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Projective-planar signed graphs and tangled signed graphs

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Abstract

A projective-planar signed graph has no two vertex-disjoint negative circles. We prove that every signed graph with no two vertex-disjoint negative circles and no balancing vertex is obtained by taking a projective-planar signed graph or a copy of $-K_5$ and then taking 1-, 2-, and 3-sums with balanced signed graphs.

Keywords: Signed graph; Projective plane; Regular matroid

1. Introduction

The reader may or may not be familiar with signed graphs as in [12]. If not, then an introduction to signed graphs is given in Section 2 and it contains all necessary information to read this paper. Most of the motivation for the study of projective-planar signed graphs comes from matroid theory; however, matroid theory is used only sparingly in this paper.

Given a graph G imbedded in the projective plane, let N be a nonseparating circle in the topological dual graph G^*. Let σ_N be a signing on the edges that is negative only on the edges of G that are dual to the edges of N. It is known that, given the imbedding of G, the signed graph (G, σ_N) is unique up to switching. That is, if N' is another nonseparating circle in G^*, then $(G, \sigma_{N'})$ and (G, σ_N) are switching equivalent. Signed graphs that may be obtained in this way up to switching are called *projective planar*. One of the first studies of projective-planar signed graphs is [16] in which Zaslavsky obtains a forbidden-minor characterization of them.

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Given the topology of the projective plane, a circle in the signed graph (G, σ_N) is negative iff it is imbedded as a nonseparating closed curve. Also any two nonseparating closed curves in the projective plane must intersect. Thus no projective-planar signed graph contains two vertex-disjoint negative circles. Furthermore, a projective-planar signed graph does not have a balancing vertex iff the associated imbedding in the projective plane has face width at least two.

A signed graph that does not contain two vertex-disjoint negative circles may or may not have a balancing vertex. If a signed graph has no two vertex-disjoint negative circles and yet has no balancing vertex, then we call this signed graph *tangled*. Any projective-planar signed graph without a balancing vertex is a tangled signed graph and $-K_5$ is a signed graph that is tangled yet not projective planar.

Given a signed graph Σ containing a balanced K_t subgraph and a balanced signed graph Υ containing a copy of K_t, we can switch the signs in Σ and Υ so that the sign pattern on the common K_t subgraph is the same. The t-sum $\Sigma \oplus_t \Upsilon$ is obtained by identifying Σ and Υ along the common K_t subgraph and then deleting the edges of K_t. Note that for $t \in \{1, 2, 3\}$ we have $M(\Sigma \oplus_t \Upsilon) = M(\Sigma) \oplus_t M(\Upsilon)$. Theorem 1.1 is found in [9, Lemma 8].

Theorem 1.1. *If Σ is unbalanced and Υ is balanced, then the following are true.*

(1) *$\Sigma \oplus_1 \Upsilon$ is tangled iff Σ is tangled.*
(2) *For each $t \in \{2, 3\}$, if Υ is vertically t-connected, then $\Sigma \oplus_t \Upsilon$ is tangled iff Σ is tangled.*

Theorem 1.1 gives us a method of constructing tangled signed graphs by starting with a copy of $-K_5$ or a projective-planar signed graph and then taking t-sums with balanced signed graphs. Theorem 1.2 (which is the main result of this paper) tells us that this method is sufficient for constructing all tangled signed graphs.

Theorem 1.2. *If Σ is a connected and tangled signed graph, then Σ is either*

(1) *projective planar,*
(2) *isomorphic to $-K_5$ possibly along with some positive loops and parallel links with the same sign, or*
(3) *a 1-sum, 2-sum, or 3-sum of a tangled signed graph and a balanced signed graph where the balanced signed graph has at least 2, 3, or 5 vertices, respectively.*

Theorem 1.2 has been known for some time to L. Lovász, A.M.H. Gerards, and others, but it has not appeared in the mathematical literature.

1.1. Tangled signed graphs and regular matroids

Tangled signed graphs arise naturally in the study of matroids coming from signed graphs because with some trivial exceptions the class of signed graphs whose frame matroids are regular is exactly the class of tangled signed graphs. Theorem 1.3 is implied from [9, Theorems 1.3 and 1.4].

Theorem 1.3. *If Σ is connected, then the following are true.*

(1) *If Σ is tangled, then $M(\Sigma)$ is regular.*
(2) *If $M(\Sigma)$ is regular and not graphic, then Σ is tangled.*

Since the classes of graphic matroids and cographic matroids are prominent within the class of regular matroids, it would also be useful to know when the frame matroid of a tangled signed graph is graphic and when it is cographic. In the cographic case, the answer is given in [4,7] where it is shown that a connected cographic matroid $M^*(G)$ satisfies $M^*(G) = M(\Sigma)$ for some signed graph Σ iff Σ is projective planar or $M(\Sigma)$ is graphic. Theorem 1.4 along with Theorem 1.1 and the fact that the class of graphic matroids is closed under k-summing give a complete construction method for tangled signed graphs whose frame matroids are graphic.

Theorem 1.4. *If Σ is a connected and tangled signed graph such that $M(\Sigma)$ is graphic, then Σ is either*

(1) *a projective planar signed graph whose topological dual graph is planar, or*
(2) *a* 1*-sum,* 2*-sum, or* 3*-sum of a tangled signed graph whose frame matroid is graphic and a balanced signed graph with at least* 2, 3*, or* 5 *vertices, respectively.*

1.2. Organization of this paper

In Section 2 we have definitions and some preliminary results. In Section 3 we will prove many lemmas for the proof of our main results in Section 4. Sections 3.1 and 3.2 are the only places in this paper that we use matroid theory.

As a final comment we would like to mention the following. Since $M(\Sigma)$ is regular when Σ is tangled (see Theorem 1.3), it would seem that the obvious place to start a proof of Theorem 1.2 would be with Seymour's Decomposition Theorem for regular matroids [5]. However it seems to the author that a proof using Seymour's theorem would likely be no easier than the direct graph-theoretical proof presented in this paper. So rather than start by quoting a deep and difficult result like Seymour's theorem, a direct proof seems preferable.

2. Definitions and background information

Graphs. A graph G consists of a collection of vertices (i.e., topological 0-cells), denoted by $V(G)$, and a set of edges (i.e., topological 1-cells), denoted by $E(G)$, where an edge has two ends each of which attached to a vertex. A *link* is an edge that has its ends incident to distinct vertices and a *loop* is an edge that has both of its ends incident to the same vertex.

A *circle* is a connected, 2-regular graph (i.e., a simple closed path). In graph theory a circle is often called a cycle, circuit, polygon, etc. If $X \subseteq E(G)$, then we denote the subgraph of G consisting of the edges in X and all vertices incident to an edge in X by $G{:}X$. The collection of vertices in $G{:}X$ is denoted by $V(X)$, the number of vertices in $G{:}X$ is denoted by v_X, and the number of connected components in $G{:}X$ is denoted by c_X.

For $k \geqslant 1$, a *k-separation* of a graph is a bipartition (A, B) of the edges of G such that $|A| \geqslant k$, $|B| \geqslant k$, and $|V(A) \cap V(B)| = k$. A *vertical* k-separation (A, B) of G is a k-separation where $V(A) \setminus V(B) \neq \emptyset$ and $V(B) \setminus V(A) \neq \emptyset$. A separation or vertical separation (A, B) is said to have *connected parts* when $G{:}A$ and $G{:}B$ are both connected. A connected graph on at least $k+1$ vertices is said to be *vertically k-connected* when there is no vertical r-separation for $r < k$. Vertical k-connectivity is usually just called k-connectivity, but here we wish to distinguish between this kind of graph connectivity and the second type used in Tutte's book on graph theory [10].

Given a subgraph H of G, an *H-bridge* is either an edge not in H whose endpoint(s) are both in H or a connected component C of $G \setminus H$ along with the links between C and H. Given an H-bridge B of G: a *foot* of B is an edge of B with an endpoint in H, a *vertex of attachment* of B is a vertex in H that is an endpoint of a foot of B, and $\overline{B}$ denotes the bridge B minus the vertices of attachment of B (i.e., either a connected component of $G \setminus H$ or $\emptyset$ when B is a single edge). An H-bridge of G with n vertices of attachment is called an *n-bridge*.

If G' is a subdivision of a graph G with minimum degree three, then a *branch vertex* of G' is a vertex of degree at least three in G' and a *branch* is a path in G' corresponding to an edge in G. A G'-bridge B is called *local* if all attachments of B are on the same branch of G'.

Graphic matroids. A matroid is said to be *graphic* if it is the cycle matroid of a graph G. A matroid is said to be *cographic* if it is the dual of the cycle matroid of an ordinary graph G. We denote the *cycle matroid* of G by $M(G)$. This is the matroid with element set $E(G)$ and circuits consisting of edge sets of circles in G. If $X \subseteq E(G)$, then $r(X) = v_X - c_X$. For each edge e in G, $M(G \setminus e) = M(G) \setminus e$ and $M(G/e) = M(G)/e$.

Signed graphs. A *signed graph* is a pair (G, σ) in which $\sigma : E(G) \to \{+1, -1\}$. A circle or path in a signed graph Σ is called *positive* if the product of signs on its edges is positive, otherwise the circle or path is called *negative*. If H is a subgraph of Σ, then H is called *balanced* when all circles in H are positive. A *balancing vertex* of an unbalanced signed graph is a vertex whose removal leaves a balanced subgraph. Not all unbalanced signed graphs have balancing vertices and balanced signed graphs do not have balancing vertices. We use $\|\Sigma\|$ to denote the underlying graph of Σ. We consider a graph G to be a signed graph with all edges signed positively. In this sense, the class of signed graphs contains the class of graphs.

When drawing signed graphs, positive edges are represented by solid curves and negative edges by dashed curves. One special convention we will also utilize is that a crosshatched curve represents a positive path that may have length zero or may have positive length.

A *switching function* on a signed graph $\Sigma = (G, \sigma)$ is a function $\eta : V(\Sigma) \to \{+, -\}$. The signed graph $\Sigma^\eta = (G, \sigma^\eta)$ has sign function σ^η defined on all edges of Σ by $\sigma^\eta(e) = \eta(v)\sigma(e)\eta(w)$ where v and w are the endpoint vertices (or endpoint vertex) of edge e. The signed graphs Σ and Σ^η have the same list of positive circles. When two signed graphs Σ_1 and Σ_2 satisfy $\Sigma_1^\eta = \Sigma_2$ for some switching function η, the two signed graphs are said to be *switching equivalent*. An important notion in the study of signed graphs is that two signed graphs with the same underlying graph are switching equivalent iff they have the same list of positive circles (see [12, Proposition 3.2]).

In a signed graph $\Sigma = (G, \sigma)$, the deletion of e from Σ is defined as $\Sigma \setminus e = (G \setminus e, \sigma)$ where σ is restricted to the domain $E(G \setminus e)$. The contraction of an edge e is defined for three distinct cases. If e is a link, then $\Sigma/e = (G/e, \sigma^\eta)$ where η is a switching function satisfying $\sigma^\eta(e) = +$, which always exists. Note that the contraction Σ/e is only well defined up to switching. If e is a positive loop, then $\Sigma/e = \Sigma \setminus e$. If e is a negative loop, then Σ/e is the signed graph obtained from Σ as follows: links incident to v become negative loops incident to their other endpoint, loops incident to v become positive loops incident to v, and edges not incident to v remain unchanged. The reason for this definition of contraction in signed graphs is so that contractions in signed graphs will correspond to contractions in their signed-graphic matroids. (See the discussion on signed-graphic matroids below.)

A *minor* of Σ is a signed graph obtained from Σ by a sequence of contractions and deletions of edges, deletions of isolated vertices, and switchings. A *link minor* of Σ is a minor obtained without contracting any negative loops. A *subdivision* of Σ is a signed graph obtained from Σ by replacing each edge by a path (between the same endpoints) whose sign is the same as the sign on the replaced edge.

A signed graph is called *tangled* if it is unbalanced, has no balancing vertex, and no two vertex-disjoint negative circles. Proposition 2.1 is from [9, Proposition 1.7]. Its proof is easy.

