Embedding theorems for HNN extensions of inverse semigroups

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Abstract

A variant of an HNN extension of an inverse semigroup introduced by Gilbert [N.D. Gilbert, HNN extensions of inverse semigroups and groupoids, J. Algebra 272 (2004) 27–45] is defined provided that associated subsemigroups are order ideals. We show this presentation still makes sense without the assumption on associated subsemigroups in the sense that it gives a semigroup deserving to be an HNN extension, and it is embedded into another variant using the automata theoretical technique based on combinatorial and geometrical properties of Schützenberger graphs.

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1. Introduction

The concept of an HNN extension of a group can be generalized to the class of semigroups in several fashions. One such generalization is studied in [1,2,6,8,9] where a stable letter belongs to the group of units. It can also be generalized to the class of inverse semigroups in a way that stable letters are not necessarily a group element. There are two such approaches by Gilbert [4] and Yamamura [15]. The first is constructed by interpreting the HNN extensions of groupoids considered by Higgins [7] under the assumption that associated subsemigroups are order ideals, and each stable letter corresponds to one of the idempotents in an associated subsemigroup. This approach has a strong connection with groupoid theory. The second has the features of a free construction in inverse semigroups. It is constructed under the assumption that associated subsemigroups are monoids, and only one stable letter is required.

In this paper, we clarify the relationship between these two variants of HNN extensions of inverse semigroups. First, we introduce several other variants of HNN extensions and generalize an HNN extension in the sense of [4] to a more general context so that the embeddability can still hold even though associated subsemigroups are not order ideals. Second, we show that every HNN extension in the sense of [4] and its generalization can be naturally embedded into another variant of HNN extensions introduced in [14]. This implies that the HNN extensions in the sense of [4] are actually subsemigroups of the other variant of HNN extensions in [14,15]. Third, we give a necessary and sufficient condition for the semilattice of idempotents of an HNN extension to coincide with that of the original inverse semigroup.
Our main tool in this paper is the automata theoretic method using Schützenberger graphs introduced by Stephen [13]. To obtain the second result above, we use the iterative production of approximate automata of Schützenberger graphs. To be precise, we establish a simulation among approximate automata. The reader is referred to [13] for Schützenberger graphs and [12] for standard terminology in semigroup theory.

2. Concepts of HNN extensions of inverse semigroups

We recall several concepts of an HNN extension of an inverse semigroup in [4,14,15], and also introduce some presentations as a candidate for an HNN extension. Then we examine their properties. In the rest of the paper, we suppose that $S$ is an inverse semigroup, $A$ and $B$ are isomorphic inverse subsemigroups of $S$, and $\phi$ is an isomorphism of $A$ onto $B$.

2.1. Presentation $S(\phi, t)$

We now suppose $e_A \in A \subseteq e_ASe_A$ and $e_B \in B \subseteq e_BSe_B$, where $e_A$ and $e_B$ are idempotents. The inverse semigroup $S(\phi, t)$ is defined by the presentation

$$\text{Inv}(S, t \mid t^{-1}at = \phi(a) \text{ for } \forall a \in A, t^{-1}t = e_B, tt^{-1} = e_A). \quad (2.1)$$

The element $t$ in $S(\phi, t)$ is called the stable letter. The most important property of this construction is that $S$ is naturally embedded into $S(\phi, t)$. This is classified as an HNN extension of type I in [14] and the restricted case that the stable letter belongs to the group of units is discussed in [1,2,6,8,9]. This construction is applied to several algorithmic problems like the undecidability of Markov properties of inverse semigroups in [15,18]. There are many concrete examples that admit a natural decomposition as an HNN extension (2.1). For example, free groups, free inverse semigroups, the bicyclic monoid, free Clifford semigroups and Bruck–Reilly extensions admit a natural HNN extension decomposition (see [15,18]). It is also clear that an HNN extension of a group is an HNN extension in the sense of (2.1). We also remark that lower bounded HNN extensions are discussed in [10] and an inverse semigroup whose defining relations have the form $d_1 = d_2$, where $d_1$ and $d_2$ are Dyck words, admits a decomposition as an HNN extension of a semilattice [16].

An HNN extension (2.1) is called full if $E(A) = E(B) = E(S)$. A full HNN extension can be characterized as a fundamental inverse monoid of a loop of inverse monoids, and this is employed to study the class of inverse monoids acting on ordered forests in [19]. This is considered as a generalization of the Bass–Serre theory. A normal form is given for locally full HNN extensions in [18]. Recall that an inverse submonoid $A (e \in A \subset eSe)$ is called locally full if $E(A) = E(eSe)$ and an HNN extension (2.1) is called locally full if $A$ and $B$ are locally full [17].

2.2. Presentation $S[\phi, t_e]$

The inverse semigroup $S[\phi, t_e]$ is defined by the presentation

$$\text{Inv}(S, t_e (e \in E(A)) \mid t^{-1}_e a^{-1}a t_a^{-1} = \phi(a) \text{ for } \forall a \in A, t^{-1}_e t_f = \phi(e)\phi(f), t_e t_f^{-1} = ef). \quad (2.2)$$

We here denote the set of idempotents of $A$ by $E(A)$. The elements $t_e (e \in E(A))$ in $S[\phi, t_e]$ are called the stable letters. Interpreting the concept of an HNN extension of a groupoid given by Higgins [7], Gilbert [4] studies the inverse semigroups presented by (2.2) provided that $A$ and $B$ are order ideals of $S$, and denotes it by $S_{\phi,A,\phi}$. As a matter of fact, he adopts the relation $t^{-1}_e t_f = \phi(e)\phi(f)$ instead of $t^{-1}_e t_f = \phi(e)\phi(f)$ in (2.2). These two relations are equivalent and one can choose either of them. However, we adopt $t^{-1}_e t_f = \phi(e)\phi(f)$ because for technical reasons as we will see later. Every element in (2.2) has a certain normal form and $S$ is naturally embedded into $S[\phi, t_e]$ provided that $A$ and $B$ are order ideals [4]. Gilbert’s perspective is groupoid theoretic, and the assumption that $A$ and $B$ are order ideals is critical to obtain a relatively easy groupoid structure. We remark that finite presentations of such a presentation is discussed in [3].

We shall show that the construction (2.2) still makes sense even though $A$ and $B$ are not order ideals in the sense that the natural mapping $s \mapsto s$ ($s \in S$) is an embedding of $S$ into $S[\phi, t_e]$. Unless $A$ and $B$ are order ideals, the inverse semigroup presented by (2.2) has more complicated groupoid structure than $S$. In fact, we shall show that in Theorem 6.4 the set of vertices of the corresponding groupoid for $S[\phi, t_e]$ is equal to that of $S$ if and only if $A$ and $B$ are order ideals.
2.3. Presentation $S(\phi, t)$

The inverse semigroup $S(\phi, t)$ is defined by the presentation

$$\text{Inv}(S, t \mid t^{-1}at = \phi(a) \text{ for } \forall a \in A, tbt^{-1} = \phi^{-1}(b) \text{ for } \forall b \in B). \quad (2.3)$$

The element $t$ in $S(\phi, t)$ is called the stable letter. This is classified as an HNN extension of type III and $S$ is shown to be naturally embedded into $S(\phi, t)$ in [14]. We shall give an alternative proof for the embeddability in Section 3.

One of our main objectives in this paper is to show that $S[\phi, t_e]$ is embedded into $S(\phi, t)$ in a natural fashion. A proof is given in Section 6. This clarifies the relationship between the two approaches [4,15] for HNN extensions of inverse semigroups.

2.4. Presentation $S(\phi, t\parallel)$

The inverse semigroup $S(\phi, t\parallel)$ is defined by the presentation

$$\text{Inv}(S, t \mid t^{-1}at = \phi(a) \text{ for } \forall a \in A). \quad (2.4)$$

The element $t$ in $S(\phi, t\parallel)$ is called the stable letter. Clearly $S(\phi, t)$ is a homomorphic image of $S(\phi, t\parallel)$. In $S(\phi, t\parallel)$, the conjugation by the stable letter $t$ does not necessarily give an isomorphism of $A$ onto $B$ as we see in Section 3.2.

2.5. Presentation $S[\phi, t]$

The inverse semigroup $S[\phi, t]$ is defined by the presentation

$$\text{Inv}(S, t \mid at = t\phi(a) \text{ for } \forall a \in A). \quad (2.5)$$

The element $t$ in $S[\phi, t]$ is called the stable letter. A slightly different presentation is given by

$$\text{Inv}(S, t \mid at = t\phi(a) \text{ for } \forall a \in A, t^{-1}\phi^{-1}(b) = bt^{-1} \text{ for } \forall b \in B). \quad (2.6)$$

It is easy to see that (2.5) and (2.6) give an isomorphic inverse semigroup since the second relation in (2.6) is derived from the first. Then these are also equivalent to

$$\text{Inv}(S, t \mid t^{-1}\phi^{-1}(b) = bt^{-1} \text{ for } \forall b \in B). \quad (2.7)$$

Clearly $S(\phi, t)$ is a homomorphic image of $S[\phi, t]$.

3. Embedding theorems

We show that an inverse semigroup $S$ is embedded into $S[\phi, t_e]$ without the assumption that $A$ and $B$ are order ideals and examine the relationship among the variants of HNN extensions.

