An Algorithm for the Symmetric Generalized Eigenvalue Problem

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ABSTRACT

A new method is presented for the solution of the matrix eigenvalue problem

\[ Ax = \lambda Bx, \]

where \( A \) and \( B \) are real symmetric square matrices and \( B \) is positive semidefinite. It reduces \( A \) and \( B \) to diagonal form by congruence transformations that preserve the symmetry of the problem. This method is closely related to the QR algorithm for real symmetric matrices.

1. INTRODUCTION

The generalized eigenvalue problem

\[ Ax = \lambda Bx, \tag{1.1} \]

where \( A \) and \( B \) are real symmetric \( n \times n \) matrices and \( B \) is positive definite and well conditioned with respect to inversion, is most frequently solved by computing the Cholesky decomposition of \( B \),

\[ B = LL^T, \tag{1.2} \]

to reduce (1.1) to the standard form

\[ L^{-1}AL^{-T}y = \lambda y \tag{1.3} \]

(see e.g. [5]). The solutions of (1.3) are then computed with one of the methods for the standard eigenvalue problem, e.g. the QL algorithm [1].
If $B$ is nearly singular, then in general $\|L^{-1}AL^{-T}\|$ will become very large and the eigenvalues will vary over many orders of magnitude. In general the small eigenvalues of (1.3) can only be computed very inaccurately from (1.3) [11, p. 337 f.] (see also Example 1 of Section 6).

If $B$ is singular or nearly singular, then the Fix-Heiberger reduction [2] may be used, which deflates those eigenvalues of $B$ from the problem (1.1) which are too small. For this purpose the spectral decomposition of $B$ is computed:

$$B = G \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} G^T,$$

where $G^TG = I$ and $\Lambda_1, \Lambda_2$ are diagonal, such that $\|\Lambda_2\| < \eta \|\Lambda_1\|$ for a chosen tolerance $\eta$. $\Lambda_2$ is then replaced by zero and the problem is reduced to the symmetric standard form $Cy = \lambda y$, where $\|C\|$ is not much larger than $\|A\|$ (see also [8] for a similar method).

To solve (1.1) the QZ algorithm [6] or the LZ algorithm [4] could be applied. These are known to be stable methods even for singular or nearly singular $B$. Starting with $A_0 = A$ and $B_0 = B$, they construct two sequences

$$A_{i+1} = K_i A_i N_i, \quad B_{i+1} = K_i B_i N_i,$$

such that each $B_{i+1}$ is upper triangular, each $A_{i+1}$ is upper Hessenberg, and under suitable conditions the $A_{i+1}$ tend to upper triangular form as $i$ tends to infinity. We do not have to distinguish between infinite and finite eigenvalues, because they are treated alike. All eigenvalues $\lambda_i$ are given as the quotient of the pairs of corresponding diagonal elements $\alpha_i, \beta_i$ in the resulting upper triangular matrices. Unfortunately these two algorithms cannot take advantage of the symmetry of $A$ and $B$. The $A_{i+1}$ and $B_{i+1}$ will be full upper Hessenberg and full triangular matrices rather than tridiagonal and diagonal respectively.

In this paper the MDR algorithm is presented, which finds the eigensystem of

$$Ax = \lambda Bx,$$

where $A, B$ are real symmetric square matrices and $B$ is positive semidefinite. It combines the advantages of the QZ or LZ method and the first-mentioned standard method. As in the QZ or LZ method, finite and infinite eigenvalues are treated alike, and the singularity or nearness to singularity of $B$ does not affect the computation of the (well-conditioned) small eigenvalues. No deci-
A necessary condition about neglectability of eigenvalues of \( B \) as in the Fix-Heiberger reduction.

As in the standard method, we can preserve the symmetry of the matrices, which reduces the amount of computational work and the size of necessary storage considerably compared with the QZ and LZ algorithm.

Starting with \( A_0 = A \) and \( B_0 = B \), we construct two sequences

\[
A_{i+1} = M_i^T A_i M_i, \quad B_{i+1} = M_i^T B_i M_i,
\]

such that each \( B_{i+1} \) is diagonal, each \( A_{i+1} \) is tridiagonal for \( i \geq 1 \), and under suitable conditions the \( A_{i+1} \) tend to diagonal form as \( i \) tends to infinity.

The \( M_i \) are chosen so that their \( \| \cdot \|_2 \)-condition number \( \| M_i \|_2 \| M_i^{-1} \|_2 \) is not too large (where \( \| x \|_2 = (\sum_{i=1}^{n} x_i^2)^{1/2} \) for \( x \in \mathbb{R}^n \), and for \( C \in \mathbb{R}^{n \times n} \), \( \| C \|_2 \) is the spectral norm—the corresponding matrix norm). The infinite and the ill-disposed eigenvalues of (1.1) will appear in the resulting pair of diagonal matrices as pairs \( a, b \) where \( b \) is zero and where \( a, b \) both are very small, respectively, just as they do in the QZ and LZ algorithm. The use of congruence transformations instead of equivalences as in the QZ and LZ algorithm preserves the symmetry of the problem and allows one to work in each step with a tridiagonal and a diagonal matrix instead of Hessenberg and triangular matrices.

As congruence transformations with nonsingular matrices do not change the inertia of a symmetric matrix, the diagonals \( B_{i+1} \) will only have nonnegative or positive diagonal entries for positive semidefinite or positive definite \( B \) respectively. This fact will be used throughout the paper without being explicitly mentioned.

In an initial step the positive semidefinite matrix \( B \) is reduced to diagonal form \( D = M^T B M \) in an appropriate way. In Section 2 we give a necessary and sufficient condition for the existence of a nonsingular matrix, which simultaneously diagonalizes the transformed matrices \( D = M^T B M \) and \( M^T A M \).

Section 3 describes elementary matrices \( M_{ji} \) which annihilate the \((j, i)\)th element of a matrix, transform a given diagonal \( D \) to diagonal form again (i.e., \( M_{ji}^T D M_{ji} \) is diagonal), and have optimal condition number. With these matrices we define the MDR algorithm for the computation of the eigenvalues of \( A x = \lambda D x \), where \( D \) is a nonnegative diagonal and \( A \) an arbitrary real square matrix. Section 4 shows that this algorithm is very closely connected with the QR algorithm for \( D^{-1/2} A D^{-1/2} \), if \( D \) is nonsingular. A proof of convergence for the algorithm is given.

Section 5 describes a further reduction of the symmetric generalized eigenproblem to condensed form and two versions of the algorithm using shifts to accelerate convergence.

