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Balanced Butler Groups

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The class of Butler groups, pure subgroups of finite rank completely decomposable groups, has been studied extensively by abelian group theorists in recent years. Classification by numerical invariants up to quasi-isomorphism and even isomorphism has been achieved for special subclasses. Here we highlight a new class in which to extend and expand classification results, the balanced Butler groups or $\mathscr{K}(1)$-groups. These are the pure balanced subgroups of finite rank completely decomposable groups. A strictly decreasing chain of classes of Butler groups, introduced by Kravchenko, is obtained by defining the $\mathscr{K}(n)$-groups $(n \geq 2)$ to be those balanced subgroups of a completely decomposable group for which the quotient is a $\mathscr{K}(n-1)$-group. We establish an internal characterization of $\mathscr{K}(n)$-groups, give a method for constructing examples, and derive decomposition results.
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A *Butler group* is a pure subgroup of a *completely decomposable group*—a finite direct sum of groups isomorphic to subgroups of the additive rationals Q. This class of groups has been widely studied in the past 10 years. The popularity of the class is due largely to the many structural properties of its members. On the other hand, it is well known that the class of Butler groups presents a hopeless classification problem [A-4]. The best results have focused on special subclasses [AV-3, FM, HM]. It is the

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purpose of this work to advertise some classes of Butler groups that seem to hold strong promise for expanding known classification results.

Recall that a sequence $0 \to H \to G \to K \to 0$ of torsion-free groups is called *balanced* if the sequence $0 \to H(\tau) \to G(\tau) \to K(\tau) \to 0$ is exact for each type τ. Here $G(\tau)$ denotes the usual τ-socle of G, that is, the set of elements in G of type τ or greater. It is well known that an exact sequence of torsion-free groups is balanced if and only if rank one groups are projective with respect to the sequence (see [F]). The classes we study are defined in terms of balanced sequences and were introduced originally by Kravchenko [Kr]. Denote the class of Butler groups by $\mathscr{K}(0)$. For $n \geq 1$, $\mathscr{K}(n)$ is the class of groups that appear as the first term in a balanced exact sequence, $0 \to H \to G \to K \to 0$, where $K \in \mathscr{K}(n-1)$ and G is a finite rank completely decomposable group. In this situation, H will be called a $\mathscr{K}(n)$-*group* and the exact sequence a $\mathscr{K}(n)$-*sequence*. Kravchenko used the notation $B(n)$ rather than $\mathscr{K}(n)$. Our feeling is that the letter B is already overworked in the literature on torsion-free groups and that it is more appropriate to use the initial of the author who introduced the groups. Among other results, Kravchenko showed that the classes $\mathscr{K}(n)$ form a strictly descending chain whose intersection is the class of completely decomposable groups. Our first section establishes the basic machinery on $\mathscr{K}(n)$-groups, including the main results of [Kr]. In Section 2, we investigate conditions under which $\mathscr{K}(n)$-groups must be completely decomposable. In Section 3 we give a characterization of when pure rank one subgroups of a completely decomposable group induce a $\mathscr{K}(n)$-sequence. Section 4 starts with some examples and discusses the connection between $\mathscr{K}(n)$-groups and balanced projective resolutions of Butler groups. Theorem 4.4 provides a necessary and sufficient condition, on the critical typeset, for $\mathscr{K}(n)$-sequences to split. Section 5 characterizes the $\mathscr{K}(n)$-groups whose critical typeset is an antichain with $n+2$ elements. We conclude with a discussion of our further research on balanced Butler groups and list some open questions.

Throughout, all groups are abelian and the torsion-free ones always have finite rank. Any unexplained notation or terminology may be found in [A-2] or [AV-1]. The paper [N] contains some of our results on $\mathscr{K}(n)$-groups for $n = 1$.

1. $\mathscr{K}(n)$-GROUPS

The paper [Kr] is basic to our study. However, the English translation is riddled with typos and extremely difficult to read. Thus, we begin with our own simplified treatment. The results from [Kr] are specifically noted, but our proofs are, for the most part, substantially different from those that appear in [Kr].

For a torsion-free group G, denote by $G^{\#}(\tau)$ the pure subgroup generated by the elements of type greater than τ. For τ and σ types, we denote by $\tau \vee \sigma$ and $\tau \wedge \sigma$ the usual sup and inf in the lattice of types.

LEMMA 1.1. *Let $A \subset B$ be pure subgroups of the torsion-free group C.*

(a) *If A is balanced in C, then A is balanced in B.*

(b) *If B is balanced in C, then B/A is balanced in C/A.*

(c) *If A is balanced in C and B/A is balanced in C/A, then B is balanced in C.*

(d) *If A is balanced in B and B is balanced in C, then A is balanced in C.*

Proof. Parts (a), (b), and (c) appear as Exercise 1 on page 115 of [F]. Part (d) may be proved exactly as in the proof of property (e) on page 79 of [F].

LEMMA 1.2 [Kr]. *For a pure exact sequence $0 \to H \to G \to K \to 0$, with G a Butler group, the following conditions are equivalent*:

(a) *H is a balanced subgroup of G;*

(b) *For all $\tau \in \text{typeset}(G)$, the sequence $0 \to H(\tau) \to G(\tau) \to K(\tau) \to 0$ is balanced exact;*

(c) *For all $\tau \in \text{typeset}(G)$, the subgroup $H + G(\tau)$ is pure in G.*

Proof. For (a) implies (b), the sequence $0 \to H(\tau) \to G(\tau) \to K(\tau) \to 0$ is exact for all τ by definition. The balancedness follows from the fact that $G(\tau)(\sigma) = G(\tau \vee \sigma)$.

For (b) implies (c), note that $H + G(\tau)$ is the preimage of $K(\tau)$ in G if (b) holds; and the preimage of a pure subgroup is pure.

For (c) implies (a), let X be a pure rank one subgroup of K and G_0 the preimage of X in G. It suffices to show that the sequence $0 \to H \to G_0 \to X \to 0$ splits. It is easy to check that condition (c) still holds if G is replaced by G_0. Among the pure rank one subgroups A of G_0 with $A \cap H = 0$, choose one so that $\sigma = \text{type}(A)$ is maximal. Then $G_0^{\#}(\sigma) \subseteq H$ by the maximality of σ, and using (c), $H + G_0(\sigma)$ is pure in G_0, hence equal to G_0. But, $G_0^{\#}(\sigma) \subseteq H$ implies that $[H + G_0(\sigma)]/H \simeq G_0(\sigma)/[G_0(\sigma) \cap H] \simeq X$ has type σ. By Baer's Lemma [F, 86.5], H is a summand of $H + G_0(\sigma) = G_0$ and we are done.

LEMMA 1.3 [Kr]. *If $n > 0$, and $0 \to H \to G \to K \to 0$ is a $\mathscr{K}(n)$-sequence, then for all types τ,*

(a) *$H(\tau) \in \mathscr{K}(n)$, and $0 \to H(\tau) \to G(\tau) \to K(\tau) \to 0$ is a $\mathscr{K}(n)$-sequence*

(b) *$H/H(\tau) \in \mathscr{K}(n-1)$ and $0 \to H/H(\tau) \to G/G(\tau) \to K/K(\tau) \to 0$ is a $\mathscr{K}(n-1)$-sequence.*

(c) *For any list of types $\tau_1, \ldots, \tau_n$, $H(\tau_1) + \cdots + H(\tau_n)$ is balanced in H.*[1]

Proof. We will first prove (a) by induction on n. If H is balanced in G, then $H(\tau)$ is balanced in $G(\tau)$ by Lemma 1.2(b). This is the case $n = 1$ for (a). Let $0 \to H \to G \to K \to 0$ be a $\mathscr{K}(n)$-sequence, where $n \geq 2$. Then by Lemma 1.2(b), $0 \to H(\tau) \to G(\tau) \to K(\tau) \to 0$ is balanced exact. Moreover, since $K \in \mathscr{K}(n-1)$ by definition, then $K(\tau) \in \mathscr{K}(n-1)$ by induction. Statement (a) follows.

For (b) we again use induction on n. This time the case $n = 1$ is trivial. For $n \geq 2$, we have $K \in \mathscr{K}(1)$ so that K is a balanced subgroup of a completely decomposable group L. Then by Lemma 1.2(b), $K(\tau)$ is balanced in L whence in K. It follows that the sequence in (b) is balanced exact. Since $K \in \mathscr{K}(n-1)$, by the induction hypothesis $K/K(\tau) \in \mathscr{K}(n-2)$. It follows that the sequence in (b) is a $\mathscr{K}(n-1)$-sequence and (b) is proved.

Finally, (c) is also proved by induction, the case $n = 1$ having already been established in (a). If $n > 1$, then $H(\tau_n)$ is balanced in H and $H/H(\tau_n) \in \mathscr{K}(n-1)$ by (b), so that $\sum_{i=1}^{n-1}(H/H(\tau_n))(\tau_i) = \sum_{i=1}^{n-1}(H(\tau_i) + H(\tau_n))/H(\tau_n)$ is balanced in $H/H(\tau_n)$ by induction. By Lemma 1.1(c), $H(\tau_1) + \cdots + H(\tau_n)$ is balanced in H.

If H is a Butler group, an element τ of typeset(H) is called *join irreducible* if whenever τ is the supremum of a finite subset S of typeset(H), then $\tau \in S$.

LEMMA 1.4. *Let H be a Butler group and τ a join irreducible element of* typeset(H). *Suppose $f\colon H \to Q$ with $fH(\tau) = 0$. Then $\tau \nleq$* type(Im f).

Proof. If $f\colon H \to Q$, then type(Im f) $= \sup\{\text{type } h \mid h \in H \setminus \operatorname{Ker} f\}$ [AV-1]. Suppose also that $fH(\tau) = 0$ and, by way of contradiction, that $\tau \leq$ type(Im f). Then $\tau = \sup\{\tau \wedge \text{type}(h) \mid h \in H \setminus \operatorname{Ker} f\}$. Since τ is join irreducible, we have $\tau \leq \text{type}(h)$ for some $h \in H \setminus \operatorname{Ker}(f)$. This says $h \in H(\tau) \setminus \operatorname{Ker} f$, a contradiction. The result follows.

LEMMA 1.5 [Kr]. *If B is a balanced subgroup of the torsion-free group A, then for all types ρ, σ,*

(a) $(B + A(\rho)) \cap (B + A(\sigma)) = B + A(\rho \vee \sigma)$

(b) $(nA + B) \cap (nA + A(\rho)) = nA + B(\rho)$.

Proof. (a) Use the fact that B is balanced in A to write $(A(\rho \vee \sigma) + B)/B = (A/B)(\rho \vee \sigma) = (A/B)(\rho) \cap (A/B)(\sigma) = [(A(\rho) + B)/B] \cap$

[1] Here the τ_i's can be equal, so $\sum_{i=1}^{k} H(\tau_i)$ is balanced in H for $k \leq n$.