3.1. Weak HNN properties

Let us define a homomorphism $\eta$ of $S[\phi, t_e]$ into $S(\phi, t)$; first, we define a mapping $\eta$ of the set of generators of $S[\phi, t_e]$ into $S(\phi, t)$ and show all the defining relations for $S[\phi, t_e]$ are preserved by $\eta$. Note that $S[\phi, t_e]$ is generated by the set $S$ and the set of the stable letters $\{t_e : e \in E(A)\}$. The mapping $\eta$ of $S \cup \{t_e : e \in E(A)\}$ into $S(\phi, t)$ is defined by

$$\eta(x) = \begin{cases} 
  x & \text{if } x \in S, \\
  et & \text{if } x = t_e(e \in E(A)). 
\end{cases} \quad (3.1)$$

**Lemma 3.1.** The mapping $\eta$ can be extended to a homomorphism of $S[\phi, t_e]$ into $S(\phi, t)$. 

Proof. Note that for any \( a \in A \), \( tt^{-1}at^{-1} = a \) in \( S(\phi, t) \). It follows that \( \eta(t_e)\eta(t_f)^{-1} = et(f) = et = ef = \eta(ef) \), \( \eta(t_e)^{-1}\eta(t_f) = (et)^{-1} = ef = \phi(ef) = \phi(e)\phi(f) = \eta(\phi(ef)) \), and \( \eta(t_{aa^{-1}})^{-1}\eta(a)\eta(t_{a^{-1}a}) = (aa^{-1}t)^{-1}a^{-1}at = t^{-1}aat = t^{-1}at = \phi(a) = \eta(\phi(a)) \) for all \( e, f \in E(A) \) and \( a \in A \). The defining relations for \( S \) are evidently preserved. Since \( S[\phi, t_e] \) is the freest inverse semigroup satisfying these relations, the mapping \( \eta \) can be extended to a homomorphism of \( S[\phi, t_e] \) into \( S(\phi, t) \). \( \square \)

The homomorphism given by (3.1) is also denoted by \( \eta \) in the rest of the paper. We now suppose that \( e \in A \subset eSe \) and \( f \in B \subset fSf \) for \( e, f \in E(S) \). We define a mapping \( \delta \) to be the natural mapping of \( S(\phi, t) \) into \( S(\phi, t) \), that is, \( \delta(s) = s \) for \( s \in S \) and \( \delta(t) = t \). It is routine to verify that \( \delta \) is a homomorphism. Let \( \varepsilon \) be the composition of \( \eta \) and \( \delta \).

\[
\begin{array}{c}
S[\phi, t_e] \xrightarrow{\eta} S(\phi, t) \\
\downarrow \varepsilon \quad \downarrow \delta \\
S(\phi, t)
\end{array}
\]

\[\text{Theorem 3.2. If } A \text{ and } B \text{ are monoids, then the homomorphism } \varepsilon \text{ is an isomorphism of } S[\phi, t_e] \text{ onto } S(\phi, t). \]

Proof. It is easy to see that \( S(\phi, t) \) is transformed to \( S[\phi, t_e] \) by a Tietze transformation [11]; we just put \( t_{\phi} = t \) and \( t_e = et \) for \( e \in E(A) \), where \( e_A \) is the identity of \( A \). Thus the two presentations give the isomorphic inverse semigroup. \( \square \)

We now give another presentation equivalent to \( S(\phi, t) \). Let \( \overline{S} \) be an inverse semigroup presented by

\[
\text{Inv}(S, 1_A, 1_B \mid 1_A^2 = 1_A, 1_B^2 = 1_B, 1_Aa = a1_A = a \text{ for } \forall a \in A, 1_bb = b1_B = b \text{ for } \forall b \in B),
\]

(3.2)

where \( 1_A \) and \( 1_B \) are newly introduced letters. Thus, \( \overline{S} \) is an inverse semigroup obtained from \( S \) by adjoining new identity elements for \( A \) and \( B \) whether or not \( A \) and \( B \) are monoids. We should note that even if \( A = B \), \( 1_A \) and \( 1_B \) are different elements in \( \overline{S} \).

\[\text{Lemma 3.3. An inverse semigroup } S \text{ is naturally embedded into } \overline{S}. \text{ Furthermore, we have } 1_A \notin S \text{ and } 1_B \notin S \text{ in } \overline{S}. \]

Proof. Note that the inverse semigroup \( A^1 \) obtained from \( A \) by adjoining a new identity element is presented by

\[
\text{Inv}(A, 1_A \mid 1_A^2 = 1_A, 1_Aa = a1_A = a \text{ for } \forall a \in A),
\]

(3.3)

Clearly \( A \) is embedded into \( A^1 \) and \( 1_A \notin A \) in \( A^1 \). It is also clear that \( \overline{S} \) is the amalgamated free product \( A^1 \ast_A S \ast_B B^1 \) by its presentation. By the strong amalgamation property of inverse semigroups [5], \( S \) is naturally embedded into \( \overline{S} \), and we have \( 1_A \notin S \) and \( 1_B \notin S \) in \( \overline{S} \) since \( 1_A \in A^1 \setminus A \) and \( 1_B \in B^1 \setminus B \). \( \square \)

Clearly \( A^1 = A \cup \{1_A\} \) and \( B^1 = B \cup \{1_B\} \), and \( A^1 \) and \( B^1 \) are inverse submonoids of \( \overline{S} \) and \( A^1 \) and \( B^1 \) are isomorphic under the correspondence \( a \to \phi(a) (a \in A) \) and \( 1_A \to 1_B \). We denote it by \( \overline{\phi} \).

\[\text{Lemma 3.4. The inverse semigroup } S(\phi, t) \text{ is isomorphic to } \overline{S}(\overline{\phi}, t) \text{ under the mapping } s \mapsto s \text{ for } s \in S \text{ and } t \mapsto t. \]

Proof. Recall that \( S(\phi, t) \) is presented by (2.3). It is also presented by

\[
\text{Inv}(S, t, 1_A, 1_B \mid t^{-1}at = \phi(a) \text{ for } \forall a \in A, tbt^{-1} = \phi^{-1}(b) \text{ for } \forall b \in B, tt^{-1} = 1_A, t^{-1}t = 1_B).
\]

Then we have \( 1_Aa1_A = tt^{-1}at^{-1} = t\phi(a)t^{-1} = \phi^{-1}(\phi(a)) = a \) for every \( a \in A \) and so \( 1_Aa = a1_A = a \). Clearly \( 1_A^2 = 1_A \). Similarly we have \( 1_bb = b1_B = b \) for \( b \in B \) and \( 1_B^2 = 1_B \). Furthermore we have \( t^{-1}1_A = t^{-1}t^{-1}t = t^{-1}t = 1_B \). Thus, (2.3) is equivalent to

\[
\text{Inv}(S, 1_A, 1_B, t \mid 1_A^2 = 1_A, 1_B^2 = 1_B, 1_Aa = a1_A = a \text{ for } \forall a \in A, 1_bb = b1_B = b \text{ for } \forall b \in B, t^{-1}at = \phi(a) \text{ for } \forall a \in A, tbt^{-1} = \phi^{-1}(b) \text{ for } \forall b \in B, t^{-1}1_A = 1_B, t^{-1}t = 1_B, tt^{-1} = 1_A).\]

This is equivalent to

\[
\text{Inv}(\overline{S}, t \mid t^{-1}at = \phi(a) \text{ for } \forall a \in A, t^{-1}1_A = 1_B, t^{-1}t = 1_B, tt^{-1} = 1_A),
\]

which is \( \overline{S}(\overline{\phi}, t) \). \( \square \)
It follows that $S\langle \phi, t \rangle$ is characterized as an HNN extension (2.1) with respect to the inverse semigroup $\tilde{S}$ associated with $\phi : A^1 \rightarrow B^1$. We also remark that when $A$ and $B$ are monoids the homomorphism $\delta$ of $S(\phi, t)$ onto $S(\phi, t)$, defined by $\delta(s) = s$ for $s \in S$ and $\delta(t) = t$, is not an isomorphism. In fact, $\delta$ maps $1_A$ to $e_A$, where $e_A$ is the identity element of $A$ and $1_A$ is the newly attached idempotent to $S$. So we have $\delta(1_A) = e_A = \delta(e_A)$ and $1_A \not= e_A$. Hence, $\delta$ is not injective.

**Theorem 3.5.** The following statements are equivalent.

(i) For every inverse semigroup $S$ and its inverse submonoids $A$ and $B$ isomorphic under $\phi$, where $e \in A \subset eSe$ and $f \in B \subset fSf$ for some $e$, $f \in E(S)$, there exists an inverse semigroup $T$ such that $S \hookrightarrow T$, $t^{-1}at = \phi(a)$ for all $a \in A$, $t^{-1}1t = f$ and $tt^{-1} = e$ for some $t \in T$.

(ii) For every inverse semigroup $S$ and its inverse subsemigroups $A$ and $B$ isomorphic under $\phi$, there exists an inverse semigroup $T$ such that $S \hookrightarrow T$, $t^{-1}at = \phi(a)$ for all $a \in A$, and $tbt^{-1} = \phi^{-1}(b)$ for all $b \in B$ for some $t \in T$.

