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Note

Determinants of box products of paths

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ABSTRACT

Suppose that G is the graph obtained by taking the box product of a path of length n and a path of length m. Let $\mathbf{M}$ be the adjacency matrix of G. In 1996, Rara showed that, if $n = m$, then $\det(\mathbf{M}) = 0$. We extend this result to allow n and m to be any positive integers, and show that

$$\det(\mathbf{M}) = \begin{cases} 0 & \text{if } \gcd(n+1, m+1) \neq 1, \\ (-1)^{nm/2} & \text{if } \gcd(n+1, m+1) = 1. \end{cases}$$

1. Introduction

Let $[n] = \{1, 2, \ldots, n\}$. We define a *graph* G to be an ordered pair of sets (V, E), where V is any set and $E \subseteq \binom{V}{2}$; we refer to V as the *vertices* and E as the *edges* of G. The *adjacency matrix* of G is denoted $\mathbf{A}(G)$, and it is a matrix with rows and columns indexed by V such that

$$\mathbf{A}(G)_{i,j} = \begin{cases} 1 & \text{if } \{i, j\} \in E, \\ 0 & \text{if } \{i, j\} \notin E. \end{cases}$$

Let $\mathbf{I}_n$ be the $n \times n$ identity matrix, and let $\mathbf{0}_n$ be the $n \times n$ matrix of all zeros. If G has n vertices, the *characteristic polynomial* of $\mathbf{A}(G)$ is defined to be $q_G(x) = \det(\mathbf{A}(G) - x\mathbf{I}_n)$.

Suppose that G_1 and G_2 are graphs with vertex sets V_1 and V_2, and edge sets E_1 and E_2, respectively. The *box product* of G_1 and G_2, denoted $G_1 \square G_2$, is the graph with vertex set $V = V_1 \times V_2$ and such that, for $i_1, j_1 \in V_1$ and $i_2, j_2 \in V_2$, $\{(i_1, i_2), (j_1, j_2)\}$ is an edge in $G_1 \square G_2$ if and only if either $i_1 = j_1$ and $\{i_2, j_2\} \in E_2$, or $i_2 = j_2$ and $\{i_1, j_1\} \in E_1$. For an in-depth look at the box product (also referred to as the *Cartesian product*) of graphs, see [3].

Let G be a graph with vertex set $[n]$ and adjacency matrix $\mathbf{A}$, and let H be a graph with vertex set $[m]$ and adjacency matrix $\mathbf{B}$. Then, the vertices of $G \square H$ can be labeled with the elements of $[nm]$, by relabeling the vertex (i, j) as $(i-1)m + j$. Under this labeling, the adjacency matrix $\mathbf{M}$ of $G \square H$ can be written as an $n \times n$ block matrix $\mathbf{M} = [\mathbf{M}_{i,j}]$, where each $\mathbf{M}_{i,j}$ is $m \times m$. Further,

$$\mathbf{M}_{i,j} = \begin{cases} \mathbf{B} & \text{if } i = j, \\ \mathbf{I}_m & \text{if } i \neq j \text{ and } i \sim j \text{ in } G, \\ \mathbf{0}_m & \text{if } i \neq j \text{ and } i \not\sim j \text{ in } G. \end{cases}$$

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The $\mathbf{M}_{i,j}$ are all elements of the commutative subring S of $\mathbb{R}^{m\times m}$ generated by $\mathbf{B}$ and $\mathbf{I}_m$. Thus, if we denote the determinant over the ring S by $\det_S$, it is not hard to see that $\det_S(\mathbf{M}) = q_G(-\mathbf{B})$, so

$$\det(\mathbf{M}) = \det\left(\det_S(\mathbf{M})\right) = \det(q_G(-\mathbf{B})).$$

We now consider the case when both G and H are *paths*.

