Graph covering projections arising from finite vector spaces over finite fields

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Abstract

It is well known that regular graph covering projections may be described by certain voltage assignments. Further investigations can be done if the voltage group is abelian. The purpose of this paper is to classify isomorphism of regular graph covering projections of a graph G that arise from finite abelian groups as voltage groups. In particular, if the voltage group is a finite-dimensional vector space over a finite field and the automorphism group is trivial, the corresponding isomorphism classes will be counted.

1. Introduction

All graphs considered in this paper are assumed to be simple; the vertex set and the edge set of the graph G are denoted by V(G) and E(G), respectively. An r-to-one graph epimorphism p: H → G is called a graph covering projection of G, if p maps the neighbors of vertex v of H bijectively to the neighbors of vertex p(v) of G, for each vertex v of H.

Now let \( \Gamma \leq \text{Aut}(G) \) be a subgroup of the automorphism group of G. A \( \Gamma \)-isomorphism of graph covering projections \( p: H \rightarrow G \) and \( \tilde{p}: \tilde{H} \rightarrow G \) is a commutative diagram

\[
\begin{array}{ccc}
H & \overset{\psi}{\longrightarrow} & \tilde{H} \\
p \downarrow & & \downarrow \tilde{p} \\
G & \overset{\gamma}{\rightarrow} & \tilde{G}
\end{array}
\]

with an isomorphism \( \psi \) and \( \gamma \in \Gamma \); we write \( p \cong, p \) for short.

Presently enumerative results are of particular interest in topological graph theory. Although the main problem of counting all r-fold graph covering projections of a graph G up to \( \Gamma \)-isomorphism is still unsolved except in the cases of \( r = 2 \) [4] or trivial group of automorphisms \( \Gamma \) [5,11], some interesting results are known. For example, Negami [15] counted the equivalence classes of embeddings of a 3-connected
nonplanar graph into a projective plane by proving a bijection between them and the isomorphism classes of planar 2-fold coverings of \( G \). Mull et al. [14] enumerated 2-cell embeddings of connected graphs. Mohar [12, 13] investigated certain graph covering projections of the complete graph on four vertices to enumerate the akempic triangulations of the 2-sphere with 4 vertices of degree 3. Algebraic properties of covering spaces of this graph are considered in [1]. Concrete graph covering projections are counted in [6].

A graph covering projection \( p : H \rightarrow G \) is regular, if there is a subgroup \( \Pi \) of the automorphism group of \( H \) acting freely on \( H \) such that \( H/\Pi \) is isomorphic to \( G \). In fact, a regular graph covering projection of \( G \) can be obtained from a so-called ordinary voltage assignment. Let \( A(G) \) be the arc set of the corresponding symmetric digraph of \( G \). An ordinary voltage assignment of \( G \) with voltage group \( \Pi \) is a mapping \( F : A(G) \rightarrow \Pi \) such that inverse arcs have inverse assignments. The voltage graph \( G_F \) is the graph with vertex set \( \mathcal{V}(G) \times \Pi \). The vertices \((x, \pi)\) and \((y, \rho)\) are adjacent in \( G_F \) iff \([x, y] \in E(G)\) and \( \rho = F(x, y)\pi \). Then the natural map \( p_F : G_F \rightarrow G_F/\Pi \cong G \) is a regular \( |\Pi|\)-fold covering projection of \( G \). Gross and Tucker [3] showed that every regular graph covering projection stems from some ordinary voltage assignment of \( G \).

Our purpose is to classify isomorphism of regular graph covering projections of a graph \( G \) that arise from finite abelian groups as voltage groups. In particular, if the voltage group is a finite-dimensional vector space over a finite field and \( I \) is the identity, the corresponding isomorphism classes will be counted.

2. Classification theorems

Let \( \mathcal{A} \) be a finite abelian group. The set of ordinary voltage assignments of the graph \( G \) with voltages in \( \mathcal{A} \) is denoted by \( \mathcal{C}^1(G; \mathcal{A}) \). This is a group via pointwise addition. We start with a characterization of \( \Gamma \)-isomorphism of regular graph covering projections that arise from assignments in \( \mathcal{C}^1(G; \mathcal{A}) \).