(iii) For every inverse semigroup $S$ and its inverse subsemigroups $A$ and $B$ isomorphic under $\phi$, there exists an inverse semigroup $T$ such that $S \hookrightarrow T$, $t^{-1}at = \phi(a)$ for all $a \in A$, $te \phi^{-1}(e) = ef$ and $t^{-1}tf = \phi(e)\phi(f)$ for some $t_e \in T$ ($e \in E(A)$).

**Proof.** (i) implies (ii). Suppose that $A$ and $B$ are inverse subsemigroups of $S$ and $\phi$ is an isomorphism of $A$ onto $B$. By Lemma 3.3, $S$ is naturally embedded into $\tilde{S}$. The inverse semigroup $\tilde{S}$ is naturally embedded into $\tilde{S}(\phi, t)$ because we assume (i). It follows that $S$ is naturally embedded into $\tilde{S}(\phi, t)$. On the other hand, $\tilde{S}(\phi, t)$ is isomorphic to $S(\phi, t)$ by the natural mapping by Lemma 3.4. Thus (i) implies (ii).

(ii) implies (iii). Let $\iota$ be the natural mapping of $S$ into $S[\phi, t_e]$. Then we have the following commutative diagram.

$$
\begin{array}{ccc}
S & \xrightarrow{\iota} & S[\phi, t_e] \\
\downarrow & & \downarrow \\
S(\phi, t) & & S(\phi, t)
\end{array}
$$

Since we assume (ii), the homomorphism $S \rightarrow S[\phi, t_e]$ is injective. Therefore, the homomorphism $\iota$ is injective. Thus, (ii) implies (iii).

By Theorem 3.2, $S(\phi, t)$ can be presented as $S[\phi, t_e]$ if $A$ and $B$ are monoids. Therefore, (iii) implies (i). \qed

It is shown in [14, 15] that the statement (i) in Theorem 3.5 holds and so do all the statements. This property is called the weak HNN property of the class of inverse semigroups; an inverse semigroup $S$ is naturally embedded into $S(\phi, t)$, $S(\phi, t)$ and $S[\phi, t_e]$, respectively. Now it is reasonable to call $S[\phi, t_e]$ an HNN extension of $S$ even though either $A$ or $B$ is not an order ideal.

**Corollary 3.6.** An inverse semigroup $S$ is naturally embedded into $S|\phi, t|$ and $S\|\phi, t\|$, respectively.

**Proof.** We have the following commutative diagram.

$$
\begin{array}{ccc}
S & \xrightarrow{\iota} & S[\phi, t] \\
\downarrow & & \downarrow \\
S(\phi, t) & & S(\phi, t)
\end{array}
$$

Since the class of inverse semigroups satisfies the weak HNN property, $S$ is naturally embedded into $S(\phi, t)$. Therefore, $S$ is naturally embedded into $S|\phi, t|$. Similarly we can show that $S$ is naturally embedded into $S\|\phi, t\|$. \qed

3.2. Non-isomorphic examples

It is evident that the presentations (2.1)–(2.5) give an isomorphic group if we consider the corresponding group presentation in the class of groups. However, in the class of inverse semigroups, these presentations give non-isomorphic inverse semigroups in general. We here exemplify such non-isomorphic semigroups that are presented as variants of HNN extensions.
We now consider (2.1) and (2.3)–(2.5) for the trivial semigroup. Let $S$ be the trivial semigroup $\{e\}$. Then $S$ has the inverse semigroup presentation $\text{Inv}(e \mid e^2 = e)$. Let $A = B = S$ and $\phi$ be the trivial mapping $A \to B$. We put $S_1 = S[\phi, t] = \text{Inv}(e, t \mid e^2 = e, et = te)$, $S_2 = S[\phi, t \parallel] = \text{Inv}(e, t \mid e^2 = e, t^{-1}et = e)$, $S_3 = S(\phi, t) = \text{Inv}(e, t \mid e^2 = e, t^{-1}et = e, t^{-1}t = tt^{-1} = e)$. It is easy to see that we have a sequence of the natural homomorphisms $S_1 \to S_2 \to S_3 \to S_4$, where $s \mapsto s$ for $s \in S$ and $t \mapsto t$. For example, since $S_2$ satisfies $et = ett^{-1} = tt^{-1}et = te$, $S_2$ is a homomorphic image of $S_1$. Note that $et = te$ holds in all of $S_1$, $S_2$, $S_3$, $S_4$. Since every relation of $S_1$ ($S_2$, $S_3$ resp.) has the letter $e$ in both left and right hand side, there are no non-trivial relations among the words in $\{t, t^{-1}\}^+$. Therefore, the inverse subsemigroup $(t)$ generated by $t$ is free.

We concisely describe each of them. The inverse semigroup $S_1$ is a disjoint union of two subsemigroups $(t)$ and $(te) \cup \{e\}$. The inverse subsemigroup $(t)$ is free of rank one and $(te) \cup \{e\}$ is a free inverse monoid of rank one. Note that both left and right side of each defining relation of $S_1$ has the same numbers of the letter $t$. It follows that the word $tt^{-1}e$ and $e$ cannot represent the same element in $S_1$. Similarly $t^{-1}te$ and $e$ cannot represent the same element in $S_1$. It follows that $S_1$ contains three maximal idempotents $tt^{-1}, t^{-1}t, e$.

The inverse semigroup $S_2$ is a disjoint union of two subsemigroups $(t)$ and $(te)$. Clearly $(t)$ is a free inverse semigroup of rank one. On the other hand, every element of $(te)$ is uniquely expressed as $t^n t^{-m} e$, where $n$ and $m$ are non-negative integers. The multiplication of these elements is given by $(t^n t^{-m} e)(t^p t^{-q} e) = t^{n+l-m-q+l-p} e$, where $l = \max(m, p)$. This implies that $S_2$ is a strong semilattice of $(t)$ and $(te)$, where $(te)$ is the bicyclic monoid. The order mapping of $(t)$ onto $(te)$ is given by $w \mapsto w e$ for $w$ in $(t)$. We note that $S_2$ contains two maximal idempotents $tt^{-1}, t^{-1}t$. Note that $e$ is the image of $t^{-1}t$ because $t^{-1}te = e$.

The inverse semigroup $S_3$ is a disjoint union of two subsemigroups $(t)$ and $(te)$. Clearly $(t)$ is a free inverse semigroup of rank one. On the other hand, every element of $(te)$ is uniquely expressed as $t^n e$, where $n$ is an integer. The multiplication is given by $(t^n e)(t^p e) = t^{n+p} e$. This implies that $S_3$ is a strong semilattice of $(t)$ and $(te)$, where $(te)$ is the free group of rank one. The order mapping of $(t)$ onto $(te)$ is given by $w \mapsto w e$ for $w$ in $(t)$. We note that $S_3$ contains two maximal idempotents $tt^{-1}, t^{-1}t$.

The inverse semigroup $S_4$ can be presented as $\text{Inv}(t \mid tt^{-1} = t^{-1}t)$ and so $S_4$ is the free group of rank one. We also remark that (2.2) is $S_4$ because of Theorem 3.2. By the examples above, $S(\phi, t), S(\phi, t), S[\phi, t]$ and $S[\phi, t \parallel]$ give non-isomorphic inverse semigroups for $S = \{e\}$. These presentations make sense in the class of groups and all of them give the free group of rank one.

In the example above, $S[\phi, t \parallel]$ is a homomorphic image of $S[\phi, t]$. This is accidental as we shall see in the next examples. Let $F = \langle g \rangle$ be a free group of rank one. Let $A = B = F$ and $\phi$ be the identity mapping of $F$. We define $S_5 = F[\phi, t] = \text{Inv}(g, t \mid gg^{-1} = g^{-1}g, gt = tg)$ and $S_6 = F[\phi, t \parallel] = \text{Inv}(g, t \mid gg^{-1} = g^{-1}g, t^{-1} gt = g)$. We can show $gt \neq tg$ in $S_6$ using the Schützenberger graphs described in the next section although we omit a proof.

Therefore, $S[\phi, t \parallel]$ is not always a homomorphic image of $S[\phi, t]$. We summarize the relationships among the variants of HNN extensions of inverse semigroups in Fig. 1.

4. Schützenberger graphs and automata

We briefly review Schützenberger graphs and approximate automata introduced by Stephen [13]. Then we look into properties of Schützenberger graphs and approximate automata of HNN extensions.
4.1. Approximate automata

Let $S$ be an inverse semigroup presented by $\text{Inv}(X \mid R)$. The Schützenberger graph $S \Gamma(X, R, u)$ (for short $S \Gamma(S, u)$ if the presentation is understood) for the word $u$ over $X \cup X^{-1}$ is a graph consisting of $\{s \mid s \in S \, sRus\}$ as the sets of vertices and $\{(s_1, a, s_2) \mid s_1a = s_2, s_1 \, R \, s_2 \, R \, u, s_1, s_2 \in S, a \in X \cup X^{-1}\}$ as the set of edges, where $\mathcal{R}$ is the Green’s $\mathcal{R}$-relation. The initial and terminal vertex of the edge $(s_1, a, s_2)$ are given by $\mathbf{d}(s_1, a, s_2) = s_1$ and $\mathbf{r}(s_1, a, s_2) = s_2$, respectively. Each edge $(s_1, a, s_2)$ is considered labeled by the letter “$a$”. It is often illustrated as $(s_1) \xrightarrow{a} s_2$. Whenever we have an edge $(s_1) \xrightarrow{a} (s_2)$, then we also have $(s_1) \xrightarrow{a^{-1}} (s_2)$. Even in the case that $a = a^{-1}$ in $S$, we distinguish the edges labeled by $a$ and $a^{-1}$.