$[(A(\sigma) + B)/B] = [(A(\rho) + B) \cap (A(\sigma) + B)]/B$.

(b) The right hand side is obviously contained in the left. To show the reverse containment, let $na + b = na' + c$ with $a, a' \in A$, $b \in B$, and $c \in A(\rho)$. Since B is balanced in A, then $B + A(\rho)$ is the preimage in A of $[A/B](\rho)$, hence is pure in A. It follows that $a - a' = b' + c'$ with $b' \in B$ and $c' \in A(\rho)$. Then $n(a - a') = -b + c = n(b' + c')$, so $nb' + b = c - nc' \in B \cap A(\rho) = B(\rho)$. Thus, $na + b = (na' + nc') + (nb' + b) \in nA + B(\rho)$.

PROPOSITION 1.6. *Let H be a Butler group and n a positive integer such that $H(\rho_1) + \cdots + H(\rho_n)$ is balanced in H for every choice of types $\rho_1, \ldots, \rho_n$. Suppose $\tau_1, \ldots, \tau_n$ are types, p is a prime, and h is an element of H such that the image of h has p-height 0 in $H/[H(\tau_1) + \cdots + H(\tau_n)]$. Then there are types $\sigma_i \leq \tau_i$ such that each σ_i is join irreducible in typeset(H) and the image of h has p-height 0 in $H/[H(\sigma_1) + \cdots + H(\sigma_n)]$.*

Proof. For any type τ, $H(\tau) = H(\tau')$ for some $\tau' \leq \tau$ with $\tau' \in$ typeset(H). Thus, it suffices to consider the case where each $\tau_i \in$ typeset(H). We proceed by induction on n, beginning with the case $n = 1$. If $\tau = \tau_1$ is join irreducible, then there is nothing to show. Otherwise, write $\tau = \sup\{\rho_1, \ldots, \rho_k\}$ with $\tau > \rho_i$ in typeset(H). Then there is a canonical embedding, $H/H(\tau) \to \oplus_{i=1}^{k} H/H(\rho_i)$, since $H(\tau) = \cap H(\rho_i)$. We will show that this embedding is pure. The element $h + H(\tau)$ maps to $\oplus h + H(\rho_i)$. Suppose m is a positive integer and $h + H(\rho_i) \in mH + H(\rho_i)$ for each i. Then $h \in \bigcap_{i=1}^{k}[mH + H(\rho_i)]$. Because each $H(\rho_i)$ is balanced in H, we may apply Lemma 1.5(b) to write

$$\left[mH + H(\rho_1)\right] \cap \left[mH + H(\rho_2)\right] = mH + H(\rho_1)(\rho_2)$$
$$= mH + H(\rho_1 \vee \rho_2).$$

Then inductively, $\cap[mH + H(\rho_i)] = mH + H(\vee \rho_i) = mH + H(\tau)$. This shows that the embedding $H/H(\tau) \to \oplus_{i=1}^{k} H/H(\rho_i)$ is pure. In particular, $h + H(\rho_i)$ has p-height 0 for some ρ_i. Relabel $\rho_i = \mu_1 < \tau$. If μ_1 is join irreducible, we are done. Otherwise, there is a $\mu_2 < \mu_1$ with $h + H(\mu_2)$ having p-height 0. Since the typeset of H is finite, this process will eventually halt and the proof is complete for $n = 1$.

For $n > 1$, let $h + \sum_{i=1}^{n} H(\tau_i)$ in $H/\sum_{i=1}^{n} H(\tau_i)$ have p-height zero and let $H' = H(\tau_2) + \cdots + H(\tau_n)$. If $\overline{H} = H/H'$, then for each type ρ_1, $\overline{H}(\rho_1)$ is balanced in $\overline{H}$ (Lemma 1.1(b)). Moreover, $\overline{h} + \overline{H}(\tau_1)$ has p-height zero in $\overline{H}/\overline{H}(\tau_1)$, where $\overline{h} = h + H'$. By the $n = 1$ case, there is a join irreducible type $\sigma_1 \leq \tau_1$ such that $\overline{h} + \overline{H}(\sigma_1)$ has p-height zero. It follows that $h + [H(\sigma_1) + H(\tau_2) + \cdots + H(\tau_n)]$ has p-height zero in the corresponding factor group. This procedure may be iterated on $\tau_2, \dots, \tau_n$ to produce join irreducible types $\sigma_i \le \tau_i$ such that $h + H(\sigma_1) + \cdots + H(\sigma_n)$ has p-height zero.

A pure subgroup H of a torsion-free group G is *cobalanced* in G if any map from H into a subgroup of Q can be lifted to G. See [AV-1] for an equivalent definition of cobalanced in the case G is a Butler group.

THEOREM 1.7. *Let $n \ge 1$ and let E: $0 \to H \to G \to K \to 0$ be a cobalanced exact sequence of Butler groups such that G is completely decomposable and $H(\rho_1) + \cdots + H(\rho_n)$ is balanced in H for all types $\rho_1, \dots, \rho_n$. Then E is balanced and, for all types $\tau_1, \dots, \tau_n$, the subgroup $K(\tau_1) + \cdots + K(\tau_n)$ is pure in K.*

Proof. We will show that for each choice of types $\tau_1, \dots, \tau_n$, the subgroup $H + G(\tau_1) + \cdots + G(\tau_n)$ is pure in G. This holding, the sequence E is balanced by Lemma 1.2(c). Then using the balanced property, $K(\tau_1) + \cdots + K(\tau_n)$ is the image of $H + G(\tau_1) + \cdots + G(\tau_n)$, hence is pure in K. Denote $G' = G(\tau_1) + \cdots + G(\tau_n)$ (a summand of G) and assume h is an element of H such that $h + G'$ has p-height zero in $(H + G')/G'$. We will show that $h + G'$ has p-height zero in G/G', from which it follows that $H + G'$ is pure in G as desired. Note that $h + [H(\tau_1) + \cdots + H(\tau_n)]$ must have p-height zero in $H/\sum_{i=1}^n H(\tau_i)$. By Proposition 1.6, there are join irreducible types $\sigma_i \le \tau_i$ such that the coset $h + [\sum_{i=1}^n H(\sigma_i)]$ has p-height zero in the corresponding factor group. Let X be a rank one factor of $H/[\sum_{i=1}^n H(\sigma_i)]$ so that under the composition

$$\varphi\colon H \to H \Big/ \left[\sum_{i=1}^n H(\tau_i) \right] \to H \Big/ \left[\sum_{i=1}^n H(\sigma_i) \right] \to X,$$

$\varphi(h)$ has p-height zero [AV-1, p. 108]. Since H is cobalanced in G, there is a lifting $\psi\colon G \to X$ of φ. By Lemma 1.4, $\sigma_i \not\le \operatorname{type}(X)$ for each i. Then $\sigma_i \le \tau_i$ implies $\tau_i \not\le \operatorname{type}(X)$, so that $\psi(G(\tau_i)) = 0$ for each i. Denoting $H' = \sum_{i=1}^n H(\tau_i)$, we have a commutative diagram

$$\begin{array}{ccc} H & \to & G \\ \downarrow & \iota & \downarrow \\ H/H' & \to & G/G' \\ \bar{\varphi}\downarrow & & \downarrow\bar{\psi} \\ X & = & X \end{array}$$

where $\bar{\varphi}$ and $\bar{\psi}$ are the maps induced by φ and ψ, and ι is induced by the inclusion of H in G. By the commutativity of the diagram $\bar{\psi}\iota(h + H')$ has

p-height zero in X. It follows that $\iota(h+H')=h+G'$ has p-height zero in G/G' as desired.

THEOREM 1.8 [Kr]. *Let H be a Butler group and n a non-negative integer. The following are equivalent*:

(a) $H \in \mathscr{K}(n)$

(b) *For all types $\tau_1, \dots, \tau_n$, the subgroup $H(\tau_1) + \cdots + H(\tau_n)$ is balanced in H.*

(c) *For all types $\tau_1, \dots, \tau_{n+1}$, the subgroup $H(\tau_1) + \cdots + H(\tau_{n+1})$ is pure in H.*

Proof. Note that the theorem is trivial for $n = 0$, so we may assume $n \geq 1$. The implication (a) → (b) is Lemma 1.3(c). To show (b) → (c), let $H' = H(\tau_1) + \cdots + H(\tau_n)$, a balanced subgroup of H. Then $(H/H')(\tau_{n+1}) = [H' + H(\tau_{n+1})]/H'$. This implies that $H' + H(\tau_{n+1})$ is pure in H. The implication (c) → (b) follows from Lemma 1.2: with the same notation, $H' + H(\tau_{n+1})$ pure in H for all τ_{n+1} implies that H' is balanced in H.

To finish our task, we prove that (b) and (c) imply (a) by induction on n, the case $n = 0$ being trivial. Let

$$E\colon 0 \to H \to G \to K \to 0$$

be a cobalanced exact sequence with G completely decomposable (see [AV-1, Theorem 1.4]). By (b), $\sum_{i=1}^{n} H(\tau_i)$ is balanced in H for all $\tau_1, \dots, \tau_n$, so by Theorem 1.7, H is balanced in G and $\sum_{i=1}^{n} K(\tau_i)$ is pure in K. It follows from the induction hypothesis that $K \in \mathscr{K}(n-1)$ and therefore that the sequence E is a $\mathscr{K}(n)$-sequence and $H \in \mathscr{K}(n)$.

COROLLARY 1.9. *Let E: $0 \to H \to G \to K \to 0$ be a cobalanced exact sequence of Butler groups such that G is completely decomposable and $H \in \mathscr{K}(n)$. Then E is a $\mathscr{K}(n)$-sequence.*

Proof. This is an immediate consequence of Theorems 1.7 and 1.8.

