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Non-separable splines and numerical computation of evolution equations by the Galerkin methods

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Abstract

We construct new non-separable splines and we apply the spline sampling approximation to the computation of numerical solutions of evolution equations. The non-separable splines are basis functions which give a fine sampling approximation which enables us to compute numerical solutions by means of the method of lines combined with the Galerkin method. To demonstrate our approach we compute numerical solutions of the Burgers equation and the Kadomtsev–Petviashvili equation.

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1. Introduction

Recently, wavelet collocation methods and wavelet Galerkin methods have been investigated and successfully applied to compute numerical solutions of partial differential equations, see e.g. [3,4,14,18,20]. Note that the collocation methods enable us to compute efficiently nonlinear terms. See also [19] where we used the Coifman scaling function.

In this paper, we construct a basis of modified B-splines which are well adapted to the sampling approximation of functions and we present an efficient numerical scheme which takes advantage of the method of lines and the Galerkin method. In particular, we shall show that our new non-separable (i.e. non-tensor product) splines of two variables can be successfully applied to the numerical simulation of the Kadomtsev–Petviashvili (KP) equation [12] which apparently needs careful computation. For the reference, Bratso–Twizell [6] and Katsis–Akylas [13] designed numerical algorithms for the KP equation where the explicit finite difference methods were effectively used to obtain numerical solutions. Also, Feng–Mitsui [11] employed the implicit finite difference method with linearization. Moreover, it is known that the Adomian decomposition method [1,2] provides the solution in a rapidly convergent series with components as a numerical scheme of partial differential equations.

This paper is organized as follows: in Section 2, we recall the properties of the Coifman scaling function and the cardinal B-splines. In Section 3, we introduce two conditions for the collocation approximation and establish error

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estimates for sampling approximation in two variables case by a linear combination of the basis function and its shifts satisfying these conditions. In Section 4, we construct new non-separable splines based on the cardinal B-splines taking into account two conditions in the previous section. In Section 5, we apply our method to compute numerical solutions of the Burgers and KP equations and illustrate the usefulness of the modified spline functions combined with the method of lines. Finally, we state the conclusion in Section 6.

2. Previous research

In this section, we briefly recall our previous results on the sampling approximation of functions by means of the Coifman scaling function in the one space dimension together with basic facts on the cardinal B-spline function.

2.1. An application of the Coifman scaling function

Let φ be the Coifman scaling function of degree r and let the sampling approximation ${}_1S_j(f)$ of f be defined by

$$ {}_1S_j(f)(x) := \sum_{k=-\infty}^{\infty} f(2^{-j}k)\,\varphi(2^j x - k), \quad j = 1, 2, \ldots, $$

which turns out to be a good quasi-interpolation of f, [3,18].

Then we have the following error estimate which is an improvement on the corresponding estimate of [16]:

Theorem 1 (*[19]*). *Let φ be given as above. Then, for a function f in the Sobolev space $W^{N,p}(\mathbb{R})$ $(N \le r)$, and for $1 \le p \le \infty$, we have*

$$ \|{}_1S_j(f) - f\|_{L^p(\mathbb{R})} \le 2^{-jN}\, {}_1C_{\varphi,p,N}\, \|f^{(N)}\|_{L^p(\mathbb{R})}, \quad j = 1, 2, \ldots, $$

where ${}_1C_{\varphi,p,N}$ can be chosen as follows:

$$ {}_1C_{\varphi,p,N} = \frac{\|\varphi\|_{L^\infty(\mathbb{R})}(K+1)^N(2K+1)N^{1/p}}{N!}, $$

with K a constant satisfying $\operatorname{supp}\varphi \subset [-K, K]$. *Moreover, the derivatives are approximated as well:*

$$ \|({}_1S_j(f) - f)^{(n)}\|_{L^p(\mathbb{R})} \le 2^{-j(N-n)}\, {}_1C_{\varphi,p,N}\, \|f^{(N)}\|_{L^p(\mathbb{R})} $$

for $j = 1, 2, \ldots,$ and $n = 1, \ldots, N-1$.

Note that this approximation ${}_1S_j(f)$ is easy to compute since we need not compute the inner products of the form $\langle f, \varphi\rangle$. Furthermore, next theorem yields a good approximation of its power $({}_1S_j(f))^2$ simply as follows:

Theorem 2 (*[19]*). *Let φ be given as above. Then for any function f in $W^{N,p}(\mathbb{R}) \cap W^{N,\infty}(\mathbb{R})(N \le r)$, and for $1 \le p \le \infty$, we have*

$$ \|({}_1S_j(f))^2 - {}_1S_j(f^2)\|_{L^p(\mathbb{R})} \le 2^{-jN}\, {}_1C'_{\varphi,p,N} \sum_{n=0}^{N} \|f^{(s)}\|_{W^{n,p}(\mathbb{R})} \|f^{(t)}\|_{W^{N-n,\infty}(\mathbb{R})} $$

for $j = 1, 2, \ldots,$ where it suffices to choose

$$ {}_1C'_{\varphi,p,N} = \left\{(2K+1)\|\varphi\|_{L^\infty(\mathbb{R})} + 1 + 2^N\right\} {}_1C_{\varphi,p,N}\,. $$

As can be seen from the theorems above, the Coifman scaling function has a nice application in the approximation of functions. However, it is not quite satisfactory for two reasons:

- As r the degree of φ increases, so does its regularity only at a low rate, although its support is enlarged at the rate proportional to $6r$ [8].
- It is not a symmetric function.

In Section 4, we shall construct new functions from the cardinal B-spline which enjoy many nice properties as φ without the above disadvantage.

2.2. The cardinal B-spline

We recall the basic facts on the cardinal B-spline (Refs. [17,9,7,10]).

We define the cardinal B-spline of order m, denoted by N_m, as the m times convolution of the indicator function of the unit interval:

$$N_m(x) := \overbrace{N_1 * \cdots * N_1}^{m}(x),$$

where

$$N_1(x) := \begin{cases} 1, & 0 \le x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

Let us recall the following well-known properties of N_m which are very useful in the numerical computation:

Theorem 3 (*[7,9]*). *The cardinal B-spline N_m satisfies the following equalities:*

(1) $\operatorname{supp} N_m(x) = [0, m]$.
(2) $N'_m(x) = N_{m-1}(x) - N_{m-1}(x-1)$.
(3) $N_m(x) = \frac{x}{m-1} N_{m-1}(x) + \frac{m-x}{m-1} N_{m-1}(x-1),\ m \ge 2$.
(4) $N_m(m/2 - x) = N_m(m/2 + x),\ x \in \mathbb{R}$.
(5) *For any polynomials p of degree $m-1$, we have*

$$\sum_{k=-\infty}^{\infty} p(k) N_m(x-k) = \sum_{k=1}^{m-1} N_m(k) p(x-k).$$

3. Sampling approximation

In this section, we describe conditions for a basis function to yield a fine sampling approximation and we examine its order of accuracy.

3.1. Conditions for the sampling approximation

We state two-dimensional case, since the general case is essentially the same.

Let φ be a smooth function with compact support in $\mathbb{R}^2$. Then, the following condition for φ is called the Strang–Fix condition of order r:

$$\text{(SF)} \quad \begin{cases} \text{(a) } \widehat{\varphi}(0,0) = 1, \\ \text{(b) } \left. \dfrac{\partial^{\alpha_1+\alpha_2}}{\partial \xi_1^{\alpha_1} \partial \xi_2^{\alpha_2}} \widehat{\varphi}(\xi_1, \xi_2) \right|_{\xi_i = 2\pi k_i} = 0, \quad \forall (k_1, k_2) \in \mathbb{Z}^2 \setminus (0,0),\ i = 1, 2, \end{cases}$$

where $\alpha_1, \alpha_2 \in \mathbb{N} \cup \{0\}$, $0 \le \alpha_1 + \alpha_2 \le r$ and $\widehat{\varphi}$ is the Fourier transform of φ.

