Available online at www.sciencedirect.com

ScienceDirect

Advances in Mathematics 218 (2008) 1512–1525

ADVANCES IN Mathematics

www.elsevier.com/locate/aim

On a problem of H.P. Rosenthal concerning operators on $C[0,1]$

I. Gasparis

Department of Mathematics, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece

Received 26 March 2006; accepted 17 March 2008

Available online 18 April 2008

Communicated by Michael. J. Hopkins

Abstract

A problem of H.P. Rosenthal asks whether every bounded linear operator $T : C[0,1] \to C[0,1]$ which is an isomorphism on a closed linear infinite-dimensional subspace X not containing any isomorph of c_0, is actually an isomorphism on a subspace isomorphic to $C[0,1]$. An affirmative answer to this problem is provided when T is a contraction whose restriction to X is an isometry.

MSC: primary 46B03; secondary 06A07, 03E02

Keywords: Operators on $C(K)$ spaces; Weakly null sequence; Boundaries

1. Introduction

A well-known property of $C(K)$ spaces, where K is compact and Hausdorff, is that every bounded linear operator defined on such a space is weakly compact if, and only if, it is strictly singular [16]. Very little is known however, about non-strictly singular operators on $C(K)$. In particular, it is an open problem whether every infinite-dimensional complemented subspace of $C(K)$ is isomorphic to $C(L)$ for some compact Hausdorff space L [5,14,16]. For an in-depth analysis of this problem, we refer to [22].

When K is metrizable the following theorem, due to H. Rosenthal, provides important information about operators on $C(K)$ whose adjoints have non-separable range.

E-mail address: ioagaspa@math.auth.gr.

doi:10.1016/j.aim.2008.03.015

Theorem 1.1. *(See Rosenthal [19].) Let K be an uncountable compact, metrizable space, X a Banach space and $T : C(K) \to X$ a bounded linear operator. Assume that $T^*(X^*)$ is non-separable. Then there exists a closed, linear subspace Y of $C(K)$, isomorphic to $C(K)$ and such that the restriction of T to Y is an into isomorphism.*

Using the preceding result, combined with those of A. Miljutin [15] and A. Pelczynski [17], Rosenthal was able to deduce that a complemented subspace X of $C(K)$ having non-separable dual is isomorphic to $C(K)$. When X^* is separable, the situation is more delicate. The results of D. Alspach and Y. Benyamini [1,3], yield a countable compact metric space L such that each one of X and $C(L)$ is isomorphic to a quotient of the other. In this case it is not even known if X is c_0-saturated, that is if every infinite-dimensional subspace of X contains an isomorph of c_0. This is a well-known property of $C(K)$ for a countable and compact metric space K [18]. Rosenthal posed the following

Problem. Let K be an uncountable compact and metrizable space and let $T : C(K) \to C(K)$ be a bounded linear operator. Assume that there exists a closed, linear, infinite-dimensional subspace X of $C(K)$ which does not contain c_0 isomorphically and such that $T|X$ is an into isomorphism. Does there exist a closed, linear subspace Y of $C(K)$, isomorphic to $C(K)$, and such that $T|Y$ is an into isomorphism?

Of course, under the assumptions of this problem and thanks to Theorem 1.1, it will be sufficient to show that T^* has non-separable range in order to solve the problem in the affirmative. Note also that a positive answer to this problem yields that complemented, infinite-dimensional subspaces of $C(K)$ with separable dual are c_0-saturated.

J. Bourgain [7] showed that Rosenthal's problem has an affirmative answer when X is assumed to have non-trivial cotype. In particular, when X is isomorphic to ℓ_p for some $1 < p < \infty$, then T^* has non-separable range.

We remark here that Rosenthal in [19], actually proved a result stronger than Theorem 1.1. Namely, he showed that if $\mathcal{M}$ is a w^*-compact subset of $B_{C(K)^*}$ which is not separable in the $C(K)^*$-norm, then $\mathcal{M}$ norms a closed linear subspace X of $C(K)$ which is isomorphic to $C(K)$.