We say that $A$ is an inverse word automaton with the input alphabet $X \cup X^{-1}$, where $X$ and $X^{-1}$ are disjoint, if the transition is consistent with the involution ($a \rightarrow a^{-1}$ and $a^{-1} \rightarrow a$). We can regard the Schützenberger graph $S \Gamma(X, R, u)$ as an inverse word automaton; the start and final state of $S \Gamma(X, R, u)$ are $wu^{-1}$ and $u$, respectively.

We call $S \Gamma(X, R, u)$ the Schützenberger automaton. Note that $S \Gamma(X, R, u)$ is a deterministic (possibly infinite) automaton. Stephen [13] proves the following.

**Lemma 4.1.** For a word $u$ in $(X \cup X^{-1})^+$, the language accepted by the Schützenberger automaton $S \Gamma(X, R, u)$ consists of the words above $u$ in $S$, that is, $L(S \Gamma(X, R, u)) = \{w \mid u \leq w \text{ in } S\}$.

It follows that two words $u$ and $w$ represent the same element in $S$ if and only if the Schützenberger automata of $u$ and $w$ accept the same languages.

Suppose that we are given an inverse semigroup presentation $\text{Inv}(X \mid R)$. An inverse word automaton $A$, which is not necessarily deterministic, with the input alphabet $X \cup X^{-1}$ is called an approximate of $S \Gamma(X, R, u)$ (or an approximate for $u$) if $u \in L(A) \subseteq L(S \Gamma(X, R, u))$. For example, the linear automaton $B_0(u)$ of the word $s_1s_2 \cdots s_k$, illustrated below, is approximate.

\[
\begin{array}{cccccccc}
q_1 & s_1 & q_2 & s_2 & q_3 & s_3 & q_4 & s_4 & \cdots & s_k & q_{k+1} & \\
\end{array}
\]

Stephen [13] introduces the two operations on inverse word automata, expansions and reductions, to derive a new approximate automaton from another. Suppose that there is a path from the state $q_1$ to $q_2$ labeled by the word $w_2$ in $A$, but there is no path from $q_1$ to $q_2$ labeled by $w_1$, where $w_1 = w_2$ is a defining relation belonging to $R$. Then an expansion $B$ is obtained from $A$ by adding a new path from $q_1$ to $q_2$ labeled by $w_1$. Suppose next that there are two distinct edges sharing the same initial state and labeled by the same letter in $X \cup X^{-1}$ in $A$. Then a reduction $B$ is obtained from $A$ by identifying these two edges. We restate Theorems 5.7 and 5.9 in [13] as follows.

**Lemma 4.2.** Let $S$ be an inverse semigroup presented by $\text{Inv}(X \mid R)$.

1. Suppose that $u$ and $w$ are words in $(X \cup X^{-1})^+$ satisfying $u \leq w$ in $S$. Let $A$ be an approximate automaton for $u$. There exists a finite sequence of approximate automata $A_0, A_1, \ldots, A_n$ such that $A_0 = A$, $w$ is accepted by $A_n$, and each $A_i$ is obtained from $A_{i-1}$ by applying either an expansion or a reduction.

2. If $B$ is obtained from an approximate automaton $A$ for $u$ by applying either an expansion or a reduction then $B$ is also an approximate automaton for $u$.

4.2. Edges labeled by idempotents and stable letters

We shall consider combinatorial and geometrical properties of approximate automata. In particular, we examine properties of edges labeled by the letters representing idempotents and the stable letters in HNN extensions.

Let us make a convention on presentations. In the rest of the paper, when we discuss an HNN extension, $S$ is presented by $\text{Inv}(S \mid T)$, where $S$ is the underlying set of $S$ and $T$ is the multiplication table for $S$. Therefore, each element $s$ in $S$ is represented by the letter “$s$”. This is an extremely redundant presentation, however, this convention makes our argument unambiguous. For example, the left-hand side and right-hand side of the relation $t^{-1}at = \phi(a)$ of (2.3) are a word of length three and of length one, respectively. In the relation $t_t t_f^{-1} = ef$ of (2.2), the left-hand side is a word of length two and the right-hand side is a word of length two. Similarly, in the relation $t_e t_f^{-1} = \phi(e)\phi(f)$ of (2.2), the left-hand side is a word of length two and the right-hand side is a word of length two. This is the reason...
that we adopted the relation \( t_e^{-1} \phi(f) = \phi(e) \phi(f) \) instead of \( t_e^{-1} \phi = \phi(e) \phi \) although they are basically equivalent. We note that the structure of an HNN extension does not depend on the choice of presentation of \( S \).

An edge \( y \) in a graph is called a loop if the initial and terminal vertices coincide, that is, \( d(y) = r(y) \). A loop \( y \) can be illustrated as \((q, y, q)\) where \( y \) is labeled by the letter “\( y \)” and its initial and terminal vertex is \( q \).

**Lemma 4.3.** Let \( u \) be a word in \((X \cup X^{-1})^+\) and \( A \) an approximate automaton of \( S\Gamma(S, u) \). Suppose an edge \( y \) in \( A \) is labeled by a letter \( e \) which represents an idempotent in \( S \) and \( y \) is not a loop, that is, \( d(y) \neq r(y) \). Let \( B \) be the automaton obtained from \( A \) by making \( y \) a loop, that is, identifying \( d(y) \) and \( r(y) \). Then \( B \) is also an approximate automaton.

**Proof.** Suppose \( y \) is illustrated as \((q_1, e, q_2)\). As we see in Fig. 2, we apply an expansion for the relation \( e = e^2 \) to the edge \( y \) and apply two reductions. Note that the relation \( e = e^2 \) belongs to the set of relations, that is, the multiplication table by our assumption. Then we obtain \( B \). By Lemma 4.2(2), \( B \) is an approximate automaton. \( \square \)

One can easily prove the following lemma by applying expansions.

**Lemma 4.4.** Let \( u \) be a word in \((X \cup X^{-1})^+\) and \( A \) an approximate automaton of \( S\Gamma(S, u) \). Suppose an edge \( y \) in \( A \) is labeled by a letter \( s \) \((s \in S)\) and there exist no loops labeled by \( e \) and \( f \) adjacent to \( d(y) \) and \( r(y) \), respectively, where \( e = ss^{-1} \) and \( f = s^{-1}s \). Let \( B \) be the automaton obtained from \( A \) by attaching loops labeled by \( e \) and \( f \) at \( d(y) \) and \( r(y) \), respectively. Then \( B \) is also an approximate automaton.

We now consider approximate automata with respect to the HNN extension \( S[\phi, t_e] \).

**Lemma 4.5.** Let \( u \) be a word in \((S \cup S^{-1} \cup \{e \in E(A) \} \cup \{e^{-1} \in E(A)\})^+ \) and \( M \) an approximate automaton of \( S\Gamma(S[\phi, t_e], u) \).

1. Suppose an edge \( y \) in \( M \) is labeled by \( t_e \), where \( e \in E(A) \), and \( d(y) \) is adjacent to loops labeled by \( e \) and \( f \), where \( f \in E(A) \). Let \( M' \) be the automaton obtained from \( M \) by attaching a new edge \( z \) labeled by \( t_f \) such that \( d(y) = d(z) \) and \( r(y) = r(z) \). Then \( M' \) is also an approximate automaton.

2. Suppose an edge \( y \) in \( M \) is labeled by \( t_e^{-1} \), where \( e \in E(A) \), and \( d(y) \) is adjacent to loops labeled by \( \phi(e) \) and \( \phi(f) \), where \( f \in E(A) \). Let \( M' \) be the automaton obtained from \( M \) by attaching a new edge \( z \) labeled by \( t_f^{-1} \) such that \( d(y) = d(z) \) and \( r(y) = r(z) \). Then \( M' \) is also an approximate automaton.

**Proof.** We prove (1) and the part (2) can be similarly shown. Suppose that the edge \( y \) is illustrated as \((q_1, e, q_2)\). First, we apply an expansion for \( t_e^{-1} t_f = e f \) in (2.2). Then we apply a reduction for the edges labeled by \( t_e \). The resulting automaton is \( M' \). By Lemma 4.2(2), \( M' \) is an approximate automaton. See Fig. 3. \( \square \)

One can similarly prove the following lemma.
Lemma 4.6. Let \( u \) be a word in \((S \cup S^{-1} \cup \{t_e \mid e \in E(A)\})^+\) and \( \mathcal{M} \) an approximate automaton of \( S\Gamma(S(\phi, t), \eta(u)) \). Suppose the edges \( y \) and \( z \) in \( \mathcal{M} \) are labeled by \( t_d \) and \( t_e \), respectively. Let \( \mathcal{M}' \) be the automaton obtained from \( \mathcal{M} \) by identifying \( r(y) \) and \( r(z) \). Then \( \mathcal{M}' \) is also an approximate automaton.