Then the following result is due to Schoenberg and de Boor, which is proved by the Poisson summation formula:

Theorem 4 (*[17,9,10]*). *The following are equivalent:*

(1) *φ satisfies the Strang–Fix condition of order r.*
(2) *There are polynomials p_{α_1,α_2} of degree r such that*

$$\sum_{k_1,k_2=-\infty}^{\infty} p_{\alpha_1,\alpha_2}(k_1,k_2)\, \varphi(x_1-k_1, x_2-k_2) = x_1^{\alpha_1} x_2^{\alpha_2}, \quad 0 \le \alpha_1 + \alpha_2 \le r,$$

where $\alpha_1, \alpha_2 \in \mathbb{N} \cup \{0\}$ and $p_{\alpha_1,\alpha_2}(x_1, x_2)$ are polynomials in two variables which consist of $x_1^{\beta_1} x_2^{\beta_2}$ with $0 \le \beta_1 \le \alpha_1$, $0 \le \beta_2 \le \alpha_2$.

Now let us call the following system of identities the moment condition of order r:

$$\text{(M)} \quad \left.\frac{\partial^{\alpha_1+\alpha_2}}{\partial\xi_1^{\alpha_1}\partial\xi_2^{\alpha_2}}\widehat{\varphi}(\xi_1,\xi_2)\right|_{\xi_1=\xi_2=0} = \delta_{\alpha_1+\alpha_2,0}, \quad 0 \le \alpha_1+\alpha_2 \le r,$$

where $\alpha_1, \alpha_2 \in \mathbb{N} \cup \{0\}$. Note that if φ satisfies (SF) and (M), then the sampling approximation by means of φ gives us the desired approximation (Refs. [17,9,7,10,18]).

Proposition 5. *The following are equivalent:*

(1) φ *satisfies* (ST) *and* (M) *of order* r.
(2) $\sum_{k_1,k_2=-\infty}^{\infty}(x_1-k_1)^{\alpha_1}(x_2-k_2)^{\alpha_2}\varphi(x_1-k_1,x_2-k_2) = \delta_{\alpha_1+\alpha_2,0}$,
(3) $\sum_{k_1,k_2=-\infty}^{\infty} k_1^{\alpha_1}k_2^{\alpha_2}\varphi(x_1-k_1,x_2-k_2) = x_1^{\alpha_1}x_2^{\alpha_2}$,

where $0 \le \alpha_1+\alpha_2 \le r$.

Proof. See [10,16] for example. □

Example 6. The Coifman scaling function of degree r satisfies the above conditions (Refs. [16,3,4]).

Since the Coifman scaling function satisfies (SF) and (M), we can define a fine sampling approximation as in Theorems 1 and 2. In Section 4 we shall construct modified B-spline functions which satisfy one of the equivalent conditions of Proposition 5.

3.2. *Error estimates for sampling approximation*

In this subsection we state error estimates for sampling approximation of functions in two variables in an analogous way as in one variable case.

Let φ be a smooth function of two variables with compact support which satisfies conditions (SF) and (M) of order r. A sampling approximation ${}_2S_j(f)$ of f is defined by

$${}_2S_j(f)(x,y) := \sum_{k,l=-\infty}^{\infty} f(2^{-j}k, 2^{-j}l)\,\varphi(2^j x-k, 2^j y-l), \quad j=1,2,\ldots.$$

Then we have the following error estimates as in the one-dimensional case.

Theorem 7. *Let* φ *satisfy* (SF) *and* (M) *of order* r *and let* $W^{N,p}(\mathbb{R}^2)$ *be the Sobolev space with* $N \le r$, *and* $1 \le p \le \infty$. *Then for any function* f *in* $W^{N,p}(\mathbb{R}^2)$, *we have*

$$\|{}_2S_j(f)-f\|_{L^p(\mathbb{R}^2)} \le 2^{-jN}\,{}_2C_{\varphi,p,N}\sum_{n=0}^{N}\left\|\frac{\partial^N f(x,y)}{\partial x^n\partial y^{N-n}}\right\|_{L^p(\mathbb{R}^2)}, \quad j=1,2,\ldots,$$

where ${}_2C_{\varphi,p,N}$ *can be chosen as follows:*

$${}_2C_{\varphi,p,N} = \frac{2^{N+2-3/p}K^{N+2-2/p}\|\varphi\|_{L^\infty(\mathbb{R}^2)}(2K+1)^{2/p}}{N!},$$

with K *a positive constant such that* $\operatorname{supp}\varphi \subset \{(x,y) | -K \le x, y \le K\}$.

Proof. It suffices to apply the following well-known lemma to $F(s) := f(x+s(2^{-j}k-x), y+s(2^{-j}l-y))$. □

Lemma 8. *If* F *belongs to* $C^N(\mathbb{R})$, *then for all* t,

$$F(t) = \sum_{n=0}^{N-1}\frac{F^{(n)}(0)}{n!}\,t^n + \int_0^t F^{(N)}(s)\frac{(t-s)^{N-1}}{(N-1)!}\mathrm{d}s.$$

The rest of the proof is omitted because the computation is simple but tedious and is almost the same as in the one-dimensional case [19].

This Theorem 7 shows the order of accuracy in the sampling approximation based on φ which satisfies (SF) and (M). Note that this approximation is well adapted to fast computation of numerical solutions of differential equations with nonlinear terms.

Indeed, we have the following theorem:

Theorem 9. *Let φ be as above and let F be a function N times continuously differentiable. Then for each f in $W^{N,p}(\mathbb{R}^2)$ with $1 \leq p \leq \infty$ and $j = 1, 2, \ldots$, we have*

$$\|F({}_2S_j(f)) - {}_2S_j(F(f))\|_{L^p(\mathbb{R}^2)} \leq 2^{-jN}(M+1)\,{}_2C_{\varphi,p,N} \sum_{n=0}^{N} \left(\left\| \frac{\partial^N f}{\partial x^n \partial y^{N-n}} \right\|_{L^p(\mathbb{R}^2)} + \left\| \frac{\partial^N (F(f))}{\partial x^n \partial y^{N-n}} \right\|_{L^p(\mathbb{R}^2)} \right),$$

where M is the Lipschitz constant of $F = F(t)$ with t in the images of f and ${}_2S_j(f)$.