Recall that a subset $\mathcal{M}$ of $B_{C(K)^*}$ is said to *norm* the closed linear subspace X of $C(K)$, provided that there exists a scalar $0 < \rho \leqslant 1$ satisfying $\sup_{\mu \in \mathcal{M}} |\int_K f \, d\mu| \geqslant \rho \|f\|$, for all $f \in X$. When $\rho = 1$, we say that $\mathcal{M}$ isometrically norms X.

Bourgain's main result was that if $\mathcal{M}$ is a w^*-compact subset of $B_{C(K)^*}$ which norms a closed, linear, infinite-dimensional subspace X of $C(K)$ having non-trivial cotype, then $\mathcal{M}$ is not separable in norm. He then derived his theorem on operators on $C(K)$ from Rosenthal's preceding result.

In the present paper we investigate the isometric version of Rosenthal's problem. Our main result is as follows:

Theorem 1.2. *Let K be an uncountable, compact and metrizable space and let X be a closed, linear, infinite-dimensional subspace of $C(K)$ containing no isomorph of c_0. Suppose that $\mathcal{M} \subset B_{C(K)^*}$ is w^*-compact and that it norms X isometrically. Then $\mathcal{M}$ is not separable in the $C(K)^*$-norm.*

The preceding result, combined with Theorem 1.1, yields that the isometric version of Rosenthal's problem holds true.

Corollary 1.3. *Let K be an uncountable, compact and metrizable space and let X be a closed, linear, infinite-dimensional subspace of $C(K)$ not containing any isomorph of c_0. Suppose that $T : C(K) \to C(K)$ is a contraction such that $T|X$ is an into isometry. Then there exists a closed, linear subspace Y of $C(K)$, isomorphic to $C(K)$, and such that $T|Y$ is an into isomorphism.*

Proof. Let $\mathcal{M} = T^* B_{C(K)^*}$. Since T is a contraction, $\mathcal{M}$ is a w^*-compact subset of $B_{C(K)^*}$. Moreover, $\mathcal{M}$ isometrically norms X as $T|X$ is an into isometry. We deduce from Theorem 1.2 that $\mathcal{M}$ is not separable in norm. Hence, T^* has non-separable range and the assertion of the corollary follows by Theorem 1.1. □

The main difficulty in proving Theorem 1.2 is to find an appropriate criterion for detecting isomorphic copies of c_0 in a given Banach space. We shall next discuss some of the existing criteria by first recalling a classical result of C. Bessaga and A. Pelczynski [4].

Theorem 1.4 *(Bessaga–Pelczynski). Let K be a compact Hausdorff space and (f_n) be a sequence in $C(K)$ with $\inf_n \|f_n\| > 0$. Assume there is a constant $C > 0$ such that $\sum_{n=1}^{\infty} |f_n(t)| \leqslant C$ for all $t \in K$. Then there exists a subsequence of (f_n) which is equivalent to the usual c_0-basis.*

The proof of Theorem 1.2 relies on a new approach towards the results of J. Elton [8] and V. Fonf [9,10] concerning extremal tests for unconditional convergence of series in Banach spaces not containing c_0 isomorphically. To explain this approach, we first recall the statements of the aforementioned results.

Theorem 1.5. *(See Fonf [9].) Let X be a Banach space with the property that the set of the extreme points of B_{X^*} is countable. Then X contains c_0 isomorphically.*

Fonf's argument makes use of an important theorem by Bessaga and Pelczynski [6] which states that in separable dual spaces, closed, convex and bounded subsets have extreme points.