5. Simulations of approximate automata

In this section we give the crucial proposition that provides simulations of approximate automata of \( S\Gamma(S(\phi, t), \eta(u)) \) by those of \( S\Gamma(S(\phi, t_e), u) \). First, we define a morphism between automata with distinct input alphabets. Suppose that \( \mathcal{A} \) and \( \mathcal{B} \) are automata with the input alphabets \( X_1 \) and \( X_2 \), respectively. Let \( \kappa \) be a mapping of \( X_1 \) into \( X_2 \). A morphism \( \xi \) of \( \mathcal{A} \) into \( \mathcal{B} \) associated with \( \kappa \) is a graph morphism of \( \mathcal{A} \) into \( \mathcal{B} \) such that \( \xi \) maps an edge \( q_1 \xrightarrow{a} q_2 \) in \( \mathcal{A} \) to an edge \( p_1 \xrightarrow{\kappa(a)} p_2 \) in \( \mathcal{B} \), where \( p_1 = \xi(q_1) \), \( p_2 = \xi(q_2) \), and \( a \) belongs to \( X_1 \). A morphism \( \xi \) of automata \( \mathcal{A} \) into \( \mathcal{B} \) is called locally surjective if for every state \( q \) of \( \mathcal{A} \) and every edge \( e \) outgoing from \( q \) in \( \mathcal{A} \) satisfying \( \xi(q) \in B \), there exists an edge \( z_1 \) outgoing from \( q \) in \( \mathcal{A} \) satisfying \( \xi(z_1) = z \). A locally surjective morphism of \( \mathcal{A} \) into \( \mathcal{B} \) is called a simulation of \( \mathcal{B} \) by \( \mathcal{A} \).

Lemma 5.1. If a morphism \( \xi \) of \( \mathcal{A} \) into \( \mathcal{B} \) is locally surjective and surjective on the set of vertices, every path in \( \mathcal{B} \) can be lifted up to \( \mathcal{A} \) by \( \xi \).

Now we define a morphism of \( S\Gamma(S(\phi, t_e), u) \) into \( S\Gamma(S(\phi, t), \eta(u)) \). Let \( X_1 \) be the set \( S \cup \{t_e \mid e \in E(A)\} \) and \( X_2 \) be the set \( S \cup \{t\} \). The mapping \( \theta_1 \) of \( X_1 \cup X_1^{-1} \) into \( (X_1 \cup X_2)^{+} \) is defined by

\[
\theta_1(x) = \begin{cases} 
  x & \text{if } x \in S \cup S^{-1}, \\
  et_\phi(e) & \text{if } x = t_e(e \in E(A)), \\
  \phi(e)t_e^{-1}e & \text{if } x = t_e^{-1}(e \in E(A)).
\end{cases} \tag{5.1}
\]

The mapping \( \theta_2 \) of \( X_1 \cup X_1^{-1} \) into \( (X_2 \cup X_2^{-1})^+ \) is defined by

\[
\theta_2(x) = \begin{cases} 
  x & \text{if } x \in S \cup S^{-1}, \\
  et_\phi(e) & \text{if } x = t_e(e \in E(A)), \\
  \phi(e)t_e^{-1}e & \text{if } x = t_e^{-1}(e \in E(A)).
\end{cases} \tag{5.2}
\]

The mapping \( \tau \) of \( X_1 \cup X_1^{-1} \) into \( X_2 \cup X_2^{-1} \) is defined by

\[
\tau(x) = \begin{cases} 
  x & \text{if } x \in S \cup S^{-1}, \\
  t & \text{if } x = t_e(e \in E(A)), \\
  t^{-1} & \text{if } x = t_e^{-1}(e \in E(A)).
\end{cases} \tag{5.3}
\]

The mapping \( \theta_1 \) and \( \theta_2 \) can be naturally extended to the homomorphisms of the free semigroup \((X_1 \cup X_1^{-1})^+\) into \((X_1 \cup X_2)^+\) and \((X_2 \cup X_2^{-1})^+\), respectively. We note that for every word \( w \in (X_1 \cup X_1^{-1})^+ \) the word \( \theta_1(w) \) represents \( w \) in \( S(\phi, t_e) \) and the word \( \theta_2(w) \) represents \( \eta(w) \) in \( S(\phi, t) \). Recall that \( \eta \) is given by (3.1).

Proposition 5.2. Let \( u \) be a word in \((X_1 \cup X_1^{-1})^+\). Suppose that \( C_0 \) is the linear automaton of \( \theta_2(u) \), namely, \( C_0 = B_0(\theta_2(u)) \) and \( C_i \) is an approximate automaton of \( S\Gamma(S(\phi, t), \theta_2(u)) \) obtained from \( C_{i-1} \) \((i = 1, 2, 3, \ldots)\) by applying either an expansion or a reduction. Then there exist automata \( M_i \) and \( N_i \) and the morphisms \( \rho_i \) and \( \lambda_i \), where \( M_i \) and \( N_i \) are approximate automata of \( S\Gamma(S(\phi, t_e), u) \) and \( S\Gamma(S(\phi, t), \theta_2(u)) \), respectively, \( \rho_i : C_i \rightarrow N_i \).
is associated with the identity mapping and $\lambda_i : M_i \rightarrow N_i$ is associated with $\tau$ satisfying the following properties for every $i = 1, 2, 3, \ldots$.

(P1) $\lambda_i$ is locally surjective and bijective on the sets of vertices.

(P2) Every edge $z$ labeled by $t_e$ ($e \in E(A)$) in $M_i$ is adjacent to loops labeled by $e$ and $\phi(e)$ at $d(z)$ and $r(z)$, respectively.

\[
e \xrightarrow{t_e} (d(z)) \quad \xrightarrow{\phi(e)} (r(z)). \quad (5.4)
\]

(P3) Every edge $x$ labeled by $t$ in $N_i$ is adjacent to loops labeled by $e$ and $\phi(e)$ at $d(x)$ and $r(x)$, respectively, for some idempotent $e \in E(A)$.

\[
e \xrightarrow{t} (d(x)) \quad \xrightarrow{\phi(e)} (r(x)). \quad (5.5)
\]

**Proof.** We remark that (P2) implies that every edge $z$ labeled by $t_e^{-1}$ in $M_i$ is adjacent to loops labeled by $\phi(e)$ and $e$ at $d(z)$ and $r(z)$ and the similar is true for (P3).

First, we construct automata $M_0$ and $N_0$. Let $M_0$ be the automaton obtained from the linear automaton $B_0(\theta_1(u))$ by making every edge labeled by an idempotent letter, that is to say, a letter representing an idempotent in $S$, a loop. By Lemma 4.3, $M_0$ is an approximate automaton of $S\Gamma(S[\phi, t_e], \theta_1(u))$ and so $L(M_0) \subseteq L(S\Gamma(S[\phi, t_e], \theta_1(u)))$. Recall that $\theta_1(u)$ is the word obtained from $u$ by substituting $e \phi(e)$ ($\phi(e) t_e^{-1} e$ resp.) for every occurrence of the letter $t_e (t_e^{-1})$ ($e \in E(A)$). Because $u$ represents $\theta_1(u)$ in $S[\phi, t_e]$, we have $L(S\Gamma(S[\phi, t_e], u)) = L(S\Gamma(S[\phi, t_e], \theta_1(u)))$ by Lemma 4.1. Therefore, $L(M_0) \subseteq L(S\Gamma(S[\phi, t_e], u))$. On the other hand, it is easy to see that $M_0$ accepts the word $u$. It follows that $M_0$ is an approximate automaton of $S\Gamma(S[\phi, t_e], u)$. Since we replaced every $t_e$ in $u$ by $e t_e \phi(e)$ in $M_0$, every edge $y$ labeled by $t_e$ satisfies (5.4).

Let $N_0$ be the automaton obtained from the linear automaton $B_0(\theta_2(u))$ by making every edge labeled by an idempotent letter a loop. By Lemma 4.3, $N_0$ is an approximate automaton of $S\Gamma(S[\phi, t], \theta_2(u))$. Since we replaced every $t_e$ in $u$ by $e t_e \phi(e)$ in $N_0$, every edge $y$ labeled by $t_e$ satisfies (5.5).

Recall that we are assuming $C_0 = B_0(\theta_2(u))$. By the construction, $N_0$ is obtained from $C_0$ by making each edge labeled by an idempotent a loop. So $N_0$ is a quotient of $C_0$. Then $\rho_0$ is defined to be the naturally induced morphism of $C_0$ onto $N_0$.