THEOREM 1.10. *Let E: $0 \to H \to G \to K \to 0$ be an exact sequence of Butler groups such that G is completely decomposable. If $\tau_1, \dots, \tau_n$ are types, $n \geq 1$, then the following are equivalent*:

(a) *E is a $\mathscr{K}(n)$-sequence.*

(b) *For $1 \leq k \leq n$, $0 \to H/\sum_{i=1}^{k} H(\tau_i) \to G/\sum_{i=1}^{k} G(\tau_i) \to K/\sum_{i=1}^{k} K(\tau_i) \to 0$ is a $\mathscr{K}(n-k)$-sequence.*

(c) *For $1 \leq k \leq n+1$, $0 \to \sum_{i=1}^{k} H(\tau_i) \to \sum_{i=1}^{k} G(\tau_i) \to \sum_{i=1}^{k} K(\tau_i) \to 0$ is a $\mathscr{K}(n-k+1)$-sequence.*

Proof. (a) implies (b) and (c). We use induction on n. For $k = 1$, (b) and (c) follow from Lemma 1.3. This, together with Theorem 1.8(c) for $k = 2$, takes care of the case $n = 1$. For $2 \leq k \leq n$, denote $\overline{H} = H/H(\tau_k)$, $\overline{G} = G/G(\tau_k)$, and $\overline{K} = K/K(\tau_k)$. The sequence $0 \to \overline{H} \to \overline{G} \to \overline{K} \to 0$ is a $\mathscr{K}(n-1)$-sequence by Lemma 1.3. By induction,

$$0 \to \overline{H} \Big/ \sum_{i=1}^{k-1} \overline{H}(\tau_i) \to \overline{G} \Big/ \sum_{i=1}^{k-1} \overline{G}(\tau_i) \to \overline{K} \Big/ \sum_{i=1}^{k-1} \overline{K}(\tau_i) \to 0$$

is a $\mathscr{K}(n-1-(k-1)) = \mathscr{K}(n-k)$-sequence. Since $H(\tau_k)$ is balanced in H by Lemma 1.3, $\overline{H}(\tau_i) = [H(\tau_i) + H(\tau_k)]/H(\tau_k)$. By the Third Isomorphism Theorem, $\overline{H}/\sum_{i=1}^{k-1}\overline{H}(\tau_i)$ is isomorphic to $H/\sum_{i=1}^{k}H(\tau_i)$; similarly for G and K. This proves (b). Again by induction,

$$\overline{E}\colon 0 \to \sum_{i=1}^{k-1} \overline{H}(\tau_i) \to \sum_{i=1}^{k-1} \overline{G}(\tau_i) \to \sum_{i=1}^{k-1} \overline{K}(\tau_i) \to 0$$

is a $\mathscr{K}(n-1-(k-1)+1) = \mathscr{K}(n-k+1)$-sequence. By hypothesis, the original sequence E is balanced, so the induced map $\sum_{i=1}^{k}G(\tau_i) \to \sum_{i=1}^{k}K(\tau_i)$ is an epimorphism. We claim the kernel is $\sum_{i=1}^{k}H(\tau_i)$. Let $x \in H \cap \sum_{i=1}^{k}G(\tau_i)$. Since $\overline{E}$ is exact, there exists $g_k \in G(\tau_k)$ and $h_i \in H(\tau_i)$, $1 \leq i < k$, such that $x - g_k = h_1 + \cdots + h_{k-1}$. Since $x \in H$, $g_k \in H \cap G(\tau_k) = H(\tau_k)$. Thus, $x \in \sum_{i=1}^{k}H(\tau_i)$. We have shown that the sequence

$$E_k\colon 0 \to \sum_{i=1}^{k} H(\tau_i) \to \sum_{i=1}^{k} G(\tau_i) \to \sum_{i=1}^{k} K(\tau_i) \to 0$$

is exact. This establishes (c) for $k = n+1$. For $k \leq n$, Theorem 1.8(b) and Lemma 1.1(d) imply that the sequence E_k is balanced. Finally, since $K \in \mathscr{K}(n-1)$, the induction hypothesis applied to a $\mathscr{K}(n-1)$-sequence $0 \to K \to G' \to K' \to 0$ shows that $\sum_{i=1}^{k}K(\tau_i)$ is the first term in a $\mathscr{K}(n-1-k+1) = \mathscr{K}(n-k)$-sequence. Thus, $\sum_{i=1}^{k}K(\tau_i) \in \mathscr{K}(n-k)$ and the proof of (c) is complete.

(b) implies (a). Taking $k = n$, condition (b) implies that $\sum_{i=1}^{n}K(\tau_i)$ is pure in K. Hence K is a $\mathscr{K}(n-1)$-group. Taking $k = 1$ and $\tau = \tau_1$, (b) implies that $H/H(\tau)$ is pure in $G/G(\tau)$ whence that $H + G(\tau)$ is pure in G. By Lemma 1.2, H is balanced in G. Thus, E is a $\mathscr{K}(n)$-sequence.

(c) implies (a). Take $\tau = \text{type}(Z)$. Condition (c) says that $0 \to H(\tau) \to G(\tau) \to K(\tau) \to 0$, which is identical with E, is a $\mathscr{K}(n)$-sequence.

The class of $\mathscr{K}(n)$-groups is not closed under quasi-isomorphism for $n > 0$. A completely decomposable group is a $\mathscr{K}(n)$-group for all n, but Example 4.1 below will show that for every $n > 0$, there are indecomposable almost completely decomposable groups that are $\mathscr{K}(n)$-groups but not $\mathscr{K}(n+1)$-groups. The last result of this section shows that the $\mathscr{K}(n)$-groups are closed under a weakening of isomorphism called near isomorphism. Recall that two torsion-free groups G, H are *nearly isomorphic* if for any prime p there is a monomorphism $\varphi: G \to H$ such that the index of $\psi(G)$ in H is finite and relatively prime to p.

COROLLARY 1.11. *For each $n \geq 0$, the class of $\mathscr{K}(n)$-groups is closed under near isomorphism.*

Proof. Suppose $G \in \mathscr{K}(n)$ and G is nearly isomorphic to H. To show $H \in \mathscr{K}(n)$, it suffices by Theorem 1.8 to show that $H(\tau_1) + H(\tau_2) + \cdots + H(\tau_{n+1})$ is pure in H for each set of types $\{\tau_1, \ldots, \tau_{n+1}\}$. The same theorem says that $G(\tau_1) + \cdots + G(\tau_{n+1})$ is pure in G. Let $\varphi: G \to H$ be a monomorphism with $[H: \varphi(G)] = m > 0$. If p is relatively prime to m and G' is a pure subgroup of G, then $\varphi(G')$ is p-pure in H. In particular,

$$\varphi\big(G(\tau_1) + \cdots + G(\tau_{n+1})\big) \subseteq H(\tau_1) + \cdots + H(\tau_{n+1})$$

implies that $H(\tau_1) + \cdots + H(\tau_{n+1})$ is p-pure in H. However, by the definition of near isomorphism we can choose φ so that m is relatively prime to any given prime p.

2. DECOMPOSITION CRITERIA

We examine conditions that force $\mathscr{K}(n)$-groups to be completely decomposable. The reference result for Butler groups is that any Butler group with linearly ordered typeset is completely decomposable [But]. If H is a Butler group, then the critical typeset of H, denoted $T_{\mathrm{cr}}(H)$, is the set of types τ such that $H(\tau)/H^{\#}(\tau) \neq 0$. Since $H(\tau)/H^{\#}(\tau)$ is a homogeneous completely decomposable group of type τ that is isomorphic to a summand of $H(\tau)$, it is clear that $T_{\mathrm{cr}}(H) \subseteq \mathrm{typeset}(H) = \{\mathrm{type}(x): 0 \neq x \in H\}$.

THEOREM 2.1. *A group H in $\mathscr{K}(n)$ is completely decomposable if either of the following hold.*

(a) $|T_{\mathrm{cr}}(H)| \leq n + 1$;

(b) $T_{\mathrm{cr}}(H) = \{\tau_1, \ldots, \tau_{n+2}\}$ *and either*

  (i) $|\{\tau_i \Lambda \tau_j: 1 \leq i < j \leq n+2\}| \geq 2$, *or*

  (ii) $\sup\{\tau_1, \ldots, \tau_{n+2}\} = \mathrm{type}(Q)$.

Proof. (a) The proof is by induction on n, the case $n = 0$ being well known. For $n \geq 1$, suppose the result holds for all groups H in $\mathscr{K}(k)$, $0 \leq k < n$, that satisfy $|T_{\mathrm{cr}}(H)| \leq k + 1$. Choose any maximal type τ in typeset(H). Then $H(\tau)$ is homogeneous completely decomposable, $H/H(\tau) \in \mathscr{K}(n-1)$ by Lemma 1.3(b), and $T_{\mathrm{cr}}(H/H(\tau)) \subseteq T_{\mathrm{cr}}(H) \setminus \{\tau\}$ since $H(\tau)$ is balanced in H (Lemma 1.3(c)). By induction, $H/H(\tau)$ is completely decomposable. It follows that $H \simeq H(\tau) \oplus H/H(\tau)$ is completely decomposable.

(b) Part (i) is similar to (a): choose τ maximal in $\{\tau_i \wedge \tau_j: 1 \le i < j \le n+2\}$. Then $2 \le |T_{\mathrm{cr}} H(\tau)| \le n+1$ and $H(\tau) \in \mathscr{K}(n)$ by Lemma 1.3(a). By part (a), $H(\tau)$ is completely decomposable. Also by Lemma 1.3(b), $H/H(\tau) \in \mathscr{K}(n-1)$. Since $|T_{\mathrm{cr}}(H/H(\tau)| \le n$, then $H/H(\tau)$ is completely decomposable by (a). Since $H(\tau)$ is balanced in H, the sequence $0 \to H(\tau) \to H \to H/H(\tau) \to 0$ is split exact and the proof of (i) is complete.

For (ii), we may assume in view of (i) that $\tau_i \wedge \tau_j = \tau_0$ for $1 \le i < j \le n+2$. Let

$$M = \{\tau \in T_{\mathrm{cr}}(H): \tau \text{ is maximal in } T_{\mathrm{cr}}(H)\},$$

a set with at most $n+2$ elements. For $\tau \in M$, $H(\tau)$ is a completely decomposable group. Since $\sum_{\tau \in I} H(\tau)$ is pure in H for every subset I of cardinality at most $n+1$, it follows that $\sum_{\tau \in M} H(\tau) = \oplus_{\tau \in M} H(\tau)$ is completely decomposable. If $M \neq T_{\mathrm{cr}}(H)$, then $\tau_0 \in T_{\mathrm{cr}}(H)$ and Butler decomposition gives $H = H(\tau_0) = H_0 \oplus H^{\#}(\tau_0)$, where H_0 is homogeneous completely decomposable of type τ_0. In this case, $H^{\#}(\tau_0) = \oplus_{\tau \in M} H(\tau)$ since $|M| = n+1$ and $H \in \mathscr{K}(n)$ imply $\sum_{\tau \in M} H(\tau) = \oplus_{\tau \in M} H(\tau)$ is pure. This shows H is completely decomposable. It remains to consider the case $M = T_{\mathrm{cr}}(H)$. In this case, $H' = \oplus_{i=1}^{n+2} H(\tau_i)$ is a subgroup of finite index in H. Suppose $h \in H$ and $ph \in H'$ for some prime p. Since $\bigvee_{i=1}^{n+2} \tau_i = \text{type}(Q)$, we may assume that τ_1 is p-divisible. Then $ph = ph_1 + h_2 + \cdots + h_{n+2}$ for some choice of $h_i \in H(\tau_i)$. Thus, $p(h - h_1) \in \oplus_{i=2}^{n+2} H(\tau_i)$, a pure subgroup of H since $H \in \mathscr{K}(n)$ (Theorem 1.8(c)). Thus, $h - h_1 \in \oplus_{i=2}^{n+2} H(\tau_i)$, so that $h \in H'$ and the proof is complete.