Proposition 2.1. *If Σ is a tangled signed graph, then Σ has exactly one unbalanced block* (*in particular, Σ does not contain any negative loops*).

Given a graph G, by $-G$ we mean the signed graph obtained from G by replacing each edge with a negative edge. By $\pm G$ we mean the signed graph obtained from G by replacing each edge with a positive edge and a negative edge on the same endpoints. Theorem 2.2 is [9, Theorem 3.16]. It follows from Theorem 1.1 and the decomposition theorem of Gerards [1, Theorem 3.2.3] for signed graphs containing no $-K_4$ nor $\pm C_3$ link minor. A more general result will be proven directly in a future paper [8].

Theorem 2.2. *If Σ is a tangled signed graph, then Σ has a $-K_4$ or $\pm C_3$ link minor.*

A signed graph is called *vertically k-connected* when its underlying graph is vertically k-connected. We say that a signed graph is *almost* 4-*connected* if it vertically 3-connected and any 3-separation (not necessarily a vertical 3-separation) of Σ does not contain a balanced part with five or more vertices. If (A, B) is a 3-separation of Σ in which $\Sigma{:}B$ is balanced with at least five vertices, then we shall call (A, B) a *bad* 3-*separation*.

Matroids of signed graphs. There are three matroids associated with a signed graph. Two of them coincide when the signed graph is tangled.

The *frame matroid* (often called the *bias matroid*) of Σ is denoted by $M(\Sigma)$. The element set of $M(\Sigma)$ is $E(\Sigma)$ and a circuit of $M(\Sigma)$ is either the edge set of a positive circle or the edge set of a subdivision of a subgraph in Fig. 1 with no positive circles.

With the definition of deletions and contractions of signed graphs above, for any $e \in E(\Sigma)$, we have that $M(\Sigma \setminus e) = M(\Sigma) \setminus e$ and $M(\Sigma/e) = M(\Sigma)/e$ (see [12, Theorem 5.2]). Given $X \subseteq E(\Sigma)$ we denote the number of balanced components of $\Sigma{:}X$ by b_X. If $X \subseteq E(\Sigma)$, then $r_M(X) = v_X - b_X$ (see [12, Theorem 5.1(j)]). If a signed graph Σ is not connected and has no isolated vertices, then $M(\Sigma)$ is not connected.

The *lift matroid* of Σ is denoted by $L(\Sigma)$. The element set of $L(\Sigma)$ is $E(\Sigma)$ and a circuit of $L(\Sigma)$ is either the edge set of a positive circle or the edge set of a subdivision of a subgraph in Fig. 2 with no positive circles.

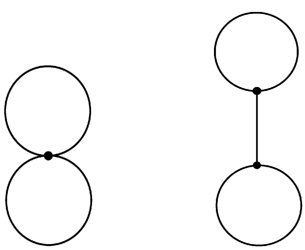

Fig. 1.

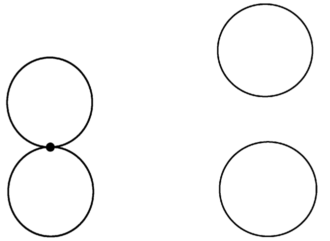

Fig. 2.

Given the definition of circuits in $M(\Sigma)$ and $L(\Sigma)$ we see that $M(\Sigma) = L(\Sigma)$ iff Σ has no two vertex-disjoint negative circles, that is, iff Σ is tangled or has a balancing vertex. Now $L(\Sigma \setminus e) = L(\Sigma) \setminus e$ for any edge e. For any link or positive loop f, $L(\Sigma/f) = L(\Sigma)/f$ and, for any negative loop f, $L(\Sigma)/f = M(\|\Sigma\| \setminus f)$ (see [15, Theorem 3.6]). If $X \subseteq E(\Sigma)$, then ϵ_X is defined to be 1 if X is unbalanced and 0 if X is balanced. So now $r_L(X) = v_X + \epsilon_X - c_X$ (see [15, Theorem 3.6]).

The *complete lift matroid* of Σ is denoted by $L_0(\Sigma)$. It is defined by $L_0(\Sigma) = L(\Sigma_0)$ where Σ_0 consists of Σ along with a negative loop, call it p_0, attached to a new vertex.

Matroid connectivity. For $k \geqslant 1$, a *k-separation* of a matroid M on E is a bipartition (X, Y) of E such that $|X|, |Y| \geqslant k$ and $r(X) + r(Y) - r(M) \leqslant k - 1$. A k-separation is *exact* if there is equality in the latter inequality. A matroid is said to be *disconnected* when it has a 1-separation. For $k \geqslant 2$, a matroid is said to be *k-connected* when k is the minimum integer for which M has a k-separation.

If (X, Y) is a k-separation of $M(\Sigma)$ (or $L(\Sigma)$) for which $\Sigma{:}X$ and $\Sigma{:}Y$ are both connected, then we call (X, Y) a k-separation of $M(\Sigma)$ (or $L(\Sigma)$) with *connected parts*.

3. Lemmas for the main results

A signed graph is said to be *simple* if it has no positive loops, no two negative loops with the same endpoint, and no two parallel links of the same sign. Note that $M(\Sigma)$ is a simple matroid iff Σ is simple and certainly Σ is projective planar iff its associated simple signed graph is projective planar. *Throughout the rest of this paper, Σ will denote a tangled simple signed graph unless otherwise stated.*

3.1. Connectivity lemmas

Lemma 3.1. *If Σ is a signed graph (not necessarily tangled) and (A, B) is a vertical t-separation with $t \in \{1, 2, 3\}$ such that both parts are balanced, then Σ is balanced or has a balancing vertex.*

Proof. The conclusion for $t \in \{1, 2\}$ is evident, so say $t = 3$. Let η be a switching function on $\Sigma{:}A$ that makes all edges positive and let ξ be a switching function on $\Sigma{:}B$ that makes all edges positive. By replacing ξ with $-\xi$ if necessary, we may assume that η and ξ disagree on at most one of the three vertices of $(\Sigma{:}A) \cap (\Sigma{:}B)$. If they agree on all vertices, then Σ is balanced. If they disagree on one vertex, then that vertex is a balancing vertex of Σ. $\square$

Lemma 3.2. *If Σ is vertically 2-connected and (X, Y) is a vertical 2-separation with both parts unbalanced, then either there is a bipartition (Y_1, Y_2) of Y such that Y_2 is balanced and $(X \cup Y_1, Y_2)$ is a vertical 2-separation of Σ or there is a bipartition (X_1, X_2) of X satisfying the corresponding conclusion.*

Proof. Let u and v be the vertices of $V(X) \cap V(Y)$. Since Σ has no balancing vertex $\Sigma \setminus u$ contains a negative circle C_1 and $\Sigma \setminus v$ contains a negative circle C_2. Each C_i must then be contained entirely in X or entirely in Y. Furthermore, it cannot be that C_1 is contained in $\Sigma{:}X$ and C_2 in $\Sigma{:}Y$ (or C_2 in $\Sigma{:}X$ and C_1 in $\Sigma{:}Y$) because then C_1 and C_2 would be vertex disjoint, a contradiction. So without loss of generality C_1 and C_2 are both contained in $\Sigma{:}X$. Since Y is unbalanced, it contains negative circles and since Σ is tangled, each negative circle must contain both u and v. Thus u and v are both balancing vertices of $\Sigma{:}Y$.

In [13, Corollary 2] it is shown then that there is a bipartition (Y_1, Y_2) of Y such that $V(Y_1) \cap V(Y_2) = \{u, v\}$ and each Y_i is balanced. Since $v_Y \geqslant 3$ we have that $v_{Y_1} \geqslant 3$ or $v_{Y_2} \geqslant 3$ (assume the latter). So now $(X \cup Y_1, Y_2)$ is our desired vertical 2-separation. □

Lemma 3.3. *$L_0(\Sigma)$ is a 3-connected matroid iff Σ is vertically 3-connected.*

Proof. Suppose that $L_0(\Sigma)$ is a 3-connected matroid and by way of contradiction, there is a vertical k-separation of (X, Y) of Σ with $k \leqslant 2$. Assuming that k is a minimum gives us that $\Sigma{:}X$ and $\Sigma{:}Y$ are both connected. So now $r_{L_0}(X \cup p_0) + r_{L_0}(Y) - r_{L_0}(\Sigma_0) = v_X + v_Y + \epsilon_Y - 1 - v_\Sigma = k + \epsilon_Y - 1 \leqslant 2$ with equality only when $k = 2$ and Y is unbalanced. It must be the case that $k = 2$ and Y is unbalanced because otherwise $(X \cup p_0, Y)$ would be a 1-separation or 2-separation of $L_0(\Sigma)$. However if Y is unbalanced and $k = 2$, then $r_{L_0}(X) + r_{L_0}(Y \cup p_0) - r_{L_0}(\Sigma_0) = v_X + \epsilon_X - 1 + v_Y - v_\Sigma = 1 + \epsilon_X \leqslant 2$ with equality only when X is unbalanced. Again, it must be the case that X is unbalanced. So (X, Y) is a vertical 2-separation of Σ with both sides unbalanced; however, Lemma 3.2 implies that there is a vertical 2-separation (A, B) of Σ with A unbalanced and B balanced. But now $r_{L_0}(A \cup p_0) + r_{L_0}(B) - r_{L_0}(\Sigma_0) \leqslant 1$ which makes $(A \cup p_0, B)$ a 1-separation or 2-separation of $L_0(\Sigma)$, a contradiction.

Conversely, assume that Σ is vertically 3-connected and yet $L_0(\Sigma)$ has a k-separation for $k \in \{1, 2\}$. Now it is either the case that there is a k-separation $(A \cup p_0, B)$ of $L_0(\Sigma)$ such that (A, B) is a vertical t-separation of Σ with connected parts or no such k-separation of $L_0(\Sigma)$ exists. Let these be Cases 1 and 2, respectively.

Case 1. If (A, B) has connected parts and is a vertical t-separation of Σ, then $t \geqslant 3$. So now $1 \geqslant k - 1 = r_{L_0}(A \cup p_0) + r_{L_0}(B) - r_{L_0}(\Sigma_0) = v_A + v_B + \epsilon_B - 1 - v_\Sigma = t + \epsilon_B - 1 \geqslant 2$, a contradiction. Thus no such k-separation of $L_0(\Sigma)$ exists.

Case 2. Either there is a k-separation $(A \cup p_0, B)$ of $L_0(\Sigma)$ such that (A, B) has connected parts or not. Let these be Cases 2.1 and 2.2, respectively. In each case, it cannot be that $|B| = 1$ because then B is a loop or coloop of $L_0(\Sigma)$ which would make B a bridge of Σ (contradicting vertical 3-connectedness) or a positive loop (contradicting simplicity).

Case 2.1. If (A, B) has connected parts, then by Case 1 it cannot be that (A, B) is a vertical t-separation of Σ. So either $V(A) = V(\Sigma)$ or $V(B) = V(\Sigma)$.

If $V(A) = V(\Sigma)$, then $1 \geqslant k - 1 = r_{L_0}(A \cup p_0) + r_{L_0}(B) - r_{L_0}(\Sigma_0) = v_B + \epsilon_B - 1$ and so $v_B = 1$ or $v_B = 2$ and B is balanced. If $v_B = 1$, then the elements of B are all loops. But Σ

has no negative loops by Proposition 2.1 and no positive loops by simplicity, a contradiction. If $v_B = 2$ and B is balanced with $|B| \geqslant 2$, then since Σ has no loops, B contains at least two links and these links all are of the same sign, a contradiction of simplicity.

If $V(B) = V(\Sigma)$, then $1 \geqslant r_{L_0}(A \cup p_0) + r_{L_0}(B) - r_{L_0}(\Sigma_0) = v_A + \epsilon_B - 1$ and so $v_A = 1$ or $v_A = 2$ and B is balanced. As in the previous paragraph we cannot have that $v_A = 1$ because Σ is loopless. If $v_A = 2$ and B is balanced, then since Σ is loopless, either vertex of $V(A)$ is a balancing vertex of Σ, a contradiction.

Case 2.2. It must be that one of $\Sigma{:}A$ or $\Sigma{:}B$ is disconnected.