Since $\theta_2(u)$ is obtained from $\theta_1(u)$ by replacing every $t_e$ ($e \in E(A)$) by $t$, the trivial graph map that replaces the label $t_e$ by $t$ is an automaton morphism of $B_0(\theta_1(u))$ onto $B_0(\theta_2(u))$. Note that $M_0$ and $N_0$ are quotients of $B_0(\theta_1(u))$ and $B_0(\theta_2(u))$, respectively; both are obtained by making an edge labeled by an idempotent letter a loop. We define $\lambda_0$ to be the morphism of $M_0$ onto $N_0$ induced from the trivial morphism of $B_0(\theta_1(u))$ onto $B_0(\theta_2(u))$ that replaces $t_e$ by $t$ for every $e \in E(A)$. See Fig. 4. The trivial morphism of $B_0(\theta_1(u))$ onto $B_0(\theta_2(u))$ is locally surjective and bijective on the set of vertices and so is $\lambda_0$. As a matter of fact, $\lambda_0$ is a graph isomorphism of $M_0$ onto $N_0$ that replaces the label $t_e$ by $t$ for every $e \in E(A)$. It follows that $M_0, N_0, \rho_0, \lambda_0$ satisfy all the properties (P1), (P2) and (P3).

Next, we inductively construct approximate automata $M_i$ and $N_i$ and morphisms $\rho_i$ and $\lambda_i$ ($i \geq 1$). The automaton $C_i$ is obtained from $C_{i-1}$ applying either an expansion or a reduction. The automata $M_i$ and $N_i$ are defined according to the operation applied to $C_{i-1}$ when $C_i$ is obtained. The possible operations applied to $C_{i-1}$ are classified in the following five cases.

- **Case 1** Expansion related to $t^{-1} a t = \phi(a)$ ($a \in A$).
- **Case 2** Expansion related to $t b t^{-1} = \phi^{-1}(b)$ ($b \in B$).
- **Case 3** Reduction for edges labeled by $t$ or $t^{-1}$.
- **Case 4** Expansion related to the relations for $S$.

![Fig. 4. Morphism $\lambda_0$.](image-url)
Lemma 5.1 and 4.4

\[
\phi(e) = \text{there is a path labeled by } t_{r}.
\]

We put \( x = \phi(d) \), \( y = \phi(f) \), and denote \( x, y \) and \( x, y \) by \( x, y \) and \( x, y \), respectively, in \( C_{t-1} \). We put \( q_1 = d(y_1), q_3 = r(y_1), q_4 = r(y_2), q_2 = r(y_3) \). Now we apply the expansion to \( C_{t-1} \); we attach a new edge labeled by \( \phi(a) \) starting from \( q_1 \) and ending at \( q_2 \). See Fig. 5.

We put \( x_1 = \rho_{i-1}(y_1), x_2 = \rho_{i-1}(y_2), x_3 = \rho_{i-1}(y_3), \) and \( p_1 = \rho_{i-1}(q_1), p_2 = \rho_{i-1}(q_2), p_3 = \rho_{i-1}(q_3), p_4 = \rho_{i-1}(q_4) \). Then the path \( x_1, x_2, x_3 \) in \( N_{i-1} \) starts from \( p_1 \) and ends at \( p_2 \), and is labeled by \( t_{r}, a, t \). Since \( \lambda_{i-1} \) is bijective on the set of vertices by (P1), there exists a vertex \( r_1 \) in \( M_{i-1} \) such that \( \lambda_{i-1}(r_1) = p_1 \). By Lemma 5.1, there exists a path \( z_1, z_2, z_3 \) starting at \( r_1 \) such that \( \lambda_{i-1}(z_1) = x_1, \lambda_{i-1}(z_2) = x_2, \lambda_{i-1}(z_3) = x_3 \) in \( M_{i-1} \). Put \( x_4 = r(z_1), x_5 = r(z_2), \) and \( x_6 = r(z_3) \). Then we have \( \lambda_{i-1}(r_2) = p_2, \lambda_{i-1}(r_3) = p_3, \) and \( \lambda_{i-1}(r_4) = p_4 \). In addition, the path \( z_1, z_2, z_3 \) is labeled by \( t^{1-a}d^{-1}a, t^{1}f^{-1}a, t^{1}f^{-1}a \) for some \( d \) in \( E(A) \), because \( \lambda_{i-1} \) is associated with \( \tau \). Note that \( d \) and \( f \) are assigned to \( a a^{-1} \) or \( a^{-1}a \). Each edge labeled by a stable letter \( e \) in \( M_{i-1} \) is adjacent to loops labeled by \( e \) and \( \phi(e) \) at its initial and terminal vertices by (P2). Thus, the edge \( z_1 \) is adjacent to a loop \( z_4 \) labeled by \( \phi(d) \) at \( r_1 \) and a loop \( z_5 \) labeled by \( d \) at \( r_3 \). The edge \( z_3 \) is adjacent to a loop \( z_6 \) labeled by \( f \) at \( r_4 \) and a loop \( z_7 \) labeled by \( \phi(f) \) at \( r_2 \). Put \( x_4 = \lambda_{i-1}(z_4), x_5 = \lambda_{i-1}(z_5), x_6 = \lambda_{i-1}(z_6), x_7 = \lambda_{i-1}(z_7) \). Then \( x_4 \) and \( x_5 \) are loops adjacent to \( x_1 \), and \( x_6 \) and \( x_7 \) are loops adjacent to \( x_3 \) in \( N_{i-1} \).

We attach a new edge \( x \) labeled by \( \phi(a) \) starting from \( p_1 \) and ending at \( p_2 \) in \( N_{i-1} \). Furthermore, we attach loops \( x_8, x_9, x_{10}, x_{11} \) labeled by \( \phi(aa^{-1}), a a^{-1}, a^{-1}a, \phi(a^{-1}a) \) at \( p_1, p_3, p_4, p_2 \), respectively. Then we obtain the new automaton \( M_{i} \). By Lemmas 4.2 and 4.4, \( M_{i} \) is an approximate automata of \( \Sigma \Gamma(S(t), \theta_{2}(u)) \). See Fig. 6.

We now attach an edge \( z_{12} \) labeled by \( t_{aa^{-1}}^{-1}a \) starting from \( r_1 \) and ending at \( r_3 \), and an edge \( z_{13} \) labeled by \( t_{a^{-1}a}^{-1}a \) starting from \( r_4 \) and ending at \( r_2 \). Furthermore, we attach loops \( z_{8}, z_{9}, z_{10}, z_{11} \) labeled by \( \phi(aa^{-1}), a a^{-1}, a^{-1}a, \phi(a^{-1}a) \) at \( r_1, r_3, r_4, r_2 \), respectively. Then we attach a new edge \( z \) labeled by \( \phi(a) \) starting from \( r_1 \) and ending at \( r_2 \) to \( M_{i-1} \). Note that the last operation is an expansion related to \( t_{aa^{-1}}^{-1}a t_{a^{-1}a}^{-1}a = \phi(a) \). Then

\[ \text{Fig. 5. Transition from } C_{t-1} \text{ to } C_{t}. \]

\[ \text{Fig. 6. Transition from } N_{i-1} \text{ to } N_{i}. \]
we obtain the new automaton $\mathcal{M}_i$. By Lemmas 4.2, 4.4 and 4.5, $\mathcal{M}_i$ is an approximate automata of $SI'(S[\phi, t_e], u)$. See Fig. 7.

We define $\rho_i$ and $\lambda_i$ to be the extension of $\rho_{i-1}$ and $\lambda_{i-1}$ by $\rho_i(y) = x$ and $\lambda_i(z) = x$, respectively. Obviously, $\rho_i$ is a morphism of $C_i$ into $\tilde{N}_i$ associated with the identity mapping, and $\lambda_i$ is a morphism of $\mathcal{M}_i$ into $\tilde{N}_i$ associated with $\tau$.

No new vertices are attached to either of $\mathcal{M}_{i-1}$ or $\tilde{N}_{i-1}$ and so $\lambda_i$ is bijective on the set of vertices since so is $\lambda_{i-1}$. Seven edges $z, z_8, z_9, z_{10}, z_{11}, z_{12}, z_{13}$ are attached to $\mathcal{M}_{i-1}$ and five edges $x, x_8, x_9, x_{10}, x_{11}$ are attached to $\tilde{N}_{i-1}$. Since $\lambda_{i-1}$ is locally surjective and bijective on the set of vertices, it is sufficient to check the surjectivity of the edges outgoing from each of $r_1, r_2, r_3, r_4$ in order to show $\lambda_i$ is locally surjective. Clearly, the local surjectivity holds at $r_1, r_2, r_3, r_4$. Therefore, (P1), (P2) and (P3) are satisfied for $i$. The proof of (Case 2) is similar and so we omit it.

(Case 3) Suppose that we apply a reduction related to $t$ or $t^{-1}$ to $C_{i-1}$. Here we consider the case of $t$. There are two distinct edges $y_1$ and $y_2$ labeled by $r$ sharing the same initial vertex in $C_{i-1}$, and we apply the reduction on these edges to obtain $C_i$. Put $q = d(y_1) = d(y_2)$, $q_1 = r(y_1)$ and $q_2 = r(y_2)$. The edges $y_1$ and $y_2$ are deleted and a new edge $y$ labeled by $t$ is created so that we can have $q = d(y)$ and $q_1 = r(y)$, where $q_1$ and $q_2$ are identified. The resulting automaton is $C_i$. See Fig. 8.