Proof. Firstly, the norm inequality gives

$$\begin{aligned} \|F({}_2S_j(f)) - {}_2S_j(F(f))\|_{L^p(\mathbb{R}^2)} &= \|F({}_2S_j(f)) - F(f) + F(f) - {}_2S_j(F(f))\|_{L^p(\mathbb{R}^2)} \\ &\leq \|F({}_2S_j(f)) - F(f)\|_{L^p(\mathbb{R}^2)} + \|F(f) - {}_2S_j(F(f))\|_{L^p(\mathbb{R}^2)}. \end{aligned} \tag{1}$$

Secondly, by the assumption on F and Theorem 7, we estimate the first term on the right side of (1) as follows:

$$\begin{aligned} \|F({}_2S_j(f)) - F(f)\|_{L^p(\mathbb{R}^2)} &\leq M\|{}_2S_j(f) - f\|_{L^p(\mathbb{R}^2)} \\ &\leq 2^{-jN} M\,{}_2C_{\varphi,p,N} \sum_{n=0}^{N} \left\| \frac{\partial^N f}{\partial x^n \partial y^{N-n}} \right\|_{L^p(\mathbb{R}^2)}. \end{aligned}$$

Furthermore, in view of Theorem 7 again applied to the second term on the right side of (1), we obtain

$$\|F(f) - {}_2S_j(F(f))\|_{L^p(\mathbb{R}^2)} \leq 2^{-jN}\,{}_2C_{\varphi,p,N} \sum_{n=0}^{N} \left\| \frac{\partial^N (F(f))}{\partial x^n \partial y^{N-n}} \right\|_{L^p(\mathbb{R}^2)}.$$

Therefore, if we choose ${}_2C_{\varphi,p,N,M} = (M+1)\,{}_2C_{\varphi,p,N}$, it follows that

$$\|F({}_2S_j(f)) - {}_2S_j(F(f))\|_{L^p(\mathbb{R}^2)} \leq 2^{-jN}\,{}_2C_{\varphi,p,N,M} \sum_{n=0}^{N} \left(\left\| \frac{\partial^N f}{\partial x^n \partial y^{N-n}} \right\|_{L^p(\mathbb{R}^2)} + \left\| \frac{\partial^N (F(f))}{\partial x^n \partial y^{N-n}} \right\|_{L^p(\mathbb{R}^2)} \right),$$

which completes the proof. □

4. Construction of non-separable splines

As mentioned in Example 6, although the Coifman scaling function satisfies conditions in Proposition 5 as well as other fine properties, it is not quite satisfactory for our purpose. On the other hand, $N_m(\cdot + m/2)$ the cardinal B-spline translated by $m/2$ satisfies the Strang–Fix condition of order $m-1$, nevertheless it does not satisfy the moment condition of order $m-1$. Therefore, let us construct non-separable splines ${}_nN_m$ which also satisfy the moment condition of order $m-1$. Here, a 'Non-separable' spline means that it is not a tensor product of spline functions of one variable but a finite linear combination of tensor products of translated cardinal B-splines, $N_m(\cdot + m/2 + q)$, $q = 0, \pm 1, \pm 2, \ldots$:

$${}_nN_m(x_1, \ldots, x_n) := \sum_{l_1,\ldots,l_n} w_{l_1,\ldots,l_n} N_m(x_1 + m/2 - l_1) \cdots N_m(x_n + m/2 - l_n),$$

where the weights $\{w_{l_1,\ldots,l_n}\}$ have to be computed in such a way that $\varphi = {}_nN_m$ satisfies (SF) and (M).

Remark 10. Note that we want to restrict the range of $(l_1, \dots, l_n)$ as small as possible because ${}_nN_m$ is expected to have minimal support and a symmetric profile, see [5] for another interesting basis function. It turns out that we can choose $0 \le |l_1| + \cdots + |l_n| \le [(m-1)/2]$, where $[x]$ is the largest integer not strictly greater than x.

To compute the weight $\{w_{l_1,\dots,l_n}\}$, it suffices to employ Proposition 5(3) as follows. From now on, let us restrict ourselves to the two-dimensional case for the sake of simplicity of presentation.

4.1. Computation of weights

We calculate the weight $\{w_{l_1,l_2}\}$ by means of Proposition 5(3). For each (i_1, i_2) satisfying $0 \le i_1+i_2 \le 2[(m-1)/2]$ and $i_1, i_2 \ge 0$, we have

$$\begin{aligned} x_1^{i_1} x_2^{i_2} &= \sum_{k_1,k_2=-\infty}^{\infty} k_1^{i_1} k_2^{i_2}\, {}_2N_m(x_1-k_1, x_2-k_2) \\ &= \sum_{k_1,k_2=-\infty}^{\infty} k_1^{i_1} k_2^{i_2} \sum_{0\le|l_1|+|l_2|\le[(m-1)/2]} w_{l_1,l_2} N_m(x_1-k_1+m/2-l_1) N_m(x_2-k_2+m/2-l_2) \\ &= \sum_{0\le|l_1|+|l_2|\le[(m-1)/2]} w_{l_1,l_2} \sum_{k_1,k_2=1}^{m-1} N_m(k_1)N_m(k_2)(x_1-k_1+m/2-l_1)^{i_1}(x_2-k_2+m/2-l_2)^{i_2}, \end{aligned} \tag{2}$$

where the last identity is valid in view of Theorem 3(5). We obtain the weights $\{w_{l_1,l_2}\}$ as a result of solving (2) and they are presented for several examples in the next subsection. See Example 12.

Remark 11. There exist other methods for computing the weights.

4.2. Examples

In this subsection, we give explicitly computed examples of weights. Let us explain the details of computation by the following example.

Example 12. We show the computation deduced from Proposition 5(3) in the case $m = 4$ and $n = 2$ (2 variables case). Since $\operatorname{supp} N_4 = [0, 4]$, it follows from (2) that

$$\begin{aligned} x_1^{i_1} x_2^{i_2} &= \sum_{k_1,k_2=-\infty}^{\infty} k_1^{i_1} k_2^{i_2} \sum_{0\le|l_1|+|l_2|\le 1} w_{l_1,l_2} N_4(x_1-k_1+2-l_1) N_4(x_2-k_2+2-l_2) \\ &= \sum_{0\le|l_1|+|l_2|\le 1} w_{l_1,l_2} \sum_{k_1,k_2=1}^{3} N_m(k_1)N_m(k_2)(x_1-k_1+m/2-l_1)^{i_1}(x_2-k_2+m/2-l_2)^{i_2}, \end{aligned} \tag{3}$$

where $0 \le i_1+i_2 \le 2$ $(i_1, i_2 \ge 0)$. Then we get the following system of equations, putting known values $N_4(2) = 2/3$, $N_4(1) = N_4(3) = 1/6$ and $N_4(n) = 0$ $(n = \mathbb{Z} \setminus \{1,2,3\})$ into this identity, and recalling, in addition, that we are looking for a symmetric function so that the identity $w_{1,0} = w_{-1,0} = w_{0,1} = w_{0,-1}$ is naturally required in this example:

$$\begin{cases} w_{0,0} + w_{1,0} + w_{-1,0} + w_{0,1} + w_{0,-1} = 1, \\ -w_{1,0} + w_{-1,0} = 0, \\ -w_{0,1} + w_{0,-1} = 0, \\ w_{0,0} + 4w_{1,0} + 4w_{-1,0} + w_{0,1} + w_{0,-1} = 0, \\ w_{0,0} + w_{1,0} + w_{-1,0} + 4w_{0,1} + 4w_{0,-1} = 0. \end{cases}$$

Therefore, we obtain

$$w_{0,0} = \frac{5}{3}, \qquad w_{1,0} = w_{-1,0} = w_{0,1} = w_{0,-1} = -\frac{1}{6}.$$

As in this example, we compute the weights for other cases in a similar way. The results of computation turn out to be the following.

Example 13. (1) One-dimensional case ($n = 1$).

In this case, the index l of the weight $\{w_l\}$ is restricted to $-[(m-1)/2] \le l \le [(m-1)/2]$ and the weight is expected to be symmetric with respect to the index.