Finally in Section 6 a few numerical examples are given.
2. INITIAL REDUCTION

Let us consider the problem
\[ \tilde{A}x - \lambda Bx, \quad (2.1) \]

where \( \tilde{A}, B \) are real symmetric \( n \times n \) matrices and \( B \) is positive semidefinite. In a preparatory step the matrix \( B \) has to be reduced to diagonal form \( D_1 \):
\[ M_0^TBM_0 = D_1 = \text{diag}(d_1, \ldots, d_n), \]
and correspondingly \( \tilde{A} \) must be transformed to
\[ M_0^T\tilde{A}M_0 = \Lambda \]
such that \( M_0 \) is nonsingular and \( \tilde{A} \) and \( B \) are not enlarged too much by this transformation. This may for instance be performed by some modified Cholesky decomposition for \( B \) with suitable pivoting.

For the computations of Section 6 the following simple elimination process was used: The nondiagonal entries of \( B \) in the \( k \)th column and \( k \)th row were eliminated by a Gaussian elimination step for \( k = n, \ldots, 2 \). In each step the greatest diagonal entry of the remaining (positive semidefinite) matrix was chosen as the pivot element. In practice this procedure tends to arrange the diagonal elements \( d_i \) in increasing order, i.e., the smallest elements will in general be on the top. This is just what the MDR algorithm turns out to do too, so that one should either use this method of diagonalizing \( B \) or connect any other method with a subsequent ordering of the diagonal elements. We have now transformed the given problem to
\[ Ay = \lambda Dy, \quad (2.2) \]
where \( A \) is symmetric and \( D \) is diagonal with nonnegative diagonal entries.

Our aim is to diagonalize \( A \) and \( D \) simultaneously.

If \( D \) is singular, there does not always exist a nonsingular matrix \( M \) for which \( M^TAM \) and \( M^TDM \) both are diagonal; e.g., for
\[ A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \]
it is easily checked that no such \( M \) exists. With the following theorem we find a necessary and sufficient condition for the existence of a simultaneously diagonalizing matrix.
THEOREM 2.1. For

\[ D = \begin{bmatrix} 0 & 0 \\ 0 & I_s \end{bmatrix}, \]

where \( I_s \) is the unit matrix of dimension \( s, 0 < s < n \), and correspondingly partitioned

\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}, \]

A symmetric, the following statements are equivalent:

(a) \( \text{rank} \left[ \begin{bmatrix} A_{11} \\ A_{12}^T \end{bmatrix} \right] = \text{rank} A_{11}. \)

(b) There exists a nonsingular matrix \( M \) such that \( M^T A M \) and \( M^T D M \) both are diagonal.

Proof. For \( M \) from (b) we may assume, without loss of generality, that

\[ M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \]

is partitioned correspondingly, we get from (2.3)

\[ M_{22}^T M_{22} = I_s \quad \text{and} \quad M_{21}^T M_{21} = 0 \]

i.e.,

\[ M_{22} \text{ is orthogonal,} \quad M_{21} = 0, \quad \text{and} \quad M_{11} \text{ is nonsingular,} \quad (2.4) \]

because \( M \) is nonsingular. Therefore

\[ M^T A M = \begin{bmatrix} C_{11} & C_{12} \\ C_{12}^T & C_{22} \end{bmatrix}, \quad \text{where} \quad C_{12}^T = M_{12}^T A_{11} M_{11} + M_{22}^T A_{12}^T M_{11}. \]

(2.5)
If $M^TAM$ is diagonal, then $C_{12}^T = 0$, i.e.,

$$0 = M_{12}^T A_{11} M_{11} + M_{22}^T A_{12}^T M_{11}.$$  

Because of (2.4) this equality may be transformed to

$$0 = M_{22} M_{12}^T A_{11} + A_{12}^T,$$

which means that

$$\text{rank} \begin{bmatrix} A_{11} \\ A_{12}^T \end{bmatrix} = \text{rank} A_{11}.$$

On the other hand, if

$$\text{rank} \begin{bmatrix} A_{11} \\ A_{12}^T \end{bmatrix} = \text{rank} A_{11},$$

then there exists a matrix $W$ such that

$$A_{12}^T + WA_{11} = 0.$$  

There exist also orthogonal matrices $M_{11}$ and $M_{22}$ such that

$$M_{11}^T A_{11} M_{11} = C_{11} \text{ is diagonal}$$

and

$$M_{22}^T \begin{bmatrix} A_{22} + WA_{12} + A_{12}^T W^T + WA_{11} W^T \\ W^T \end{bmatrix} M_{22} = C_{22} \text{ is diagonal}.$$  

If in addition we define $M_{12} = W^T M_{22}$, then the matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}$$

simultaneously diagonalizes $A$ and $D$.  

For the problem (2.2) we may assume without loss of generality that

$$D = \begin{bmatrix} 0 & 0 \\ 0 & D_s \end{bmatrix},$$
where $D_s$ is an $s \times s$ diagonal matrix with positive diagonal entries. If $s = 0$ or $s = n$, then in any case $A$ and $D$ are simultaneously diagonalizable by a congruence transformation. For $0 < s < n$ we find in particular, by Theorem 2.1, that if the $s \times s$ leading principal minor of $A$ is nonzero, then there is a simultaneously diagonalizing matrix

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}.$$ 

In this case we see by (2.5), where $C_{11} = M_{11}^T A_{11} M_{11}$, that the problem has $s$ infinite and $n - s$ finite eigenvalues.

For all further transformations of (2.2) we will use matrices which do not destroy the diagonal form of the right-hand side of (2.1). In addition these congruence transformations should not change the sensitivity of the eigenvalues to perturbations too much.

For real symmetric $A$, $B$ and a simple eigenvalue $\lambda$ with normalized eigenvector $x$, i.e. $Ax = \lambda Bx$ and $\|x\|_2 = 1$,

$$s(A, B, \lambda) = \left(\frac{\|A\|_2^2 + \|B\|_2^2}{(x^T Ax)^2 + (x^T Bx)^2}\right)^{1/2}$$

is a measure of the sensitivity of $\lambda$ to perturbations in $A$ and $B$ [9, 10]. The bigger $s(A, B, \lambda)$, the more sensitive $\lambda$ will be to perturbations in $A$ and $B$. If $A = M^T A M$ and $B = M^T B M$ for a nonsingular $M$, then it is easily seen that

$$\frac{1}{\|M\|_2 \|M^{-1}\|_2} s(A, B, \lambda) \leq s(\tilde{A}, \tilde{B}, \lambda) \leq s(A, B, \lambda).$$

Therefore the free parameters in the following elementary elimination matrices $M_i$ are chosen to minimize the $\|\cdot\|_2$-condition number $\text{cond}_2(M_i) = \|M_i\|_2 \|M_i^{-1}\|_2$.