LEMMA 2.2. *Let* $0 \to H \to G \to K \to 0$ *be a* $\mathscr{K}(n)$*-sequence. For* $n \ge 1$, *we have* $T_{\mathrm{cr}}(K) \subseteq T_{\mathrm{cr}}(G)$. *Moreover, for* $n \ge 0$, *the sequence splits whenever* $|T_{\mathrm{cr}}(G)| \le n+1$.

Proof. Let $\tau \in T_{\mathrm{cr}}(K)$. By balancedness, the sequence $0 \to H(\tau) \to G(\tau) \to K(\tau) \to 0$ is balanced exact. By Butler decomposition, $K(\tau) = K_0 \oplus K^{\#}(\tau)$ with K_0 homogeneous completely decomposable of type τ. Note that K_0 is a homomorphic image of $G(\tau)/G^{\#}(\tau)$. That is, $\tau \in T_{\mathrm{cr}}(G)$. We prove the last sentence of the lemma by induction on n, the case $n = 0$ being well known. In general, choose τ maximal in $T_{\mathrm{cr}}(G)$. Then the sequence $0 \to H(\tau) \to G(\tau) \to K(\tau) \to 0$ is split exact. In particular, $H(\tau)$ is a summand of $G(\tau)$, therefore of G and then of H by the modular law.

By Lemma 1.3(b) and induction, the sequence $0 \to H/H(\tau) \to G/G(\tau) \to K/K(\tau) \to 0$ is split exact. It follows that $H \simeq H(\tau) \oplus H/H(\tau)$ is a summand of $G \simeq G(\tau) \oplus G/G(\tau)$.

PROPOSITION 2.3. *Let* $0 \to H \to G \to K \to 0$ *be a* $\mathscr{K}(n)$*-sequence. If* $|T_{\mathrm{cr}}(G)| < \binom{n+3}{2}$, *then H is completely decomposable.*

Proof. We use induction on n and $|\mathrm{typeset}(H)|$. The result is true for $n = 0$; and it is true for all n if $|\mathrm{typeset}(H)| = 1$. We assume the proposition holds for $\mathscr{K}(k)$-sequences with $k \leq n - 1$ and for $\mathscr{K}(n)$-sequences where the cardinality of the typeset of the first term is less than $|\mathrm{typeset}(H)|$. Let τ be any maximal type in typeset(H), so that $H(\tau)$ is completely decomposable. If $H(\tau)$ is a summand of $G(\tau)$, then $G = H(\tau) \oplus G'$, so $H = H(\tau) \oplus H \cap G'$ and $0 \to H \cap G' \to G \to G/H \cap G' \to 0$ is a $\mathscr{K}(n)$-sequence. Induction finishes this case. If $H(\tau)$ is not a summand of $G(\tau)$, then $|T_{\mathrm{cr}}[G(\tau)]| \geq n + 2$ by Lemma 2.2. It follows that $|T_{\mathrm{cr}}(G/G(\tau)| \leq \binom{n+3}{2} - 1 - (n+2) = \binom{n+2}{2} - 1$. Since the sequence

$$0 \to H/H(\tau) \to G/G(\tau) \to K/K(\tau) \to 0$$

is a $\mathscr{K}(n-1)$-sequence (Lemma 1.3(b)), induction implies that $H/H(\tau)$ is completely decomposable. Since $H(\tau)$ is balanced in H (Lemma 1.3(c)), then $H \simeq H(\tau) \oplus H/H(\tau)$ is completely decomposable.

3. BALANCED RANK-ONE SUBGROUPS

We will see first that to determine the splitting of a $\mathscr{K}(n)$-sequence, it suffices to assume the first term is homogeneous. In particular, this is the case when the first term has rank one. The third term of the sequence then becomes a bracket group, or a $B^{(1)}$-group in the terminology of Fuchs and Metelli [FM]. The main result of this section characterizes which of the sequences $0 \to H \to G \to K \to 0$, with G completely decomposable and H rank-one, are $\mathscr{K}(n)$-sequences.

PROPOSITION 3.1. *Let G be a finite rank completely decomposable group. Then every* $\mathscr{K}(n)$*-sequence* E: $0 \to H \to G \to K \to 0$, *with H homogeneous, splits if and only if every* $\mathscr{K}(n)$*-sequence of the form E splits.*

Proof. One direction is clear. For the converse, assume all sequences E split if H is homogeneous and use induction on $t = |\mathrm{typeset}(H)|$. The case $t = 1$ is by hypothesis. In general, let τ be maximal in typeset(H). By Lemma 1.3(a), the exact sequence $0 \to H(\tau) \to G(\tau) \to K(\tau) \to 0$ is a $\mathscr{K}(n)$-sequence. Hence, so is $0 \to H(\tau) \to G \to G/H(\tau) \to 0$. The latter splits by hypothesis and we have $G = H(\tau) \oplus G'$, $H = H(\tau) \oplus H'$ where $H' = H \cap G'$. Then $0 \to H' \to G \to K \oplus H(\tau) \to 0$ is a $\mathscr{K}(n)$-sequence. But typeset H' is properly contained in typeset H so the induction hypothesis implies that H' is a summand of G.

The next lemma was originally due to Lee [Le] and appears in various forms elsewhere, for example [FM].

LEMMA 3.2. *Let $A_1, \ldots, A_m$ be subgroups of Q and H the subgroup of $G = \oplus A_i$ defined by $H = \{(a, a, \ldots, a) \mid a \in \cap A_i\}$. Let $g = (a_1, \ldots, a_m)$ be a non-zero element of G and let $I_1, \ldots, I_n$ be the partition of $\{1, \ldots, m\}$ determined by the equality classes of g; that is, $a_i = a_j$ for $i, j \in I_k$ and $a_i \neq a_j$ for $i \in I_k$, $j \notin I_k$. Then in G/H,*

$$\text{type}\,(g + H) = \sup_j\left\{\inf\left\{\text{type}(A_i) \mid i \notin I_j\right\}\right\}.$$

Prior to our next lemma, it is convenient to introduce some new terminology. We will call an exact sequence $0 \to H \to G \to K \to 0$ *quasi-balanced* if for each $k \in K$ there is a positive integer m such that mk has a preimage in G of the same type. The notion of quasi-balancedness is equivalent to balancedness in the quasi-homomorphism category of Butler groups. This will be treated in detail in a forthcoming paper.

LEMMA 3.3. *Let $A_1, \ldots, A_m$ be subgroups of Q containing Z, H the subgroup of $G = \oplus A_i$ defined by $H = \{(a, a, \ldots, a) | a \in \cap A_i\}$, and E: $0 \to H \to G \to G/H \to 0$ the induced exact sequence. Then*

(a) *E is a quasi-balanced exact sequence if and only if for any proper subset I of $\{1, 2, \ldots, m\}$, $\bigcap_{i \in I} A_i$ and $\bigcap_{i \notin I} A_i$ have comparable types.*

(b) *E is balanced if and only if* (a) *holds and, in addition, if* $\text{type}(\bigcap_{i \in I} A_i) < \text{type}(\bigcap_{i \notin I} A_i)$, *then $\bigcap_{i \in I} A_i \subseteq \bigcap_{i \notin I} A_i$.*

Proof. (a) Assume the sequence is quasi-balanced and let I be a proper subset of $\{1, \ldots, n\}$. Define $g = g(I) = (a_1, \ldots, a_m) \in G$ by setting $a_i = 1$ if $i \in I$, $a_i = 0$ otherwise. By Lemma 3.2,

$$(*) \qquad \text{type}(g + H) = \sup\left\{\text{type}\left(\textstyle\bigcap_{i \in I} A_i\right), \text{type}\left(\textstyle\bigcap_{i \notin I} A_i\right)\right\}.$$

Each preimage in G of $g + H$ has the form $(a_1 + a, \ldots, a_m + a)$ for some $a \in \cap A_i$. Moreover,

$$\text{type}(a_1 + a, \ldots, a_m + a) = \begin{cases} \text{type}\left(\bigcap_{i \notin I} A_i\right) & \text{if } a = -1 \\ \text{type}\left(\bigcap_{i \in I} A_i\right) & \text{if } a = 0. \\ \text{type}\left(\bigcap_{i=1}^{m} A_i\right) & \text{otherwise.} \end{cases}$$

Since the sequence is quasi-balanced, replacing g by an integer multiple if necessary, $\operatorname{type}(g+H) = \operatorname{type}(a_1 + a, \dots, a_m + a)$ for some a. It follows from $(*)$ that $\operatorname{type}(\bigcap_{i \in I} A_i)$ and $\operatorname{type}(\bigcap_{i \notin I} A_i)$ are comparable.

Conversely, to show that the sequence E is quasi-balanced, let $x = (b_1, \dots, b_m) \in G$ and let $I_1, \dots, I_k$ be the partition of $\{1, \dots, m\}$ determined by the equality classes of the b_i's. For example, I_1 is the set of all indices j such that $b_j = b_1$. By Lemma 3.2,

$$\operatorname{type}(x+H) = \operatorname{type}\left(\bigcap_{i \notin I_1} A_i + \bigcap_{i \notin I_2} A_i + \cdots + \bigcap_{i \notin I_k} A_i\right).$$

Denote $\sigma_j = \operatorname{type}(\bigcap_{i \notin I_j} A_i)$. We will show that there is an index h with $\sigma_h \geq \sigma_i$ for $1 \leq i \leq k$, that is, $\operatorname{type}(x+H) = \sigma_h$. Assuming this for the moment, denote $b = b_i$ for any $i \in I_h$. By multiplying x by a nonzero integer if necessary, we may assume $b \in \bigcap_{i=1}^{n} A_i$. Then $y = x - (b, b, \dots, b) = (c_1, \dots, c_m)$ is a preimage in G of $x + H$, with $c_i = 0$ for $i \in I_h$. In particular, $\operatorname{type} y = \sigma_h = \operatorname{type}(x+H)$ and the sequence is quasi-balanced. Returning to the search for σ_h, use the condition in (a) to deduce that σ_j and $\rho_j = \operatorname{type}(\bigcap_{i \in I_j} A_i)$ are comparable. If $\sigma_j \geq \rho_j$, then $\sigma_j \geq \sigma_i$ for all i and we are done. If $\sigma_j \leq \rho_j$, then $\sigma_j \leq \sigma_i$ for all i. We have shown that, for each j, either $\sigma_j \geq \sigma_i$ for all i, or $\sigma_j \leq \sigma_i$ for all i. It follows immediately that for some index h, $\sigma_h \geq \sigma_i$ for all i.