Later on, Elton showed that if the set of the extreme points of B_{X^*} can be covered by a countable union of norm-compact subsets of X^*, then X can be given an equivalent norm such that the set of the extreme points of the renormed dual ball is countable. He then applied Fonf's preceding result to deduce that X contains c_0 isomorphically. As a consequence, he obtained the following theorem:

Theorem 1.6. *(See Elton [8].) Let X be a Banach space and suppose that (x_n) is a normalized sequence in X satisfying $\sum_n |e^*(x_n)| < \infty$ for every extreme point e^* of B_{X^*}. Then X contains c_0 isomorphically.*

Fonf [10] strengthened Elton's theorem by replacing the set of the extreme points of B_{X^*} in the hypothesis of Theorem 1.6 by any *boundary* of X. We recall here that a subset B of B_{X^*} is called a boundary for X provided that every $x \in X$ attains its norm at an element of $B \cup -B$. Fonf proved a more general result which may be stated as follows:

Theorem 1.7. *(See Fonf [10].) Let X be a closed linear subspace of $C(K)$, where K is compact and Hausdorff, containing no isomorph of c_0. Suppose that (K_n) is an increasing sequence of* *subsets of K such that K_n does not norm X (i.e., there is no $\rho > 0$ satisfying $\sup_{t \in K_n} |f(t)| \geqslant \rho \|f\|$ for all $f \in X$) for all $n \in \mathbb{N}$. Then there exists some $f \in X$ with $\|f\| = 1$ and such that*

$$\{t \in K: |f(t)| = 1\} \cap K_n = \emptyset, \quad \forall n \in \mathbb{N}.$$

The next corollary to Theorem 1.7, also due to Fonf, generalizes Theorem 1.6.

Corollary 1.8. *Let X be a Banach space and let $B \subset B_{X^*}$ be a boundary for X. Suppose that (x_n) is a sequence in X with $\limsup_n \|x_n\| > 0$ and so that $\sum_n |b^*(x_n)| < \infty$ for all $b^* \in B$. Then X contains c_0 isomorphically.*

Fonf's argument was based on finding a renorming of X so that the new dual ball has countably many extreme points and thus he reduced the problem of embedding c_0 in X to the case dealt in [9].

G. Androulakis [2] and P. Hajek [11] have discovered further extensions of the results of Elton and Fonf in the context of isomorphically polyhedral Banach spaces. A result deeper than Corollary 1.8 was obtained in [2]. It was shown in that paper that, under the assumptions of Corollary 1.8, there is a subsequence of (x_n) spanning an isomorphically polyhedral Banach space. The latter class of spaces are known to be c_0-saturated.

We recall here Rosenthal's deep characterization of Banach spaces not containing c_0 isomorphically:

Theorem 1.9. *(See Rosenthal [21].) Let X be a Banach space. Then X contains no isomorph of c_0 if, and only if, every non-trivial weak-Cauchy sequence in X admits a subsequence which is strongly summing.*

Recall that a weak-Cauchy basic sequence (e_n) is non-trivial, if it is non-weakly convergent. It is strongly summing provided that whenever (c_n) is a scalar sequence satisfying $\sup_n \|\sum_{i=1}^n c_i e_i\| < \infty$, then the series $\sum_{n=1}^\infty c_n$ converges.

In Section 2, we give an alternative proof of the results of Elton and Fonf which is quite elementary. It only uses Theorem 1.4. The main tool is the following characterization of Banach spaces not containing an isomorph of c_0:

Theorem 1.10. *Let K be a compact Hausdorff space and $X \subset C(K)$ be a closed linear subspace. Let (δ_n) be a summable sequence in $(0, 1)$. Then X does not contain c_0 isomorphically if, and only if, for every sequence (ϵ_n) in $(0, 1)$ and every sequence (f_n) in X with $\inf_n \|f_n\| \geqslant 1$, the following property is satisfied: there exist $f \in X$ with $\|f\| = 1$, a sequence of non-zero scalars (a_n) in $(-1, 1)$ and an infinite sequence of integers $1 = m_1 < m_2 < \cdots$ such that*

(1) *$f = \sum_{n=1}^\infty a_n f_n$, uniformly on K. Moreover, the series $\sum_{n=1}^\infty |a_n||f_n|$ is uniformly convergent on K.*
(2) *$\|\sum_{i=1}^{m_n} |a_i||f_i|\| \leqslant (1 + \delta_n)^{-1} \|\sum_{i=1}^{m_{n+1}} |a_i||f_i|\| < 1$, for all $n \in \mathbb{N}$.*
(3) *$|a_i| < \epsilon_n$ for each $i \in (m_n, m_{n+1}]$ and all $n \in \mathbb{N}$.*

Theorem 1.10 will be mostly applied to a normalized weakly null sequence (f_n) in a Banach space not containing c_0 isomorphically, in contrast to Rosenthal's characterization which applies to non-trivial weak-Cauchy sequences.