**Theorem 1.** The following are equivalent for \( F, \tilde{F} \in \mathcal{C}^1(G; \mathcal{A}) \):

1. \( p_F \cong_I p_{\tilde{F}} \).
2. There is a \( \gamma \in \Gamma \) and a family \( (\varphi_x)_{x \in \mathcal{V}(G)} \) of permutations of \( \mathcal{A} \), such that

\[
\tilde{F}(\gamma(x), \gamma(y)) = \varphi_y(\xi + F(x, y)) - \varphi_x(\xi)
\]

for each arc \((x, y) \in A(G)\) and \( \xi \in \mathcal{A} \).

**Proof.** Assume that \( p_F \) and \( p_{\tilde{F}} \) are \( \Gamma \)-isomorphic. Then there is an isomorphism \( \psi \) such that the diagram

\[
\begin{array}{ccc}
G_F & \xrightarrow{\psi} & G_{\tilde{F}} \\
p_F \downarrow & & \downarrow p_{\tilde{F}} \\
G & \xrightarrow{\gamma} & G
\end{array}
\]
commutes. It follows that
\[
\psi(x, \xi) = (\gamma(x), \varphi_x(\xi))
\]
for some \( \varphi_x \in \mathcal{S}_\omega, \) where \( \mathcal{S}_\omega \) is the symmetric group of the members of \( \omega. \)

Now let \((x, \xi)\) and \((y, \xi + F(x, y))\) be adjacent vertices in \( G_F. \) Then \((\gamma(x), \varphi_x(\xi))\) is adjacent to \((\gamma(y), \varphi_x(\xi + F(x, y)))\) in \( G_F \) by Eq. (4). On the other hand, we obtain from the construction of \( G_F \) that \((\gamma(x), \varphi_x(\xi))\) is adjacent to \((\gamma(y), \varphi_x(\xi) + \bar{F}(\gamma(x), \gamma(y)))\).

We conclude that
\[
\varphi_y(\xi + F(x, y)) = \varphi_x(\xi) + \bar{F}(\gamma(x), \gamma(y), (5)
\]

since liftings of walks are unique. (See e.g. [3, Theorem 2.1.1].)

Conversely, if (2) holds, then
\[
\psi(x, \xi) = (\gamma(x), \varphi_x(\xi))
\]

is the desired isomorphism. □

Theorem 1 gives a classification of \( I \)-isomorphism of graph covering projections arising from the abelian group \( \omega. \) This classification is quite complicated; it investigates a family \((\varphi_x)_{x \in V(G)}, \) where \( \varphi_x \in \mathcal{S}_\omega \) for each \( x \in V(G). \) It will turn out that the \( I \)-isomorphism classes of such graph covering projections (where \( I \) is the identity) contain representants that are easier to handle. These representants are described in the next theorem.

**Theorem 2.** Let \( T \) be a spanning forest of \( G. \)

1. For each \( F \in \mathfrak{C}^1(G; \omega) \) there exists an \( F_T \in \mathfrak{C}^1(G; \omega) \) such that \( F_T \equiv 0 \) on the arcs of \( T \) and \( p_T \cong p_{F_T}. \)

2. Let \( F, \bar{F} \in \mathfrak{C}^1(G; \omega) \) such that \( F, \bar{F} \equiv 0 \) on the arcs of \( T. \) Furthermore, let \( p_F \cong p_{\bar{F}}. \)

Then, for every component of \( G, \) there exists a \( \varphi \in \mathcal{S}_\omega \) such that
\[
\bar{F}(x, y) = \varphi(\xi + F(x, y)) - \varphi(\xi)
\]

for every \( \xi \in \omega. \)

**Proof.** Without loss of generality we may assume that the graph \( G \) is connected. Fix a vertex \( u \) of \( G. \) For every vertex \( v \) of \( G \) there exists a unique \( uv \)-path in \( T, \) which we denote by \( P_v. \) Set
\[
F(P_v) = \sum_{(x, y) \in P_v} F(x, y).
\]

Define \( F_T \in \mathfrak{C}^1(G; \omega) \) by setting
\[
F_T(x, y) = F(P_x) + F(x, y) - F(P_y)
\]
for each arc \((x, y) \in A(G)\). Then \(F_T \equiv 0\) on the arcs of \(T\). For \(v \in V(G)\) define a permutation \(\varphi_v \in \mathcal{S}_n\) by setting \(\varphi_v(\xi) = \xi - F(P_v)\) \((\xi \in \mathcal{A})\). Then we have \(F_T(x, y) = \varphi_y(\xi + F(x, y)) - \varphi_x(\xi)\) for every arc \((x, y) \in A(G)\), hence \(p_F \cong_1 p_F\) by Theorem 1, which proves the first part of the theorem.