2. Paths and products of paths

The *path with n vertices*, denoted P_n, is the graph with vertex set $V = [n]$ and edge set $E = \{(i, i+1) : i \in [n-1]\}$. Let $q_n(x)$ be the characteristic polynomial of P_n. In [4], it was shown that $\det\left(\mathbf{A}\left(P_n \,\square\, P_n\right)\right) = 0$. We extend this result, and compute the value of $\det\left(\mathbf{A}\left(P_n \,\square\, P_m\right)\right)$ for all positive integers n and m. We do this first by looking at $q_n(x)$. Note that, since $\mathbf{A}(P_n)$ is a tridiagonal matrix and has a very simple structure, many of the properties, including the roots, of $q_n(x)$ are explicitly known; for example, see [2,1]. We will take advantage of a few particularly nice properties of $q_n(x)$. First, we will use the following theorem from [5]. We add our own corollary below.

Theorem 2.1. *For $n \geq 2$, $q_n(x) = -xq_{n-1}(x) - q_{n-2}(x)$.* □

Corollary 2.2. *Let $n \geq 0$. If n is even, $q_n(x)$ is an even polynomial. If n is odd, $q_n(x)$ is an odd polynomial.*

Proof. By inspection, Corollary 2.2 is true for $n \leq 2$. Assume that it is true for all $n' < n$ for some $n > 2$. This implies that $q_{n-1}(x)$ and $q_{n-2}(x)$ have opposite parities as polynomials, so $xq_{n-1}(x)$ and $q_{n-2}(x)$ have the same parity; hence, using Theorem 2.1, we see that $q_n(x)$ and $q_{n-2}(x)$ have the same parity. The result follows. □

We will also use the following lemma; for a proof, see [6].

Lemma 2.3. *For any $k \geq 1$, if $i \in [k-1]$, then*

$$q_k(x) = q_i(x)q_{k-i}(x) - q_{i-1}(x)q_{k-i-1}(x).$$

Further, if $q_k(\lambda) = 0$, then the following statements are true as well.

(a) *If $0 \leq s \leq k$, then $q_{k+s}(\lambda) = -q_{k-s}(\lambda)$.*
(b) *If $t \geq 1$, then $q_{t(k+1)-1}(\lambda) = 0$.* □

We are now ready to prove the following theorem.

Theorem 2.4. *Suppose that $q_k(\lambda) = 0$. Then, for all $a \geq 1$ and $0 \leq b \leq k$, $q_{a(k+1)+b}(\lambda) = \left(q_{k+1}(\lambda)\right)^a q_b(\lambda)$.*

Proof. Note that Theorem 2.4 trivially holds when $a = 1$ and $b = 0$. Suppose that $1 \leq b \leq k$. Applying Lemma 2.3 shows that

$$q_{k+1+b} = q_{k+1}(\lambda)q_b(\lambda) - q_k(\lambda)q_{b-1}(\lambda) = q_{k+1}(\lambda)q_b(\lambda),$$

and thus Theorem 2.4 holds when $a = 1$ and $0 \leq b \leq k$. Suppose it holds when $1 \leq a < a'$ and $0 \leq b \leq k$, for some $a' > 1$. Suppose that $0 \leq b \leq k$. Then, by Lemma 2.3,

$$\begin{aligned} q_{a'(k+1)+b}(\lambda) &= q_{(a'-1)(k+1)+b+k+1}(\lambda) \\ &= q_{(a'-1)(k+1)+b}(\lambda)q_{k+1}(\lambda) + q_{(a'-1)(k+1)+b-1}(\lambda)q_k(\lambda) \\ &= q_{(a'-1)(k+1)+b}(\lambda)q_{k+1}(\lambda) = \left(q_{k+1}(\lambda)\right)^{a'-1} q_b(\lambda)q_{k+1}(\lambda) \\ &= \left(q_{k+1}(\lambda)\right)^{a'} q_b(\lambda). \quad \square \end{aligned}$$