For our purposes, a weaker statement than that of the preceding theorem will actually suffice. We state below an immediate consequence of Theorem 1.10.

Corollary 1.11. *Let K be a compact Hausdorff space and $X \subset C(K)$ be a closed linear subspace not containing any isomorph of c_0. Assume that (f_n) is a sequence in X with $\inf_n \|f_n\| \geqslant 1$, and that (δ_n) is a summable sequence of scalars in $(0,1)$. Then there exist $f \in X$ with $\|f\| = 1$ and a sequence of non-zero scalars (a_n) in $(-1,1)$ so that the following properties are satisfied:*

(1) *$f = \sum_{n=1}^{\infty} a_n f_n$, uniformly on K. Moreover, the series $\sum_{n=1}^{\infty} |a_n||f_n|$ is uniformly convergent on K.*
(2) *$\|\sum_{i=1}^{n} |a_i||f_i|\| \leqslant (1+\delta_n)^{-1}$, for all $n \in \mathbb{N}$.*

We give two applications of Corollary 1.11. The first one is a direct proof of Theorems 1.7 and 1.6. The second application, given in Section 3, is Theorem 1.2 discussed in the preceding paragraphs.

Our notation is standard as may be found in [13]. We shall consider Banach spaces over the real field. If X is a Banach space then B_X stands for its closed unit ball. X is said to contain an *isomorph* of the Banach space Y (or, equivalently, that X contains Y isomorphically), if there exists a bounded linear injection from Y into X having closed range. If $Y \subset X$, then it is called *complemented* provided it is the range of a bounded, linear, idempotent operator on X.

Given sequences (x_n) and (u_n) of non-zero vectors in X, then we say that (u_n) is a *block subsequence* of (x_n) if there exist a sequence of non-zero scalars (a_n) and a sequence (F_n) of successive finite subsets of $\mathbb{N}$ (i.e., $\max F_n < \min F_{n+1}$ for all $n \in \mathbb{N}$), so that $u_n = \sum_{i \in F_n} a_i x_i$, for all $n \in \mathbb{N}$. We then call F_n the *support* of u_n for all $n \in \mathbb{N}$. Any member of a block subsequence of (x_n) will be called a *block* of (x_n).

The sequence (x_n) is said to be *semi-normalized* if $\inf_n \|x_n\| > 0$ and $\sup_n \|x_n\| < \infty$. It is called a *basic* sequence provided it is a Schauder basis for its closed linear span in X. (x_n) is *equivalent* to the usual c_0-basis, if there exist positive constants $\lambda_1 \leqslant \lambda_2$ such that $\lambda_1 \max_{i \leqslant n} |a_i| \leqslant \|\sum_{i=1}^{n} a_i x_i\| \leqslant \lambda_2 \max_{i \leqslant n} |a_i|$ for every choice of scalars $(a_i)_{i=1}^{n}$ and all $n \in \mathbb{N}$.

Finally, for an infinite subset M of $\mathbb{N}$ we let $[M]$ denote the set of its infinite subsets, while $[M]^2$ stands for the set of all doubletons of M, that is the collection $\{(m,n)\colon m,n \in M,\ m<n\}$.

2. A proof of the Elton–Fonf theorems

This section is devoted to the proof of Theorem 1.10. Before giving the details, let us see how this implies the results of Elton and Fonf described in the previous section.