In order to complete the proof, let \((x, y)\) be an arc \(T\). Then \(F(x, y) = \tilde{F}(x, y) = 0\). From Theorem 1 we obtain \(\varphi_y(\xi) = \varphi_x(\xi)\) for every \(\xi \in \mathcal{A}\), hence \(\varphi_y = \varphi_x\).

As suggested by Theorem 2(2), we restrict attention to connected graphs. The easy exercise to formulate all of the following results (particularly the enumeration formulas of Section 4) for graphs which are not connected is left to the reader.

From now on, let \(\mathcal{A} = \mathbb{F}_p^r\) be the \(r\)-dimensional vector space over the field \(\mathbb{F}_p\) with \(p\) elements, where \(p\) is a prime number. The next theorem presents a reformulation of Theorem 2 investigating the algebraic structure of vector spaces.

**Theorem 3.** Let \(T\) be a spanning forest of \(G\), and let \(F, \tilde{F} \in \mathcal{C}^1(G; \mathbb{F}_p^r)\) such that \(F, \tilde{F} \equiv 0\) n the arcs of \(T\). The following are equivalent:

1. \(p_F \cong_1 p_{\tilde{F}}\).
2. There exists an \(A \in \text{GL}_r(\mathbb{F}_p)\), such that \(F(x, y) = AF(x, y)\) for every arc \((x, y)\) of \(G\).

**Proof.** If (2) holds, then simply set \(\varphi_x(\xi) = A\xi\) for each \(x \in V(G)\) and apply Theorem 1.

Now assume that \(p_F \cong_1 p_{\tilde{F}}\). From Theorem 2(2) we conclude that there is a \(\varphi \in \mathcal{S}_{\mathbb{F}_p^r}\) such that Eq. (2) holds. Set \(\mathcal{V} = \langle \text{im}(F) \rangle\) and \(\tilde{\mathcal{V}} = \langle \text{im}(\tilde{F}) \rangle\). It follows from Eq. (2) by induction, that

\[
\varphi(jF(x, y)) = j\tilde{F}(x, y) + \varphi(0),
\]

for all \(j \in \mathbb{N}\). Furthermore, for two arcs \((x_1, y_1), (x_2, y_2)\) of \(G\),

\[
\varphi(F(x_1, y_1) + F(x_2, y_2)) = \tilde{F}(x_2, y_2) + \varphi(F(x_1, y_1))
\]

\[
= \tilde{F}(x_2, y_2) + \tilde{F}(x_1, y_1) + \varphi(0).
\]

Hence \(\varphi\) is an affine epimorphism from \(\mathcal{V}^r\) into \(\varphi(0) + \tilde{\mathcal{V}}\). Since \(\varphi\) is bijective, it can be extended to an affine automorphism \(\varphi : \mathbb{F}_p^r \to \mathbb{F}_p^r\), hence there exists \(A \in \text{GL}_r(\mathbb{F}_p)\) and \(b \in \mathbb{F}_p^r\) such that \(\varphi(\xi) = A\xi + b\) for \(\xi \in \mathbb{F}_p^r\). Now the assertion follows from Eq. (7).