(b) Assuming the sequence E is balanced, the condition in (a) is necessary. Suppose that I is a proper subset of $\{1, \dots, n\}$ with $\operatorname{type}(\bigcap_{i \in I} A_i) < \operatorname{type}(\bigcap_{i \notin I} A_i)$ and let $\alpha \in \bigcap_{i \in I} A_i$. Consider the element $g = (a_1, \dots, a_m) \in G$ defined by $a_i = \alpha$ for $i \in I$, $a_i = 0$ otherwise. Note that $\operatorname{type}(g) = \operatorname{type}(\bigcap_{i \in I} A_i)$. Since the sequence is balanced, there exists $a \in \bigcap_{i=1}^{n} A_i$ so that $(a_1 + a, \dots a_m + a) \in G$ has type equal to $\operatorname{type}(g+H)$. The formula $(*)$ above and the fact that $\operatorname{type}(\bigcap_{i \in I} A_i) < \operatorname{type}(\bigcap_{i \notin I} A_i)$ imply that $a = -\alpha$ and $\alpha \in \bigcap_{i=1}^{n} A_i \subseteq \bigcap_{i \notin I} A_i$.

Conversely, assuming the inequality conditions of (b), we may conclude that the sequence E is quasi-balanced by (a). Recalling the notation from (a), let $x = (a_1, \dots, a_m) \in G$, $I_1, \dots, I_k$ be the partition of $\{1, \dots, m\}$ determined by the equality classes of a_i's, and $\sigma_j = \operatorname{type}(\bigcap_{i \notin I_j} A_i)$. Also, let h be the index for which $\operatorname{type}(x+H) = \sigma_h \geq \sigma_i$ for $1 \leq i \leq k$. We want to show $x + H$ has a preimage (in G) of the same type. As in (a), the types σ_h and $\rho_h = \operatorname{type}(\bigcap_{i \in I_h} A_i)$ are comparable. If $\rho_h \geq \sigma_h$, then $\operatorname{type}(x) = \sigma_h = \inf\{\sigma_h, \rho_h\} = \operatorname{type}(\bigcap_{i=1}^{n} A_i) = \operatorname{type}(x+H)$. In this case, any preimage of $x + H$ has the same type. On the other hand, if $\rho_h < \sigma_h$, then $\bigcap_{i \in I_h} A_i \subseteq \bigcap_{i \notin I_h} A_i$ by the hypotheses of (b). Let $a = a_i$ for some $i \in I_h$. Then $a \in \bigcap_{i \notin I_h} A_i$ and $(a_1 - a, \dots, a_m - a)$ is a preimage for $x + H$ with $a_i - a = 0$ for $i \in I_h$. Thus, $\operatorname{type}(a_1 - a, \dots, a_m - a) \geq \operatorname{type}(\bigcap_{i \notin I_h} A_i) = \operatorname{type}(x+H)$ and the proof is complete.

We can now prove the $\mathscr{K}(n)$ version of Lemma 3.3. This result is quite easy to use, despite its apparent complexity.

THEOREM 3.4. *Let $A_1, \dots, A_m$ be subgroups of Q and let H be the subgroup of $G = \oplus A_i$ defined by $H = \{(a, a, \dots, a) | a \in \cap A_i\}$. Then $0 \to H \to G \to G/H \to 0$ is a $\mathscr{K}(n)$-sequence if and only if for each partition $S(1) \cup S(2) \cup \cdots \cup S(n+1)$ of $\{1, 2, \dots, m\}$, the collection of subgroups of Q of the form $B_k = \cap \{A_i | i \in S(k)\}$ contains an element B_h such that, for $\tau_k = \text{type}(B_k)$,*

(a) $\tau_h \leq \wedge_{k \neq h} \tau_k$; *and*

(b) *if $\tau_h < \tau_k$ for all $k \neq h$ and $G \neq \sum_{k \neq h} G(\tau_k)$, then $B_h \subseteq \bigcap_{k \neq h} B_k$.*

Proof. Without loss of generality, we may assume $1 \in A_i$ for each i. Starting with a $\mathscr{K}(n)$-sequence E: $0 \to H \to G \to K \to 0$, we will use induction on n to show that if $S(1), \dots, S(n+1)$ is a partition of $\{1, 2, \dots, m\}$, then (a) and (b) hold. The case $n = 0$ is vacuously true. The case $n = 1$ follows from Lemma 3.3(b). Let $B_k = \cap \{A_i | i \in S(k)\}$ and $\tau_k = \text{type}(B_k)$, $1 \leq k \leq n+1$. We allow $S(k)$ to be empty, in which case $B_k = Q$. To show that condition (a) holds, assume that $\tau_1 = \text{type}(B_1) \nleq \wedge_{i \neq 1} \text{type}(B_i) = \sigma_1$ (if inequality holds, we are done). Define elements $e_i = (e_{i1}, \dots, e_{im}) \in G$ by setting $e_{ij} = 1$ if $j \in S(i)$ and $e_{ij} = 0$ otherwise. By Lemma 3.3 and the fact that the sequence is balanced, $\tau_1 > \sigma_1$. By Lemma 3.2, $\text{type}(e_1 + H) = \tau_1 = \text{type}(e_1)$. Note that

$$e_1 + H = (-e_2 - e_3 - \cdots - e_{n+1} + H) \in \sum_{j=2}^{n+1} K(\tau_j).$$

By Theorem 1.10(c), the map $\sum_{j=2}^{n+1} G(\tau_j) \to \sum_{j=2}^{n+1} K(\tau_j)$ is a balanced epimorphism. Moreover, $\sum_{j=2}^{n+1} K(\tau_j)$ is pure in K since $K \in \mathscr{K}(n-1)$. Thus, the element $e_1 + H$ of $\sum_{j=2}^{n+1} K(\tau_j)$ has a preimage in $\sum_{j=2}^{n+1} G(\tau_j)$ of the same height. That is, there exist elements $x_j \in G(\tau_j)$, $2 \leq i \leq n+1$, so that $x = x_2 + x_3 + \cdots + x_{n+1}$ satisfies $x + H = e_1 + H$, $\text{height}(x) = \text{height}(e_1 + H)$, and consequently, $\text{type}(x) = \tau_1$. Now $x - e_1 \in H$ implies that for some $a \in \bigcap_{i=1}^m A_i$, the projection $x(i)$ of x onto A_i equals $1 + a$ if $i \in S(1)$ and a otherwise. Since $\text{type}(B_1) \nleq \wedge_{i \neq 1} \text{type}(B_i)$, it follows that $a = 0$ and $x = e_1$. Define subsets $S(1j)$ of $S(1)$ as follows:

$$S(12) = \{i \in S(1) \,|\, x_2(i) \neq 0\}$$

and, for $j > 2$,

$$S(1j) = \{i \in S(1) \,|\, x_2(i) = x_3(i) = \cdots = x_{j-1}(i) = 0;\ x_j(i) \neq 0\}.$$

Since $e_1 = x_2 + \cdots + x_n$, we have $S(1) \subseteq \cup S(1, j)$. Thus, the n sets $T(j) = S(j) \cup S(1j)$, $2 \le j \le n + 1$, form a partition of $\{1, 2, \ldots, m\}$. If $C_j = \bigcap_{i \in T(j)} A_i$, then $\text{type}(C_j) = \bigcap_{i \in S(j)} A_i = \tau_j$ because if $i \in S(1j)$ then $\tau_j \le \text{type}(x_j) \le \text{type}(A_i)$ since $x_j(i) \neq 0$. To apply the induction hypothesis, we regard E as a $\mathscr{K}(n-1)$-sequence and use (a) on the partition $\{T(j)\}$. Then there exists j, for convenience say $j = 2$, such that $\text{type}(C_2) \le \wedge_{j > 2} \text{type}(C_j)$. That is, $\tau_2 = \text{type}(C_2) = \wedge_{j=2}^{n+1} \text{type}(C_j) = \wedge_{j=2}^{n+1} \tau_j$. But $\cup\{T(j)\} = \{1, \ldots, m\}$ implies $\wedge_{j=2}^{n+1} \text{type}(C_j) = \text{type}(\bigcap_{i=1}^{m} A_i) = \wedge_{j=1}^{n+1} \tau_j$. Thus, $\text{type}(B_2) \le \wedge_{j \neq 2} \text{type}(B_j)$ and (a) holds.

To show (b) for $n \ge 1$, assume for convenience that $\tau_{n+1} = \text{type}(B_{n+1}) < \text{type}(B_j) = \tau_j$ for $j \le n$ and $G \neq \sum_{j=1}^{n} G(\tau_j)$. Since E is a $\mathscr{K}(n)$-sequence, K is in $\mathscr{K}(n-1)$ and $L = K(\tau_1) + \cdots + K(\tau_n)$ is pure in K. Since $G \to K$ is a balanced epimorphism, the preimage of L, $H + G(\tau_1) + \cdots + G(\tau_n)$, must be pure in G. Because $\text{type}(\bigcap_{j=1}^{n+1} B_j) = \text{type}(B_{n+1})$, there is a minimal positive integer u such that $uB_{n+1} = \bigcap_{j=1}^{n+1} B_j = \bigcap_{i=1}^{m} A_i$. Define $x = (x_1, \ldots, x_m) \in G$ by $x_i = 1$ if $\tau_j \nleq \text{type}(A_i)$ for all $1 \le j \le n$; and $x_i = 0$ otherwise. Note that $x \neq 0$ since $G \neq \sum_{j=1}^{n} G(\tau_j)$. Then,

$$
\begin{aligned}
&H + G(\tau_1) + \cdots + G(\tau_n) \\
&\quad = \left[\bigcap_{i=1}^{m} A_i\right](1, 1, \ldots, 1) + \sum_{j=1}^{n} G(\tau_j) \\
&\quad = uB_{n+1}(1, 1, \ldots, 1) + \sum_{j=1}^{n} G(\tau_j) \\
&\quad = uB_{n+1} x \oplus \left[\oplus\left\{A_i \mid \text{type}(A_i) \ge \tau_j \text{ for some } j \le n\right\}\right].
\end{aligned}
$$

Here we are working inside the vector space $QG = Q^m$, where it makes sense to multiply vectors by subgroups of Q. Thus, for example, $[\bigcap_{i=1}^{m} A_i](1, 1, \ldots, 1) = \{(a, a, \ldots, a) \mid a \in \bigcap_{i=1}^{m} A_i\}$. Plainly, $S(1) \cup \cdots \cup S(n) \subseteq \{i \mid \text{type}(A_i) \ge \tau_j$ for some $j \le n\}$ so that $x_i \neq 0$ implies $i \in S(n+1)$. Thus, $B_{n+1} x \in G$; and by purity of uB_{n+1}, $u = 1$ and $B_{n+1} \subseteq \bigcap_{j=1}^{n} B_j$, as desired.