Label the roots of $q_n(x)$ as $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,n}$. Using our result from the previous section, $\det(\mathbf{A}(P_n \,\square\, P_m)) = \det(q_n(-\mathbf{A}_m))$. Corollary 2.2 implies that $q_n(-\mathbf{A}_m) = (-1)^n q_n(\mathbf{A}_m)$. Further, we can factor $q_n(x)$ as

$$q_n(x) = \prod_{i=1}^{n} \left(\lambda_{n,i} - x\right) = (-1)^n \prod_{i=1}^{n} \left(x - \lambda_{n,i}\right).$$

Thus,

$$\begin{aligned} \det(\mathbf{A}(P_n \,\square\, P_m)) &= \det(q_n(-\mathbf{A}_m)) = \det\left((-1)^n q_n(\mathbf{A}_m)\right) \\ &= \det\left((-1)^n(-1)^n \prod_{i=1}^{n} \left(\mathbf{A}_m - \lambda_{n,i}\mathbf{I}_m\right)\right) \\ &= \det\left(\prod_{i=1}^{n} (\mathbf{A}_m - \lambda_{n,i}\mathbf{I}_m)\right) \\ &= \prod_{i=1}^{n} \det(\mathbf{A}_m - \lambda_{n,i}\mathbf{I}_m) = \prod_{i=1}^{n} q_m(\lambda_{n,i}). \end{aligned}$$

Since, by definition, it is immediately evident that $P_n \square P_m$ and $P_m \square P_n$ are isomorphic as graphs, it follows that $\det(\mathbf{A}(P_n \square P_m)) = \det(\mathbf{A}(P_m \square P_n))$. Thus,

$$\prod_{i=1}^{n} q_m(\lambda_{n,i}) = \prod_{i=1}^{m} q_n(\lambda_{m,i}).$$

This leads to the following results.

Theorem 2.5. *Suppose that $n \geq 1$. Then, $\prod_{i=1}^{n} q_{n+1}(\lambda_{n,i}) = (-1)^{n(n+1)/2}$.*

Proof. By inspection, Theorem 2.5 is true for $n = 1$. Suppose that it is true for some $n \geq 1$. Then, by Lemma 2.3, $q_{n+2}(\lambda_{n+1,i}) = -q_n(\lambda_{n+1,i})$ for any $i \in [n+1]$, so

$$\begin{aligned}\prod_{i=1}^{n+1} q_{n+2}(\lambda_{n+1,i}) &= \prod_{i=1}^{n+1} -q_n(\lambda_{n+1,i}) = (-1)^{n+1}\prod_{i=1}^{n+1} q_n(\lambda_{n+1,i}) \\ &= (-1)^{n+1}\prod_{i=1}^{n} q_{n+1}(\lambda_{n,i}) = (-1)^{n+1}(-1)^{n(n+1)/2} \\ &= (-1)^{n+1+n(n+1)/2} = (-1)^{(n+1)(1+n/2)} = (-1)^{(n+1)(n+2)/2}. \quad \square\end{aligned}$$

Theorem 2.6. *Suppose that $n, m \geq 1$. Then,*

$$\prod_{i=1}^{n} q_m(\lambda_{n,i}) = \begin{cases} 0 & \text{if } \gcd(n+1, m+1) \neq 1, \\ (-1)^{nm/2} & \text{if } \gcd(n+1, m+1) = 1. \end{cases}$$

Proof. Note that the product in the statement of Theorem 2.6 is the determinant of $\mathbf{A}(P_n \square P_m)$, which, as discussed above, is equal to the determinant of $\mathbf{A}(P_m \square P_n)$. Thus, without loss of generality, we may assume that $n \leq m$. We will induct on the remainder when $m+1$ is divided by $n+1$. Suppose that this remainder is 0. Then, $\gcd(n+1, m+1) \neq 1$, and $m+1 = k(n+1)$ for some $k \geq 1$, so $m = k(n+1) - 1$. Thus, by Lemma 2.3, $q_m(\lambda_{n,i}) = 0$ for $i \in [n]$, and it follows that the product of these terms is zero. This verifies Theorem 2.6 for this case.