Claim 1. *If $(A \cup p_0, B)$ is a k-separation of $L_0(\Sigma)$ for $k \in \{1, 2\}$, then A is unbalanced.*

Proof. Suppose that A is balanced, then $r_{L_0}(A \cup p_0) + r_{L_0}(B) - r_{L_0}(\Sigma_0) \leqslant 1$ implies that $r_L(A) + r_L(B) - r_L(\Sigma) = 0$ and so $L(\Sigma)$ is not connected. Thus by [3, Theorem 1.2.2], there is a 1-separation (X, Y) of $L(\Sigma)$ with connected parts. The equation $r_L(X) + r_L(Y) - r_L(\Sigma) = 0$ will now imply that $|V(X) \cap V(Y)| \leqslant 2$. So, since Σ is 3-connected, it cannot be that (X, Y) is a vertical separation of Σ. Thus either $V(X) = V(\Sigma)$ or $V(Y) = V(\Sigma)$ (without loss of generality assume that $V(X) = V(\Sigma)$). So now $0 = r_L(X) + r_L(Y) - r_L(\Sigma) = v_Y + \epsilon_Y + \epsilon_X - 2$ and so $1 \leqslant v_Y = 2 - (\epsilon_X + \epsilon_Y) \leqslant 2$. But $v_Y \neq 1$ because Σ is loopless and $v_Y = 2$ would imply that X and Y are both balanced and so each of the two vertices of $V(Y)$ is a balancing vertex of Σ, a contradiction. □

Claim 2. *If $A_1, \ldots, A_n$ are the edge sets of the connected components of $\Sigma{:}A$, then at most one A_i is unbalanced and $r_L(A) = r_L(A_1 \cup \cdots \cup A_n) = \sum_i r_L(A_i)$.*

Proof. This is evident because Σ has no two vertex-disjoint negative circles and from the form of the rank function. □

Now since A is unbalanced (by Claim 1), $r_{L_0}(A \cup p_0) + r_{L_0}(B) - r_{L_0}(\Sigma_0) = r_L(A) + r_L(B) - r_L(\Sigma)$. Let $A_1, \ldots, A_n$ be the edge sets of the connected components of $\Sigma{:}A$. Since A is unbalanced, we may assume that A_1 is unbalanced and so $|A_1| \geqslant 2$ because Σ is loopless. So now it follows from Claim 2, submodularity of the rank function, and the fact that $r_L(A) + r_L(B) - r_L(\Sigma) = t \in \{0, 1\}$ that

$$\begin{aligned} t + r_L(\Sigma) &= r_L(A) + r_L(B) \\ &= r_L(A_1) + r_L(A_2 \cup \cdots \cup A_n) + r(B) \\ &\geqslant r_L(A_1) + r_L(B \cup A_2 \cup \cdots \cup A_n) \\ &\geqslant r_L(\Sigma). \end{aligned}$$

Use $\overline{X}$ to denote the set complement of X in $E(\Sigma)$. So writing $\overline{A}_1 = B \cup A_2 \cup \cdots \cup A_n$ we get that $(A_1, \overline{A}_1)$ is an m-separation of $L(\Sigma)$ for $m \in \{1, 2\}$ with $\Sigma{:}A_1$ connected and unbalanced. Now let $B_1, \ldots, B_m$ be the edge sets of the connected components of $\Sigma{:}\overline{A}_1$. Either there is some $|B_i| \geqslant 2$ or each $|B_i| = 1$. In the latter case, we can perform a rank calculation as above to get a g-separation $(\overline{B}_1, B_1)$ of $L(\Sigma)$ with $g \in \{1, 2\}$ and $A_1 \subseteq \overline{B}_1$. Since $\Sigma{:}A_1$ and Σ are connected, $\Sigma{:}\overline{B}_1$ is connected and so $(\overline{B}_1, B_1)$ is a g-separation of $L(\Sigma)$ with connected parts and $\overline{B}_1$ unbalanced. Thus $(\overline{B}_1 \cup p_0, B_1)$ would be a g-separation of $L_0(\Sigma)$ with connected parts which can be used in Case 1 or Case 2.1 and we are done. So assume that each $|B_i| = 1$. Since

Σ is vertically 3-connected and loopless, $V(A_1) = V(\Sigma)$ and $r_L(\bar{A}_1) = |\bar{A}_1|$. So since A_1 is unbalanced $1 \geqslant r_L(A_1) + r_L(\bar{A}_1) - r_L(\Sigma) = |\bar{A}_1| \geqslant |B| \geqslant 2$, a contradiction. □

3.2. Splitter lemmas

Lemma 3.4. *If Σ is vertically 3-connected, then $\Sigma \cong \pm C_3$ or Σ contains a subdivision of $-K_4$.*

Proof. Since Σ is tangled, Σ contains a $\pm C_3$ or $-K_4$ link minor by Theorem 2.2. If Σ contains a $-K_4$ link minor, then since every vertex of $-K_4$ is of degree three, Σ contains a subdivision of $-K_4$. So assume that Σ does not contain a $-K_4$ minor. Since Σ must now contain a $\pm C_3$ minor, we can assume Σ contains $\pm C_3$ as a proper minor or we are done.

First, since Σ contains a $\pm C_3$ minor and since $L_0(\Upsilon) \cong F_7$ iff $\Upsilon \cong \pm C_3$ (see [14, §3]), $L_0(\Sigma)$ contains an F_7 minor. Also in [14, §3] it is shown that $L(\Upsilon) \cong F_7$ iff Υ is the disjoint union of $\pm C_3$ and a negative loop or the one-vertex join of $\pm C_3$ and a negative loop.

Second, since Σ is vertically 3-connected, $L_0(\Sigma)$ is 3-connected by Lemma 3.3. Thus using the splitter theorem, there are edges $e_1, \ldots, e_n$ and operations $\wr_1, \ldots, \wr_n \in \{/, \backslash\}$ such that $L_0(\Sigma) \wr_1 e_1 \cdots \wr_n e_n \cong F_7$ and each $L_0(\Sigma) \wr_1 e_1 \cdots \wr_i e_i$ is 3-connected. So now $L_0(\Sigma) \wr_1 e_1 \cdots \wr_n e_n = L(\Sigma_0) \wr_1 e_1 \cdots \wr_n e_n = L(\Sigma_0 \wr_1 e_1 \cdots \wr_n e_n) \cong F_7$ and since Σ is tangled and $\pm C_3$ is tangled, the negative loop of $\Sigma_0 \wr_1 e_1 \cdots \wr_n e_n$ must be the edge p_0 of Σ_0 and $\Sigma_0 \wr_1 e_1 \cdots \wr_n e_n$ is obtained by contracting and deleting links only. So now $\Sigma_0 \wr_1 e_1 \cdots \wr_{n-1} e_{n-1}$ is obtained from $\Sigma_0 \wr_1 e_1 \cdots \wr_n e_n$ by adding a link or decontracting a link. Since $L(\Sigma_0 \wr_1 e_1 \cdots \wr_{n-1} e_{n-1}) = L_0(\Sigma) \wr_1 e_1 \cdots \wr_{n-1} e_{n-1}$ is 3-connected, there is no way to add a link to $\Sigma_0 \wr_1 e_1 \cdots \wr_n e_n$ without loosing simplicity. Thus $\Sigma_0 \wr_1 e_1 \cdots \wr_{n-1} e_{n-1}$ is obtained from $\Sigma_0 \wr_1 e_1 \cdots \wr_n e_n$ by decontracting a link. One can check that, up to isomorphism, there is only one way to decontract a link from $\pm C_3$ without ruining cosimplicity and without creating two vertex-disjoint negative circles; however, this way of decontracting a link leaves a signed graph with a $-K_4$ subgraph, a contradiction. □

Lemma 3.5. *If Σ is vertically 3-connected and contains $-K_5$ as a link minor, then $\Sigma \cong -K_5$.*

Proof. Suppose by way of contradiction, that Σ properly contains $-K_5$ as a link minor. Thus $L_0(\Sigma)$ has a $L_0(-K_5)$ minor. Since Σ and $-K_5$ are vertically 3-connected, Lemma 3.3 implies that $L_0(\Sigma)$ and $L_0(-K_5)$ are 3-connected. So by the Splitter Theorem, there are edges $e_1, \ldots, e_n$ in Σ_0 and operations $\wr_1, \ldots, \wr_n \in \{/, \backslash\}$ such that $L_0(\Sigma) \wr_1 e_1 \cdots \wr_n e_n \cong L_0(-K_5)$ and each $L_0(\Sigma) \wr_1 e_1 \cdots \wr_i e_i$ is 3-connected. So now $L(\Sigma_0 \wr_1 e_1 \cdots \wr_n e_n) \cong L_0(-K_5)$.

Claim 1. *If $L(\Upsilon) \cong L_0(K_5)$, then Υ is a disjoint union or one-vertex join of $-K_5$ and a negative loop.*

Proof. Since $L(\Upsilon) \cong L_0(-K_5)$, there is an edge f in Υ such that $L(\Upsilon \setminus f) = L(\Upsilon) \setminus f \cong L_0(-K_5) \setminus p_0 = L(-K_5)$. From [14, §6], $L(\Sigma') \cong R_{10}$ iff $\Sigma' \cong -K_5$. Thus $\Upsilon \setminus f \cong -K_5$. Now if f is a negative loop in Υ, then we are done. So suppose that f is a link. Thus f must have both endpoints in $\Upsilon \setminus f \cong -K_5$ or else f would be a coloop of $L_0(\Upsilon)$, a contradiction of 3-connectedness. Thus f is parallel to a link f' in $\Upsilon \cong -K_5$. Since $L_0(-K_5)$ is a simple matroid, it cannot be that f and f' have the same sign. Evidently f along with any three edges of $\Upsilon \setminus f \cong -K_5$ do not form a circuit in $L(\Upsilon)$. However, p_0 along with the edges of any triangle is a circuit in $L_0(-K_5)$, contradicting that $L(\Upsilon) \cong L_0(-K_5)$. □

By Claim 1, $\Sigma_0 \wr_1 e_1 \cdots \wr_n e_n$ is a disjoint union or one-vertex join of $-K_5$ and a negative loop. However, since Σ and $-K_5$ are both tangled, it must be that $\Sigma_0 \wr_1 e_1 \cdots \wr_n e_n$ contains the negative loop p_0 of Σ_0, $e_1, \ldots, e_n$ are links in Σ, and $\Sigma \wr_1 e_1 \cdots \wr_n e_n \cong -K_5$. So now since each $L(\Sigma_0 \wr_1 e_1 \cdots \wr_i e_i) = L_0(\Sigma) \wr_1 e_1 \cdots \wr_i e_i$ is 3-connected, each $\Sigma \wr_1 e_1 \cdots \wr_i e_i$ is vertically 3-connected by Lemma 3.3. Thus $\Sigma \wr_1 e_1 \cdots \wr_{n-1} e_{n-1}$ is vertically 3-connected and tangled. However one can easily check that there is no way to decontract a link or add a link to $-K_5$ and preserve all of the following properties: vertical 3-connectedness of Σ, simplicity and cosimplicity of $L_0(\Sigma)$, and the property of containing no two vertex disjoint negative circles, a contradiction. □

3.3. Lemmas on subdivisions of $-K_4$ in Σ

In the remainder of Section 3, $\mathcal{S}$ will denote a subdivision of $-K_4$ in Σ. A basic fact that will be used repeatedly without further reference is Lemma 3.6.

Lemma 3.6. *If B is an $\mathcal{S}$-bridge in Σ, then B is balanced.*

Proof. First, $\overline{B}$ must be balanced because any negative circle in $\overline{B}$ will be vertex disjoint from $\mathcal{S}$, a contradiction. So let η be a switching function on Σ so that $\overline{B}$ is all positive. If B were to contain a negative circle it would be because there is a vertex of attachment v of B and two feet e and f of v with different signs in Σ^η. Since $\overline{B}$ is connected there is a path γ in $\overline{B}$ joining the endpoints of e and f. So now $e \cup f \cup \gamma$ is a negative circle that intersects $\mathcal{S}$ is a single vertex. Thus we have two vertex disjoint negative circles in $\mathcal{S} \cup B$, a contradiction. □

Consider the subdivision of $-K_4$ labeled and signed as in Fig. 3. Let T denote the *triad* $\tau_1 \cup \tau_2 \cup \tau_3$ and let L denote the *lower triangle* $\theta_1 \cup \theta_2 \cup \theta_3$. For the rest of Section 3, when working with a subdivision of $-K_4$, we will usually assume by switching that the signing and labeling is as in Fig. 3 unless otherwise specified. This signing has all edges on the triad signed positively. The terms above, below, higher and lower will also be applied to describe the relative position of vertices on the triad T.