**Remark.** In the proof of Theorem 3 only the bijectivity of \(\varphi\) and the \(\mathbb{Z}\)-module property of \(\mathbb{F}_p^r\) is needed. Since every finite field is itself a vector space over its prime field, Theorem 3 classifies \(I\)-isomorphism for all voltage groups that are finite-dimensional vector spaces over finite fields \(\mathbb{F}_q\).
Let $\mathcal{C}^0(G; F_p^p)$ be the vector space of functions $f: V(G) \to F_p^p$. Introduce the coboundary operator
\begin{equation}
\delta: \mathcal{C}^0(G; F_p^p) \to \mathcal{C}^1(G; F_p^p)
\end{equation}
by setting
\begin{equation}
\delta(f)(x, y) = f(x) - f(y) \quad ((x, y) \in A(G)).
\end{equation}
Using this vector space homomorphism, the classification theorem for graph covering projections of $G$ arising from $F_p^p$ as voltage group looks as follows.

**Theorem 4.** The following are equivalent for $F, \bar{F} \in \mathcal{C}^1(G; F_p^p)$:

1. $PF \equiv_f P\bar{F}$.
2. There exist $f \in \mathcal{C}^0(G; F_p^p)$, $\gamma \in \Gamma$ and $A \in GL_r(F_p)$, such that
\begin{equation}
\bar{F}(\gamma(x), \gamma(y)) = AF(x, y) + \delta(f)(x, y)
\end{equation}
for each arc $(x, y) \in A(G)$.

**Proof.** Assume that $PF \equiv_f P\bar{F}$ by $\gamma \in \Gamma$. Define $\gamma^{-1}(\bar{F})$ by setting $\gamma^{-1}(\bar{F})(x, y) = \bar{F}(\gamma(x), \gamma(y))$. As in the proof of Theorem 2, define $F_T$ and $\gamma^{-1}(\bar{F})_T$ by setting
\begin{equation}
\begin{aligned}
F_I(x, y) &= F(P_x) + F(x, y) - F(P_y), \\
\gamma^{-1}(\bar{F})_T(x, y) &= \gamma^{-1}(\bar{F})(P_x) + \gamma^{-1}(\bar{F})(x, y) - \gamma^{-1}(\bar{F})(P_y).
\end{aligned}
\end{equation}
It follows from Theorem 3 that
\begin{equation}
\gamma^{-1}(\bar{F})_T = AF_T
\end{equation}
for some $A \in GL_r(F_p)$. Using Eqs (15) one obtains
\begin{equation}
\begin{aligned}
\gamma^{-1}(\bar{F})(x, y) &= (AF(P_x) - \gamma^{-1}(\bar{F})(P_x)) + F(x, y) \\
&\quad - (AF(P_y) - \gamma^{-1}(\bar{F})(P_y)).
\end{aligned}
\end{equation}
The assertion follows by setting
\begin{equation}
f(v) = AF(P_v) - \gamma^{-1}(\bar{F})(P_v)
\end{equation}
for each $v \in V(G)$.

Conversely, set $\varphi_v(\xi) = A\xi - f(v)$ for each $v \in V(G)$ and apply Theorem 1. \qed

### 3. Cohomology

The coboundary operator $\delta$ gives rise to an exact sequence of vector spaces over $F_p$.

Let $\mathcal{H}^0(G; F_p^p)$ be the kernel of $\delta$, and set $\mathcal{H}^1(G; F_p^p) = \mathcal{C}^1(G; F_p^p)/\text{im}(\delta)$. Then the
sequence
\[
0 \rightarrow \mathcal{H}^0(G; F_p^r) \xrightarrow{\delta^0} \mathcal{C}^0(G; F_p^r) \xrightarrow{\delta} \mathcal{C}^1(G; F_p^r) \xrightarrow{\delta^1} \mathcal{H}^1(G; F_p^r) \rightarrow 0 \quad (19)
\]
is exact, where $\delta^0$ and $\delta^1$ are the canonical monomorphism and epimorphism, respectively. Some properties of this sequence are investigated in [7, 8]. The group $GL_n(F)$ acts on $\mathcal{C}^0(G; F_p^r)$ and $\mathcal{C}^1(G; F_p^r)$ via
\[
(Af)(v) = Af(v) \quad \text{for } f \in \mathcal{C}^0(G; F_p^r),
\]
\[
(AF)(x,y) = AF(x,y) \quad \text{for } F \in \mathcal{C}^1(G; F_p^r).
\]
Since every $A \in GL_n(F)$ commutes with $\delta$, $GL_n(F)$ acts on $\text{im}(\delta)$ and $\mathcal{H}^1(G; F_p^r)$. Obviously, $GL_n(F)$ acts on $\mathcal{H}^0(G; F_p^r)$, too. Summarizing, we state that the diagram
\[
0 \rightarrow \mathcal{H}^0(G; F_p^r) \xrightarrow{\delta^0} \mathcal{C}^0(G; F_p^r) \xrightarrow{\delta} \mathcal{C}^1(G; F_p^r) \xrightarrow{\delta^1} \mathcal{H}^1(G; F_p^r) \rightarrow 0
\]
\[
A \downarrow \quad A \downarrow \quad A \downarrow \quad A \downarrow \quad A \downarrow \quad (21)
\]
is commutative. We obtain the simple but important following result.