Proof of Theorem 1.7. Choose a summable sequence (δ_n) of scalars in $(0,1)$. Then choose another such sequence (ϵ_n) satisfying $2\sum_{i>n} \epsilon_i < \delta_n$, for all $n \in \mathbb{N}$. Since none of the K_n's norms X, we can find a normalized sequence (f_n) in X satisfying

$$\bigl|f_n(t)\bigr| < \epsilon_n, \quad \forall t \in K_n,\ \forall n \in \mathbb{N}. \tag{2.1}$$

We next apply Corollary 1.11 to the sequences (f_n) and (δ_n), to obtain a sequence of scalars (a_n) in $(-1,1)$ and an $f \in X$ so that

(1) $f = \sum_{n=1}^{\infty} a_n f_n$, uniformly on K, and $\|f\| = 1$.
(2) $\| \sum_{i=1}^{n} |a_i||f_i| \| \leqslant (1+\delta_n)^{-1}$, for all $n \in \mathbb{N}$.

Now let $t \in K$ satisfy $|f(t)| = 1$. We claim $t \notin \bigcup_{n=1}^{\infty} K_n$. Indeed, assuming otherwise, there is some $n \in \mathbb{N}$ with $t \in K_n$. Since (K_n) is increasing, we infer from (2.1) that

$$\left|f_i(t)\right| < \epsilon_i, \quad \forall i > n.$$

We finally have the estimate

$$\begin{aligned} 1 = \left|f(t)\right| &\leqslant \sum_{i=1}^{\infty} |a_i|\left|f_i(t)\right| \\ &\leqslant \sum_{i=1}^{n} |a_i|\left|f_i(t)\right| + \sum_{i>n} |a_i|\left|f_i(t)\right| \\ &\leqslant (1+\delta_n)^{-1} + \sum_{i>n} \epsilon_i \quad \bigl(\text{by (2) and (2.1)}\bigr) \\ &< (1+\delta_n)^{-1} + (1/2)\delta_n. \end{aligned}$$

We deduce from the above, that $\delta_n > 1$ contradicting our choice of the sequence (δ_n). It is clear now, that f satisfies the conclusion of the theorem. □

Proof of Corollary 1.8. Assume on the contrary, that X contains no isomorph of c_0. Let K denote the compact space B_{X^*} endowed with the w^*-topology. Set

$$K_n = \left\{ x^* \in K\colon \sum_{i=1}^{\infty} \left|x^*(x_i)\right| \leqslant n \right\}, \quad \forall n \in \mathbb{N}.$$

It is clear that each K_n is a w^*-closed subset of K. Moreover, our hypotheses yield that $B \subset \bigcup_{n=1}^{\infty} K_n$. We can of course identify X with a subspace of $C(K)$. If there is some $n \in \mathbb{N}$ such that K_n norms X, then Theorem 1.4 would imply that some subsequence of (x_n) is equivalent to the c_0-basis. This contradicts our assumption on X.

Therefore, K_n norms X for no $n \in \mathbb{N}$. Since (K_n) is increasing, Theorem 1.7 yields that $\bigcup_{n=1}^{\infty} K_n$ is not a boundary for X, despite the fact that it contains B as a subset. This contradiction proves the corollary. □

Remark 2.1. If X satisfies the hypotheses of Corollary 1.8 then, evidently, so does every closed linear subspace Y of X containing a subsequence of (x_n) which does not converge to zero in norm. It follows from this that c_0 is contained isomorphically in the closed linear span of the x_n's in X. Moreover, there is a subsequence of (x_n) spanning a closed linear subspace Z of X which is c_0-saturated. In fact, a stronger result is obtained in [2] where it is proved that Z can be chosen to be isomorphically polyhedral.

To find a c_0-saturated Z, spanned by a subsequence, we can assume that (x_n) is bounded away from zero. Letting $y_n = \|x_n\|^{-1} x_n$ for all $n \in \mathbb{N}$, one checks that $(b^*(y_n))$ is absolutely summable for all $b^* \in B$. It follows that some subsequence of (y_n) is basic. To see this, observe that no subsequence of (y_n) is equivalent to the usual ℓ_1-basis as c_0 is contained in the closed linear span of the y_n's in X, thanks to Corollary 1.8 and the preceding comment. We now deduce from Rosenthal's ℓ_1-Theorem [20], that there is a subsequence of (y_n) which is weak-Cauchy. Without loss of generality, assume that (y_n) is itself weak-Cauchy.