Every path in Σ that has its endpoints on $\mathcal{S}$ and is internally disjoint from $\mathcal{S}$ is called an *$\mathcal{S}$-path*. Note that every $\mathcal{S}$-path is contained in a unique $\mathcal{S}$-bridge and when Σ is vertically 2-connected, every edge in Σ that is not in $\mathcal{S}$ is contained in some $\mathcal{S}$-path. An $\mathcal{S}$-path is called *positive* if the product of signs on its edges is positive, otherwise it is called *negative*.

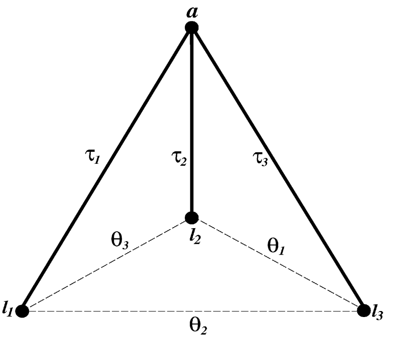

Fig. 3.

3.3.1. $\mathcal{S}$-bridges that force bad 3-separations

Given a collection $\mathcal{H}$ of subgraphs of G, by $\bigcup\mathcal{H}$ we mean the subgraph that is the union of subgraphs in $\mathcal{H}$. Lemma 3.7 is a technical result that will be very useful in subsequent proofs.

Lemma 3.7. *Let Σ be vertically 3-connected, $\mathcal{A}$ a collection of $\mathcal{S}$-bridges, and a_1, a_2, a_3 vertices on T. If the following are all satisfied, then Σ is not almost 4-connected.*

(1) *a_1, a_2, a_3 are pairwise distinct,*
(2) *a_i is the lowest attachment of a bridge in $\mathcal{A}$ on τ_i or if no bridge in $\mathcal{A}$ has an attachment on τ_i then $a_i = a$,*
(3) *bridges in $\mathcal{A}$ only have attachments on the triad T of $\mathcal{S}$,*
(4) *$T \cup (\bigcup\mathcal{A})$ is balanced, and*
(5) *if v is a vertex above a_i on τ_i, $v \notin \{a_1, a_2, a_3\}$, and $\overline{T}$ is the part of the triad at and above $\{a_1, a_2, a_3\}$, then we may mark any two vertices from $\{v, a_1, a_2, a_3\}$ with an x and the other two vertices with a y and there are two disjoint paths in $\overline{T} \cup (\bigcup\mathcal{A})$, one with the x-labeled vertices as endpoints and one with the y-labeled vertices as endpoints.*

Proof. By way of contradiction, assume that Σ is almost 4-connected. By switching assume that $\mathcal{S}$ is signed as in Fig. 3 with all edges on the triad signed positively. Let $\mathcal{L}$ the collection of $\mathcal{S}$-bridges containing negative $\mathcal{S}$-paths and/or with attachments on the interior of the lower triangle L. Note that a negative $\mathcal{S}$-path cannot have both endpoints off the lower triangle or we would create two vertex-disjoint negative circles. If must be that $\mathcal{L} \neq \emptyset$ or we will have a bad 3-separation of Σ at $\{l_1, l_1, l_3\}$ because when $\mathcal{L} = \emptyset$ all $\mathcal{S}$-bridges of Σ have attachments on the triad T and their union with the triad is balanced because they contain no negative $\mathcal{S}$-paths. Let $\hat{l}_i$ either be l_i or if there is a bridge in $\mathcal{L}$ with an attachment on τ_i, then let $\hat{l}_i$ be the highest such attachment.

Claim 1. *Let v be a vertex on τ_i below $\hat{l}_i$ in Σ and let $\underline{S}$ be the subgraph of $\mathcal{S}$ consisting of the lower triangle along with each $l_i\hat{l}_i$-subpath of τ_i. If we label $\hat{l}_i$ with x and v with y, then there is one vertex of $\{\hat{l}_1, \hat{l}_2, \hat{l}_3\} \setminus \{\hat{l}_i\}$ that may be labeled with x and the other is then labeled with y and there are two disjoint negative paths in the subgraph $\underline{S} \cup (\bigcup\mathcal{L})$, one with the x-labeled vertices as endpoints and the other with the y-labeled vertices as endpoints.*

Proof. Without loss of generality, let $i = 2$. By the definition of $\hat{l}_2$, there is some $L \in \mathcal{L}$ that has $\hat{l}_2$ as an attachment. Now either L has an attachment in the interior of the lower triangle or not.

In the former case, there is an $\mathcal{S}$-path δ' in L that connects $\hat{l}_2$ to some vertex w in the interior of the lower triangle. The vertex w subdivides its branch into two subpaths. Since the branch of w is negative, exactly one of these subpaths may be appended to δ' to obtain a negative path δ. Note that the other endpoint of δ must be l_1 or l_3 or there will be two vertex-disjoint negative circles in $\mathcal{S} \cup (\bigcup\mathcal{L}) \cup \delta'$, a contradiction. If this other endpoint of δ is l_j then label $\hat{l}_j$ with an x and the other vertex of $\{\hat{l}_1, \hat{l}_3\}$ with a y. One can now easily construct the two desired disjoint negative paths in $\underline{S} \cup (\bigcup\mathcal{L}) \cup \delta'$ (see Fig. 4).

In the latter case, L must contain a negative $\mathcal{S}$-path and so there is a negative $\mathcal{S}$-path δ containing $\hat{l}_2$. The second endpoint of δ must be either l_1 or l_3 or there will be two vertex-disjoint negative circles in $\mathcal{S} \cup (\bigcup\mathcal{A}) \cup \delta$, a contradiction. If this other endpoint of δ is l_j then label $\hat{l}_j$ with an x and the other vertex of $\{\hat{l}_1, \hat{l}_3\}$ with a y. One can now easily construct the desired disjoint negative paths in $\underline{S} \cup (\bigcup\mathcal{L}) \cup \delta$ (see Fig. 4). □

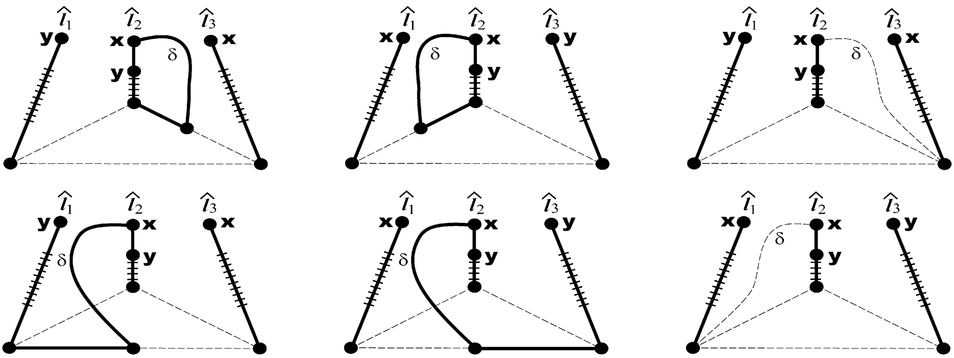

Fig. 4.

Claim 2. *$\hat{l}_i$ is not above a_i on τ_i unless $\hat{l}_i = a$ and $a \in \{a_1, a_2, a_3\}$.*

Proof. By way of contradiction, assume that $\hat{l}_i$ is above a_i on τ_i and if $\hat{l}_i = a$, then $a \notin \{a_1, a_2, a_3\}$. Label the vertices in $\{a_i, \hat{l}_1, \hat{l}_2, \hat{l}_3\}$ as in Claim 1. Now using part (5) of the hypothesis and the corresponding labels on $\{\hat{l}_i, a_1, a_2, a_3\}$ and we can use the disjoint paths in Claim 1 and (5) to obtain two vertex-disjoint negative circles in $\mathcal{S} \cup (\bigcup \mathcal{A}) \cup (\bigcup \mathcal{L})$, a contradiction. □

Now, let $\Sigma_0 = \mathcal{S}$ and $\Sigma_1 = \mathcal{S} \cup (\bigcup \mathcal{A}) \cup (\bigcup \mathcal{L})$. By Claim 2 and parts (3) and (4) of the hypothesis, Σ_1 has a bad 3-separation at $\{a_1, a_2, a_3\}$. Let $\mathcal{D}_1$ be the collection of $\mathcal{S}$-bridges that do not appear in Σ_1. Note that all bridges in $\mathcal{D}_1$ have attachments only on the triad T and only contain positive $\mathcal{S}$-paths. Thus $T \cup (\bigcup \mathcal{A}) \cup (\bigcup \mathcal{D}_1)$ is balanced. Also let $T_1 = \overline{T}$, $a_{i,1} = a_i$, $a_{1,0} = a_{2,0} = a_{3,0} = a$, and $\mathcal{B}_0 = \mathcal{A}$. Thus Σ_1 satisfies the $P_1(1) \wedge P_2(1) \wedge P_3(1) \wedge P_4(1)$.

$P_1(m)$ There is a bad 3-separation of Σ_m at $\{a_{1,m}, a_{2,m}, a_{3,m}\}$.
$P_2(m)$ Each $a_{i,m}$ is not above $a_{i,m-1}$ and not below $\hat{l}_i$.
$P_3(m)$ No bridge in $\mathcal{D}_m$ has an attachment above $a_{i,m-1}$ on τ_i.
$P_4(m)$ If v is a vertex on τ_i above $a_{i,m}$ but not above $a_{i,m-1}$, then we can label any two vertices from $\{v, a_{1,m}, a_{2,m}, a_{3,m}\}$ with an x and the other two vertices with y and there are two disjoint paths in $T_m \cup (\bigcup \mathcal{B}_0) \cup \cdots \cup (\bigcup \mathcal{B}_{m-1})$, one path connecting the x-labeled vertices and the other connecting the y-labeled vertices.

Now suppose for some $n \geqslant 1$ that Σ_n, $\{a_{1,n}, a_{2,n}, a_{3,n}\}$, T_n, and $\mathcal{D}_n$ are all defined and that $P_1(n) \wedge P_2(n) \wedge P_3(n) \wedge P_4(n)$ is true. We will show that $P_1(n) \wedge P_2(n) \wedge P_3(n) \wedge P_4(n)$ guarantees that there is a subgraph Σ_{n+1} along with $\{a_{1,n+1}, a_{2,n+1}, a_{3,n+1}\}$, T_{n+1}, and $\mathcal{D}_{n+1}$ such that $\Sigma_n \subsetneq \Sigma_{n+1} \subseteq \Sigma$ and $P_1(n+1) \wedge P_2(n+1) \wedge P_3(n+1) \wedge P_4(n+1)$ is true. This will give us a contradiction as Σ is a finite graph.

Let $\mathcal{B}_n$ be the $\mathcal{S}$-bridges in $\mathcal{D}_n$ that have attachments above $\{a_{1,n}, a_{2,n}, a_{3,n}\}$. It must be that $\mathcal{B}_n \neq \emptyset$ because otherwise Σ has a bad 3-separation at $\{a_{1,n}, a_{2,n}, a_{3,n}\}$ by $P_1(n)$.