3. MDR REDUCTION

We would like to transform (2.2) by matrices which eliminate certain elements of $A$ and do not destroy the diagonal form of $D = \text{diag}(d_1, \ldots, d_n)$. 

SYMMETRIC GENERALIZED EIGENVALUE PROBLEM
In Sections 3 and 4 we will not need the symmetry of $A$, so for these two sections we replace it by an arbitrary real $n \times n$ matrix $A = (a_{ij})_{i, j \in \{1, \ldots, n\}}$. The problem of annihilating the $(j, i)$th element of $A$, $j \neq i$, is essentially two-dimensional. For the $(i, j)$ plane we have to solve the

**Basic elimination problem.** Let

$$a = \begin{bmatrix} a_{ii} \\ a_{ji} \end{bmatrix}, \quad a_{ji} \neq 0, \quad D = \begin{bmatrix} d_i & 0 \\ 0 & d_j \end{bmatrix} \geq 0.$$

Find a nonsingular matrix $M$ such that

$$M^T a = \begin{bmatrix} k \\ 0 \end{bmatrix}, \quad k > 0, \quad M^T D M = \begin{bmatrix} c_i & 0 \\ 0 & c_j \end{bmatrix}$$

and $\text{cond}_2(M) = \|M\|_2\|M^{-1}\|_2$ is as small as possible.

For the solution we distinguish three cases.

**Case 1:** $a_{ii} = 0$. The simplest solution is then

$$M^T = \text{sgn}(a_{ji}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where $k = |a_{ji}|$, $c_i = d_j$, $c_j = d_i$, and $\text{cond}_2(M) = 1$.

**Case 2:** $a_{ii} \neq 0$, $d_i = d_j = 0$. The problem is then solved by the orthogonal matrix

$$M^T = \frac{1}{\|a\|_2} \begin{bmatrix} a_{ii} & a_{ji} \\ -a_{ji} & a_{ii} \end{bmatrix}$$

Then we have $k = \|a\|_2$, $c_i = c_j = 0$, and $\text{cond}_2(M) = 1$.

**Case 3:** $a_{ii} \neq 0$, $d_i + d_j \neq 0$. We find the solution

$$M^T = \begin{bmatrix} b_{ii} & b_{ji} \\ \|b\|_2 & \|b\|_2 \\ a_{ji} & a_{ii} \\ \|a\|_2 & \|a\|_2 \end{bmatrix}, \quad \text{where} \quad b = \begin{bmatrix} b_{ii} \\ b_{ji} \end{bmatrix} = \begin{bmatrix} d_i a_{ii} \\ d_j a_{ij} \end{bmatrix}.$$
Then we have $k = (b, a) / \|b\|_2$, where $0 < k \leq \|a\|_2$ and $(\cdot, \cdot)$ denotes the standard scalar product,

$$c_i = d_i d_j \frac{(b, a)}{\|b\|_2^2} \left( = \frac{d_i b_{ii}^2}{\|b\|_2^2} + \frac{d_j b_{jj}^2}{\|b\|_2^2} \right),$$

$$c_j = \frac{(b, a)}{\|a\|_2^2} \left( = \frac{d_i a_{ji}^2}{\|a\|_2^2} + \frac{d_j a_{ij}^2}{\|a\|_2^2} \right).$$

and

$$\min \{ d_i, d_j \} \leq c_i \leq \max \{ d_i, d_j \} \quad \text{for} \quad l \in \{ i, j \}.$$ 

(If $d_i = d_j$ then $c_i = c_j$ and $M$ is orthogonal.) In addition we find that

$$1 \leq \|M\|_2 = (1 + \alpha)^{1/2} \leq \sqrt{2}$$

and

$$1 \leq \|M^{-1}\|_2 = \frac{1}{(1 - \alpha)^{1/2}},$$

where

$$\alpha = \left| \frac{a_{ii} a_{ji} (d_i - d_j)}{\|b\|_2 \|a\|_2} \right| < 1.$$ 

The conditions

$$M^T a = \begin{pmatrix} k \\ 0 \end{pmatrix} \quad \text{and} \quad M^T D M = \begin{pmatrix} c_i & 0 \\ 0 & c_j \end{pmatrix}$$

determine $M^T$ uniquely up to a premultiplied diagonal matrix, which is here chosen to normalize the rows of $M^T$ with respect to $\|\cdot\|_2$. This choice of the diagonal matrix is known to give the minimal $\|\cdot\|_2$ condition number [3].
**Example 3.1.**

(1) For

\[ a = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 10^{-20} \end{bmatrix}, \]

we get

\[ M^T = \begin{bmatrix} 10^{-20}/(10^{-40} + 9)^{1/2} & 3/(10^{-40} + 9)^{1/2} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}, \]

\[ c_i = 10^{-20} \frac{10^{-20} + 9}{10^{-40} + 9}, \quad c_j = \frac{9}{10} + \frac{10^{-20}}{10}, \]

\[ k = \frac{9 + 10^{-20}}{(10^{-40} + 9)^{1/2}} \quad \text{and} \quad \text{cond}_2(M) < \sqrt{2}. \]

(2) For

\[ a = \begin{bmatrix} 10^{-4} \\ 1 \end{bmatrix}, \quad D = \begin{bmatrix} 10^{-8} & 0 \\ 0 & 1 \end{bmatrix}, \]

we get

\[ M^T = \frac{1}{(1 + 10^{-8})^{1/2}} \begin{bmatrix} 1 & 10^{-4} \\ -1 & 10^{-4} \end{bmatrix}, \]

\[ c_i = c_j = \frac{2 \times 10^{-8}}{1 + 10^{-8}}, \]

\[ k = 2 \times 10^{-4}/(1 + 10^{-8})^{1/2} \quad \text{and} \quad \text{cond}_2(M) = 10^4. \]

Note that in this case the symmetric 2×2 problem in the (i, j) plane,

\[ \begin{bmatrix} 10^{-4} & 1 \\ 1 & a_{jj} \end{bmatrix} x = \lambda \begin{bmatrix} 10^{-8} & 0 \\ 0 & 1 \end{bmatrix} x, \]

is near to a problem where the matrices cannot be simultaneously diagonalized (see Section 2).
For $i < j$ the $(j, i)$th element of $A$ is annihilated by premultiplying with the matrix

\[
M_{j,i}^T = \begin{bmatrix}
1 & & & & & \\
& \ddots & & & & \\
& & 1 & & & \\
& & m_{ii} & 0 & \cdots & 0 \\
& & 0 & 1 & \cdots & 0 \\
& & \vdots & \ddots & \ddots & \ddots \\
& & 0 & \cdots & 1 & 0 \\
& m_{ji} & 0 & \cdots & 0 & m_{jj}
\end{bmatrix}
\]

where

\[
M^T = \begin{bmatrix}
m_{ii} & m_{ij} \\
m_{ji} & m_{jj}
\end{bmatrix}
\]

is the solution of the basic elimination problem for

\[
\begin{bmatrix}
a_{ii} \\
a_{ji}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
d_i & 0 \\
0 & d_j
\end{bmatrix}
\]

The elements $d_i$ and $d_j$ of the diagonal matrix $D$ then have to be replaced by $c_i$ and $c_j$, respectively.