- $m = 3$.

$$(w_{-1}, w_0, w_1) = \left(-\frac{1}{8}, \frac{5}{4}, -\frac{1}{8}\right).$$

- $m = 4$.

$$(w_{-1}, w_0, w_1) = \left(-\frac{1}{6}, \frac{4}{3}, -\frac{1}{6}\right).$$

- $m = 5$.

$$(w_{-2}, w_{-1}, w_0, w_1, w_2) = \left(\frac{47}{1152}, -\frac{107}{288}, \frac{319}{192}, -\frac{107}{288}, \frac{47}{1152}\right).$$

- $m = 6$.

$$(w_{-2}, w_{-1}, w_0, w_1, w_2) = \left(\frac{13}{240}, -\frac{7}{15}, \frac{73}{40}, -\frac{7}{15}, \frac{13}{240}\right).$$

- $m = 7$.

$$(w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3) = \left(-\frac{2159}{138\,240}, \frac{751}{4608}, -\frac{37\,003}{46\,080}, \frac{79\,879}{34\,560}, -\frac{37\,003}{46\,080}, \frac{751}{4608}, -\frac{2159}{138\,240}\right).$$

- $m = 8$.

$$(w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3) = \left(-\frac{311}{15\,120}, \frac{22}{105}, -\frac{1657}{1680}, \frac{2452}{945}, -\frac{1657}{1680}, \frac{22}{105}, -\frac{311}{15\,120}\right).$$

(2) Two-dimensional case ($n = 2$).

In this case, the indices l_1 and l_2 of the weight $\{w_{l_1,l_2}\}$ are restricted to $-[(m-1)/2] \le l_1 + l_2 \le [(m-1)/2]$ and the symmetry is also taken into account. Then we have

- $m = 3$.

$$\begin{pmatrix} & w_{-1,0} & \\ w_{0,-1} & w_{0,0} & w_{0,1} \\ & w_{1,0} & \end{pmatrix} = \begin{pmatrix} & -\frac{1}{8} & \\ -\frac{1}{8} & \frac{3}{2} & -\frac{1}{8} \\ & -\frac{1}{8} & \end{pmatrix}.$$

- $m = 4$.

$$\begin{pmatrix} & w_{-1,0} & \\ w_{0,-1} & w_{0,0} & w_{0,1} \\ & w_{1,0} & \end{pmatrix} = \begin{pmatrix} & -\frac{1}{6} & \\ -\frac{1}{6} & \frac{5}{3} & -\frac{1}{6} \\ & -\frac{1}{6} & \end{pmatrix}.$$

- $m = 5$.

$$\begin{pmatrix} & & w_{-2,0} & & \\ & w_{-1,-1} & w_{-1,0} & w_{-1,1} & \\ w_{0,-2} & w_{0,-1} & w_{0,0} & w_{0,1} & w_{0,2} \\ & w_{1,-1} & w_{1,0} & w_{1,1} & \\ & & w_{2,0} & & \end{pmatrix} = \begin{pmatrix} & & \frac{47}{1152} & & \\ & \frac{25}{576} & -\frac{11}{24} & \frac{25}{276} & \\ \frac{47}{1152} & -\frac{11}{24} & \frac{719}{288} & -\frac{11}{24} & \frac{47}{1152} \\ & \frac{25}{576} & -\frac{11}{24} & \frac{25}{576} & \\ & & \frac{47}{1152} & & \end{pmatrix}.$$

- $m = 6$.

$$\begin{pmatrix} & & w_{-2,0} & & \\ & w_{-1,-1} & w_{-1,0} & w_{-1,1} & \\ w_{0,-2} & w_{0,-1} & w_{0,0} & w_{0,1} & w_{0,2} \\ & w_{1,-1} & w_{1,0} & w_{1,1} & \\ & & w_{2,0} & & \end{pmatrix} = \begin{pmatrix} & & \frac{13}{240} & & \\ & \frac{1}{16} & -\frac{71}{120} & \frac{1}{16} & \\ \frac{13}{240} & -\frac{71}{120} & \frac{29}{10} & -\frac{71}{120} & \frac{13}{240} \\ & \frac{1}{16} & -\frac{71}{120} & \frac{1}{16} & \\ & & \frac{13}{240} & & \end{pmatrix}.$$

- $m = 7$.

$$\begin{pmatrix} & & & w_{-3,0} & & & \\ & & w_{-2,-1} & w_{-2,0} & w_{-2,1} & & \\ & w_{-1,-2} & w_{-1,-1} & w_{-1,0} & w_{-1,1} & w_{-1,2} & \\ w_{0,-3} & w_{0,-2} & w_{0,-1} & w_{0,0} & w_{0,1} & w_{0,2} & w_{0,3} \\ & w_{1,-2} & w_{1,-1} & w_{1,0} & w_{1,1} & w_{1,2} & \\ & & w_{2,-1} & w_{2,0} & w_{2,1} & & \\ & & & w_{3,0} & & & \end{pmatrix}$$

$$= \begin{pmatrix} & & & -\frac{2159}{138\,240} & & & \\ & & -\frac{931}{46\,080} & \frac{781}{3840} & -\frac{931}{46\,080} & & \\ & -\frac{931}{46\,080} & \frac{1421}{5760} & -\frac{57\,877}{46\,080} & \frac{1421}{5760} & -\frac{931}{46\,080} & \\ -\frac{2159}{138\,240} & \frac{781}{3840} & -\frac{57\,877}{46\,080} & \frac{38\,429}{8640} & -\frac{57\,877}{46\,080} & \frac{781}{3840} & -\frac{2159}{138\,240} \\ & -\frac{931}{46\,080} & \frac{1421}{5760} & -\frac{57\,877}{46\,080} & \frac{1421}{5760} & -\frac{931}{46\,080} & \\ & & -\frac{931}{46\,080} & \frac{781}{3840} & -\frac{931}{46\,080} & & \\ & & & -\frac{2159}{138\,240} & & & \end{pmatrix}.$$

• $m = 8$.

$$\begin{pmatrix} & & & w_{-3,0} & & & \\ & & w_{-2,-1} & w_{-2,0} & w_{-2,1} & & \\ & w_{-1,-2} & w_{-1,-1} & w_{-1,0} & w_{-1,1} & w_{-1,2} & \\ w_{0,-3} & w_{0,-2} & w_{0,-1} & w_{0,0} & w_{0,1} & w_{0,2} & w_{0,3} \\ & w_{1,-2} & w_{1,-1} & w_{1,0} & w_{1,1} & w_{1,2} & \\ & & w_{2,-1} & w_{2,0} & w_{2,1} & & \\ & & & w_{3,0} & & & \end{pmatrix}$$

$$= \begin{pmatrix} & & & -\frac{311}{15\,120} & & & \\ & & -\frac{31}{1080} & \frac{1009}{3780} & -\frac{31}{1080} & & \\ & -\frac{31}{1080} & \frac{46}{135} & -\frac{24\,349}{15\,120} & \frac{46}{135} & -\frac{31}{1080} & \\ -\frac{311}{15\,120} & \frac{1009}{3780} & -\frac{24\,349}{15\,120} & \frac{1006}{189} & -\frac{24\,349}{15\,120} & \frac{1009}{3780} & -\frac{311}{15\,120} \\ & -\frac{31}{1080} & \frac{46}{135} & -\frac{24\,349}{15\,120} & \frac{46}{135} & -\frac{31}{1080} & \\ & & -\frac{31}{1080} & \frac{1009}{3780} & -\frac{31}{1080} & & \\ & & & -\frac{311}{15\,120} & & & \end{pmatrix}.$$

(3) Three-dimensional case ($n = 3$).