Conversely, assume (a) and (b) hold. These two conditions imply the same two statements with n replaced by $n - 1$, since we allow $S(i)$ to be empty. We wish to show that E is a $\mathscr{K}(n)$-sequence. The case $n = 0$ is trivial. For $n \ge 1$, the conditions (a) and (b) imply that the sequence is balanced by Lemma 3.3. It remains to show that $K \in \mathscr{K}(n-1)$, which by Theorem 1.8 is equivalent to showing that $L = K(\tau_1) + \cdots + K(\tau_n)$ is pure in K for any list of types $\tau_1, \ldots, \tau_n$.

For $n \geq 1$, we may assume by induction on n that E is a $\mathscr{K}(n-1)$-sequence. In particular, the sum of any proper subset of the $K(\tau_i)$'s is pure in K. Form a partition by defining

$$S(1) = \{i \mid \operatorname{type}(A_i) \geq \tau_1\},$$
$$S(k) = \{i \mid \operatorname{type}(A_i) \geq \tau_k\} \setminus \bigcup_{j=1}^{k-1} S(j), \qquad \text{for } 2 \leq k \leq n.$$

Note that if $S(k)$ is empty, then $G(\tau_k) \subseteq \sum_{j=1}^{k-1} G(\tau_j)$. This implies $K(\tau_k) \subseteq \sum_{j=1}^{k-1} K(\tau_j)$ and purity of L follows by induction. Let

$$S(n+1) = \{1, 2, \ldots, m\} \setminus \bigcup_{k=1}^{n} S(k).$$

Denote $B_k = \cap\{A_i | i \in S(k)\}$ and apply condition (a). If $\operatorname{type}(B_j) \leq \inf_{k \neq j} \operatorname{type}(B_k)$ for some $1 \leq j \leq n$, then $\tau_j \leq \tau_k$ for all $j \neq k$ and $L = K(\tau_j)$ is pure. Thus, we have only to deal with the case $\operatorname{type}(B_{n+1}) < \operatorname{type}(B_k)$ for $1 \leq k \leq n$. If $G = \sum_{j=1}^{n} G(\tau_j)$, then $K = \sum_{j=1}^{n} K(\tau_j)$ and purity is trivial. Thus, we may assume $G \neq \sum_{j=1}^{n} G(\tau_j)$ and apply (b). As above, define an element $x = (x_1, \ldots, x_m) \in G$ by $x_j = 1$ if $j \in S(n+1)$, $x_j = 0$ otherwise. Condition (b) says $B_{n+1} \subseteq \bigcap_{i=1}^{n} B_k$. In this case,

$$H = \left[\bigcap_{i=1}^{m} A_i\right](1, 1, \ldots 1) = \left[\bigcap_{j=1}^{n+1} B_j\right](1, 1, \ldots 1) = B_{n+1}(1, 1, \ldots 1);$$

and

$$H + G(\tau_1) + \cdots + G(\tau_n) = B_{n+1}x \oplus \left[\oplus\left\{A_i \mid i \in \bigcup_{k=1}^{n} S(k)\right\}\right.$$

is pure in G. It follows that $K(\tau_1) + \cdots + K(\tau_n)$ is pure in K as desired.

4. SPLITTING $\mathscr{K}(n)$-SEQUENCES

In this section we examine conditions that guarantee when a $\mathscr{K}(n)$-sequence will split. Reference points are the well-known results that any pure subgroup of a homogeneous completely decomposable group is a summand; and any Butler group with linearly ordered typeset is completely decomposable. We begin with some examples.

EXAMPLES 4.1. Let $A_1, \ldots, A_{n+3}$ be subgroups of Q containing Z such that any two A_i, A_j have incomparable types and common intersection $A_i \cap A_j = A_0$.

(a) The group $G[A_1, \ldots, A_{n+3}]$ is a strongly indecomposable element of $\mathscr{K}(n) \setminus \mathscr{K}(n+1)$.

(b) If m is a positive integer and $(A_i)_p = (A_j)_p \neq Q$ for all $0 \leq i, j \leq n+2$ and some prime p dividing m, then $G[m^{-1}A_0, A_1, \ldots, A_{n+2}]$ is an indecomposable almost completely decomposable element of $\mathscr{K}(n) \setminus \mathscr{K}(n+1)$.

Proof. The fact that the group in (a) is strongly indecomposable follows from standard arguments about this type of group. See [AV-3] for example. The indecomposability of the group in (b) is also a standard argument—see [A-2]. That these groups belong to $\mathscr{K}(n)$ is a straightforward consequence of Theorem 3.4. Moreover, if H is the group in either (a) or (b), it is routine to check that $H(\tau_1) + \cdots + H(\tau_{n+2})$ is not pure in H for $\tau_i =$ type(A_i). That is, $H \notin \mathscr{K}(n+1)$.

It turns out that the examples in 4.1 are exhaustive in the following sense. Every strongly indecomposable element of $\mathscr{K}(n)$ that has critical typeset of size not exceeding $n+3$ is either of rank one or has the form in (a). The proof of this fact appears in a forthcoming paper that studies balanced representations of partially ordered sets [NV]. Analogously, we show in 5.4 below that an indecomposable element of $\mathscr{K}(n)$ that has at most $n+2$ elements in its critical typeset is either of rank one or nearly isomorphic to a group of the form in (b).

Using methods from Theorem 1.4 of [AV-1], each group in 4.1(a) can be embedded as a balanced subgroup of a completely decomposable group G with rank $G = \binom{n+3}{2} = |T_{\mathrm{cr}}(G)|$. Specifically, there is a cobalanced exact sequence,

$$0 \to G[A_1, \ldots, A_{n+3}] \to \bigoplus_{i<j}(A_i + A_j) \to K \to 0,$$

that is, a $\mathscr{K}(n)$-sequence by Corollary 1.9. This shows that the bound of Proposition 2.3 is sharp. Compare also with Theorem 2.1(b).

The methods from [AV-1] provide another view of $\mathscr{K}(n)$-groups, one that can be used to generate examples. Given a Butler group G, we can form a balanced projective resolution of G:

$$\cdots \to C_n \xrightarrow{\varphi_n} C_{n-1} \to \cdots \to C_2 \xrightarrow{\varphi_2} C_1 \xrightarrow{\varphi_1} G \to 0.$$

In this exact sequence each C_n is completely decomposable and each epimorphism $\varphi_n\colon C_n \to \varphi_n(C_n)$ is balanced. It is easy to see that Ker φ_n is an element of $\mathscr{K}(n)$, and that each element of $\mathscr{K}(n)$ can be realized in this way.

If T is a poset and $\tau, \sigma \in T$, then we say τ is a cover of σ in T if $\tau > \sigma$ and $\{\rho \in T: \sigma \leq \rho \leq \tau\} = \{\tau, \sigma\}$.

PROPOSITION 4.2. *Let T be a finite lattice of types with* $\sup T < \text{type}(Q)$ *and n a nonnegative integer. Then every $H \in \mathscr{K}(n)$ with* $\text{typeset}(H) \subseteq T$ *is completely decomposable if and only if every element of T has at most $n+1$ covers in T.*

Proof. The case $n = 0$ is the reference result of Butler on linearly ordered typesets. If T contains a type τ_0 with covers $\tau_1, \dots, \tau_{n+2}$, then we can construct a group in $\mathscr{K}(n)$ as in Example 4.1(b) that is not completely decomposable. This shows the $n+1$ covers condition is necessary.

To prove the converse, we induct on $t = |\text{typeset}(H)|$, where $H \in \mathscr{K}(n)$ has typeset contained in a lattice T in which every element has at most $n+1$ covers. The conclusion holds for $t \leq n+1$ by Theorem 2.1(a). If $\tau_0 = \inf\{\tau \in \text{typeset}(H)\}$, then $H(\tau_0) = H_0 \oplus H^{\#}(\tau_0)$, where H_0 is homogeneous completely decomposable of type τ_0. Thus, we may assume $H_0 = 0$. Let $\tau_1, \dots, \tau_k$ be the covers of τ_0 in T. Note that $k \geq 2$ since $H_0 = 0$. By assumption, $k \leq n+1$, and since $H \in \mathscr{K}(n)$, $H = H^{\#}(\tau_0) = \sum_{i=1}^{k} H(\tau_i)$, the latter being pure by Theorem 1.8(c). There is no loss of generality in assuming that no $H(\tau_j)$ is contained in the sum of the remaining $k-1$ socles, $\sum_{i \neq j} H(\tau_j)$. Now $H' = \sum_{i=1}^{k-1} H(\tau_i)$ is balanced in H by Theorem 1.8(b). Thus, H/H' has typeset properly contained in $\text{typeset}(H)$. By induction, H/H' is completely decomposable and is therefore a summand of H because H' is balanced. Again by induction, H' is completely decomposable and the proof is complete.

PROPOSITION 4.3. *Let E: $0 \to H \to G \to K \to 0$ be a $\mathscr{K}(n)$-sequence, $n \geq 1$, in which H is 0 or homogeneous completely decomposable of type* $\tau_0 = \inf[\text{typeset}(G)]$ *and* $K = \sum_{i=1}^{n} K(\sigma_i)$ *with* $\tau_0 < \sigma_i \in \text{typeset}(G)$ *for each i. Then the sequence E splits.*

Proof. We show that the hypotheses imply that K is completely decomposable by induction on n. If $n = 1$, then $K = K(\sigma_1)$ with $\tau_0 < \sigma_1$. By balancedness $G(\sigma_1) \to K(\sigma_1)$ is an epimorphism that must be an isomorphism since $H \cap G(\sigma_1) = 0$. Thus, $K(\sigma_1)$ is completely decomposable. Suppose that the proposition is true for $n-1$, with $n \geq 2$. By Lemma 1.3(b), the sequence $0 \to H/H(\sigma_n) \to G/G(\sigma_n) \to K/K(\sigma_n) \to 0$ is a $\mathscr{K}(n-1)$ sequence. Moreover, $H(\sigma_n) = 0$ implies $K(\sigma_n) \simeq G(\sigma_n)$ which is completely decomposable. Also, $K(\sigma_n)$ is balanced in K by Theorem 1.3(c). The induction hypothesis implies that $K/K(\sigma_n)$ is completely decomposable. Since completely decomposables are balanced projective and $K \to K/K(\sigma_n)$ is balanced, $K \simeq K/K(\sigma_n) \oplus K(\sigma_n)$ and the proof is complete.

We can now characterize those critical typesets that permit only trivial (i.e., split) $\mathscr{K}(n)$-sequences.