With these elementary eliminating matrices we may construct a special reduction for a given matrix described in the following lemma.

**Lemma 3.2.** Let $D$ be a given nonnegative diagonal matrix of dimension $n$. For every real $n \times n$ matrix $A$ there exists a nonsingular matrix $M$ such that

1. $M^T A = R$ is an upper triangular matrix with nonnegative diagonal entries;
2. $M^T D M = C$ is a diagonal matrix;
(3) if $P, Q$ are permutations for which

$$PDP^T = \begin{bmatrix} 0 & \cdots & 0 \\ & \ddots & \\ & & 0 \end{bmatrix} \quad \text{and} \quad QCQ^T = \begin{bmatrix} 0 & \cdots & 0 \\ & \ddots & \\ & & 0 \end{bmatrix},$$

where $\tilde{D}$ and $\tilde{C}$ are nonsingular, then we have with corresponding partitioning

$$PMQT = \begin{bmatrix} M_{11} \\ M_{12} \\ 0 \\ M_{22} \end{bmatrix}$$

and $M_{11}$ is orthogonal.

If $D$ has only positive diagonal entries and $\Lambda$ is nonsingular, then $M$ and $R$ are essentially uniquely determined, i.e., if $N$ and $S$ is another pair of matrices for which (1) and (2) hold, then there exists a diagonal matrix $J$ with positive diagonal entries such that

$$N = MJ \quad \text{and} \quad S = JR.$$

**Proof.** For nonsingular $D$ the lemma is of course trivial, because then $D^{-1/2}$ exists and $D^{-1/2}D$ has a QR decomposition

$$D^{-1/2}D = QR,$$

where $Q$ is orthogonal and $R$ is an upper triangular matrix. For any diagonal matrix $C$ with positive diagonal entries, $M = D^{-1/2}QC^{1/2}$ is a matrix for which the lemma holds. This reduction of $\Lambda$ is essentially unique because of the uniqueness of the QR decomposition for nonsingular matrices.

In the general case the reduction may be constructed in the following way. With matrices $M_{j,i}^T$ from (3.1) we annihilate columnwise, as described above, the elements of $\Lambda$ below the diagonal in the order

$$(j, i) = (2, 1), (3, 1), \ldots, (n, 1), (3, 2), \ldots, (n, n-1).$$

The product of these matrices,

$$M^T = M_{n,n-1}^T \cdots M_{3,2}^T M_{n,1}^T \cdots M_{3,1}^T M_{2,1}^T,$$

satisfies (1) and (2) by construction.
Let $P$ and $Q$ be permutations as in 3. Let

$$
\tilde{M} = PMQ = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix}
$$

be partitioned correspondingly. Then

$$
\tilde{M}^T \begin{bmatrix}
0 & 0 \\
0 & \tilde{D}
\end{bmatrix} \tilde{M} = \begin{bmatrix}
0 & 0 \\
0 & \tilde{C}
\end{bmatrix}
$$

implies $M_{21}^T \tilde{D} M_{21} = 0$.

Because $\tilde{D}$ is positive definite, $M_{21}$ has to be the zero matrix. $M_{11}$ is orthogonal by the special construction. \hfill \blacksquare

The reduction $M^T A = R$ from Lemma 3.2, which is constructed in the special way described in the proof, will be called \textit{MDR reduction of} $A$, where the $D$ will always refer to the given diagonal matrix.

If $D$ is a multiple of the unit matrix, then this reduction is the QR decomposition of $A$ constructed with Givens rotations.

4. \textit{MDR ALGORITHM}

With these reductions an eigenvalue algorithm can be established.

MDR algorithm (Basic form). Let $D$ be a given nonnegative diagonal matrix and $A$ a real square matrix. Define

$$
A_1 = A \quad \text{and} \quad D_1 = D.
$$

For $i = 1, 2, 3, \ldots$ find the $MD_i R$ reduction of $A_i$:

$$
M_i^T A_i = R_i, \quad \text{where} \quad M_i^T D_i M_i = D_{i+1},
$$

and compute

$$
A_{i+1} = R_i M_i.
$$

Remark 4.1.

(1) If $A$ is an upper Hessenberg matrix, then by looking at the construction of the $MD_i R$ reduction it can be seen that all $A_i$ are upper Hessenberg.
(2) If \( A \) is symmetric, then obviously all \( A_i \) are symmetric. Therefore all \( A_i \) are symmetric and tridiagonal if the first matrix \( A \) is symmetric and tridiagonal.

(3) If \( D \) is a multiple of the unit matrix, then the MDR algorithm reduces to the QR algorithm in its basic form for \( A \), i.e., for \( A_1 = A \) we compute for \( i = 1, 2, 3, \ldots \) in the \( i \)th step,

\[
A_i = Q_i R_i,
\]

the QR decomposition with \( r_{jj} > 0 \), and

\[
A_{i+1} = R_i Q_i.
\]

The MDR algorithm is very closely connected with the QR algorithm.

**Lemma 4.2.** Let the diagonal matrix \( D \) have only positive diagonal entries, let the matrix \( A \) be nonsingular, and let \( (A_i)_{i \in \mathbb{N}_+}, (D_i)_{i \in \mathbb{N}_+} \) be the matrix sequences produced by the MDR algorithm for \( A \) and \( D \). If \( (C_i)_{i \in \mathbb{N}_+} \) is the sequence which the QR algorithm in its basic form (see Remark 4.1.3) generates for \( C_1 = D^{-1/2}AD^{-1/2} \), then

\[
A_{i+1} = D_{i+1}^{1/2}C_{i+1}D_{i+1}^{1/2}
\]

holds for all \( i \in \mathbb{N} \).