The indices l_1, l_2 and l_3 of the weight $\{w_{l_1,l_2,l_3}\}$ are restricted to $-[(m-1)/2] \le l_1 + l_2 + l_3 \le [(m-1)/2]$. Also we have computed weights only for even m since they have less complicated values than weights for odd m. Of course the computation is done by computer taking into account the symmetry.

• $m = 4$.

$$w_{0,0,0} = 2,$$

$$w_{1,0,0} = w_{-1,0,0} = w_{0,1,0} = w_{0,-1,0} = w_{0,0,1} = w_{0,0,-1} = -\frac{1}{6}.$$

• $m = 6$.

$$w_{0,0,0} = \frac{169}{40},$$

$$w_{1,0,0} = w_{-1,0,0} = w_{0,1,0} = w_{0,-1,0} = w_{0,0,1} = w_{0,0,-1} = -\frac{43}{60},$$

$$w_{1,1,0} = w_{1,-1,0} = w_{1,0,1} = w_{1,0,-1} = w_{0,1,1} = w_{0,1,-1} = w_{-1,1,0}$$
$$= w_{-1,-1,0} = w_{-1,0,1} = w_{-1,0,-1} = w_{0,-1,1} = w_{0,-1,-1} = \frac{1}{16},$$

$$w_{2,0,0} = w_{-2,0,0} = w_{0,2,0} = w_{0,-2,0} = w_{0,0,2} = w_{0,0,-2} = \frac{13}{240}.$$

• $m = 8$.

$$w_{0,0,0} = \frac{8959}{945},$$

$$w_{1,0,0} = w_{-1,0,0} = w_{0,1,0} = w_{0,-1,0} = w_{0,0,1} = w_{0,0,-1} = -\frac{7205}{3024},$$

$$w_{1,1,0} = w_{1,-1,0} = w_{1,0,1} = w_{1,0,-1} = w_{0,1,1} = w_{0,1,-1} = w_{-1,1,0}$$
$$= w_{-1,-1,0} = w_{-1,0,1} = w_{-1,0,-1} = w_{0,-1,1} = w_{0,-1,-1} = \frac{56}{135},$$

$$w_{2,0,0} = w_{-2,0,0} = w_{0,2,0} = w_{0,-2,0} = w_{0,0,2} = w_{0,0,-2} = \frac{613}{1890},$$

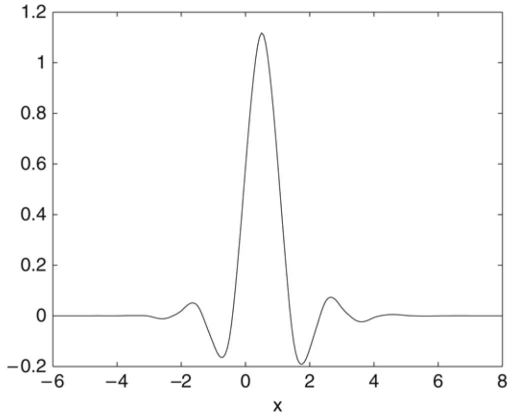

Fig. 1. The Coifman scaling function of degree 8.

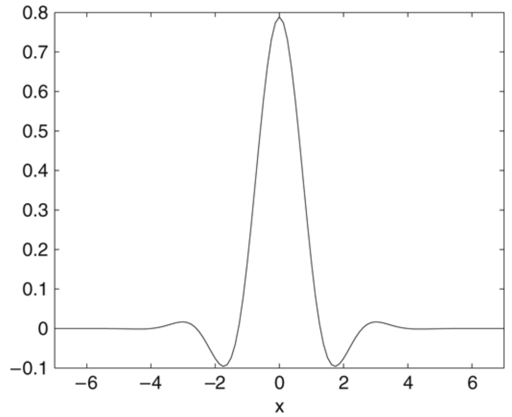

Fig. 2. The (non-separable) spline of degree 8 of one variable (${}_1N_8$).

$$\begin{aligned}
w_{1,1,1} &= w_{1,-1,1} = w_{1,1,-1} = w_{1,-1,-1} = w_{-1,1,1} = w_{-1,-1,1} \\
&= w_{-1,1,-1} = w_{-1,-1,-1} = -\frac{1}{27}, \\
w_{2,1,0} &= w_{2,-1,0} = w_{2,0,1} = w_{2,0,-1} = w_{1,2,0} = w_{1,-2,0} = w_{1,0,2} \\
&= w_{1,0,-2} = w_{0,2,1} = w_{0,2,-1} = w_{0,1,2} = w_{0,1,-2} = w_{-2,1,0} \\
&= w_{-2,-1,0} = w_{-2,0,1} = w_{-2,0,-1} = w_{-1,2,0} = w_{-1,-2,0} \\
&= w_{-1,0,2} = w_{-1,0,-2} = w_{0,-2,1} = w_{0,-2,-1} = w_{0,-1,2} \\
&= w_{0,-1,-2} = -\frac{31}{1080}, \\
w_{3,0,0} &= w_{-3,0,0} = w_{0,3,0} = w_{0,-3,0} = w_{0,0,3} = w_{0,0,-3} = -\frac{311}{15120}.
\end{aligned}$$

Remark 14. See Fig. 2 the profile of computed one dimensional modified spline of degree 8 compared to Fig. 1 the profile of Coifman scaling function. The computed function ${}_2N_m(x, y)$ has smaller support than that of the tensor product ${}_2N_m(x)\,{}_2N_m(y)$ (almost half) and it turns out to be very close to a radial function to our good surprise. See Fig. 3.

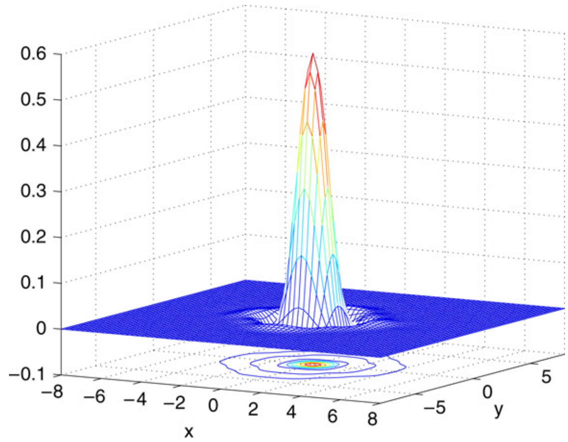

Fig. 3. The non-separable spline of degree 8 of two variables (${}_2N_8$).

5. Numerical simulations for evolution equations

In this section, we apply the method of lines to evolution equations using the non-separable splines constructed in the preceding section, [15]. Let us consider the Burgers equation in one-dimensional case and the KP equation in two-dimensional case. In particular, we compare our spline Galerkin method to the one based on the Coifman scaling function and to the finite difference method in the numerical computation of solutions to Burgers equation.