Claim 3. *No bridge in $\mathcal{B}_n$ has an attachment below $\hat{l}_i$ on τ_i.*

Proof. Suppose by way of contradiction that $B \in \mathcal{B}_n$ has an attachment b below $\hat{l}_i$ on τ_i. By the definition of $\mathcal{B}_n$, the bridge B also has an attachment v above $a_{j,n}$ on τ_j but not above $a_{j,n-1}$ by

$P_2(n)$. Thus there is a positive $\mathcal{S}$-path δ with endpoints b and v. So now using $P_4(n)$ and Claim 1 we can construct two vertex-disjoint negative circles in $\Sigma_n \cup \delta$, a contradiction. □

So now let $\Sigma_{n+1} = \Sigma_n \cup (\bigcup \mathcal{B}_n)$. Since $\mathcal{B}_n \neq \emptyset$, $\Sigma_n \subsetneqq \Sigma_{n+1} \subseteq \Sigma$. Let $a_{i,n+1} = a_{i,n}$ or if there is a bridge in $\mathcal{B}_n$ with an attachment on τ_i that is lower than $a_{i,n}$, then let $a_{i,n+1}$ be the lowest such attachment. By Claim 3, Σ_{n+1} has a bad 3-separation at $\{a_{1,n+1}, a_{2,n+1}, a_{3,n+1}\}$. Thus Σ_{n+1} satisfies $P_1(n+1)$. Evidently Σ_{n+1} satisfies $P_2(n+1)$. Now let $\mathcal{D}_{n+1}$ be the $\mathcal{S}$-bridges of Σ that do not appear in Σ_{n+1}. Since each $B \in \mathcal{D}_{n+1}$ did not appear in Σ_{n+1} it must be that Σ_{n+1} and $\mathcal{D}_{n+1}$ satisfy $P_3(n+1)$. Now let T_{n+1} be the part of the triad T at and above $\{a_{1,n+1}, a_{2,n+1}, a_{3,n+1}\}$. We need only show that Σ_{n+1}, $\mathcal{D}_{n+1}$, and T_{n+1} satisfy $P_4(n+1)$ and we will complete our proof.

So, without loss of generality, let v be a vertex on τ_2 above $a_{2,n+1}$ but not above $a_{2,n}$. Label v with an x. If we label $a_{2,n+1}$ with an x, then our conclusion follows easily. So label $a_{2,n+1}$ with y. Without loss of generality then label $a_{1,n+1}$ with an x and $a_{3,n+1}$ with y. By the definition of $a_{2,n+1}$ and by $P_2(n)$ there is a positive $\mathcal{S}$-path in some bridge of $\mathcal{B}_n$ from $a_{2,n+1}$ to some vertex above $a_{i,n}$ on τ_i but not above $a_{i,n-1}$ on τ_i (see Fig. 5). We can now easily construct the two desired disjoint paths using property $P_4(n)$, δ, and three paths from the subgraphs of Σ_{n+1} shown in Fig. 5. □

3.3.2. Lemmas on belonging

We say that an $\mathcal{S}$-bridge B of Σ *belongs* to a subdivided quadrilateral Q of $\mathcal{S}$ when all vertices of attachment of B are on Q and $Q \cup B$ is balanced.

Lemma 3.8. *If B is an $\mathcal{S}$-bridge of Σ and R is a subdivided triangle of $\mathcal{S}$, then B does not have an attachment on the interior of each branch of R.*

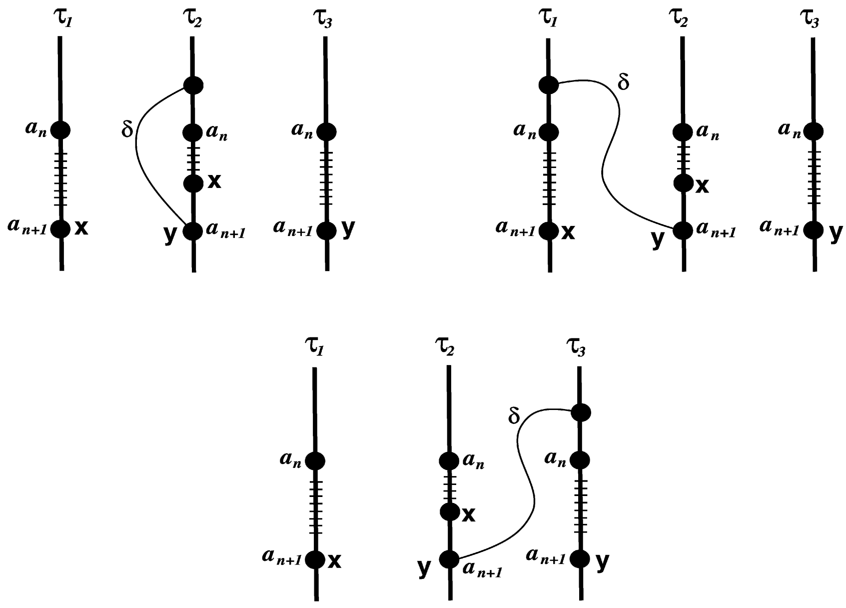

Fig. 5.

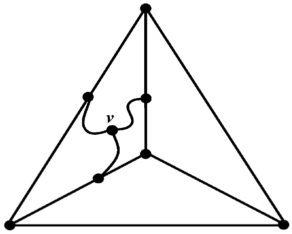

Fig. 6.

Proof. By way of contradiction, if B has an attachments v_1, v_2, v_3 on the interiors of the three branches of R, then there is a vertex $v \in \overline{B}$ and three internally disjoint paths γ_1, γ_2, γ_3 in B connecting v to v_1, v_2, v_3. (See Fig. 6. Signs on edges are suppressed in Fig. 6.)

However, since all subdivided triangles of $-K_4$ are negative, there is a negative circle besides R in $R \cup \gamma_1 \cup \gamma_2 \cup \gamma_3$ and this negative circle would be vertex disjoint from some negative circle in $\mathcal{S}$, a contradiction. □

Lemma 3.9. *If Σ is almost 4-connected, then no $\mathcal{S}$-bridge has a vertex of attachment in the interior of each branch τ_i of the triad of $\mathcal{S}$.*

Proof. By way of contradiction, assume that such a bridge B exists. The strategy of the proof will be to let $\mathcal{A} = \{B\}$ and show that this satisfies all of the properties in the hypothesis of Lemma 3.7. This will contradict that Σ is almost 4-connected and complete our proof.

For each $i \in \{1,2,3\}$, let $\hat{a}_i$ be the lowest attachments of B in the interior of τ_i and let a_i be the lowest attachment of B on τ_i. Note that $\hat{a}_i \neq a$ and $\hat{a}_i = a_i$ iff B does not have l_i as a vertex of attachment. Evidently Σ, $\mathcal{A}$, and a_1, a_2, a_3 satisfy parts (1) and (2) of the hypothesis of Lemma 3.7. Part (3) follows from Lemma 3.8. It remains only to show that they satisfy parts (4) and (5).

For part (4) suppose, by way of contradiction, that $T \cup B$ is unbalanced. By switching we may assume that all edges of T and $\overline{B}$ are positive. Thus B has two feet of different signs; however, if two feet off the lower triangle have different signs, then we would have two vertex-disjoint negative circles in $\mathcal{S} \cup B$, a contradiction. Thus we may assume by switching that all feet of B off the lower triangle are positive. So now there must be a negative foot of B incident to one of $\{l_1, l_2, l_3\}$, say l_1, without loss of generality. So if δ is an $\mathcal{S}$-path in B from l_1 to $\hat{a}_1$, then there are two vertex-disjoint negative circles in $\mathcal{S} \cup \delta$, a contradiction. Thus $T \cup B$ is balanced.

To prove part (5), recall that $\overline{T}$ is the part of the triad at and above $\{a_1, a_2, a_3\}$. Now let v be a vertex on τ_i above a_i (say, without loss of generality, $i = 1$). If we mark v with an x, then our conclusion easily follows if a_1 is marked with an x. So assume that a_1 is marked with y and, without loss of generality, a_2 with an x and a_3 with y. Now there is a path in $\overline{T}$ from v to a_2 and there is an $\mathcal{S}$-path in B from a_1 to a_3, as required. □

Lemma 3.10. *If Σ is almost 4-connected, then every $\mathcal{S}$-bridge has all of its attachments on at least one of the three subdivided quadrilaterals of $\mathcal{S}$.*

Proof. Every branch vertex of $\mathcal{S}$ is on all three subdivided quadrilaterals of $\mathcal{S}$ and every vertex in the interior of a branch is on exactly two subdivided quadrilaterals. So say there is some $\mathcal{S}$-bridge B whose vertices of attachment are not all on some subdivided quadrilateral. Thus B has three vertices of attachment on the interiors of three branches that form a subdivided triangle of $\mathcal{S}$ or a subdivided triad of $\mathcal{S}$. The latter case cannot happen because of Lemma 3.9 and in the former case cannot happen because of Lemma 3.8. Our conclusion follows. □

Lemma 3.11. *If Σ is almost 4-connected and $\Sigma \not\cong -K_5$, then every $\mathcal{S}$-bridge B has a unique subdivided quadrilateral of $\mathcal{S}$ to which it belongs.*

Proof. Let B be an $\mathcal{S}$-bridge. If B is a 2-bridge, then the conclusion is easy to verify. So assume that B is an $n \geqslant 3$-bridge. By Lemma 3.10, all attachments of B are on a subdivided quadrilateral Q of $\mathcal{S}$. Label the branch vertices of $\mathcal{S}$ and the branches of $\mathcal{S}$ on Q as shown in Fig. 7. Switch Σ so that all edges of Q are positive and so $\mathcal{S}$ is signed as shown in Fig. 7. Let Q_α be the other quadrilateral of $\mathcal{S}$ that contains α_1 and α_2.

Now switch $\overline{B}$ so that all of its edges are positive. So now $Q \cup B$ is balanced (and so B belongs to Q) unless B has a positive foot and a negative foot. Since the feet of B form an edge separation of Σ, we may switch the feet of B so that B has at least as many positive feet as negative feet. Let f_- be a negative foot of B and since B is a $n \geqslant 3$-bridge, there are at least two positive feet of B. Now either we can choose f_- so that it is attached to the interior of a branch of Q or all negative feet of B are on branch vertices of $\mathcal{S}$. Let these be Cases 1 and 2, respectively.

Claim 1. *If B has two feet on the same branch of Q, then these feet have the same sign save when they are attached to the endpoints of the branch. If B has two feet on the interiors of two adjacent branches of Q, then these feet must have the same sign.*

Proof. If the two feet mentioned have different signs, then there would be two vertex-disjoint negative circles in $\mathcal{S} \cup B$, a contradiction. □

Case 1. Without loss of generality, say that f_- is attached to a vertex in the interior of α_1. By Claim 1 any positive feet of B must now be attached to α_2. Now since B has at least two positive feet and all positive feet are attached to α_2, every negative foot of B must then be attached to α_1

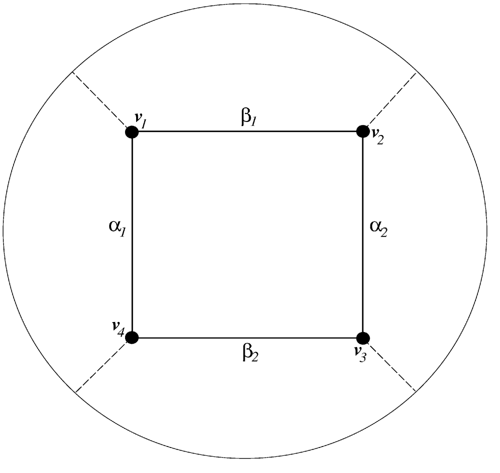

Fig. 7.

by Claim 1. Thus all attachments of B are on Q_α and $B \cup Q_\alpha$ is balanced hence B belongs to Q_α.

Case 2. Without loss of generality say that f_- is attached to v_1. It follows by Claim 1 that any positive foot of B must then be attached to a vertex on $\alpha_2 \cup \beta_2$. Now either all positive feet are contained on one of α_2 and β_2 or not. Let these be Cases 2.1 and 2.2, respectively.

Case 2.1. Without loss of generality we have that all positive feet of B are on α_2 and then since there are at least two positive feet, all negative feet must be contained on α_1 and so B belongs to Q_α as in Case 1.