**Theorem 5.** The $I$-isomorphism classes of graph covering projections of $G$ arising from $F_p^r$ as voltage group are in one-to-one correspondence with the orbits of the operation of $GL_n(F)$ on $\mathcal{C}^1(G; F_p^r)$.

4. Enumeration of $I$-isomorphism classes

From now on, we assume that $r = I$, i.e., we consider $G$ as a labeled graph. The purpose of this section is to count the orbits of the operation of $GL_n(F)$ on the space $\mathcal{C}^1(G; F_p^r)$ in order to obtain the number of $I$-isomorphism classes of the corresponding graph covering projections of $G$.

For each $A \in GL_n(F)$, the mapping $\mu_A : F \mapsto (A - I)F$ is an endomorphism of $\mathcal{C}^1(G; F_p^r)$ that induces an endomorphism $\nu_A$ of $\mathcal{H}^1(G; F_p^r)$ via the commutativity of Diagram (21); in particular, $\nu_A(F + \text{im}(\delta)) = \mu_A(F) + \text{im}(\delta)$. In order to apply Burnside's lemma (which is, in fact, due to Cauchy–Forbenius, as we know from [10]), we have to determine the size of $\text{ke}(\nu_A)$ for each $A \in GL_n(F)$, since this space contains exactly the fixed points of $A$ in $\mathcal{H}^1(G; F_p^r)$. Set
\[
\mathcal{C}^1_A(G; F_p^r) = \{ f \in \mathcal{C}^1(G; F_p^r) | \exists f \in \mathcal{C}^0(G; F_p^r) : (A - I)F = \delta(f) \}. \quad (22)
\]
The space $\mathcal{C}^1_A(G; F_p^r)$ is the preimage of $\text{ke}(\nu_A)$ under $\delta^1$. Since $\text{im}(\delta) \subseteq \mathcal{C}^1_A(G; F_p^r)$, it suffices to compute the size of $\mathcal{C}^1_A(G; F_p^r)$ in order to obtain the size of $\text{ke}(\nu_A)$. Note that
dim(\text{im}(\delta)) = r(n - 1), where \(n\) is the number of vertices of \(G\); this follows immediately from exactness of Sequence (19).

Let \(A \in \text{GL}_n(\mathbb{F}_p)\), and let \(\text{Eig}(A; 1)\) be the eigenspace of \(A\) belonging to the eigenvalue 1; if 1 is not an eigenvalue, we set \(\text{Eig}(A; 1) = \{0\}\). For a walk \(W\) in \(G\), let \(F(W)\) be the sum of voltages given by \(F\) along \(W\). We start with a criterion for \(F\) to be in \(\mathcal{V}^1_A(G; \mathbb{F}_p)\).

**Theorem 6.** The following are equivalent for \(F \in \mathcal{V}^1(G; \mathbb{F}_p)\):

1. \(F \in \mathcal{V}^1_A(G; \mathbb{F}_p)\).
2. \(F(C) \in \text{Eig}(A; 1)\) for every closed walk \(C\) of \(G\).