**Proof.** (By induction.) Suppose \( A_s = D_s^{1/2}C_s D_s^{1/2} \). Then in the \( s \)th step of the QR algorithm we decompose

\[
C_s = Q_s \tilde{R}_s
\]

and compute

\[
C_{s+1} = \tilde{R}_s Q_{s}.
\]

For the matrices from the \( s \)th step of the MDR algorithm we have

\[
R_s = M_s^T A_s = D_{s+1}^{1/2}D_s^{-1/2}M_s^T D_s^{1/3}D_s^{1/3}D_s^{-1/3}A_s D_s^{-1/2}D_s^{1/2}
\]

\[
= D_{s+1}^{1/2}U_s^T C_s D_s^{1/2}, \tag{4.1}
\]

where

\[
U_s^T = D_{s+1}^{-1/2}M_s^T D_s^{1/2}
\]
and
\[ U_s^T U_s = D_{s+1}^{-1/2} M_s^T D_s^{1/2} D_s^{1/2} M_s D_{s+1}^{-1/2} = I. \]

(4.1) now gives another QR decomposition of \( C_s \):
\[ C_s = U_s D_s^{-1/2} R_s D_s^{-1/2}. \]

Because the QR decomposition is uniquely determined, we find that
\[ \tilde{R}_s = D_{s+1}^{-1/2} R_s D_s^{-1/2} \quad \text{and} \quad Q_s = U_s D_{s+1}^{1/2} M_s D_s^{1/2}. \]

So finally we get
\[ A_{s+1} = R_s M_s = D_{s+1}^{1/2} \tilde{R}_s D_s^{1/2} D_s^{-1/2} Q_s D_{s+1}^{1/2} = D_{s+1}^{1/2} C_{s+1} D_{s+1}^{1/2}. \]

This relation may be used to prove that under certain conditions the MDR algorithm converges.

**Theorem 4.3** (Convergence of the MDR algorithm). Let \( D = \text{diag}(d_1, \ldots, d_n) \) be a nonnegative diagonal matrix such that \( d_1 = d_2 = \cdots = d_k = 0 \) and \( d_{k+1} d_{k+2} \cdots d_n \neq 0 \) (\( k \) may be zero). Let the leading \( k \times k \) principal submatrix of the \( n \times n \) matrix \( A \) have eigenvalues \( |\alpha_1| > |\alpha_2| > \cdots > |\alpha_k| > 0 \).

If for the \( n-k \) finite eigenvalues \( \lambda_{k+1}, \ldots, \lambda_n \) of \( Ax = \lambda Dx \) we have \( |\lambda_{k+1}| > |\lambda_{k+2}| > \cdots > |\lambda_n| > 0 \), then under the conditions (4.2) and (4.4) given below the \( A_{i+1} \) generated by the MDR algorithm for \( A \) and \( D \) tend to upper triangular form, and for the \( s \)th diagonal element \( a_{ss}^{(i)} \) of \( A_i \) and the \( s \)th diagonal element \( d_s^{(i)} \) of \( D_i \) we have
\[ \lim_{i \to \infty} a_{ss}^{(i)} = \alpha_s \quad \text{for} \quad s \in \{1, \ldots, k\}, \]
\[ \lim_{i \to \infty} \frac{a_{ss}^{(i)}}{d_s^{(i)}} = \lambda_s \quad \text{for} \quad s \in \{k+1, \ldots, n\}. \]

**Proof.** First it should be noted that \( Ax = \lambda Dx \) has \( k \) infinite and \( n-k \) finite eigenvalues, because the first \( k \times k \) principal minor of \( A \) is nonzero. This may easily be shown just as in Lemma 2.1.
For $A_1 = A$ and $D_1 = D$ we compute, in the first step of the MDR algorithm, the $MD_1R$ reduction of $A_1$:

$$M_1^T A_1 = R_1 \quad \text{and} \quad D_2 = M_1^T D_1 M_1.$$ 

Observing this reduction step by step, we notice that, because the leading $k \times k$ principal submatrix $A_{11}^{(1)}$ of $A_1$ is nonsingular, the zero diagonal entries of the new diagonal $D_2$ are still in top position, and according to Lemma 3.2(3)

$$M_1 = \begin{bmatrix} M_{11}^{(1)} & M_{12}^{(1)} \\ 0 & M_{22}^{(1)} \end{bmatrix}, \quad \text{where} \quad M_{11}^{(1)} \text{ is orthogonal.}$$

Therefore $A_2 = R_1 M_1$ is a block upper triangular matrix

$$A_2 = \begin{bmatrix} A_{11}^{(2)} & A_{12}^{(2)} \\ 0 & A_{22}^{(2)} \end{bmatrix}, \quad \text{where} \quad A_{11}^{(2)} = M_{11}^{(1)^T} A_{11}^{(1)} M_{11}^{(1)}.$$ 

It is easily seen that from now on all $M_i$ are block diagonal matrices:

$$M_i = \begin{bmatrix} M_{11}^{(i)} & 0 \\ 0 & M_{22}^{(i)} \end{bmatrix}, \quad \text{where} \quad M_{11}^{(i)} \text{ is orthogonal},$$

all $A_i$ are of the same block triangular form as $A_2$, and for all $D_i$ the zero entries on the diagonal are kept in the first $k$ positions, i.e.,

$$A_i = \begin{bmatrix} A_{11}^{(i)} & A_{12}^{(i)} \\ 0 & A_{22}^{(i)} \end{bmatrix} \quad \text{and} \quad D_i = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{D}_i \end{bmatrix}$$

where $\tilde{D}_i$ is nonsingular.

The algorithm now acts independently on the two diagonal blocks of $A_i$. In particular $(A_{11}^{(i+1)})_{i \in \mathbb{N}}$ is the sequence generated by the QR algorithm in its basic form for $A_{11}^{(i)}$.

Because of our assumptions there exists a nonsingular $X$ such that

$$A_{11}^{(i)} = X \text{diag}(\alpha_1, \ldots, \alpha_k) X^{-1}.$$ 

If

all leading principal minors of $X^{-1}$ are nonzero, \hspace{1cm} (4.2)
then the QR algorithm for $A_{11}^{(i)}$ converges in the sense that the $A_{11}^{(i)}$ tend to upper triangular form, and the diagonal elements converge to the $\alpha_1, \ldots, \alpha_k$ respectively [11, p. 517 ff.]. On the lower diagonal blocks of $A_i$ the algorithm now acts just as the MDR algorithm starting with $A_{22}^{(i)}$ and $D_2$.