In case (y_n) is weakly convergent then, since B is a boundary for X and $(b^*(y_n))$ is absolutely summable for all $b^* \in B$, we obtain that (y_n) is a normalized weakly null sequence in X. Hence, by a classical result [4], there is a subsequence of (y_n) which is basic. When (y_n) is a non-trivial weak-Cauchy sequence (i.e., non-weakly convergent) then, again, it admits a subsequence which is basic (in fact dominating the summing basis of c_0). The latter is shown in [21]. Concluding, every subsequence of (y_n) admits a basic subsequence. Once again, there is no loss of generality in assuming that (y_n) is itself a basic sequence. Let Z denote the closed linear span of (y_n) in X.

We next observe that $(b^*(u_n))$ is absolutely summable for every normalized block basis (u_n) of (y_n) and each $b^* \in B$. Therefore, c_0 is contained isomorphically in the closed linear span in X of every normalized block basis of (y_n). Standard permanence properties of basic sequences now yield that every infinite-dimensional subspace of Z contains a basic sequence equivalent to a normalized block basis of (y_n) and thus Z is c_0-saturated.

The proof of Theorem 1.10 requires two lemmas.

Terminology. If (x_n) is a sequence of non-zero vectors in a Banach space, then a positive block of (x_n) is a vector of the form $u = \sum_{i \in F} a_i x_i$, where F is a finite subset of $\mathbb{N}$ and $a_i > 0$ for all $i \in F$. We also set $\|u\|_{c_0} = \max_{i \in F} a_i$.

Lemma 2.2. *Suppose that X is a closed linear subspace of $C(K)$ not containing any isomorph of c_0 and let (f_n) be a sequence of non-zero elements of X. Set $g_n = |f_n|$, for all $n \in \mathbb{N}$, and consider a block subsequence (u_n) of (g_n) consisting of positive blocks. Then, the following properties hold:*

(1) *If (u_n) is a basic sequence in $C(K)$, then it is not equivalent to the c_0-basis.*
(2) *If $\sup_n \| \sum_{i=1}^n u_i \| < \infty$, then the series $\sum_{n=1}^\infty u_n$ is uniformly convergent on K.*

Proof. (1) Assume, on the contrary, that (u_n) is equivalent to the c_0-basis. It follows that (u_n) is semi-normalized and there is some $C > 0$ such that

$$\sum_{n=1}^{\infty} u_n(t) \leqslant C, \quad \forall t \in K.$$

We may write $u_n = \sum_{i \in F_n} a_i g_i$ for all $n \in \mathbb{N}$, where $F_1 < F_2 < \cdots$ form a sequence of successive, finite subsets of $\mathbb{N}$, and the a_i's are all positive.

Let $n \in \mathbb{N}$ and choose $t_n \in K$ such that $\|u_n\| = u_n(t_n)$. Then choose signs $(\sigma_i)_{i \in F_n}$ so that $\sigma_i f_i(t_n) = g_i(t_n)$, for all $i \in F_n$. Set $v_n = \sum_{i \in F_n} \sigma_i a_i f_i$. Clearly, (v_n) is a block subsequence of (f_n). Note also that

$$\left|v_n(t)\right| \leqslant \sum_{i \in F_n} a_i \left|f_i(t)\right| = u_n(t), \quad \forall t \in K, \ \forall n \in \mathbb{N}. \tag{2.2}$$

Since $v_n(t_n) = u_n(t_n) = \|u_n\|$, we conclude that $\|v_n\| = \|u_n\|$ for all $n \in \mathbb{N}$. In particular, (v_n) is semi-normalized. Moreover, (2.2) implies that

$$\sum_{n=1}^{\infty} \left|v_n(t)\right| \leqslant C, \quad \forall t \in K.$$

Therefore, some subsequence of (v_n) is equivalent to the c_0-basis, by Theorem 1.4. Hence, c_0 is contained isomorphically in X contradicting our assumptions. Thus, (1) holds.