**Proof.** Let \(F \in \mathcal{V}^1_A(G; \mathbb{F}_p)\), i.e. \((A - I)F = \delta(f)\) for some \(f \in \mathcal{V}^0(G; \mathbb{F}_p)\). Then we have for every closed walk \(C = x_0x_1 \ldots x_1x_0\) of \(G\)

\[
\begin{align*}
(A - I)F(C) &= \sum_{i=0}^{l-1} (A - I)F(x_i,x_{i+1}) + (A - I)F(x_0,x_0) \\
&= \sum_{i=0}^{l-1} (f(x_i) - f(x_{i+1})) + (x_1 - x_0) = 0. 
\end{align*}
\]

(23)

Now assume that \(F(C) \in \text{Eig}(A; 1)\) for every closed walk \(C\) of \(G\). Fix a vertex \(u\) of \(G\) and define \(f \in \mathcal{V}^0(G; \mathbb{F}_p)\) by setting

\[
f(x) = - (A - I)F(P_x),
\]

(24)

where \(P_x\) is any \(ux\)-path in \(G\). Obviously, \(f\) is well defined. For the arc \((x, y) \in A(G)\) consider the closed walk \(C = uP_xxy(-P_y)u\), where \((-P_y)\) is \(P_y\) to the reverse; we obtain

\[
(A - I)F(x, y) = (A - I)F(C) - (A - I)F(P_x) + (A - I)F(P_y)
= f(x) - f(y),
\]

(25)

hence \(F \in \mathcal{V}^1_A(G; \mathbb{F}_p)\). \(\square\)

Let \(m\) be the number of edges of \(G\). Recall that \(n\) is the number of vertices of \(G\). Set \(B(G) = m - n + 1\), the Betti number of \(G\).

**Theorem 7.** If \(A \in \text{GL}_r(\mathbb{F}_p)\) such that \(\dim(\text{Eig}(A; 1)) = l (0 \leq l \leq r)\), then \(\dim(\text{ke}(v_\lambda)) = lB(G)\).

**Proof.** Consider the space \(\mathcal{V}^1_A(G; \mathbb{F}_p)\). Let \(T\) be a spanning tree of \(G\). We count the number of voltage assignments of \(G\) with elements in \(\mathbb{F}_p\), such that \(F(C) \in \text{Eig}(A; 1)\) for every closed walk \(C\) of \(G\).

First assign values to the arcs of \(T\); there are \(p^{r(n-1)}\) possible choices. Now consider an arc \((x, y) \in A(G) - A(T)\). Let \(P_{yx}\) be the unique \(yx\)-path in \(T\). The circuit
$C_{xy} = xyP_{yx}$ is called elementary. Since $F(C_{xy})$ has to be in $\text{Eig}(A; 1)$, we obtain

$$F(x, y) \in F(P_{yx}) + \text{Eig}(A; 1), \quad (26)$$

hence there are $p^t$ possible choices for $F(x, y)$. So we obtain $p^{IB(G)}$ possible choices for the edges in $A(G) - A(T)$. Altogether, it follows that $\dim(\mathcal{C}_A(G; \mathbb{F}_p)) = IB(G) - r(n - 1)$. The assertion now follows from $(\text{im} \delta^1 | \mathcal{C}_A(G; \mathbb{F}_p)) = \text{ke}(\nu_A), \text{ke}(\delta^1 | \mathcal{C}_A(G; \mathbb{F}_p)) = \text{im}(\delta)$, and $\dim(\text{im}(\delta)) = r(n - 1)$. □

We conclude from Theorem 7 that $A \in \text{GL}_r(\mathbb{F}_p)$ has exactly $p^{IB(G)}$ fixed points in $\mathcal{W}^1(G; \mathbb{F}_p)$, provided that $\dim(\text{Eig}(A; 1)) = l$. The posed counting problem can be solved by investigating the numbers of $l$-dimensional subspaces of $\mathbb{F}_p^r$, and of all $A \in \text{GL}_r(\mathbb{F}_p)$ such that $\text{Eig}(A; 1)$ is a given $l$-dimensional subspace of $\mathbb{F}_p^r$. The first of these numbers is well known. For a natural number $n$, set $[n]_p = 1 + p + \cdots + p^{n - 1}$, and define the $p$-factorial of $n$ by $[n]_p! = [1]_p[2]_p\cdots[n]_p$, where $[0]_p = 1$. Then the $p$-binomial numbers are defined by

$$\binom{m}{n}_p = \frac{[m]_p!}{[n]_p![m - n]_p!} \quad (27)$$

if $n \leq m$, and 0 otherwise. The following lemma can be found e.g. in [10].