THEOREM 4.4. *Let T be a finite set of types with* $\sup T < \text{type}(Q)$. *The following are equivalent for $n \geq 1$:*

(a) *Each $\mathscr{K}(n)$-sequence $0 \to H \to G \to K \to 0$ with $T_{\text{cr}}(G) \subseteq T$ is split exact.*

(b) *For each subset S of T let $\tau_S = \inf\{\tau \mid \tau \in S\}$. Then,*

(i) *if $\tau_S \in S$, there is a partition $S(1) \cup \cdots \cup S(n)$ of $S \setminus \{\tau_S\}$ such that $\tau_S < \inf S(i)$ for $1 \leq i \leq n$.*

(ii) *if $\tau_S \notin S$, there is a partition $S(1) \cup \cdots \cup S(n+1)$ of S such that $\tau_S < \inf S(i)$ for $1 \leq i \leq n+1$.*

Proof. (a) implies (b). By way of contradiction suppose $\tau_S \in S$ and (i) fails. Then for every partition $P(1) \cup \cdots \cup P(n)$ of $S \setminus \{\tau_S\}$, $\tau_S = \inf P(i)$ for some i. For each $\tau \in S$, choose a subgroup A_τ of Q such that $Z \subseteq A_\tau$ and $\text{type}(A_\tau) = \tau$. These groups can also be chosen so that the finite lattice of subgroups of Q generated by the A_τ is isomorphic to the lattice of types generated by S under the map $\tau \to A_\tau$ (see [AV-4]). Since $\sup T < \text{type}(Q)$, we can also assume that for some prime p, $1 \in A_\tau$ has p-height 0 for each $\tau \in S$. Denote $A = p^{-1}A_{\tau_S} \oplus \{\oplus\{A_\tau \mid \tau \in S \setminus \{\tau_S\}\}$. Then there is an exact sequence E: $0 \to H \to A \to K \to 0$ where H is the pure subgroup of A generated by $(1, 1, \ldots, 1)$. We use Theorem 3.4 to show that this is a $\mathscr{K}(n)$-sequence. To this end, let $S(1) \cup \cdots \cup S(n+1)$ be a partition of S with, say, $\tau_S \in S(n+1)$. We consider two cases. First, if $S(n+1) = \{\tau_S\}$, then $S(1) \cup \cdots \cup S(n)$ is a partition of $S \setminus \{\tau_S\}$. By hypothesis, $\tau_S = \inf S(i)$ for some i, so that $\bigcap_{\tau \in S(i)} A_\tau = \bigcap_{\tau \in S} A_t$ by choice of the A_τ's. Conditions (a) and (b) of 3.4 follow immediately. The second case is where $S(n+1)$ contains at least two elements. In this case, the intersection of $p^{-1}A_{\tau_S}$ with the other groups in $S(n+1)$ equals $\bigcap_{\tau \in S} A_\tau$. Finally, note that E is not split exact (it is quasi-split), as in Example 4.1(b). This contradicts (a) so that (i) must hold.

Next assume that (ii) fails for some $S \subseteq T$. Then $\tau_S \notin S$ and for every partition $S(1) \cup \cdots S(n+1)$ of S, $\tau_S = \inf S(h)$ for some h. Choose subgroups A_τ of Q for each $\tau \in S$ that satisfy all the conditions in the previous paragraph. Form $G = \bigoplus_{\tau \in S} A_\tau$ and an exact sequence E: $0 \to H \to G \to K \to 0$ with H the pure subgroup generated by $(1, 1, \ldots, 1)$. The choice of the A_τ's guarantees that if $\tau_S = \inf S(h)$, then $\bigcap_{\tau \in S(h)} A_\tau = \bigcap_{\tau \in S} A_\tau$. Theorem 3.4 shows that E is a $\mathscr{K}(n)$-sequence. The sequence is clearly not split, since $\text{type}(H) = \tau_S \notin T_{\text{cr}}(G) = S$.

(b) implies (a). Given a $\mathscr{K}(n)$-sequence E: $0 \to H \to G \to K \to 0$, it is enough to assume H is homogeneous by Proposition 3.1; and that H has nonzero projection onto each rank-one summand in a decomposition of G. Let $S = T_{cr}(G)$ and assume $\tau_S \in S$. By b(i), there is a partition $S(1) \cup \cdots \cup S(n)$ of $S \setminus \{\tau_S\}$ such that $\tau_S < \inf S(i) = \tau_i$ for each i. We show that the balanced sequence E splits by showing that K is completely decomposable. First write $K = K_0 \oplus K^{\#}(\tau_S)$ with K_0 homogeneous completely decomposable of type τ_S. Next note that $G^{\#}(\tau_S) = \sum_{i=1}^{n} G(\tau_i)$ so that $K^{\#}(\tau_S) = \sum_{i=1}^{n} K(\tau_i)$, the latter being pure by Theorem 1.8(c). By Theorem 1.10(c) the sequence $0 \to \sum_{i=1}^{n} H(\tau_i) \to \sum_{i=1}^{n} G(\tau_i) \to \sum_{i=1}^{n} K(\tau_i) \to 0$ is exact. However, $\sum_{i=1}^{n} H(\tau_i) = 0$ since $\tau_S < \tau_i$ for each i. It follows that $K^{\#}(\tau_S) \simeq G^{\#}(\tau_S)$ whence K is completely decomposable.

It remains to treat the case $\tau_S \notin S$. By b(ii), there is a partition $S(1) \cup \cdots \cup S(n+1)$ of S with $\tau_S < \inf S(i) = \tau_i$ for each i. Under the assumption that H is homogeneous, we will show by induction on n that $H = 0$. By Lemma 1.3(b),

$$0 \to H/H(\tau_{n+1}) \to G/G(\tau_{n+1}) \to K/K(\tau_{n+1}) \to 0$$

is a $\mathscr{K}(n-1)$-sequence. Moreover, $H(\tau_{n+1}) = 0$ since $\operatorname{type}(H) = \tau_S < \tau_{n+1}$. That is,

$$0 \to H \to G/G(\tau_{n+1}) \to K/K(\tau_{n+1}) \to 0$$

is a $\mathscr{K}(n-1)$-sequence. Let $T = T_{cr}(G/G(\tau_{n+1}))$. If $n = 1$, then $G = G(\tau_2) + G(\tau_1)$ and $\inf T \geq \tau_1 > \tau_S$. This implies that $H = 0$ as desired. For the general case, $\{T \cap S(j) \colon 1 \leq j \leq n\}$ is a partition of T. Moreover, $\inf[T \cap S(j)] \geq \inf S(j) = \tau_j > \tau_S$. By induction, $H = 0$ and we are done.

5. DECOMPOSITIONS OF $\mathscr{K}(n)$-GROUPS WITH TYPESET OF SIZE $n+2$

In this section we give the $\mathscr{K}(n)$ version of a theorem originally stated by Arnold [A-1]: a Butler group with critical typeset of size two is a direct sum of rank one groups and indecomposable rank two groups. We will need two additional definitions. If H is an almost completely decomposable group, then any completely decomposable subgroup of minimal index in H is called a *regulating subgroup* of H [L-1]. As shown by Burkhardt [Bur], the intersection of all regulating subgroups of H is a fully invariant completely decomposable subgroup of finite index, called the *regulator*, $R(H)$, of H.

LEMMA 5.1. *Let $H \in \mathscr{K}(n)$ with $|T_{cr}(H)| = n+2$ and assume that H is not completely decomposable. Then the following hold.*

(a) *H is an almost completely decomposable group*;

(b) *$T_{\rm cr}(H)$ is an antichain in which the meet of any two distinct elements is the same*;

(c) *if $H/R(H) = \bigoplus_{i=1}^{k}\langle h_i + R(H)\rangle$ where $\langle h_i + R(H)\rangle$ is a nonzero cyclic group of order m_i with $m_i|m_{i+1}$, then* rank $H(\tau) \geq k$ *for each* $\tau \in T_{\rm cr}(H)$.

Proof. If H is not completely decomposable and $T_{\rm cr}(H) = \{\tau_1, \dots, \tau_{n+2}\}$, then by Theorem 2.1(b), $\tau_i \wedge \tau_j = \tau_0$ is constant for all $i \neq j$ and $\sup T_{\rm cr}(H) < \text{type}(Q)$. Also, from the proof of 2.1(b), $T_{\rm cr}(H)$ is an antichain and $R(H) = \bigoplus_{i=1}^{n+2}H(\tau_i)$ is a completely decomposable subgroup of finite index in H. In particular, H is almost completely decomposable. This shows (a) and (b). Given the hypotheses in (c), write $m_i h_i = \sum_{j=1}^{n+2}h_{ij}$, with $h_{ij} \in H(\tau_j)$, and observe that by Theorem 1.8(c) each $h_{ij} \neq 0$. We will show that $\{h_{ij}\}$ is Z-independent. Suppose

$$0 = \sum_{i=1}^{k}\sum_{j=1}^{n+2}\alpha_{ij}h_{ij} \qquad \text{with } \alpha_{ij} \in Z.$$

Then, for a fixed r, $\sum_{i=1}^{k}\alpha_{ir}h_{ir} = -\sum_{j\neq r}\sum_{i=1}^{k}\alpha_{ij}h_{ij} \in H(\tau_r) \cap \sum_{j\neq r}H(\tau_j) = 0$, since $\sum_{j\neq r}H(\tau_j) = \bigoplus_{j\neq r}H(\tau_j)$ is a pure subgroup of H containing no element of type τ_r. Thus, $\sum_{i=1}^{k}\alpha_{ir}h_{ir} = 0$ for each r. Since $\alpha_{ir}m_i h_i = \sum_{j=1}^{n+2}\alpha_{ir}h_{ij}$, we have $\sum_{i=1}^{k}\alpha_{ir}m_i h_i = \sum_{i=1}^{k}\sum_{j=1}^{n+2}\alpha_{ir}h_{ij} = \sum_{i=1}^{k}\alpha_{ir}h_{ir} + \sum_{i=1}^{k}\sum_{j\neq r}\alpha_{ir}h_{ij} \in \bigoplus_{j\neq r}H(\tau_j)$ because the first term is 0. Let $d = \gcd\{\alpha_{ir}m_i \mid 1 \leq i \leq k\}$ and write $\alpha_{ir}m_i = du_i$ so that $\gcd\{u_1, \dots, u_k\} = 1$. Then $d(\sum_{i=1}^{k}u_i h_i) \in \bigoplus_{j\neq r}H(\tau_j)$ and if $d \neq 0$, then purity implies that $\sum_{i=1}^{k}u_i h_i \in \bigoplus_{j\neq r}H(\tau_j) \subseteq R(H)$. Thus, $m_i|u_i$ for $1 \leq i \leq k$ and $m_1|u_i$ for all i. This contradiction shows $d = 0$ and each $\alpha_{ir} = 0$. Consequently, $\{h_{ij}\}$ is independent. In particular, rank $H(\tau_j) \geq k$ for $1 \leq j \leq n + 2$.