(2) Let $d > 0$ satisfy $\sum_{n=1}^{\infty} u_n(t) \leqslant d$, for all $t \in K$. Suppose that the series $\sum_{n=1}^{\infty} u_n$ is not uniformly convergent on K. Then there exist $\epsilon > 0$ and a sequence of successive, finite subsets of $\mathbb{N}$ $I_1 < I_2 < \cdots$, such that

$$\left\|\sum_{i \in I_n} u_i\right\| \geqslant \epsilon, \quad \forall n \in \mathbb{N}.$$

Set $w_n = \sum_{i \in I_n} u_i$, for all $n \in \mathbb{N}$. It follows now that $\epsilon \leqslant \|w_n\| \leqslant d$, for all $n \in \mathbb{N}$, and that $\sum_{n=1}^{\infty} w_n(t) \leqslant d$, for all $t \in K$. We infer from Theorem 1.4, that (w_n) has a subsequence equivalent to the c_0-basis. Since (w_n) is a block subsequence of (g_n) consisting of positive blocks of (g_n), we have contradicted (1). Thus, (2) holds as well. □

We thank the referee for simplifying the proof of the next lemma.

Lemma 2.3. *Suppose that (g_n) is a sequence of non-negative and non-zero functions in $C(K)$ such that $\sup_n \|\sum_{i=1}^{n} g_i\| = \infty$. Let λ_1, λ_2 be scalars in $(0, 1]$, and let u be a block of (g_n) satisfying $\|u\| < \lambda_1\lambda_2$. Then, given $\epsilon > 0$, there exists a positive block v of (g_n), $u < v$ (i.e., $\max \operatorname{supp} u < \min \operatorname{supp} v$), satisfying*

$$\lambda_1^{-1}\|u\| < \|u + v\| < \lambda_2 \quad \textit{and} \quad \|v\|_{c_0} < \epsilon.$$

Proof. Set $m = \max \operatorname{supp} u$. Given $n \in \mathbb{N}$, $n > m$, put $k_n = n - m$ and define a map $\phi_n : [0, \epsilon]^{k_n} \to \mathbb{R}$ by the rule

$$\phi_n(t_1, \dots, t_{k_n}) = \left\|u + \sum_{i=1}^{k_n} t_i g_{m+i}\right\|.$$

Clearly, ϕ_n is continuous for all $n > m$. We claim that there exist $n > m$ and $(h_1, \dots, h_{k_n}) \in (0, \epsilon)^{k_n}$ so that

$$\lambda_1^{-1}\|u\| < \phi_n(h_1, \dots, h_{k_n}) < \lambda_2.$$

Once our claim is established, then $v = \sum_{i=1}^{k_n} h_i g_{m+i}$ will be the required positive block of (g_n).

To prove the claim, we choose $n > m$ so that

$$\phi_n(\epsilon, \dots, \epsilon) > \lambda_1^{-1}\|u\|. \tag{2.3}$$

Note that if such a choice of n was not possible, then

$$\left\| u + \sum_{i=m+1}^{k} \epsilon g_i \right\| \leqslant \lambda_1^{-1} \|u\|, \quad \forall k > m.$$

Therefore, $\| \sum_{i=1}^{k} g_i \| \leqslant \| \sum_{i=1}^{m} g_i \| + \epsilon^{-1}(1+\lambda_1^{-1})\|u\|$, for all $k \in \mathbb{N}$. This of course contradicts our assumption that the sequence of partial sums of (g_n) is unbounded in the $C(K)$-norm. Hence, there is some $n > m$ so that (2.3) is satisfied. But since $[0, \epsilon]^{k_n}$ is connected, ϕ_n is continuous, and $\phi_n(0, \dots, 0) = \|u\| \leqslant \lambda_1^{-1}\|u\| < \phi_n(\epsilon, \dots, \epsilon)$, the intermediate value theorem readily yields the existence of some $(h_1, \dots, h_{k_n}) \in (0, \epsilon)^{k_n}$ satisfying the claim. □

Remark 2.4. The proof of Lemma 2.3 shows that v can be chosen such that $\operatorname{supp} v$ is an interval in $\mathbb{N}$ whose left endpoint is $\max \operatorname{supp} u + 1$.