**Lemma 1.** The number of $l$-dimensional subspaces of $\mathbb{F}_p^r$ is $[r]_p! (0 \leq l \leq r)$.

We obtain the second number by computing first the number $\beta(p^t; l)$ of $A \in \text{GL}_r(\mathbb{F}_p)$ that fix a given $l$-dimensional subspace of $\mathbb{F}_p^r$ (but possibly more) and then applying Moebius inversion.

**Lemma 2.** $\beta(p^t; l) = p^{\binom{l}{2} - \binom{1}{2}}(p^l - 1)^{r - l}[r - l]_p!$.  

**Proof.** Let $\mathcal{V}$ be an $l$-dimensional subspace of $\mathbb{F}_p^r$ with base $\mathcal{V} = (v_1, \ldots, v_l)$. Extend $\mathcal{V}$ to a base $\mathcal{B} = (v_1, \ldots, v_l, v_{l+1}, \ldots, v_r)$ of $\mathbb{F}_p^r$. Each base $\mathcal{C} = (c_1, \ldots, c_r)$ of $\mathbb{F}_p^r$ determines an $A \in \text{GL}_r(\mathbb{F}_p)$ by setting $A v_i = c_i \quad (1 \leq i \leq r)$. Since we assume that $\mathcal{V} \subseteq \text{Eig}(A; 1)$, $c_i = v_i$ for $1 \leq i \leq l$. We conclude that

$$\beta(p^t; l) = (p^l - p^t)(p^l - p^{l+1})\cdots(p^l - p^{t-1}) = p^{\binom{l}{2} - \binom{1}{2}}(p^{r - 1} - 1)\cdots(p - 1)$$

$$= p^{\binom{l}{2} - \binom{1}{2}}(p - 1)^{r - l}[r - l]_p!, \quad (28)$$

which completes the proof. □
Corollary 1. \(|\text{GL}_r(\mathbb{F}_p)| = p^\binom{r}{2}(p - 1)^r [r]_p.\)

Proof. Obvious, since \(|\text{GL}_r(\mathbb{F}_p)| = \beta(p^r; 0).\) □

Lemma 3. Let \(\mathcal{L}(r; p)\) be the subspace lattice of \(\mathbb{F}_p\) with Moebius function \(\mu(U, V)\). 

1. The Moebius function \(\mu(U, V)\) of \(\mathcal{L}(r; p)\) depends only on the dimensions of \(U\) and \(V\). (Hence it makes sense to speak about \(\mu(k, l) := \mu(U, V)\), where \(\dim(U) = k\) and \(\dim(V) = l\).) 

2. \(\mu(k, k + 1)\) depends only on \(l\). (Therefore it is useful to set \(\mu(l) := \mu(k, k + l)\).) 

3. \(\mu\) satisfies the recursion 

\[\mu(l) = - \sum_{j=0}^{l-1} \binom{l}{j}_p \mu(j)\]

for \(l \geq 1; \mu(0) = 1.\) 

4. For each \(l \geq 0,\)

\[\mu(l) = (-1)^l p^\binom{l}{2}.\]

Proof. Let \(V\) be a subspace of \(\mathbb{F}_p\) of dimension \(l\), and let \(U\) be a proper subspace of \(V\) of dimension \(k\). Then \(\mu(U, V)\) satisfies the recursion formula

\[\mu(U, V) = - \sum_{U < X < V} \mu(U, X).\]  

Now (1) follows by induction observing that \(\mu(U, U) = 1\) and that the number of spaces \(X\) between \(U\) and \(V\) of dimension \(j\) is \(\binom{j-1}{k-1}_p\), which depends only on the dimensions of \(U\) and \(V\). From Eq. (31) one obtains

\[\mu(k, k + 1) = - \sum_{j=0}^{l-1} \binom{l}{j}_p \mu(k, k + j);\]

part (2) of the lemma follows by induction, while part (3) is Eq. (32) again. In order to prove the equation of part (4), consider the \(q\)-binomial theorem, that states that

\[\forall n > l \geq 0: \sum_{k=0}^{n} \binom{n}{k}_q q^{\binom{n-k}{2}} (-1)^{n-k} q^k = 0.\] 

The assertion can be proved by induction using this theorem. □

Now we are ready to compute the numbers \(\alpha(p^l, 1) (0 \leq l \leq r)\) of \(A \in \text{GL}_r(\mathbb{F}_p)\), such that \(\text{Eig}(A; 1) = V\) for a given \(l\)-dimensional subspace \(V\) of \(\mathbb{F}_p^r\).