Case 2.2. We must have one positive foot f_α on $\alpha_2 \setminus v_3$ and one positive foot f_β on $\beta_2 \setminus v_3$. By Claim 1 there can be no feet, positive or negative, of B on the interiors of α_1 and β_1. Thus all attachments of B are on $v_1 \cup \alpha_2 \cup \beta_2$ and any other negative feet of B must be attached to v_3. However, if there is a negative foot f'_- attached to v_3, then $\mathcal{S} \cup \overline{B} \cup \{f_-, f'_-, f_\alpha, f_\beta\}$ contains $-K_5$ as a link minor. Thus $\Sigma \cong -K_5$ by Lemma 3.5, a contradiction of our hypothesis. Thus all feet of B besides f_- are positive.

Now switch Σ on vertex v_3 and now $\mathcal{S} \cup B$ is signed as in Fig. 8. (Each crosshatched path in Fig. 8 may have length zero.)

By the previous paragraph B only has attachments on the triad of $\mathcal{S}$ at v_3 and the union of the triad at v_3 with B is balanced. We can now apply Lemma 3.7 in a similar fashion as in the proof of Lemma 3.9 and we will contradict that Σ is almost 4-connected. □

Lemma 3.12. *Say that Σ is almost 4-connected, $\Sigma \ncong -K_5$, and $B_1, \ldots, B_n$ are the $\mathcal{S}$-bridges of Σ that belong to Q, then $Q \cup B_1 \cup \cdots \cup B_n$ is balanced.*

Proof. We may switch Σ so that Q has only positive edges. Now we may switch on the vertices in each $\overline{B}_i$ so that the edges in B_i are all positive. Thus $Q \cup B_1 \cup \cdots \cup B_n$ is balanced. □

3.3.3. Lemmas on noncrossing $\mathcal{S}$-paths

Lemma 3.13. *If Σ is almost 4-connected, then there does not exist two vertex-disjoint and positive $\mathcal{S}$-paths β and γ as in Fig. 9. (Each crosshatched path in the figure may be of length zero.)*

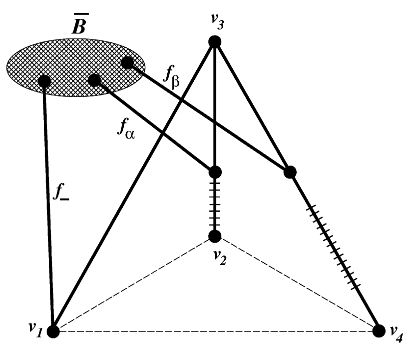

Fig. 8.

Proof. By way of contradiction, assume that the $\mathcal{S}$-paths β and γ exist. Let B and C be $\mathcal{S}$-bridges, containing β and γ, respectively. The strategy of the proof will be to let $\mathcal{A} = \{B, C\}$ and show that this satisfies all of the properties in the hypothesis of Lemma 3.7. This will contradict that Σ is almost 4-connected and complete our proof.

Let $\hat{a}_2 = a$ and for each $i \in \{1, 3\}$, let $\hat{a}_i$ be the lowest endpoint of $\beta \cup \gamma$ on τ_i. Let a_i be the lowest attachment of $B \cup C$ on τ_i. Evidently Σ, $\mathcal{A}$, and a_1, a_2, a_3 satisfy parts (1) and (2) of the hypothesis of Lemma 3.7. It remains only to show that they satisfy each of parts (3)–(5).

For part (3) suppose, by way of contradiction, that v is an attachment of B or C on the interior of the lower triangle. Thus there is a path δ in $B \cup C$ that is internally disjoint from $\mathcal{S} \cup \beta \cup \gamma$ with one endpoint at v and the other endpoint in the interior of β or γ. It is easy to check that $\mathcal{S} \cup \beta \cup \gamma \cup \delta$ must now have two vertex-disjoint negative circles, a contradiction.

For part (4) suppose, by way of contradiction, that $T \cup B \cup C$ is unbalanced. Since $\overline{B}$ and $\overline{C}$ are balanced we may assume by switching that all edges in $\overline{B}$ and $\overline{C}$ are positive. So one of B and C has both a positive foot and a negative foot. Since β and γ are both positive $\mathcal{S}$-paths, the feet of β have the same sign and the feet of γ have the same sign. By switching we may furthermore assume that the feet of both β and γ are all positive unless possibly when $B = C$. In the latter case, however, we would have a negative $\mathcal{S}$-path between the endpoints of β and γ that are off the lower triangle which would create two vertex-disjoint negative circles, a contradiction. So now if all of the feet of β and γ are positive, then there is a negative path δ that is internally disjoint from $\mathcal{S} \cup \beta \cup \gamma$ and has one endpoint in the interior of β or γ and the other endpoint somewhere on the triad of $\mathcal{S}$. One can now check that $\mathcal{S} \cup \beta \cup \gamma \cup \delta$ has two vertex-disjoint negative circles, a contradiction.

For part (5), let $v \notin \{a_1, a_2, a_3\}$ be above a_i on τ_i. There are three cases to consider: $v \neq a$ and v is above $\hat{a}_i$ on τ_i for $i \in \{1, 3\}$, $a_2 \neq a$ and v is above a_2 on τ_2, and v is above a_i but not above $\hat{a}_i$ for $i \in \{1, 3\}$. In each case, one can check that (5) is satisfied. □

Lemma 3.14. *Let Σ be almost 4-connected and $\mathcal{A}$ be a collection of $\mathcal{S}$-bridges with endpoints only on the triad T of $\mathcal{S}$ and such that $T \cup (\bigcup \mathcal{A})$ is balanced. Then there does not exist three internally disjoint $\mathcal{S}$-paths α, β, δ in $\mathcal{A}$ as shown in one of the signed graphs in Fig. 10.*

Proof. Again apply Lemma 3.7 to contradict almost 4-connectedness. □

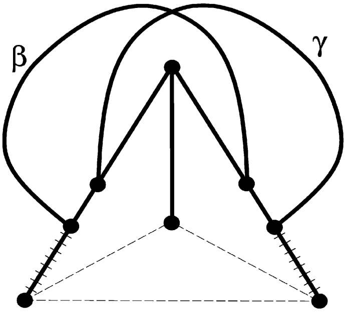

Fig. 9.

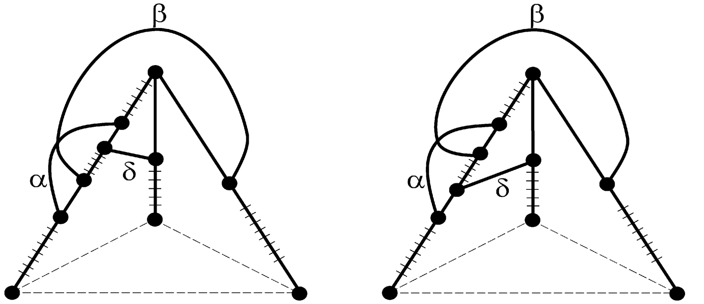

Fig. 10.

Now say that α and β are vertex-disjoint $\mathcal{S}$-paths that both belong to the same subdivided quadrilateral Q of $\mathcal{S}$. If the endpoints of α and β alternate in cyclic order around Q, then we say that α and β are *crossing* $\mathcal{S}$-paths on Q.

Lemma 3.15. *Let Σ be almost* 4*-connected. If α and β are crossing $\mathcal{S}$-paths on Q, then at least three of the four endpoints of α and β must all be contained on one branch of Q.*

Proof. Suppose by way of contradiction that no three of the four endpoints of α and β appear on the same branch of Q. Using Lemma 3.13 we cannot have that all four of the endpoints of α and β appear on two adjacent branches of Q. We can now easily conclude that $\mathcal{S} \cup \alpha \cup \beta$ contains two vertex-disjoint negative circles, a contradiction. □

Lemma 3.16 is an adaptation of [2, Lemma 6.2.1]. Note that if $\Sigma \ncong -K_5$ and Σ is almost 4-connected, then each $\mathcal{S}$-bridge B belongs to a subdivided quadrilateral Q by Lemma 3.11. Now choose $\mathcal{S}$ as in Lemma 3.16 so that there are no local bridges save possibly for a link B with endpoints the same as some branch β of $\mathcal{S}$ and with the opposite sign as β. Now then this special bridge B is local on β but does not belong to either subdivided quadrilateral of $\mathcal{S}$ that contains β. So if $\mathcal{S}$ is chosen as in Lemma 3.16, then the bridges belonging to a given Q are not local any of the branches of Q.

Lemma 3.16. *Let Σ be vertically* 3*-connected. If $\mathcal{S}$ has local bridges, then there is another subdivision $\mathcal{S}'$ of $-K_4$ such that*

(1) *the branch vertices of $\mathcal{S}'$ are the same as the branch vertices of $\mathcal{S}$, and*
(2) *if B is a local bridge on a branch β' of $\mathcal{S}'$, then B is a single edge whose endpoints are the endpoints of β' and whose sign is the opposite of the sign of β'.*

Proof. Since Σ is vertically 3-connected and simple, all $\mathcal{S}$-bridges have at least two attachments and any 2-bridge consists of a single link. Note that if β is a local $\mathcal{S}$-bridge on β, then $B \cup \beta$ will be balanced if B has an attachment on the interior of β because otherwise there would be two vertex-disjoint negative circles in $\mathcal{S} \cup B$, a contradiction. So the only possibility for $B \cup \beta$ to be unbalanced is if B is a link whose only attachments are the endpoints of β.

Now it suffices to show how to replace each branch β of $\mathcal{S}$ with a branch β' that satisfies (1) and (2). So now let $\ell(\beta)$ be the union of β and all its local $\mathcal{S}$-bridges besides the special links from the previous paragraph. So now $\ell(\beta)$ is balanced and β' may be chosen from $\ell(\beta)$ as in the proof from [2, Lemma 6.2.1]. $\square$

Lemma 3.17. *Let Σ be almost 4-connected, not isomorphic to $-K_5$, and say that $\mathcal{S}$ has no local bridges in the sense of Lemma 3.16. Suppose that α and β are crossing $\mathcal{S}$-paths on Q. Furthermore assume that any crossing $\mathcal{S}$-paths on Q have at least three of four endpoints on the same branch of Q. Then α and β do not belong to different $\mathcal{S}$-bridges.*

Proof. Suppose by way of contradiction that α and β belong to distinct $\mathcal{S}$-bridges A and B, respectively. Let τ_1 be the branch of Q, that contains three or four endpoints of α and β. If τ_1 contains four endpoints of α and β, then since B is not a local bridge, we can rechoose β so that τ_1 contains exactly three of their endpoints. Since A is not a local bridge, there is some attachment of A off τ_1. But since Q does not have two crossing $\mathcal{S}$-paths without three or four endpoints on the same branch, the only attachment of A off τ_1 is the same as the endpoint of β off τ_1, call it d. Similarly, B now only has the attachment d off τ_1.

Let b_0 and c_0 be the extreme attachments of A on τ_1. Let $\mathcal{C}_1$ be the collection of $\mathcal{S}$-bridges belonging to Q that have attachments strictly between b_0 and c_0 on τ_1. Note that $\mathcal{C}_1 \neq \emptyset$ as $B \in \mathcal{C}_1$, no bridge in $\mathcal{C}_1$ has attachments off τ_1 except for d (by our hypothesis on crossing paths), and all bridges in $\mathcal{C}_1$ have d as an attachment (because there are no local bridges for Q). Let $b_1 = b_0$ or if there is an attachment of a bridge in $\mathcal{C}_1$ further out on τ_1, then let b_1 be the attachment furthest out along τ_1. Similarly define c_1. Now suppose that for some $n \geqslant 1$ that $\mathcal{C}_1, \dots, \mathcal{C}_n$ are collections of $\mathcal{S}$-bridges belonging to Q and $(b_0, c_0), \dots, (b_n, c_n)$ are pairs of vertices on τ_1 such that $P_1(n) \wedge P_2(n)$ is true.

$P_1(m)$ All bridges in $\mathcal{C}_1 \cup \dots \cup \mathcal{C}_m$ have d as an attachment and all other attachments are on τ_1.

$P_2(m)$ For each $i \in \{1, \dots, m\}$, the vertices $b_0, \dots, b_{i-1}, c_0, \dots, c_{i-1}$ are attachments from $\mathcal{C}_1, \dots, \mathcal{C}_{i-1}$; b_i and c_i are attachments from $\mathcal{C}_i$; and $b_0, \dots, b_{i-1}, c_0, \dots, c_{i-1}$ are contained between b_i and c_i on τ_1.