Suppose that the remainder when $m+1$ is divided by $n+1$ is 1; we then have $m+1 = k(n+1)+1$ for some $k \geq 1$. Note that this implies that $\gcd(n+1, m+1) = 1$ and $m = k(n+1)$, so, by Theorem 2.4, for $i \in [n]$,

$$q_m(\lambda_{n,i}) = q_{k(n+1)}(\lambda_{n,i}) = (q_{n+1}(\lambda_{n,i}))^k q_0(\lambda_{n,i}) = (q_{n+1}(\lambda_{n,i}))^k.$$

Thus,

$$\prod_{i=1}^{n} q_m(\lambda_{n,i}) = \prod_{i=1}^{n} (q_{n+1}(\lambda_{n,i}))^k = \left(\prod_{i=1}^{n} q_{n+1}(\lambda_{n,i})\right)^k = \left((-1)^{n(n+1)/2}\right)^k,$$

by Theorem 2.5. Further,

$$\left((-1)^{n(n+1)/2}\right)^k = (-1)^{nk(n+1)/2} = (-1)^{nm/2}.$$

Thus, Theorem 2.6 is true in this case.

Finally, suppose that Theorem 2.6 is true whenever the remainder when $(m+1)$ is divided by $(n+1)$ is less than r, for some $r > 1$. Then, consider any $(m+1)$ and $(n+1)$ with $(m+1)$ having remainder r when divided by $(n+1)$. It follows that there exists $k \geq 1$ such that $m+1 = k(n+1)+r$, implying that $m = k(n+1)+r-1$. Then, once again applying Theorem 2.4,

$$\begin{aligned}\prod_{i=1}^{n} q_m(\lambda_{n,i}) &= \prod_{i=1}^{n} (q_{n+1}(\lambda_{n,i}))^k q_{r-1}(\lambda_{n,i}) = \prod_{i=1}^{n} (q_{n+1}(\lambda_{n,i}))^k \prod_{i=1}^{n} q_{r-1}(\lambda_{n,i}) \\ &= \left(\prod_{i=1}^{n} q_{n+1}(\lambda_{n,i})\right)^k \prod_{i=1}^{n} q_{r-1}(\lambda_{n,i}) = (-1)^{nk(n+1)/2}\prod_{i=1}^{n} q_{r-1}(\lambda_{n,i})\end{aligned}$$

Note that $\gcd(n+1, m+1) = \gcd(r, n+1)$, by construction. Further, the remainder when $(n+1)$ is divided by r is less than r. Thus, by our induction hypothesis, if $\gcd(n+1, m+1) \neq 1$, then $\gcd(r, n+1) \neq 1$, so,

$$\prod_{i=1}^{n} q_m(\lambda_{n,i}) = (-1)^{nk(n+1)/2}\prod_{i=1}^{n} q_{r-1}(\lambda_{n,i}) = 0.$$

Otherwise, $\gcd(n+1, m+1) = 1$, so $\gcd(r, n+1) = 1$, implying that

$$\begin{aligned}\prod_{i=1}^{n} q_m(\lambda_{n,i}) &= (-1)^{nk(n+1)/2} \prod_{i=1}^{n} q_{r-1}(\lambda_{n,i}) = (-1)^{nk(n+1)/2}(-1)^{n(r-1)/2} \\ &= (-1)^{n(k(n+1)+r-1)/2} = (-1)^{nm/2}. \quad \square\end{aligned}$$

The following corollary to Theorem 2.6 follows immediately.

Corollary 2.7. *Suppose that n and m are positive integers. Then,*

$$\det(\mathbf{A}(P_n \,\square\, P_m)) = \begin{cases} 0 & \textit{if } \gcd(n+1, m+1) \neq 1, \\ (-1)^{nm/2} & \textit{if } \gcd(n+1, m+1) = 1. \end{cases} \quad \square$$

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