Lemma 4.

\[\alpha(p^l, 1) = \sum_{i=0}^{r-l} \binom{r-l}{i}_p (-1)^i (p - 1)^{r-l-i} q^\binom{i}{2} q^{-\binom{i}{2}}.\]
Proof. It is clear that, for every \( l \) with \( 0 \leq l \leq r \),

\[
\beta(p', l) = \sum_{j=1}^{r} \sum_{\dim(U) = j} \alpha(p', l),
\]

where \( V \) is of dimension \( l \). Applying M"obius inversion we obtain the result from Lemmas 2 and 3 after a straightforward calculation. 

After these preparations it is no more difficult to solve the counting problem posed at the beginning of this section. For any finite group \( \mathcal{A} \), let \( \text{cov}(G, \Gamma, \mathcal{A}) \) be the number of \( \Gamma \)-isomorphism classes of graph covering projections of \( G \) that stem from the voltage group \( \mathcal{A} \).

**Theorem 8.** For every connected graph \( G \),

\[
\text{cov}(G, l, \mathbb{F}_p) = \frac{1}{|\text{GL}_l(\mathbb{F}_p)|} \sum_{j=0}^{r} \left[ \begin{array}{c} r \\ l \end{array} \right] \alpha(p', l) p^{\beta(G)}(p' - 1),
\]

where \( |\text{GL}_l(\mathbb{F}_p)| \) can be computed by Corollary 1 and \( \alpha(p', l) \) is given by Lemma 4.

Proof. We conclude from Theorem 7 that the number of fixed points of \( A \in \text{GL}_r(\mathbb{F}_p) \) equals \( p^{\beta(G)} \), where \( l \) is the dimension of the eigenspace of \( A \) with respect to the eigenvalue 1. For \( 0 \leq l \leq r \), there are \( \left[ \begin{array}{c} r \\ l \end{array} \right] \) subspaces \( V \) of \( \mathbb{F}_p \) of dimension \( l \) by Lemma 1, and there are exactly \( \alpha(p', l) \) members \( A \in \text{GL}_r(\mathbb{F}_p) \) with the property that \( \text{Eig}(A; 1) = V \). Now the theorem follows by applying Burnside’s lemma.

The formula of Theorem 8 is quite complicated. Restricting it to particular values of \( p \) and \( r \) gives rise to further formulas that are useful for concrete computations.

**Corollary 2.** (1) For every prime number \( p \),

\[
\text{cov}(G, l, \mathbb{F}_p) = \frac{p - 2 + p^{\beta(G)}}{p - 1}.
\]

(2) For every prime number \( p \),

\[
\text{cov}(G, l, \mathbb{F}_p^2) = \frac{(p^4 - 2p^3 - p^2 + 3p) + (p + 1)(p^2 - p - 1)p^{\beta(G)} + p^{2\beta(G)}}{p(p - 1)^2(p + 1)}.
\]

(3) For \( p = 2 \) and \( r = 3 \),

\[
\text{cov}(G, l, \mathbb{F}_2^3) = \frac{1}{21}(6 + 49 \cdot 2^{\beta(G)} - 2 + 21 \cdot 2^{\beta(G)} - 3 + 2^{3\beta(G)} - 3).
\]

Note that Theorem 8 generalizes some well known results. For example, Corollary 2(1) is contained in [11] and is an extension of the particular case of \( p = 2 \), which is
Table 1

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<th>$p'$</th>
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classical (see, for example, [4,16]). Furthermore, the formula for $\text{cov}(G,I,\mathbb{F}_2^2)$ is a partial result of [9].

The formulas of Corollary 2 are used to compute the values of Table 1, which presents the numbers $\text{cov}(G,I,\mathbb{F}_2^2)$ for some small values of $p'$ and $B(G)$.

References