By Lemma 4.2 we know that for all $i \geq 2$

$$A_{22}^{(i)} = \hat{D}_i^{1/2} C_i \hat{D}_i^{1/2}, \quad (4.3)$$

where the $C_i$ are those matrices which are generated by the QR algorithm in its basic form starting with $C_2 = \hat{D}_2^{-1/2} A_{22}^{(2)} \hat{D}_2^{-1/2}$. For $C_2$ there exists a nonsingular $Y$ such that

$$C_2 = Y \text{diag}(\lambda_{k+1}, \ldots, \lambda_n) Y^{-1}.$$  

If

$$\text{all leading principal minors of } Y^{-1} \text{ are nonzero,} \quad (4.4)$$

then, as above, the $C_i$ tend to upper triangular form and the $s$th diagonal elements $c_{ss}^{(i)}$ converge to $\lambda_s$.

Because the $\hat{D}_i$ are bounded [we have $\|\hat{D}_i\|_2 \leq \|\hat{D}_2\|_2$ for all $i \geq 2$ by construction of the decomposition (see Section 3)], we see, by (4.3), that the $A_{22}^{(i)}$ tend to upper triangular form and

$$\lim_{i \to \infty} a_{ss}^{(i)} / d_{ss}^{(i)} = \lim_{i \to \infty} c_{ss}^{(i)} = \lambda_s.$$  

5. PRACTICAL CONSIDERATIONS

We now return to our problem (2.2): $Ay = \lambda Dy$, where $A$ is symmetric and $D = \text{diag}(d_1, \ldots, d_n)$, with nonnegative $d_i$. $D$ was computed from the positive semidefinite matrix $B$ in (2.1), and in practice it will not often occur that diagonal entries of $D$ are exactly zero. If it does occur, we should perform one step of the MDR algorithm for $A$ and $D$ to split the problem as described in Section 4 into one with a zero diagonal matrix and one with a nonsingular matrix.

It will simplify the program for our algorithm considerably if we can work with the nonsingular diagonal and the zero diagonal (i.e. with the QR algorithm) separately. So for the following we will assume that $D$ has only positive but possibly extremely small diagonal entries.

Reduction to Tridiagonal Form

In a first step $A = (a_{ij})_{i,j \in \{1, \ldots, n\}}$ is reduced to symmetric tridiagonal form. For $i + 1 < j$ the $(j, i)$th element in $A$ is annihilated by premultiplying
with

\[
M_{j,i+1}^T = \begin{bmatrix}
1 & \cdots & & & \\
& 1 & \cdots & & \\
& & \ddots & \cdots & \\
& & & \ddots & \cdots \\
& & & & 1
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
m_{i+1,i+1} & m_{i+1,j} \\
m_{j,i+1} & m_{j,j}
\end{bmatrix}
\]

solves the basic elimination problem for

\[
\begin{bmatrix}
a_{i+1,i} \\
s_{j,i}
\end{bmatrix}
\text{ and } \begin{bmatrix}
d_{i+1} & 0 \\
0 & d_j
\end{bmatrix},
\]

giving two new diagonal elements \(c_{i+1}\) and \(c_j\).

Because \(A\) is symmetric and the \(j\)th column of \(M_{j,i+1}\) is the \(j\)th unit vector, \(M_{j,i+1}^TAM_{j,i+1}\) has zero elements in positions \((j,i)\) and \((i,j)\). In the diagonal matrix \(D\) the \((i+1)\)th and \(j\)th diagonal entries have to be replaced by \(c_{i+1}\) and \(c_j\). To reduce \(A\) to tridiagonal form the elements \((j,i)\) and \((i,j)\) are annihilated in the order

\[(j,i) = (3,1)(4,1),\ldots,(n,1)(4,3),\ldots,(n,n-2),\]

as described above. If the resulting tridiagonal matrix is irreducible (i.e., the subdiagonal elements are all nonzero), then this transformation is uniquely determined up to a diagonal matrix by fixing the first column of the transforming matrix.

**Lemma 5.1.** Let \(A\) be an \(n \times n\) symmetric matrix, let \(D\) be an \(n \times n\) diagonal matrix with positive diagonal entries, let \(M, N\) be \(n \times n\) nonsingular
matrices for which

\[ M^TAM = T \text{ is an irreducible tridiagonal matrix}, \]
\[ N^TAN = \bar{T} \text{ is a tridiagonal matrix}, \]
\[ M^TDM = C \text{ and } N^TDN = G \text{ both are diagonal}, \]

and let the first column of \( M \) and the first column of \( N \) be linearly dependent. Then there exists an orthogonal diagonal matrix \( Q \) such that

\[ N = MG^{-1/2}QTC^{1/2} \quad \text{and} \quad \bar{T} = C^{1/2}QG^{-1/2}TG^{-1/2}Q^TC^{1/2}. \]

**Proof.** We have \( M^TAN^{-1}M = T \) and \( M^TN^{-1}CM^{-1}M = G \). \( Q = C^{1/2}N^{-1}MG^{-1/2} \) is an orthogonal matrix, and its first column is a multiple of the first unit vector. Now \( Q \) transforms the tridiagonal \( C^{-1/2}\bar{T}C^{-1/2} \) to the irreducible tridiagonal \( C^{1/2}Tk^{-1}2C^{1/2} \), and therefore \( Q \) must be an orthogonal diagonal matrix [11, p. 352 ff.].

The original problem (2.1) is now transformed to

\[ T_1z = \lambda D_1z, \quad (5.1) \]

where \( T_1 \) is a symmetric tridiagonal matrix and \( D_1 \) is a diagonal matrix with positive diagonal entries. We may assume that \( T_1 \) is irreducible, because if it is not, we can split (5.1) into problems of smaller dimension.

To accelerate the convergence of the \( MDR \) algorithm for (5.1), shifts of origin can be used as in the \( QR \) and related algorithms.

**MDR Algorithm with Explicit Shift**

Let \( T_1 \) and \( D_1 \) be given as in (5.1).