Now let $\mathcal{C}_{n+1}$ be the collection of $\mathcal{S}$-bridges belonging to Q that have an attachment on the interior of the $b_n c_n$-subpath on τ_1. If $\mathcal{C}_{n+1} \neq \emptyset$, then by our hypothesis on crossing $\mathcal{S}$-paths, each bridge in $\mathcal{C}_{n+1}$ has no attachments other than on τ_1 and d. Also, since $\mathcal{S}$ has no local bridges, d is an attachment of each bridge in $\mathcal{C}_{n+1}$. So now define b_{n+1} and c_{n+1} as before and we have that $P_1(n+1) \wedge P_2(n+1)$ is true. If $\mathcal{C}_{n+1} = \emptyset$, then this process halts. Since Σ is finite, the process halts at some iteration, say $n \geqslant 1$, with $\mathcal{C}_{n+1} = \emptyset$. Now there are no $\mathcal{S}$-bridges belonging to Q that have attachments strictly between b_n and c_n on τ_1. However, now note that there is a bad 3-separation of $\mathcal{S} \cup A \cup (\bigcup(\mathcal{C}_1 \cup \dots \cup \mathcal{C}_n))$ at $\{b_n, c_n, d\}$. Since Σ has no bad 3-separation, there is an $\mathcal{S}$-bridge $C \notin \{A\} \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_n$ that has an attachment, call it m, strictly between b_n and c_n on τ_1. Now C must belong to Q_2 where Q_2 is the other quadrilateral of $\mathcal{S}$ that has τ_1 as a branch.

By the construction of $\mathcal{C}_1, \dots, \mathcal{C}_n$ there must be an $\mathcal{S}$-bridge, call it A', in $\{A\} \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_n$ such that m is strictly between the extreme attachments of A' on τ_1. Call these extreme attachments b and c. Also by the construction of $\mathcal{C}_1, \dots, \mathcal{C}_n$, there is some $\mathcal{S}$-bridge, call it B', in $\{A\} \cup \mathcal{C}_1 \cup \dots \cup \mathcal{C}_n$ that has an attachment in the interior of the bc-subpath of τ_1. Let α' be an bc-path in A' and let β' be a path in B from the interior of the bc-subpath of τ_1 to d. So now $\mathcal{S} \cup \alpha' \cup \beta'$ after switching is a subdivision of one of the signed graphs in Fig. 11 minus the $\mathcal{S}$-path in Q_2.

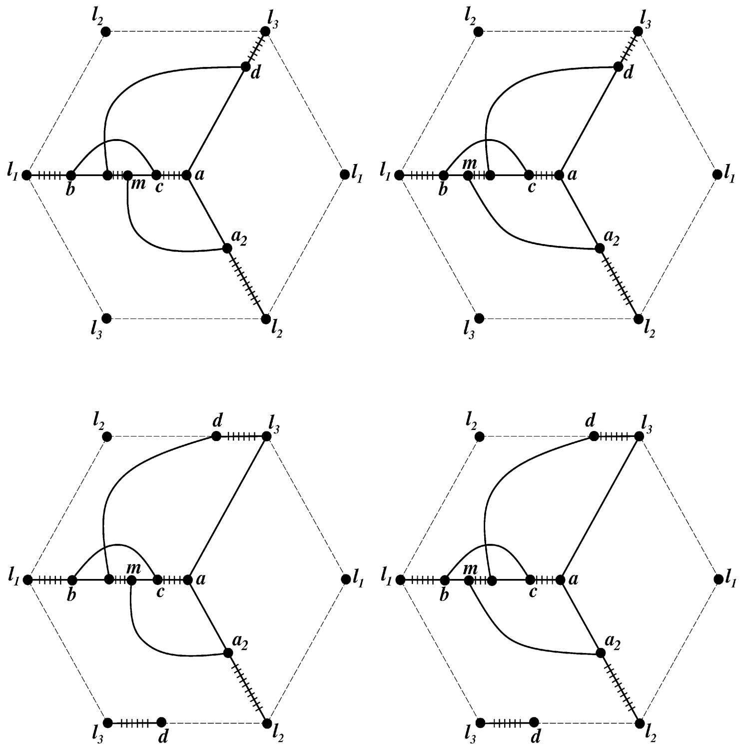

Fig. 11.

Now since C is not a local bridge on Q_2, there is an attachment of C off τ_1. It can be shown that C does not have attachments on the interior of the $l_1l_3l_2$-path on the lower triangle, because such an attachment would create two vertex-disjoint negative circles in $\mathcal{S} \cup \alpha' \cup \beta' \cup C$, a contradiction. So C must have an attachment on $\tau_2 \setminus a$. Let a_2 be the lowest such attachment and let γ be an $\mathcal{S}$-path in C connecting a_2 to m. So now $\mathcal{S} \cup \alpha' \cup \beta' \cup \gamma$ is a subdivision of one of the signed graphs in Fig. 11.

If d is on the interior of the l_2l_3-branch of $\mathcal{S}$, then one can check that $\mathcal{S} \cup \alpha' \cup \beta' \cup \gamma$ contains two vertex-disjoint negative circles, a contradiction. Thus d is on τ_3 and so A', B', C only have attachments on the triad $T = \tau_1 \cup \tau_2 \cup \tau_3$ and $T \cup A' \cup B' \cup C$ is balanced. Thus Σ is not almost 4-connected by Lemma 3.14, a contradiction. □

Lemma 3.18. *Let Σ be almost* 4*-connected, not isomorphic to* $-K_5$*, and say that $\mathcal{S}$ has no local bridges in the sense of Lemma* 3.16*. Suppose that α and β are crossing $\mathcal{S}$-paths on Q. Furthermore assume that any crossing $\mathcal{S}$-paths on Q have at least three of four endpoints on the same branch of Q. Then α and β do not belong to the same $\mathcal{S}$-bridge.*

Proof. Suppose by way of contradiction that α and β belong to the same $\mathcal{S}$-bridge B. Let τ_1 be the branch of Q containing three or four endpoints of α and β. If τ_1 contains four endpoints of α and β, then since B is not a local bridge, then we can rechoose α and β so that τ_1 contains exactly three of their endpoints. By symmetry assume that the endpoints of α are both on τ_1.

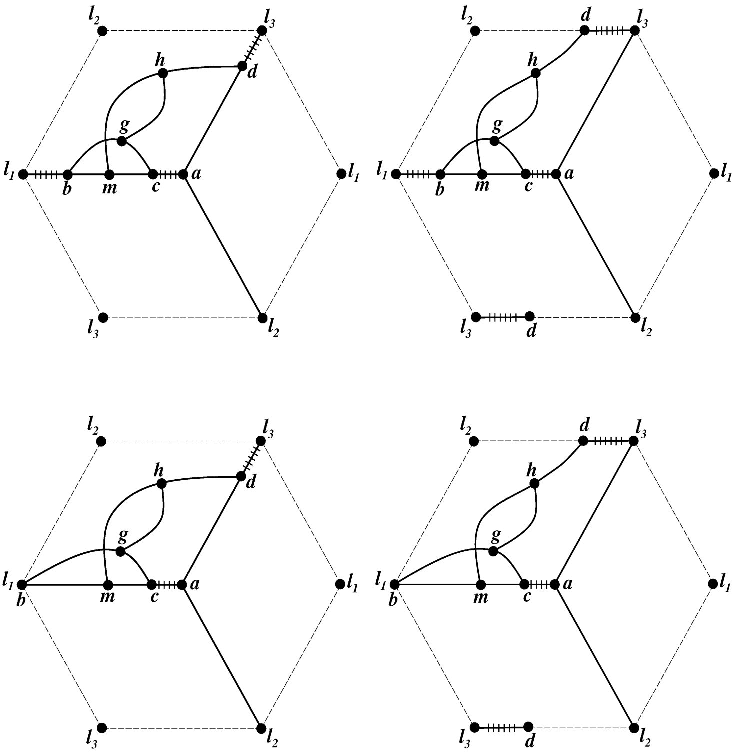

Fig. 12.

Since $\overline{B}$ is connected, there is a path γ in $\overline{B}$ that is internally disjoint from $\alpha \cup \beta$ and with one endpoint on the interior of α and the other on the interior of β (see the two upper graphs in Fig. 12).

Note that, by definition, the hd-path on β must have at least one edge. We claim that the $l_1 b$-path on τ_1 has length zero. If it did not, then there is a new subdivision $\mathcal{S}_g$ of $-K_4$ consisting of the same lower triangle as $\mathcal{S}$ and then the triad with apex vertex g and the three branches from g to the lower triangle shown in Fig. 12. But then there is an $\mathcal{S}_g$-bridge B_g containing the vertex m and B_g has three attachments b, c, h on the interiors of the three branches of the triad of $\mathcal{S}_g$. This is a contradiction of Lemma 3.9. So $\mathcal{S} \cup B$ contains a subdivision of one of the two lower graphs in Fig. 12.

Rechoose α and β so that the length of the ac-path on τ_1 is minimal and $\alpha \cup \beta$ has three of four endpoints on τ_1. Now leaving α fixed, rechoose β and γ so that the length of the hd-path on β is a minimum including length zero (if possible). Let β_d be the hd-subpath of β, let β_m be the hm-subpath of β, and let $\bar{\tau}$ be the bc-subpath on τ_1.

Note that $(\bar{\tau} \cup \beta_m \cup \alpha \cup \gamma) \setminus \{b, c, h\}$ has two connected components, one containing m and another containing g. Let X be the union of the connected components of $(\mathcal{S} \cup B) \setminus \{b, c, h\}$ that contain m and g.

Claim 1. *There is a vertical* 3*-separation* (R, S) *of* $\mathcal{S} \cup B$ *at* $\{b, c, h\}$ *with* $X \subseteq R \subseteq (B \cup \bar{\tau})$.

Proof. Let v be a vertex of attachment of B off $\bar{\tau}$. To prove our claim we need only show that there is no path from v to m or g that does not pass through $\{b, h, c\}$.

First suppose v is on τ_1 and suppose by way of contradiction that there is a path from v to $\{m, g\}$ that avoids $\{b, h, c\}$. Thus there is a path δ in B such that is internally disjoint from $\mathcal{S}$, is internally disjoint from $(\bar{\tau} \cup \beta_m \cup \alpha \cup \gamma) \setminus \{b, c, h\}$, has v as one endpoint, and has its second endpoint z somewhere on $(\bar{\tau} \cup \beta_m \cup \alpha \cup \gamma) \setminus \{b, c, h\}$. It cannot be that z is in the interior of α or γ because then we would contradict the minimality of the length of the ac-path on τ_1. It cannot be that the z is on the interior of β_m or $\bar{\tau}$ else $\mathcal{S} \cup B$ would contain a subdivision of one of the two upper graphs in Fig. 12 where the l_1b-subpath of τ_1 has nonzero length, which would contradict Lemma 3.9 as in the second to last paragraph before this claim.

