)$. The proof of the following theorem is trivial and therefore omitted.

THEOREM 4.4. *Let cone $P \subset X$ have property (P). Assume that $A: P \to X$ is completely continuous, weakly inward with respect to P, differentiable along P at $x = 0$ and at $x = \infty$, $A0 = 0$, and $\lambda = 1$ is not an eigenvalue of a* *positive eigenfunction for $A'(0)$. Then, A has a nontrivial positive fixed point, provided that one of the following two conditions is satisfied:*

(a) *$A'(0)$ has a positive eigenvalue larger than 1 with a positive eigenfunction but $A'(\infty)$ does not;*

(b) *$A'(\infty)$ has a positive eigenvalue larger than 1 with a positive eigenfunction, but $A'(0)$ does not.*

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