For \( i = 1, 2, 3, \ldots \), find for a given real number \( k_i \) the \( MDR \) reduction of \( T_i - k_iD_i \):

\[ M_i^T(T_i - k_iD_i) = R_i, \quad \text{where} \quad M_i^TD_iM_i = D_{i+1}, \]

and compute

\[ T_{i+1} = R_iM_i + k_iD_{i+1}. \]

The relation between the \( T_i \) and the matrices \( C_i \), which the \( QR \) algorithm with these same shifts \( k_i \) generates for \( C_i = D_{i-1}^{-1/2}T_1D_1^{-1/2} \), is still

\[ T_i = D_i^{1/2}C_iD_i^{1/2}, \quad (5.2) \]
as in Lemma 4.2. If

\[
T_i = \begin{bmatrix}
\alpha_1 & \beta_1 \\
\beta_1 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & & \beta_{n-1} \\
& & & & \alpha_n
\end{bmatrix}
\]

and

\[D_i = \text{diag}(d_1, \ldots, d_n),\]

then the simplest choice of \(k_i\) would be \(k_i = \alpha_n / d_n\). For the computations of Section 6, \(k_i\) was chosen as

\[k_i = \gamma / d_n, \quad (5.3)\]

where

\[\gamma = \alpha_n - \frac{\text{sgn}(\delta) \beta_{n-1}^2 d_n}{|\delta| + \left( \delta^2 + \beta_{n-1}^2 d_n d_{n-1} \right)^{1/2}}\]

and

\[\delta = \frac{d_n \alpha_{n-1} - d_n \alpha_n}{2}.
\]

In this case \(k_i\) is the eigenvalue of

\[
\begin{bmatrix}
\alpha_{n-1} & \beta_{n-1} \\
\beta_{n-1} & \alpha_n
\end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \lambda \begin{bmatrix} d_{n-1} & 0 \\
0 & d_n
\end{bmatrix}
\]

which is nearest to \(\alpha_n / d_n\). This is Wilkinson's shift for \(D_i^{-1/2} T_i D_i^{-1/2}\), and it can be shown that the QR algorithm with these shifts always converges, i.e., the last subdiagonal element tends to zero [7]. By (5.2) this carries over to our algorithm because the \(D_i\) are bounded [we have \(\|D_i\|_2 \leq \|D\|_2\) by construction of the decomposition (see Section 3)].
In the ith step of the MDR algorithm with explicit shift the tridiagonal $T_i$ is transformed to the tridiagonal $T_{i+1} = M_i^T T_i M_i$. According to Lemma 5.1, for our irreducible tridiagonals this transformation is essentially uniquely determined by the first column of $M_i$. Therefore, as in the QR algorithm, we may perform the algorithm with implicit shift.

**MDR Algorithm with Implicit Shift**

Let $T_1$ and $D_1$ be given as in (5.1).

For $i = 1, 2, 3, \ldots$ compute for a given real number $k_i$ the first column vector $a$ of $T_i - k_i D_i$. Find $M_i^{(1)}$ for which $M_i^{(1)} a = c e_1$ and $M_i^{(1)} D_i M_i^{(1)} = D_i^{(1)}$ is diagonal, by solving the basic elimination problem for the two nonzero entries of $a$. Compute $M_i^{(1)} T_i M_i^{(1)}$ and reduce it to tridiagonal form with respect to $D_i^{(1)}$ as described above.

It can be shown with Lemma 5.1 that the matrices $T_i, D_i$ produced by the MDR algorithm with implicit shift differ from the corresponding $T_i, D_i$ from MDR with explicit shift only by a diagonal $\Lambda_i$, i.e.,

$$T_i = \Lambda_i T_i \Lambda_i \quad \text{and} \quad D_i = \Lambda_i D_i \Lambda_i \quad \text{for all } i,$$

and the $\Lambda_i$ and $\Lambda_i^{-1}$ are bounded.

6. **NUMERICAL EXAMPLES**

For our examples we reduced the problem (2.1), $Ax = \lambda Bx$, as described in Section 2 and Section 5, to $Tz = \lambda Dz$, where $T$ is an irreducible tridiagonal matrix and $D$ is a diagonal matrix with positive diagonal entries. The MDR algorithm with implicit shift was used, where the shifts were determined according to (5.3). For the iterates

$$T_i = \begin{bmatrix}
\alpha_1 & \beta_1 \\
\beta_1 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \beta_{n-1} \\
& & & \beta_{n-1} & \alpha_n
\end{bmatrix} \quad \text{and} \quad D_i = \text{diag}(d_1, \ldots, d_n)$$

a subdiagonal element $\beta_j$ was neglected if

$$|\beta_j| \leq 0.5 \times 10^{-10} \frac{|\alpha_j| + |\alpha_{j+1}|}{2}.$$
i.e., in this case the problem was split into two problems of dimension $j$ and $n - j$ respectively.

The examples give a list of the final diagonals replacing $A$ and $B$ respectively and of the corresponding eigenvalues.

The calculations were done on a TR440 computer.

**Example 1.**

$$A = \begin{bmatrix}
10 & 1100 \\
-100 & 11 \\
10 & 330 & -33 & 77.4 \\
10 & 110 & 11 & -25.8 & 7.6 \\
-10 & 110 & 11 & 23.8 & -7.6 & 12.6
\end{bmatrix}$$

$$B = \begin{bmatrix}
0.2 & \text{symm.} \\
-2 & 10^{-8} \\
0.2 & -2 \cdot 10^{-9} & 10.4 \\
-0.6 & 6 + 3 \times 10^{-9} & -31.2 & 104.4 \\
0.2 & -2 \cdot 10^{-9} & 10.4 & -34.8 & 12.6 \\
-0.2 & 2 + 10^{-9} & -10.4 & 34.8 & -12.6 & 12.8
\end{bmatrix}$$

Eigenvalues of $Ax = \lambda Bx$:

$$\begin{bmatrix}
10^{10} \\
50 \\
25 \\
0 \\
-1 \\
-2
\end{bmatrix}$$

The MDR algorithm gives the following results:

<table>
<thead>
<tr>
<th>Diagonal of $A$</th>
<th>Diagonal of $B$</th>
<th>Computed eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000000000</td>
<td>1.055013855E-10</td>
<td>9.478548512E9</td>
</tr>
<tr>
<td>9.900666376</td>
<td>1.980133275E-1</td>
<td>5.000000000E1</td>
</tr>
<tr>
<td>4.880210305</td>
<td>1.952084122E-1</td>
<td>2.500000000E1</td>
</tr>
<tr>
<td>-1.76273658E-11</td>
<td>2.736668979</td>
<td>-6.439484176E-12</td>
</tr>
<tr>
<td>-1.011799572</td>
<td>1.011799573</td>
<td>-9.999999994E-1</td>
</tr>
<tr>
<td>-1.540214239</td>
<td>7.701071197E-1</td>
<td>-2.000000000</td>
</tr>
</tbody>
</table>
Two NAG library programs have been used on the problem for comparison. The first one transforms $Ax = \lambda Bx$ to standard form (1.3) and applies the QL algorithm; the second one is the QZ algorithm.