We put $x_1 = \rho_{i-1}(y_1), x_2 = \rho_{i-1}(y_2), p = \rho_{i-1}(q), p_1 = \rho_{i-1}(q_1)$ and $p_2 = \rho_{i-1}(q_2)$. If $x_1 = x_2$, we do not apply any operations to either $\mathcal{M}_{i-1}$ or $\tilde{N}_{i-1}$ and put $\mathcal{M}_i = \mathcal{M}_{i-1}$ and $\tilde{N}_i = \tilde{N}_{i-1}$. We define $\rho_i$ and $\lambda_i$ to be $\rho_{i-1}$ and $\lambda_{i-1}$, respectively. Obviously, (P1), (P2) and (P3) are satisfied for $i$.

We now suppose $x_1 \neq x_2$. Since $\lambda_{i-1}$ is bijective on the set of vertices, there exists the vertex $r$ in $\mathcal{M}_{i-1}$ such that $\lambda_{i-1}(r) = p$. In addition, since $\lambda_{i-1}$ is locally surjective, there are $z_1$ and $z_2$ in $\mathcal{M}_{i-1}$ such that $d(z_1) = d(z_2) = r$, $\lambda_{i-1}(z_1) = x_1$ and $\lambda_{i-1}(z_2) = x_2$ by Lemma 5.1. We note that $z_1 \neq z_2$ because $x_1 \neq x_2$. We put $r_1 = r(z_1)$ and $r_2 = r(z_2)$.

We apply a reduction to $x_1$ and $x_2$ in $\tilde{N}_{i-1}$; the edges $x_1$ and $x_2$ are deleted and a new edge $x$ labeled by $t$ is created so that we can have $p = d(x)$ and $p_1 = r(y)$, where $p_1$ and $p_2$ are identified. The resulting automaton is $C_i$.

The edges $z_1$ and $z_2$ are labeled by stable letters (say $t_d$ and $t_e$ for some $d, e$ in $E(A)$). There are loops $z_3$ and $z_4$ at $r$ labeled by $d$ and $e$, respectively, by (P2). We put $x_3 = \lambda_{i-1}(z_3)$ and $x_4 = \lambda_{i-1}(z_4)$. We identify $r_1$ and $r_2$ and obtain $\mathcal{M}_i$. See Fig. 9. The morphism $\lambda_i$ is defined to be the extension of $\lambda_{i-1}$ by $\lambda_i(z_1) = \lambda_i(z_2) = x$. We should note that even though $e = d$, there exists two edges labeled by $e (= d)$ starting from $r$ and ending at $r_1$ in $\mathcal{M}_i$.

It is clear that $\lambda_i$ is locally surjective and bijective on the set of vertices. Furthermore, $\mathcal{M}_i$ and $\tilde{N}_i$ are approximate automata of $SI'(S[\phi, t_e], u)$ and $SI'(S[\phi, t], \theta_2(u))$, respectively, by Lemmas 4.2 and 4.6. Therefore, (P1), (P2), and (P3) are satisfied for $i$.

(Case 4, 5) Suppose that we apply an expansion related to a relation $w_1 = w_2$ in $S$ to $C_{i-1}$. We assume that there exists a path labeled by $w_2$ but no path labeled by $w_1$ in $C_{i-1}$, and attach a new path labeled by $w_1$ to obtain $C_i$.
If the same expansion is applicable to $N_{i-1}$ at the corresponding position via $\rho_{i-1}$, we apply it; we attach a new path labeled by $w_1$ to obtain $N_i$. In such a case, we also apply the same expansion to $M_{i-1}$ at the corresponding position via $\lambda_{i-1}$ as well. We note that if the expansion is applicable to $N_{i-1}$, then so is to $M_{i-1}$, that is, there does not exist a path labeled by $w_1$.

The morphisms $\rho_i$ and $\lambda_i$ are defined accordingly; the new paths in $C_i$ and $M_i$ are mapped onto the new path in $N_i$ by $\rho_i$ and $\lambda_i$. Clearly the morphism $\lambda_i$ is locally surjective and bijective on the sets of the vertices. Furthermore, $M_i$ and $N_i$ are approximate automata of $S\Gamma(S[\phi, t_e], u)$ and $S\Gamma(S[\phi, t], \theta_2(u))$, respectively, by Lemma 4.2(2). Therefore, (P1), (P2) and (P3) are satisfied for $i$.

If the expansion is inapplicable to $N_{i-1}$, then there exists a path labeled by $w_1$ at the corresponding position via $\rho_{i-1}$ in $N_{i-1}$. By Lemma 5.1, there also exists a path labeled by $w_1$ in $M_{i-1}$ at the corresponding position, and so, the expansion is inapplicable to $M_{i-1}$. Then we define $\lambda_i$ to be $\lambda_{i-1}$, and $N_i$ and $M_i$ to be $N_{i-1}$ and $M_{i-1}$, respectively. On the other hand, to obtain $\rho_i$ we extend $\rho_{i-1}$ by mapping the new path attached to $C_{i-1}$ to the path labeled by $w_1$ that already exists in $N_i$. Obviously, $M_i$ and $N_i$ are approximate automata of $S\Gamma(S[\phi, t_e], u)$ and $S\Gamma(S[\phi, t], \theta_2(u))$, respectively, and (P1), (P2) and (P3) are clearly satisfied for $i$.

Suppose next that we apply a reduction related to the alphabet in $S$ to $C_{i-1}$. We apply the same reduction to $N_{i-1}$ at the corresponding position via $\rho_{i-1}$ if the reduction is applicable. In such a case, we apply the same reduction to $M_{i-1}$ at the corresponding position via $\lambda_{i-1}$ as well. The morphisms $\rho_i$ and $\lambda_i$ are defined accordingly.

Clearly the morphism $\lambda_i$ is locally surjective and bijective on the sets of the vertices. Furthermore, $M_i$ and $N_i$ are approximate automata of $S\Gamma(S[\phi, t_e], u)$ and $S\Gamma(S[\phi, t], \theta_2(u))$, respectively, by Lemmas 4.5 and 4.6. Therefore, (P1), (P2), and (P3) are satisfied for $i$. $\square$

6. Relationship between $S[\phi, t]$ and $S[\phi, t_e]$

We shall show that the homomorphism $\eta$ given by Lemma 3.1 is an embedding of $S[\phi, t_e]$ into $S[\phi, t]$ using Proposition 5.2. In addition, we shall give a necessary and sufficient condition for $E(S)$ to coincide with $E(S[\phi, t_e])$.

**Theorem 6.1.** The homomorphism $\eta$ is an embedding of $S[\phi, t_e]$ into $S[\phi, t]$. The semigroup $S[\phi, t_e]$ is isomorphic to an inverse subsemigroup of $S[\phi, t]$ generated by $S$ and the set $\{ e \in E(A) \}$.

**Proof.** Suppose that $u$ and $w$ are words in $(X_1 \cup X_1^{-1})^+$, where $X_1 = S \cup \{ t_e \mid e \in E(A) \}$, and $\eta(u) = \eta(w)$ in $S[\phi, t]$. We shall show that $w$ is accepted by a certain approximate automaton of $S\Gamma(S[\phi, t_e], u)$ and $S\Gamma(S[\phi, t], \theta_2(u))$ and vice versa. The words $\theta_2(u)$ and $\theta_2(w)$ represent the same element in $S[\phi, t]$ because $\eta(u) = \eta(w)$. Recall that $\theta_2$ is defined by (5.2). Therefore, $\theta_2(u) \leq \theta_2(w)$ in $S[\phi, t]$. By Lemma 4.2(1), there exists a sequence of approximate automata $C_i$ ($i = 0, 1, 2, \ldots, n$) for $\theta_2(u)$ such that $C_0$ is the linear automaton of $\theta_2(u)$, $C_n$ accepts $\theta_2(u)$ and each $C_i$ is obtained from $C_{i-1}$ by either an expansion or a reduction.

By Proposition 5.2, we have approximate automata $N_i$ of $S\Gamma(S[\phi, t], \theta_2(u))$ and $M_i$ of $S\Gamma(S[\phi, t_e], u)$ and the morphisms $\rho_i$ and $\lambda_i$ satisfying the properties (P1), (P2), and (P3) for $i = 0, 1, 2, \ldots, n$. Since the morphism $\rho_n : C_n \to N_n$ is associated with the identity mapping, we have $L(C_n) \subset L(N_n)$. Hence, $\theta_2(u)$ is also accepted by
By Lemma 5.1, every path in $N_\tau$ can be lifted up to $\mathcal{M}_n$. Hence, there exists a word $w_1$ in $\tau^{-1}(\theta_2(w))$ that is accepted by $\mathcal{M}_n$ since $\lambda_\eta$ is associated with $\tau$. Recall that $\tau$ is the mapping defined in (5.3).

Suppose that the word $\theta_1(w)$ is of the form

$$p_1l_1p_2l_2p_3\cdots l_{k-1}p_kl_kp_{k+1},$$

where $l_i = e_it_{d_i}\phi(e_i)$ or $l_i = \phi(e_i)t^{-1}_{d_i}e_i$ for some $e_i \in E(A)$ ($i = 1, 2, \ldots, k$) and $p_i \in (S \cup S^{-1})^\ast$ ($i = 1, 2, \ldots, k, k + 1$). Recall that $\theta_1$ is defined by (5.1). Then the word $w_1$ is of the form

$$p_1l'_1p_2l'_2p_3\cdots l'_{k-1}p_kl'_kp_{k+1},$$

and

$$l'_i = \begin{cases} e_it_{d_i}\phi(e_i) & \text{if } l_i = e_it_{d_i}\phi(e_i) \\ \phi(e_i)t^{-1}_{d_i}e_i & \text{if } l_i = \phi(e_i)t^{-1}_{d_i}e_i \end{cases}$$

where $d_i \in E(A)$ for every $i = 1, 2, \ldots, k$. We should note that $d_i$ may have nothing to do with $e_i$.