COROLLARY 5.2. *Under the hypotheses of* 5.1, *assume* rank $H = n + 2$ *and H is indecomposable. Then $H/R(H)$ is cyclic and* $\gcd(|H/R(H)|, p) = 1$ *for any prime p such that some $\tau \in T_{\rm cr}(H)$ is p-divisible.*

Proof. Using the notation of Lemma 5.1 and its proof, the set $\{h_{ij}\}$ is of cardinality $k(n+2)$. Thus, if rank $H = n + 2$, then $k = 1$ and $H/R(H) = \langle h_1 + R(H)\rangle$ is cyclic of order $m = m_1$. If $p|m$ and τ_1 is infinite at p, say, then $m = pm'$ and

$$mh_1 = pm'h_1 = \sum_{i=1}^{n+2} g_i = pg_1' + \sum_{i=2}^{n+2} g_i \qquad \text{with } g_i \in H(\tau_i) \text{ and } pg_1' = g_1.$$

This implies that $m'h_1 - g_1' \in \bigoplus_{i=2}^{n+2}H(\tau_i)$, a pure subgroup of H. Therefore, $m'(h_1 + R(H)) = 0$, a contradiction, since $m' < m$.

Remark. If $H \in \mathscr{K}(n)$, $|T_{\text{cr}}(H)| = n + 2 = \operatorname{rank}(H)$ and H is not completely decomposable, then H is indecomposable—any proper summand of H is completely decomposable by 2.1(a).

Recall (Section 1) that two torsion-free groups G and H of finite rank are *nearly isomorphic*, written $G \simeq_n H$, if for any nonzero integer m there is a monomorphism φ: $G \to H$ such that $H/\varphi(G)$ is finite of order relatively prime to m. We will make use of a result due to Lady [L-2], or see [A-2], that if $G \simeq_n H_1 \oplus H_2$, then $G = G_1 \oplus G_2$ where $G_i \simeq_n H_i$ for $i = 1, 2$.

THEOREM 5.3. *If* $X, Y \in \mathscr{K}(n)$ *and* $|T_{\text{cr}}(X)| = n + 2 = |T_{\text{cr}}(Y)|$, *then* $X \simeq_n Y$ *if and only if* $R(X) \simeq R(Y)$ *and* $X/R(X) \simeq X/R(Y)$.

Proof. If X and Y are nearly isomorphic, then X, Y, $R(X)$, and $R(Y)$ are all quasi-isomorphic. It follows that $R(X) \simeq R(Y)$ since both are completely decomposable. If the critical typeset is not an anti-chain, then by Lemma 5.1 X and Y are completely decomposable. In this case, $X = R(X)$ and $Y = R(Y)$, so the quotients are trivially isomorphic. If the critical typeset is an antichain, then X has a unique regulating subgroup $R(X) = \sum X(\tau)$, where the sum is over $\tau \in T_{\text{cr}}(X)$. In this case, let φ: $X \to Y$ be any homomorphism. Then $\varphi R(X) = \sum \varphi X(\tau) \subseteq \sum Y(\tau) = R(Y)$. If we choose φ to be a monomorphism with $|Y/\varphi X|$ relatively prime to $|Y/R(Y)|$, then there is an induced epimorphism of $X/R(X)$ onto $Y/R(Y)$. By symmetry, there is an epimorphism of $Y/R(Y)$ onto $X/R(X)$. Since both groups are finite, it follows that $X/R(X) \simeq Y/R(Y)$.

Conversely, suppose $R(X) \simeq R(Y)$ and $X/R(X) \simeq Y/R(Y)$. To show X and Y are nearly isomorphic, first suppose p doesn't divide $m = |X/R(X)|$. In this case, the composition φ: $X \overset{m}{\to} R(X) \simeq R(Y) \subseteq Y$ provides a monomorphism with $|Y/\varphi(X)|$ prime to p. Next suppose p divides $|X/R(X)|$. As in Lemma 5.1, $X/R(X) = \bigoplus_{i=1}^{k} \langle x_i + R(X) \rangle$ where $\langle x_i + X \rangle$ is cyclic of order m_i and m_i divides m_{i+1}. Moreover, $m_i x_i = \sum_{j=1}^{n+2} x_{ij}$ with $x_{ij} \in X(\tau_j)$ and $\{x_{ij}\}$ independent. Similarly, $Y/R(Y) = \bigoplus_{i=1}^{k} \langle y_i + R(Y) \rangle$ and $m_i y_i = \sum_{j=1}^{n+2} y_{ij}$. For each j, the pure subgroup of $X(\tau_j)$ generated by $\{x_{1j}, \ldots, x_{kj}\}$ is a summand of $X(\tau)$. Plainly, the complementary summand will be a completely decomposable summand of X, isomorphic to the analogous completely decomposable summand of Y. Thus, it is sufficient to assume $X(\tau_j) = \langle x_{1j}, \ldots, x_{kj} \rangle_* = \bigoplus_{i=1}^{k} \langle x_{ij} \rangle_*$. We will need one additional fact about the elements x_{ij}, namely, that p-height$(x_{ij}) = 0$ for each p dividing m_i. To see this, fix i and p dividing m_i, and let $J(i, p)$ be the set of all indices j such that p-height$(x_{ij}) = 0$. If $|J(i, p)| \le n + 1$, then $\sum_{j \in J(i,p)} X(\tau_j)$ is pure in X because $X \in \mathscr{K}(n)$. But then $m_i x_i - \sum_{j \notin J(i,p)} x_{ij} = \sum_{j \in J(i,p)} x_{ij}$ and the left hand side is divisible by p while the right hand side is not. It follows that $|J(i, p)| = n + 2$, as desired. The discussion to this point allows us to define a map φ: $X \to Y$ by sending x_{ij} to y_{ij}. This map is a quasi-isomorphism with $Y_p = \varphi(X)_p$ for each p dividing m_k and the proof is complete.

COROLLARY 5.4. *Let $H \in \mathscr{K}(n)$ have critical typeset which is an antichain containing $n+2$ elements. Then H is a direct sum of rank one groups and indecomposable groups nearly isomorphic to groups of the form in Example 4.1(b).*

Proof. The idea comes from [MV]. We will construct a group H that is nearly isomorphic to G and is a direct sum of rank one groups and indecomposable groups of the form in 4.1(b). The result then follows from Lady's theorem on near-isomorphism uniqueness of decompositions [L-2]. Write $T_{\mathrm{cr}}(G) = \{\tau_1, \dots, \tau_{n+2}\}$. We need only consider the case where G satisfies the conditions of Lemma 5.1: $T_{\mathrm{cr}}(G)$ is an antichain in which the meet of any two distinct elements is the same; $G/R(G) = \bigoplus_{i=1}^{k} C_i$, where C_i is a nontrivial finite cyclic group of order m_i; and m_i divides m_{i+1} for $1 \le i < k$. In particular, $m_1 > 1$ and $m_k(G/R(G)) = 0$. If p is any prime such that τ_i is infinite at p for some i, then we have by Corollary 5.2 that $\gcd(m_k, p) = 1$ so that $\gcd(m_i, p) = 1$ for each i. We can use Example 4.1(b) to construct an indecomposable $\mathscr{K}(n)$-group H_i with $H_i/R(H_i) \simeq C_i$ for $1 \le i \le k$. Specifically, for $1 \le i \le n+2$, choose subgroups $Z \subseteq A_i \subseteq Q$ so that $\mathrm{type}(A_i) = \tau_i$ and so that the lattice of subgroups of Q generated by the A_i's is isomorphic to the lattice of types generated by the τ_i's under the map that sends A_i to τ_i. Let $A_0 = A_i \cap A_j$ for any $i \neq j$. By construction, A_0 is independent of the choice of i and j. Let K_i be the pure subgroup of $G_i = m_i^{-1}A_0 \oplus A_1 \oplus \cdots \oplus A_{n+2}$ generated by $(1, 1, \dots, 1)$. By Example 4.1(b), $H_i = G_i/K_i$ is an indecomposable almost completely decomposable $\mathscr{K}(n)$-group. It is easy to see that $H_i/R(H_i)$ is cyclic of order m_i. Let $H = \bigoplus_{i=1}^{k} H_i \oplus [\bigoplus_{j=1}^{n+2} X_j]$, where X_j is a homogeneous completely decomposable group of type τ_j with $\mathrm{rank}(X_j) = \mathrm{rank}(G(\tau_j)) - k$ (non-negative by 5.1), $1 \le j \le n+2$. Then H is a direct sum of $\mathscr{K}(n)$-groups, whence a $\mathscr{K}(n)$-group. Moreover,

$$R(H) = \bigoplus_{i=1}^{k} R(H_i) \oplus \left[\bigoplus_{j=1}^{n+2} X_j\right] \simeq R(G);$$

and

$$H/R(H) \simeq \bigoplus_{i=1}^{k} H_i/R(H_i) \simeq \bigoplus_{i=1}^{k} C_i = G/R(G).$$

By Theorem 5.3, $G \simeq_n H$ so that, by Lady's theorem, $G = \bigoplus_{i=1}^{k} G_i \oplus [\bigoplus_{j=1}^{n+2} Y_j]$ with $G_i \simeq_n H_i$ and $Y_j \simeq_n X_j$. For homogeneous completely decomposable groups, quasi-isomorphism implies isomorphism. Thus, $Y_j \simeq X_j$ is homogeneous completely decomposable. Since each H_i has the form in 4.1(b), the proof is complete.

We conclude the paper with some remarks and open questions. First, all of our results here can be dualized to the cobalanced Butler groups defined by Kravchenko [Kr]. In the quasi-homomorphism setting, the standard Butler duality applies (see [AV-2]). In the locally free setting, Warfield duality can be used [W]. Details will be provided in future work.

Second, there is an intriguing behavior associated with $\mathscr{K}(n)$ groups, where results on the class $\mathscr{K}(n)$ are "linear in n." See 1.8, 2.1, 2.3, 3.4, 4.1, 4.4, and 5.4 for examples. It would be nice to have a metatheorem explaining this phenomenon. Short of that, one can look to exploit the linearity to extend known classification of Butler groups to classes of $\mathscr{K}(n)$-groups, as was done in Theorem 5.4.

Our work on balanced representations of posets [NV] has already been mentioned. The well-known result of Butler allows us to apply any results in the representation setting to the quasi-homomorphism category of Butler groups. Those familiar with the theory of representations of posets know that, in general, 3-element antichains have finite representation type, 4-element antichains have tame representation type, and 5-element antichains have wild representation type. In the balanced case, we discussed after 4.1 how the balanced Butler groups whose critical typeset is a four-element antichain are quasi-isomorphic to direct sums of strongly indecomposable groups of rank one and rank three (finite representation type). Work in progress indicates that the balanced Butler groups with critical typeset that is an antichain of cardinality 5 or greater have wild representation type. This intriguing behavior can also be investigated in $\mathscr{K}(n)$ for $n > 1$.

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