*Transformation to Standard Form and QL Algorithm:*

<table>
<thead>
<tr>
<th>Computed eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>$9.761298308E9$</td>
</tr>
<tr>
<td>$5.000000001E1$</td>
</tr>
<tr>
<td>$2.500001490E1$</td>
</tr>
<tr>
<td>$2.910365282E-10$</td>
</tr>
<tr>
<td>$-1.000357189$</td>
</tr>
<tr>
<td>$-1.999999998$</td>
</tr>
</tbody>
</table>

*QZ Algorithm:*

<table>
<thead>
<tr>
<th>Diagonal of $A$</th>
<th>Diagonal of $B$</th>
<th>Computed eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.053048381E1$</td>
<td>$1.035007694E-9$</td>
<td>$1.017430485E10$</td>
</tr>
<tr>
<td>$1.004987564E2$</td>
<td>$2.009975128$</td>
<td>$5.000000001E1$</td>
</tr>
<tr>
<td>$3.535533906$</td>
<td>$1.414213562E1$</td>
<td>$2.500000000E1$</td>
</tr>
<tr>
<td>$3.277674285E-10$</td>
<td>$3.366168312E1$</td>
<td>$9.737107542E-12$</td>
</tr>
<tr>
<td>$-7.171013081$</td>
<td>$3.585506543$</td>
<td>$-1.999999999$</td>
</tr>
<tr>
<td>$-1.219673442$</td>
<td>$1.219673443$</td>
<td>$-9.999999997E-1$</td>
</tr>
</tbody>
</table>

The approximation to the eigenvalue $-1$ is, even in this small-dimensional example, far less accurate for the second method than for the two others.

**Example 2.** We have constructed many problems by choosing two diagonal matrices $D_A$ and $D_B$ and multiplying them with an elementary orthogonal matrix $Q = (I - 2vv^T)$, where $\|v\|_2 = 1$, giving $A = Q^T D_A Q$ and $B = Q^T D_B Q$. A typical example of this kind is the $20 \times 20$ problem

$$A = (I - 2vv^T)D_A(I - 2vv^T)$$ and $$B = (I - 2vv^T)D_B(I - 2vv^T),$$

where

$$v^T = \left( \frac{1}{\sqrt{20}}, \frac{1}{\sqrt{20}}, \ldots, \frac{1}{\sqrt{20}} \right)$$

and the eigenvalues are given in Table 1.
TABLE 1

<table>
<thead>
<tr>
<th>$D_A$</th>
<th>$D_B$</th>
<th>Eigenvalues of $Ax = \lambda Bx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
<td>-0.5</td>
</tr>
<tr>
<td>1.1</td>
<td>1</td>
<td>1.1</td>
</tr>
<tr>
<td>-1.1</td>
<td>2</td>
<td>-0.55</td>
</tr>
<tr>
<td>1.2</td>
<td>1</td>
<td>1.2</td>
</tr>
<tr>
<td>-1.2</td>
<td>2</td>
<td>-0.6</td>
</tr>
<tr>
<td>1.3</td>
<td>1</td>
<td>1.3</td>
</tr>
<tr>
<td>-1.3</td>
<td>2</td>
<td>-0.65</td>
</tr>
<tr>
<td>1.4</td>
<td>0</td>
<td>$\infty$</td>
</tr>
<tr>
<td>-1.4</td>
<td>1</td>
<td>-1.4</td>
</tr>
<tr>
<td>1.5</td>
<td>2</td>
<td>0.75</td>
</tr>
<tr>
<td>-1.5</td>
<td>1</td>
<td>-1.5</td>
</tr>
<tr>
<td>1.6</td>
<td>2</td>
<td>0.8</td>
</tr>
<tr>
<td>-1.6</td>
<td>1</td>
<td>-1.6</td>
</tr>
<tr>
<td>1.7</td>
<td>2</td>
<td>0.85</td>
</tr>
<tr>
<td>-1.7</td>
<td>0</td>
<td>$\infty$</td>
</tr>
<tr>
<td>1.8</td>
<td>1</td>
<td>1.8</td>
</tr>
<tr>
<td>-1.8</td>
<td>2</td>
<td>-0.9</td>
</tr>
<tr>
<td>1.9</td>
<td>1</td>
<td>1.9</td>
</tr>
<tr>
<td>-1.9</td>
<td>2</td>
<td>-0.95</td>
</tr>
</tbody>
</table>

The MDR algorithm gives the following results shown in Table 2. The approximations to the finite eigenvalues of the problem are of very high precision, which in our examples always occurred when the eigenvalues were well conditioned.

If $D_A$ is modified by changing the ninth entry 1.4 to 0, then for all $\lambda$ we have $Ax_0 = \lambda Bx_0$, where $x_0 = (I - 2vv^T)e_9$, and $e_9$ is the ninth unit vector. But in addition we still have the other eigenvalues corresponding to eigenvectors $x_j = (I - 2vv^T)e_j$ for $j \neq 9$. The MDR algorithm now gives in the first positions of the corresponding tabulation:

<table>
<thead>
<tr>
<th>Diagonal of $A$</th>
<th>Diagonal of $B$</th>
<th>Computed Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1.700,000,000\times10$</td>
<td>$1.058,791,184\times-22$</td>
<td>$-1.605,604,604\times22$</td>
</tr>
<tr>
<td>$-2.546,585,165\times11$</td>
<td>$3.913,963,406\times-12$</td>
<td>$-6.506,410,256\times0$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

The finite eigenvalues corresponding to $x_j$ for $j \neq 9$ are still given just as precisely as in the first case.
TABLE 2

<table>
<thead>
<tr>
<th>Diagonal of A</th>
<th>Diagonal of B</th>
<th>Computed eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3.000000002</td>
<td>1.058791185</td>
<td>-2.833419890</td>
</tr>
<tr>
<td>2.508343579</td>
<td>1.794386876</td>
<td>1.397883373</td>
</tr>
<tr>
<td>1.874758658</td>
<td>9.867150828</td>
<td>1.900000000</td>
</tr>
<tr>
<td>1.976846170</td>
<td>1.098247871</td>
<td>1.600000001</td>
</tr>
<tr>
<td>-1.698422413</td>
<td>1.061514008</td>
<td>-1.600000000</td>
</tr>
<tr>
<td>-1.728343689</td>
<td>1.152229126</td>
<td>-1.500000000</td>
</tr>
<tr>
<td>-1.705487166</td>
<td>1.216205118</td>
<td>-1.400000000</td>
</tr>
<tr>
<td>1.523555711</td>
<td>1.171965931</td>
<td>1.300000000</td>
</tr>
<tr>
<td>1.488703948</td>
<td>1.240586623</td>
<td>1.200000000</td>
</tr>
<tr>
<td>1.366424372</td>
<td>1.242039756</td>
<td>1.100000000</td>
</tr>
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I would like to thank Professor L. Elsner for his valuable comments and suggestions.

REFERENCES


*Received 5 March 1983, revised 18 July 1983*