5.1. The Burgers equation

We consider the initial value problem of the following Burgers equation:

$$\begin{cases} u_t + uu_x = \nu u_{xx}, & x \in \mathbb{R}, t > 0, \nu > 0, \\ u(x,0) = u_0(x), & u(x,t) \to 0 \text{ (as } |x| \to \infty). \end{cases} \tag{4}$$

Let us suppose that $u(x,t)$ belong to $W^{N,p}(\mathbb{R}) \cap W^{N,\infty}(\mathbb{R})$. Then by Theorem 1, with φ defined as before, we have

$$\left\| u(\cdot,t) - \sum_{k=-\infty}^{\infty} u(2^{-J}k,t)\varphi(2^J \cdot -k) \right\|_{L^p(\mathbb{R})} \le 2^{-JN} C \|u^{(N)}(\cdot,t)\|_{L^p(\mathbb{R})},$$

where J is a large fixed integer and C is a constant depending only on φ, p and N. This fact enables us to get an accurate approximation of $u(\cdot,t)$ by a linear sum of $\{\varphi(2^J \cdot -k)\}$. Therefore, we expect to be able to approximate the unknown solution u of the Burgers equation (4) by the well-known method of lines, i.e. by a semi-discretization $u_{S,J}$ defined as follows:

$$u_{S,J}(x,t) := \sum_{k=-\infty}^{\infty} s_k(t)\,{}_1N_m(2^J x - k),$$

where the unknown coefficients $\{s_k(t)\}$ are computed by means of the Galerkin method. Let us call this method the spline Galerkin method (SGM). In addition, in order to compare the (non-separable) spline with other basis, let $u_{C,J}$ be another semi-discretization based on the Coifman scaling function φ_m in Example 6:

$$u_{C,J}(x,t) := \sum_{k=-\infty}^{\infty} c_k(t)\varphi_m(2^J x - k),$$

where the unknown coefficients $\{s_k(t)\}$ are computed by the same method. We call this the wavelet Galerkin method (WGM) [19].

Remark 15. For the approximation of uu_x, we change uu_x into $(u^2)_x/2$. In view of Theorem 2, note that $u^2_{S,J}$ and $u^2_{C,J}$ are approximated respectively by

$$\sum_{k=-\infty}^{\infty} s_k^2(t)\, {}_1N_m(2^J x - k) \quad \text{and} \quad \sum_{k=-\infty}^{\infty} c_k^2(t)\varphi_m(2^J x - k).$$

Now, let us sketch the numerical computation. As explained above, we apply the Galerkin method to two approximations for all $n \in \mathbb{Z}$ as follows:

- The spline Galerkin method.
  Applying the Galerkin method to (4) with $u = u_{S,J}$, we have
  $$\int_{-\infty}^{\infty} \left\{ (u_{S,J})_t + \frac{1}{2}(u^2_{S,J})_x - \nu (u_{S,J})_{xx} \right\} {}_1N_m(2^J x - n)\mathrm{d}x = 0.$$
  Therefore, we obtain
  $$\sum_{k=-\infty}^{\infty} \dot{s}_k(t)\alpha_{k-n} + 2^{J-1} \sum_{k=-\infty}^{\infty} s_k^2(t)\beta_{k-n} + 2^{2J} \sum_{k=-\infty}^{\infty} s_k(t)\gamma_{k-n} = 0, \tag{5}$$
  where the coefficients can be computed beforehand as
  $$\alpha_a := \int_{-\infty}^{\infty} {}_1N_m(x)\, {}_1N_m(x+a)\mathrm{d}x, \qquad \beta_a := \int_{-\infty}^{\infty} {}_1N'_{m-1}(x)\, {}_1N_m(x+a)\mathrm{d}x,$$
  $$\gamma_a := \int_{-\infty}^{\infty} {}_1N'_m(x)\, {}_1N'_m(x+a)\mathrm{d}x, \qquad s_n(0) = u_0(2^{-J}n).$$
- The wavelet Galerkin method.
  In the same manner, we have
  $$\int_{-\infty}^{\infty} \left\{ (u_{C,J})_t + \frac{1}{2}(u^2_{C,J})_x - \nu (u_{C,J})_{xx} \right\} \varphi_m(2^J x - n)\mathrm{d}x = 0,$$
  to find
  $$\dot{c}_n(t) + 2^{J-1} \sum_{k=-\infty}^{\infty} c_k^2(t)\iota_{k-n} + 2^{2J}\nu \sum_{k=-\infty}^{\infty} c_k(t)\mu_{k-n} = 0, \tag{6}$$
  where we have defined the coefficients as follows by taking into account the orthonormality of the Coifman scaling function:
  $$\iota_a := \int_{-\infty}^{\infty} \varphi'_m(y)\varphi_m(y+a)\mathrm{d}y, \qquad \mu_a := \int_{-\infty}^{\infty} \varphi'_m(y)\varphi'_m(y+a)\mathrm{d}y.$$
  $$c_n(0) = u_0(2^{-J}n).$$

Eqs. (5) and (6) are systems of ordinary differential equations with respect to the time variable t. Then we compute numerical solutions of (5) and (6) by the Runge–Kutta method of order 4. Also finally we put $s_n(\cdot)$ and $c_n(\cdot)$ in the definition of $u_{S,J}$ and $u_{C,J}$ to obtain numerical solutions of u.

Remark 16. Note that the method proposed in this paper is in fact equivalent to the one obtained in [19] by a variational argument for the generalized energy integrals.

Example 17. Figs. 4–7 illustrate the evolution of profiles of the solutions at $t = 0, 1, 2, 3, 4$ respectively with a Gaussian initial function $u_0(x) = \mathrm{e}^{-8(x-1)^2}$ with $\Delta x = 2^{-4}$ $(J = 4)$, $\Delta t = 2^{-8}$ and a viscosity coefficient $\nu = 1/32$. We compute numerical solutions in the domain $\Omega = [-3, 5]$, matrices for α_a, β_a and γ_a in a larger domain $\Omega' = [-3 - (m-1)/2^4, 5 + (m-1)/2^4]$ $(m = 8)$ and matrices for ι_a and μ_a in a larger domain $\Omega'' = [-3 - 15/2, 5 + 31/2]$. The boundary conditions are $u_{S,J}|_{\partial\Omega'} = 0$ for SGM and $u_{C,J}|_{\partial\Omega''} = 0$ for WGM respectively.

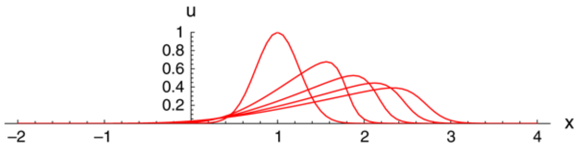

Fig. 4. SGM ($\Delta x = 2^{-4}$, $\Delta t = 2^{-8}$).

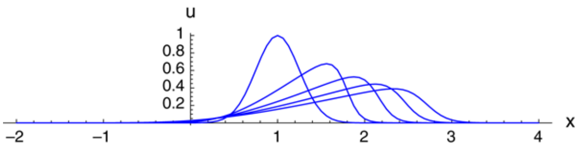

Fig. 5. WGM ($\Delta x = 2^{-4}$, $\Delta t = 2^{-8}$).

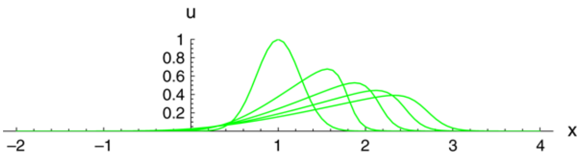

Fig. 6. FDM ($\Delta x = 2^{-4}$, $\Delta t = 2^{-8}$).

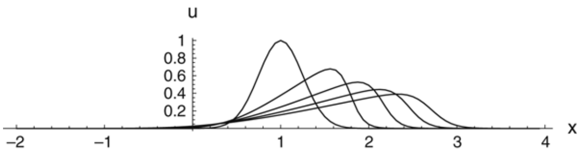

Fig. 7. Exact solution.