Second suppose that v is not on τ_1. Thus there is a path δ in B such that is internally disjoint from $\mathcal{S}$, is internally disjoint from $(\bar{\tau} \cup \beta_m \cup \alpha \cup \gamma) \setminus \{b, c, h\}$, has v as one endpoint, and has its second endpoint z somewhere on $(\bar{\tau} \cup \beta_m \cup \alpha \cup \gamma) \setminus \{b, c, h\}$. If $v \neq d$, then we easily get a contradiction of our hypothesis about crossing $\mathcal{S}$-paths having at least three of four vertices on the same branch of $\mathcal{S}$. Thus $v = d$ and either $v = d = h$ or $v = d \neq h$. The former case is not possible because we did not avoid $\{b, h, c\}$ and the latter case is not possible because β and γ were chosen to minimize the length of the hd-subpath of β. $\square$

By Claim 1 there is a bad 3-separation of $\mathcal{S} \cup B$ at $\{b, c, h\}$ and since Σ is almost 4-connected, there is an $\mathcal{S}$-bridge C such that $\mathcal{S} \cup B \cup C$ does not have a bad 3-separation. Thus C has an attachment on the interior of $\bar{\tau}$ and so C belongs either to Q or the other quadrilateral of $\mathcal{S}$ that has τ_1 as a branch, call it Q_2. If C belongs to Q, then since C is not local on Q, C has an attachment off τ_1. But then we would have two crossing $\mathcal{S}$-paths on Q with three or four endpoints on τ_1 that are in different bridges, a contradiction of Lemma 3.17. Thus C belongs to Q_2 and since C is not local on Q_2, there is an attachment of C off τ_1. It cannot be, however, that C has an attachment on the interior of the $l_1l_3l_2$-path on the lower triangle as this would create two vertex-disjoint negative circles, a contradiction. So C has all attachments off τ_1 on $\tau_2 \setminus a$. Let a_2 be the lowest attachment of C on τ_2 and let χ be an $\mathcal{S}$-path in C from an interior vertex of $\bar{\tau}$ to a_2. Now one can show that B has no attachments on the interior of the $l_1l_2l_3$-path of the lower triangle as this would create two vertex-disjoint negative circles, a contradiction. Thus d is on the triad $T = \tau_1 \cup \tau_2 \cup \tau_3$ of $\mathcal{S}$, all attachments of $B \cup C$ are on the triad T of $\mathcal{S}$, and $T \cup B \cup C$ is balanced. Now $\mathcal{S} \cup \alpha \cup \beta \cup \chi$ is a subdivision of one of the two upper graphs in Fig. 11 and so Σ is not almost 4-connected by Lemma 3.14, a contradiction. $\square$

3.3.4. Lemmas on tripods

Given a circle C in Σ, a *tripod* of C is a subdivision of the graph shown in Fig. 13.

Lemma 3.19. *Let Σ be almost 4-connected and not isomorphic to $-K_5$. Furthermore assume that there are no crossing $\mathcal{S}$-paths on any of the subdivided quadrilaterals of $\mathcal{S}$. Then the collection of bridges belonging to any subdivided quadrilateral Q does not contain a tripod of Q.*

Proof. Let $\mathcal{B}$ be the collection of $\mathcal{S}$-bridges belonging to Q. Suppose by way of contradiction that $\bigcup \mathcal{B}$ contains a tripod T on Q. Let b_1, b_2, b_3 be the vertices of the tripod on Q, let χ_i be the crosshatched path of the tripod that has b_i as an endpoint, and let $a_{i,0}$ be the other endpoint of χ_i. Let t_1 and t_2 be the top vertices of the tripod and let T_0 be the tripod T minus the edges and vertices below $\{a_{1,0}, a_{2,0}, a_{3,0}\}$.

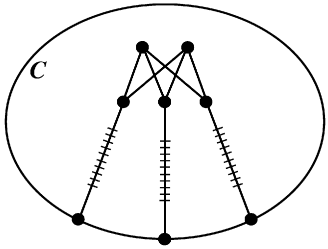

Fig. 13.

If t_1 and t_2 are separated from Q in Σ by $\{a_{1,0}, a_{2,0}, a_{3,0}\}$, then there is a bad 3-separation of Σ at $\{a_{1,0}, a_{2,0}, a_{3,0}\}$, a contradiction. Thus there are paths in $\bigcup \mathcal{B}$ from $\{t_1, t_2\}$ to Q that avoid $\{a_{1,0}, a_{2,0}, a_{3,0}\}$. However, such a path cannot be internally disjoint from the tripod unless one endpoint of such a path is from $\{b_1, b_2, b_3\}$ because otherwise we would have two crossing $\mathcal{S}$-paths on Q, a contradiction. So any path from $\{t_1, t_2\}$ to Q that avoids $\{a_{1,0}, a_{2,0}, a_{3,0}\}$ includes a subpath in $\bigcup \mathcal{B}$ that is internally disjoint from the tripod, has one endpoint above $\{a_{1,0}, a_{2,0}, a_{3,0}\}$ on the tripod, and the other endpoint below $\{a_{1,0}, a_{2,0}, a_{3,0}\}$ on the tripod. Let $\mathcal{P}_0$ be the collection of all such paths.

Let $a_{i,1} = a_{i,0}$ or if there is a path in $\mathcal{P}_0$ with an endpoint below $a_{i,0}$ on χ_i, then let $a_{i,1}$ be the lowest such endpoint. Let T_1 be the subgraph of the tripod with all of the edges and vertices below $\{a_{1,1}, a_{2,1}, a_{3,1}\}$ removed. It is easy to verify that $P_1(1) \wedge P_2(1)$ is satisfied.

$P_1(m)$ $T_{m-1} \subsetneqq T_m$.

$P_2(m)$ If v is a vertex on χ_i above $a_{i,m}$ but not above $a_{i,m-1}$ then we can label any two vertices from $\{v, a_{1,m}, a_{2,m}, a_{3,m}\}$ with an x and the other two vertices with y and there are two vertex-disjoint paths in $T_m \cup (\bigcup \mathcal{P}_0) \cup \cdots \cup (\bigcup \mathcal{P}_{m-1})$, one with the x-labeled vertices as endpoints and one with the y-labeled vertices as endpoints.

Now for some $n \geqslant 1$ suppose that $\{a_{1,n}, a_{2,n}, a_{3,n}\}$ and T_n are defined and satisfy $P_1(n) \wedge P_2(n)$. We will show that this suffices to define $\{a_{1,n+1}, a_{2,n+1}, a_{3,n+1}\}$ and T_{n+1} that satisfy $P_1(n+1) \wedge P_2(n+1)$. This however is not possible because Σ is a finite graph and this will complete our proof.

Now $\{a_{1,n}, a_{2,n}, a_{3,n}\}$ cannot separate $\{t_1, t_2\}$ from Q or there would be a bad 3-separation of Σ at $\{a_{1,n}, a_{2,n}, a_{3,n}\}$. Thus there are paths from $\{t_1, t_2\}$ to the attachments of $\bigcup \mathcal{B}$ on Q that avoid $\{a_{1,n}, a_{2,n}, a_{3,n}\}$. Such a path however cannot be internally disjoint from T unless one of their endpoints is in $\{b_1, b_2, b_3\}$ because otherwise, using such a path and $P_2(n)$ we could construct two disjoint crossing paths on Q which by the hypothesis of the lemma do not exist. So any such path from $\{t_1, t_2\}$ to Q must use a subpath that is internally disjoint from T, has one endpoint above $\{a_{1,n}, a_{2,n}, a_{3,n}\}$ on T and one endpoint below $\{a_{1,n}, a_{2,n}, a_{3,n}\}$ on T. Note that any such path cannot have an endpoint above $\{a_{1,n-1}, a_{2,n-1}, a_{3,n-1}\}$ on T by the construction of $\mathcal{P}_0, \ldots, \mathcal{P}_{n-1}$. So let $\mathcal{P}_n$ be the collection of such paths. Also note that by our construction of $\mathcal{P}_0, \ldots, \mathcal{P}_{n-1}$, any path in $\mathcal{P}_n$ must be internally disjoint from all paths in $\mathcal{P}_0, \ldots, \mathcal{P}_{n-1}$. Now let $a_{i,n+1} = a_{i,n}$ or if there is a path in $\mathcal{P}_n$ with an endpoint below $a_{i,n}$ on χ_i, then let $a_{i,n+1}$ be the lowest such endpoint. Let T_{n+1} be the subgraph of the tripod with all of the edges and vertices below $\{a_{1,n+1}, a_{2,n+1}, a_{3,n+1}\}$ removed. Evidently $P_1(n+1)$ is satisfied. To show that $P_2(n+1)$ is satisfied we would use an argument very similar to the one in the last paragraph of the proof of Lemma 3.7. $\square$

4. Proofs of our main results

The following theorem is from [2, Theorem 6.3.1]. Note that if $\widehat{G}$ is planar, then G has a planar imbedding with C as a facial boundary cycle.

Theorem 4.1. *Let G be a graph and C a circle in G. Let $\widehat{G}$ be the graph obtained from G by attaching a new vertex v adjacent to the vertices of C. If $\widehat{G}$ is vertically 3-connected, then $\widehat{G}$ is planar iff G does not contain two crossing C-paths and does not contain a tripod on C.*

Proof of Theorem 1.2. Certainly Σ satisfies our conclusion iff the associated simple signed graphs satisfies the conclusion. So assume that Σ is a simple tangled signed graph.

First, suppose that Σ is almost 4-connected. In this case we may also assume that $\Sigma \ncong -K_5$ because otherwise Σ satisfies part (2) and we are done. By Lemma 3.4, $\Sigma = \pm C_3$ or contains a subdivision $\mathcal{S}$ of $-K_4$. In the former case, Σ is projective planar. In the latter case, choose $\mathcal{S}$ so that it does not have local bridges in the sense of Lemma 3.16. So now each $\mathcal{S}$-bridge belongs to a quadrilateral of $\mathcal{S}$ by Lemma 3.11. Furthermore, by Lemma 3.12, if $\mathcal{B}$ is the collection of all bridges belonging to a quadrilateral Q of $\mathcal{S}$, then $Q \cup (\bigcup \mathcal{B})$ is balanced. Now there are no crossing $\mathcal{S}$-paths on Q by Lemmas 3.15, 3.17, and 3.18. So now there are no tripods on Q by Lemma 3.19. Now it is easy to check that $\widehat{Q \cup (\bigcup \mathcal{B})}$ (as in Theorem 4.1) is vertically 3-connected. So by Theorem 4.1, $Q \cup (\bigcup \mathcal{B})$ has a planar imbedding with Q on the outside face. Thus we can extend the unique imbedding of $\mathcal{S}$ in the projective plane to an imbedding of the entire signed graph Σ in which each facial boundary is a positive circle in Σ. Given the topology of the projective plane, this makes a circle in Σ negative iff it is imbedded as a nonseparating closed curve. Thus Σ is projective planar and so satisfies part (1).

Second, suppose that Σ is vertically 3-connected but not almost 4-connected. Thus there is a vertical 3-separation (A, B) of Σ with $\Sigma{:}B$ balanced and containing at least five vertices. Since Σ is tangled, Lemma 3.1 implies that A is unbalanced. Switch Σ so that all of the edges of B are positive and attach an all-positive triangle T to $\Sigma{:}A$ and $\Sigma{:}B$ on the three vertices in $(\Sigma{:}A) \cap (\Sigma{:}B)$. Write $\Upsilon = (\Sigma{:}A) \cup T$ and $\Omega = (\Sigma{:}B) \cup T$. Evidently $\Sigma = \Upsilon \oplus_3 \Omega$ with Υ unbalanced and Ω balanced. One can show that Ω must be vertically 3-connected and so by Theorem 1.1, Υ is tangled. Thus Σ satisfies part (3).

Lastly, suppose that Σ is connected but not vertically 3-connected. Thus by Proposition 2.1 and Lemmas 3.1 and 3.2, there is a vertical t-separation (A, B) of Σ with $t \in \{1, 2\}$, A unbalanced, and B balanced. In a similar fashion as in the previous paragraph we get that Σ satisfies part (3). $\square$

Theorem 4.2. *(See H. Whitney [11].) If G is a connected graph, then $M^*(G)$ is graphic iff G is planar.*

Proof of Theorem 1.4. Again we may assume that Σ is simple. By Theorem 1.2, $\Sigma \cong -K_5$, Σ is projective planar, or $\Sigma = \Upsilon \oplus_t \Omega$ where Υ is tangled and Ω is balanced. Since $M(-K_5) \cong R_{10}$ is not graphic, $\Sigma \ncong -K_5$. Also, if $\Sigma = \Upsilon \oplus_t \Omega$, then assume that k is minimal.

If Σ is projective planar, then let G be the topological dual graph of Σ in the projective plane. In [6, §2] it is shown that $M^*(G) = M(\Sigma)$. So $M(\Sigma) = M^*(G)$ is graphic and cographic and so Theorem 4.2 implies that G is planar. Thus Σ satisfies part (1).

If $\Sigma = \Upsilon \oplus_k \Omega$ where Υ is tangled and Ω is balanced, then it must be that $M(\Upsilon)$ is graphic because Υ may be obtained as a minor of Σ. Thus Σ satisfies part (2). $\square$

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