Since $w_1$ is accepted by $\mathcal{M}_n$, there exists a path labeled by the word (6.2) starting from the start state and ending at the final state of $\mathcal{M}_n$. We look into the local structure of $\mathcal{M}_n$ along the path labeled by $w_1$. Let us consider the subword $l'_1$ given in (6.3). By the property (P2), the edge labeled by $t_{d_i}$ (or $t^{-1}_{d_i}$) in $l'_1$, let us say $y$, is adjacent to loops labeled by $d_i$ and $\phi(d_i)$ (res. $\phi(d_i)$ and $d_i$) at $d(y)$ and $r(y)$. We put $q_1 = d(y)$ and $r_1 = r(y)$. We induce a new approximate automaton by making the edges labeled by $e_i$ and $\phi(e_i)$ loops by Lemma 4.3. Furthermore, we add a new edge labeled by $r_{d_i}$ whose initial and terminal vertices are $q_1$ and $r_1$, then the new automaton is also approximate of $\mathcal{M}_n$. In Fig. 10, we illustrate the local structure of $\mathcal{M}_n$ along the path labeled by $l'_1$ and operations on it in the case of $l'_1 = e_1t_{d_1}\phi(e_1)$.

Iteratively constructing approximate automata of $S\Gamma(S[\phi, t_e], u)$, we obtain an approximate automaton accepting the word $p_1l_1p_2l_2p_3\cdots l_{k-1}p_kl_kp_{k+1}$. See Fig. 11 for an example of a path in an approximate automaton where $l_1 = e_1t_{e_1}\phi(e_1), l_2 = e_2t_{e_2}\phi(e_2)$, and $l_k = e_kt_{e_k}\phi(e_k)$.

By (6.1), we have $\theta_1(w) = p_1l_1p_2l_2p_3\cdots l_{k-1}p_kl_kp_{k+1}$, and hence, $u \leq \theta_1(w)$. On the other hand, we have $\theta_1(w) = w$ in $S[\phi, t_e]$. Thus, $u \leq w$ in $S[\phi, t_e]$. Similarly, we can show $w \leq u$ in $S[\phi, t_e]$. Therefore, $u = w$ in $S[\phi, t_e]$ and $\eta$ is an embedding. Clearly, $S[\phi, t_e]$ is isomorphic to the inverse subsemigroup generated by $S$ and $et$ under $\eta$. □

We have shown that the class of inverse semigroups has the weak HNN property; $S$ is naturally embedded into $S(\phi, t), S(\phi, t)$ and $S[\phi, t_e]$, respectively. It is also proved in [14,15] that the class of inverse semigroups has the strong HNN property, that is, for every $S$ and its HNN extension $S(\phi, t)$ we have

$$t^{-1}St \cap S = B \quad \text{and} \quad tSt^{-1} \cap S = A,$$

in $S(\phi, t)$. We next obtain similar results for $S(\phi, t)$ and $S[\phi, t_e]$. Recall that $S(\phi, t)$ is isomorphic to $S(\phi, t)$ by Lemma 3.4. In the following theorem, we regard $\overline{S(\phi, t)}$ as $S(\phi, t)$. By (3.2), the inverse semigroup $S$ is generated

\begin{align*}
\overline{S(\phi, t)} & = B \quad \text{and} \quad \overline{tSt^{-1} \cap S} = A,
\end{align*}

in $S(\phi, t)$. We next obtain similar results for $S(\phi, t)$ and $S[\phi, t_e]$. Recall that $S(\phi, t)$ is isomorphic to $S(\phi, t)$ by Lemma 3.4. In the following theorem, we regard $\overline{S(\phi, t)}$ as $S(\phi, t)$. By (3.2), the inverse semigroup $S$ is generated
by $S$ and $1_A$ and $1_B$. Since $tt^{-1} = 1_A$ and $t^{-1}t = 1_B$ in $S(\phi, t)$, we denote the inverse subsemigroup generated by $S \cup \{t^{-1}t, tt^{-1}\}$ in $S(\phi, t)$ by $\bar{S}$.

**Theorem 6.2.** We have $t^{-1}\bar{S}t \cap S = B \cup \{1_B\}$, $t\bar{S}t^{-1} \cap S = A \cup \{1_A\}$, $t^{-1}St \cap S = B$, and $tSt^{-1} \cap A = A \in S(\phi, t)$.

**Proof.** Since $\phi$ is an isomorphism of $A \cup \{1_A\}$ onto $B \cup \{1_B\}$, we have $t^{-1}\bar{S}t \cap S = B \cup \{1_B\}$ and $t\bar{S}t^{-1} \cap S = A \cup \{1_A\}$ by (6.4). For every $b \in B$, we have $b = t^{-1}\phi^{-1}(b)t$. Therefore, $B \subseteq t^{-1}St \cap S$. On the other hand, take an arbitrary element $s \in t^{-1}St \cap S$. Then $s \in t^{-1}\bar{S}t \cap S$. Hence, $s \in B \cup \{1_B\}$. Since $1_B \notin S \in \bar{S}$ by Lemma 3.3 and $s \in S$, we have $s \in B$. Hence, we have $t^{-1}St \cap S \subseteq B$. Similarly we can show that $tSt^{-1} \cap S = A$.

**Corollary 6.3.** Suppose $e, f \in E(A)$. We have $t_e^{-1}St_f \cap S = \phi(e)B\phi(f)$ and $t_eSt_f^{-1} \cap S = eAf$ in $S(\phi, t_e)$.

**Proof.** First, we note that $t_e = et\phi(e) = et \phi(e)$ up to $\eta$ for every $e \in E(A)$. Therefore, $t_e^{-1}St_f = \phi(e)t^{-1}St\phi(f)$. By Theorem 6.2, we have $t_e^{-1}St_f \cap S = \phi(e)\phi(t^{-1}St\phi(f)) \supset \phi(e)t^{-1}St\phi(f) \supset \phi(e)(t^{-1}St \cap S)\phi(f) = \phi(e)B\phi(f)$ in $S(\phi, t)$.

Conversely, we take an arbitrary element $x$ from $t_e^{-1}St_f \cap S$. We note that $t_e^{-1}St_f \cap S = t^{-1}eSt\cap S \subset t^{-1}St \cap S = B$ by Theorem 6.2. Hence, $x$ belongs to $B$. Since $x$ belongs to $\phi(e)t^{-1}St\phi(f)$, $x = \phi(e)x\phi(f) \in \phi(e)B\phi(f)$. It follows that $t_e^{-1}St_f \cap S = \phi(e)B\phi(f)$. Similarly we can show $t_eSt_f^{-1} \cap S = eAf$.

Using the strong HNN embeddability, we can characterize an HNN extension $S[\phi, t_e]$ whose set of idempotents coincides with that of $S$. It is shown in [17] that $E(S) = E(S(\phi, t))$ if and only if the associated subsemigroups $A$ and $B$ are locally full. It is easy to see that $A$ is locally full if and only if $A$ is an order ideal. We show that a similar result holds for $S[\phi, t_e]$.

**Theorem 6.4.** We have $E(S[\phi, t_e]) = E(S)$ if and only if $A$ and $B$ are order ideals of $S$.

**Proof.** If $A$ and $B$ are monoids, then $S[\phi, t_e]$ coincides with $S(\phi, t)$ by Theorem 3.2. Thus, the claim holds by Corollary 3.1 in [17], and so we suppose that $A$ and $B$ are not monoids.

Suppose that $E(S[\phi, t_e]) = E(S)$. We shall show that $E(A)$ and $E(B)$ are order ideals in $E(S)$. Suppose that $e$ is an idempotent in $A$ and $f \leq e$, where $f$ is in $E(S)$. We shall show that $f$ belongs to $E(A)$. We have $t^{-1}ft = t^{-1}efet = \phi(e)t^{-1}efet\phi(e) = t_e^{-1}f t_e \in E(S[\phi, t_e]) = E(S)$. Hence, we have $t^{-1}ft \in E(S)$. On the other hand, we have $t^{-1}ft \in t^{-1}St \cap S = B = \phi(A) = t^{-1}At$ by Theorem 6.2. Thus we have $t^{-1}ft = t^{-1}at$ for some $a \in A$. Then $f = ef e = tt^{-1}ef t t^{-1} = tt^{-1}ft t^{-1} = tt^{-1}at t^{-1} = a$. Hence, $f \in E(A)$ and so $E(A)$ is an order ideal. This implies $A$ is an order ideal. Similarly we can show that $B$ is an order ideal.

If $A$ and $B$ are order ideals, then $S[\phi, t_e]$ is the HNN extension $S_{sA, \phi}$ in the sense of Gilbert, and we have $E(S[\phi, t_e]) = E(S)$ as seen in Corollary 2.5 in [4].

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**References**