Figs. 4 and 5 illustrate the numerical solutions computed by SGM and WGM respectively combined with the Runge–Kutta method of order 4. SGM uses the (non-separable) spline of order 8 ($m = 8$) and WGM uses the Coifman scaling function of the same order for the discretization of the space. In addition, we compute numerical solutions by still other method, i.e. the finite difference method (FDM) of the same order 8 in Fig. 6.

Furthermore, the analytic solution to the Burgers equation given by the Cole–Hope transformation is employed to compare with the numerical solutions. See Fig. 7 illustrating the profile of the analytic solution and Fig. 8 for the comparison of numerical errors. We can observe that with a not too small viscosity coefficient ν, three methods yield stable and accurate numerical solutions. However, note that SGM is more accurate than others.

Example 18. In this example, we show corresponding results of numerical simulations for a smaller viscosity coefficient $\nu = 1/144$ in Figs. 9–13. The initial condition and step sizes are the same as in Example 17. It is known that a solution of the Burgers equation after a certain time, converges to a shock wave as $\nu \to 0$. This must explain the fact that with the small ν, numerical solutions are accompanied by a kind of numerical oscillations. Nevertheless, we can say that SGM provides us with the most accurate solutions.

5.2. *The KP equation*

Let us consider the following KP equation as a two-space-dimensional important nonlinear evolution equation:

$$\begin{cases} u_{tx} + (3u^2)_{xx} + u_{xxxx} - 3u_{yy} = 0, & (x, y) \in \mathbb{R}^2, t > 0, \\ u(x, y, 0) = u_0(x, y), & u(x, y, t) \to 0 \text{ (as } |x| + |y| \to \infty). \end{cases} \tag{7}$$

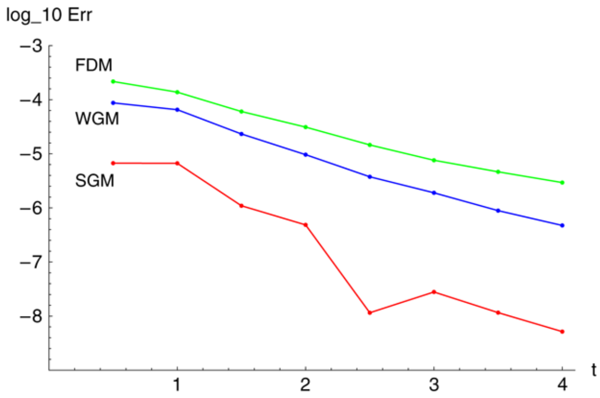

Fig. 8. Maximum errors.

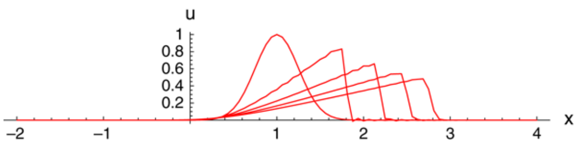

Fig. 9. SGM ($\Delta x = 2^{-4}$, $\Delta t = 2^{-8}$).

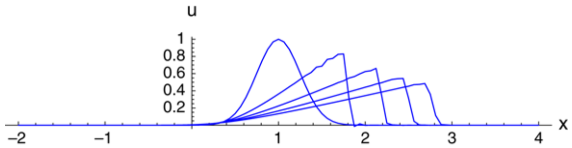

Fig. 10. WGM ($\Delta x = 2^{-4}$, $\Delta t = 2^{-8}$).

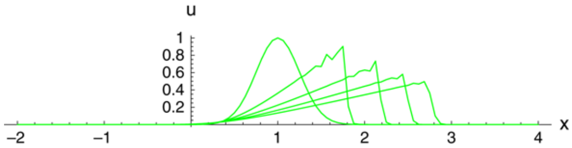

Fig. 11. FDM ($\Delta x = 2^{-4}$, $\Delta t = 2^{-8}$).

As in Section 5.1, we compute the semi-discretized numerical approximation u_J of the solution u to the KP equation (7) by means of the non-separable spline ${}_2N_m$ as follows:

$$u_J(x, y, t) := \sum_{k,l=-\infty}^{\infty} s_{k,l}(t)\,{}_2N_m(2^J x - k, 2^J y - l).$$

Then, let us derive the system of equations which satisfy the unknown coefficients $\{s_{k,l}(t)\}$ from (7) due to the Galerkin method.

Remark 19. As in Remark 15, by Theorem 2, we note that u^2 is approximated by

$$\sum_{k,l=-\infty}^{\infty} s_{k,l}^2(t)\,{}_2N_m(2^J x - k, 2^J y - l).$$

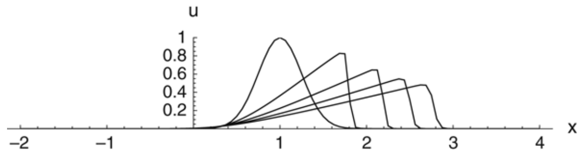

Fig. 12. Exact solution.

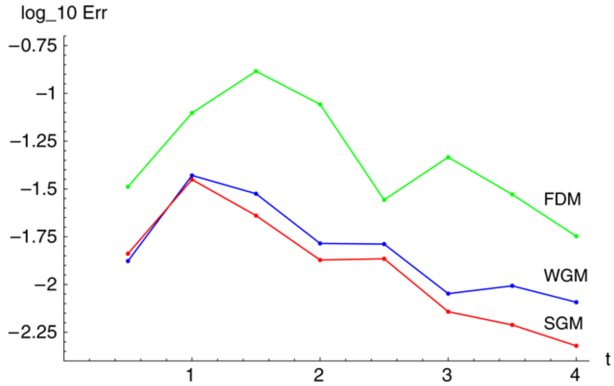

Fig. 13. Maximum errors.

In fact, applying the Galerkin method to (7) with $u = u_J$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{(u_J)_{tx} + 3(u_J^2)_{xx} + (u_J)_{xxxx} - 3(u_J)_{yy}\right\}{}_2N_m(2^Jx - a, 2^Jy - b)\mathrm{d}x\mathrm{d}y = 0, \tag{8}$$

for all $a, b \in \mathbb{Z}$. However, since the pivotal elements of a matrix computed from $\int\int({}_2N_m)_x(2^Jx - k, 2^Jy - l)\,{}_2N_m(2^Jx - a, 2^Jy - b)\mathrm{d}x\,\mathrm{d}y$ are nearly zero, a matrix obtained from (8) has a bad condition number. Therefore this method is complicated since we need pivoting in the computation. Then there are two possible methods for us to overcome this difficulty:

(1) Integrating (7) with respect to the x variable:

$$\int_{-\infty}^{x}\{u_{t\xi} + (3u^2)_{\xi\xi} + u_{\xi\xi\xi\xi} - 3u_{yy}\}\mathrm{d}\xi = 0. \tag{9}$$

(2) Differentiating again (7) with respect to the x variable:

$$\{u_{tx} + (3u^2)_{xx} + u_{xxxx} - 3u_{yy}\}_x = 0. \tag{10}$$

If we apply the Galerkin method to (9), the resulting system of equations is not easy to handle because there appears a non-sparse matrix in the numerical computation. Therefore we choose the second method which provides a sparse matrix in order to avoid a constant solution with respect to the space variable x.

Now for all $a, b \in \mathbb{Z}$, the Galerkin method applied to (10) yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{(u_J)_{tx} + 3(u_J^2)_{xx} + (u_J)_{xxxx} - 3(u_J)_{yy}\right\}_x{}_2N_m(2^Jx - a, 2^Jy - b)\mathrm{d}x\,\mathrm{d}y = 0. \tag{11}$$

Consequently, we obtain as before

$$\sum_{k,l=-\infty}^{\infty}\left\{\dot{s}_{k,l}(t)\theta_{k-a,l-b} + 2^J\cdot 3s_{k,l}^2(t)\rho_{k-a,l-b} - 2^{3J}s_{k,l}(t)\tau_{k-a,l-b} + 2^J\cdot 3s_{k,l}(t)\omega_{k-a,l-b}\right\} = 0, \tag{12}$$

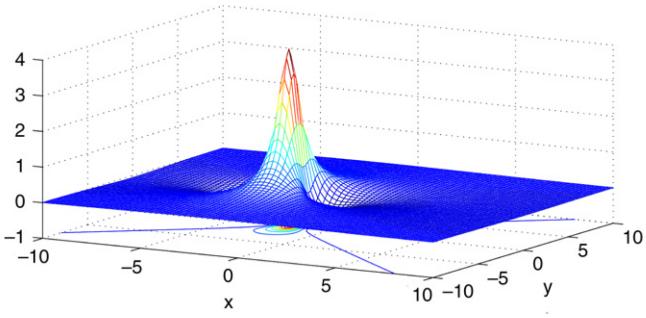

Fig. 14. $t = 0$.

where the coefficients are given by

$$\theta_{a,b} := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ({}_2N_m)_x(x,y)({}_2N_m)_x(x+a,y+b)\mathrm{d}x\mathrm{d}y,$$

$$\rho_{a,b} := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ({}_2N_m)_{xx}(x,y)({}_2N_m)_x(x+a,y+b)\mathrm{d}x\mathrm{d}y,$$

$$\tau_{a,b} := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ({}_2N_m)_{xxx}(x,y)({}_2N_m)_{xx}(x+a,y+b)\mathrm{d}x\mathrm{d}y,$$

$$\omega_{a,b} := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ({}_2N_m)_{xy}(x,y)({}_2N_m)_y(x+a,y+b)\mathrm{d}x\mathrm{d}y.$$

Eq. (12) is a system of ordinary differential equation with respect to the time variable t. We compute the numerical solution of (12) by the Euler backward-difference method using the semi-Newton method. Then we get the numerical solution u_J, putting the obtained values $s_{k,l}(\cdot)$ in the definition of u_J.

Example 20. Figs. 14–17 illustrate the evolution of the profiles of solutions to (7) with the lump-type initial condition

$$u_0(x,y) = \frac{4\{-(x+2)^2+y^2+1\}}{\{(x+2)^2+y^2+1\}^2}, \tag{13}$$

with $\Delta x = \Delta y = 2^{-2}$ ($J = 2$) and $\Delta t = 2^{-9}$ ($T = 9$). Actually, we solve numerical solutions in the domain

$$\Omega = \{(x,y) | -10 \le x, y \le 10\},$$

computing $\theta_{a,b}$, $\rho_{a,b}\tau_{a,b}$ and $\omega_{a,b}$ in a larger domain

$$\Omega' = \{(x,y) | -10-(m-1)/2^J \le x, y \le 10+(m-1)/2^J\} \quad (m = 6).$$

In the computation we use $\varphi = {}_2N_6$ ($m = 6$). Note that (7) with the initial condition (13) is known to have an analytic solution written as

$$u(x,y,t) = \frac{4\{-(x+2-3t)^2+y^2+1\}}{\{(x+2-3t)^2+y^2+1\}^2},$$

(e.g. [11]). Therefore, taking this into account, we can define the relative errors by

$$\text{Error} = \frac{\left\|u(k\Delta x, l\Delta y, n\Delta t) - s_{k,l}(n\Delta t)\right\|_{l^\infty}}{\|u(k\Delta x, l\Delta y, n\Delta t)\|_{l^\infty}}.$$

See Table 1 which exhibits the computed relative errors for each J and T at time $t = 1$.

6. Conclusion

In this paper, we have proposed an efficient as well as accurate sampling approximation of functions for computing numerical solutions of evolution equations.

Table 1
Relative errors

$\Delta t/(\Delta x)^3$	T	Error ($J = 1, t = 1$)	T	Error ($J = 2, t = 1$)
$1/2$	4	0.9031	7	0.3256
$1/2^2$	5	0.6367	8	0.1305
$1/2^3$	6	0.1163	9	0.0526

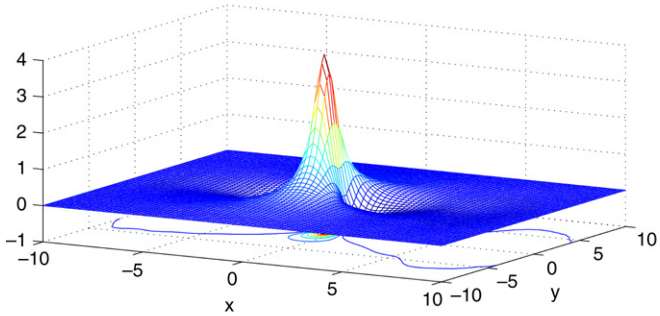

Fig. 15. $t = 0.5$.

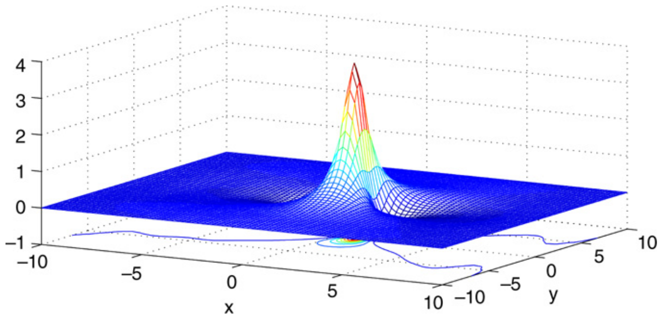

Fig. 16. $t = 1$.

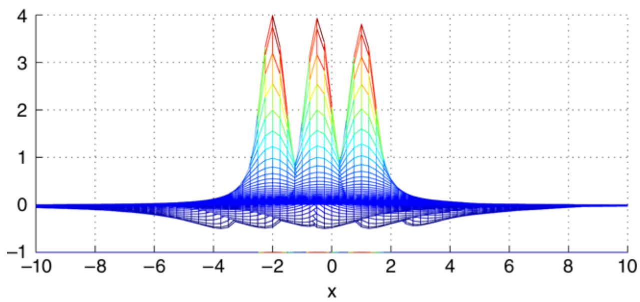

Fig. 17. $t = 0, 0.5, 1$.

First of all, we have recalled the Strang–Fix condition and the moment condition for a function to form a basis for a good sampling approximation of functions in Sobolev spaces. Moreover, we have generalized the error estimates to several dimensional case including a key application to handle nonlinearity.

Next, we have constructed the non-separable (i.e. non-tensor product) splines in two-dimensional case based on the cardinal B-spline of one variable, which turn out to be close to radially symmetric functions.

Then, we have applied the Galerkin method with the non-separable splines to the computation of numerical solutions of the Burgers and KP equations resulting in the following procedure:

Firstly, we express numerical solutions by the method of lines using the non-separable spline with respect to the space variable and secondly, a system of ordinary differential equations is obtained via the Galerkin method. Then, the system of ordinary differential equations is numerically solved by a suitable method such as the Runge–Kutta method or the semi-Newton method. Finally, we have shown that our numerical scheme based on the spline Galerkin method is sufficiently stable and accurate in the computation of numerical solutions of the Burgers and KP equations.

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