Proof of Theorem 1.10. We first assume that X does not contain any isomorph of c_0. Let (δ_n), (ϵ_n) be scalar sequences and (f_n) be a sequence in X according to the hypotheses of the theorem. Set $g_n = |f_n|$ for all $n \in \mathbb{N}$, and $d = \sup_n \| \sum_{i=1}^{n} g_i \|$. Note that $d = \infty$ or else, $\sum_{n=1}^{\infty} |f_n(t)| \leqslant d$ for all $t \in K$. Since $\inf_n \|f_n\| \geqslant 1$, Theorem 1.4 yields that some subsequence of (f_n) is equivalent to the c_0-basis. This contradicts the hypothesis that c_0 is not contained isomorphically in X.

Choose a scalar b_1 with $0 < b_1\|g_1\| < \prod_{i=1}^{\infty}(1+\delta_i)^{-1}$ and set $u_1 = b_1 g_1$. Since $d = \infty$, we can apply Lemma 2.3 to the block u_1 and the scalars "λ_1" $= (1+\delta_1)^{-1}$, "λ_2" $= \prod_{i=2}^{\infty}(1+\delta_i)^{-1}$ and "ϵ" $= \|u_1\|\epsilon_1$ (observe that $\|u_1\| = b_1\|g_1\| < \lambda_1\lambda_2$), to obtain a positive block u_2 of (g_n) with $u_1 < u_2$ and so that

$$(1+\delta_1)\|u_1\| < \|u_1 + u_2\| < \prod_{i=2}^{\infty}(1+\delta_i)^{-1}, \quad \text{and} \quad \|u_2\|_{c_0} < \|u_1\|\epsilon_1.$$

Next, apply Lemma 2.3 to the positive block $u_1 + u_2$ of (g_n) and the scalars "λ_1" $= (1+\delta_2)^{-1}$, "λ_2" $= \prod_{i=3}^{\infty}(1+\delta_i)^{-1}$ and "ϵ" $= \|u_1\|\epsilon_2$, to obtain a positive block u_3 of (g_n) with $u_2 < u_3$ and such that

$$(1+\delta_2)\|u_1 + u_2\| < \|u_1 + u_2 + u_3\| < \prod_{i=3}^{\infty}(1+\delta_i)^{-1}, \quad \text{and} \quad \|u_3\|_{c_0} < \|u_1\|\epsilon_2.$$

We continue this process in the same fashion, applying Lemma 2.3 to the sum of the blocks of (g_n) produced at the nth stage of the process with "λ_1" $= (1+\delta_n)^{-1}$, "λ_2" $= \prod_{i=n+1}^{\infty}(1+\delta_i)^{-1}$ and "ϵ" $= \|u_1\|\epsilon_n$ and inductively construct a block subsequence (u_n) of (g_n), consisting of positive blocks, so that $\|u_1\| > 0$ and

$$(1+\delta_n)\left\| \sum_{i=1}^{n} u_i \right\| < \left\| \sum_{i=1}^{n+1} u_i \right\| < \prod_{i=n+1}^{\infty}(1+\delta_i)^{-1}, \quad \text{and} \quad \|u_{n+1}\|_{c_0} < \|u_1\|\epsilon_n, \quad \forall n \in \mathbb{N}. \tag{2.4}$$

We obtain in particular, that $\sup_n \| \sum_{i=1}^{n} u_i \| \leqslant 1$. Since X contains no isomorph of c_0, part (2) of

Lemma 2.2 yields that the series $\sum_{n=1}^{\infty} u_n$ converges uniformly on K to a non-negative function u in $C(K)$. Moreover, $\|u\| \geqslant \|u_1\| > 0$. We set $g = \sum_{n=1}^{\infty} \|u\|^{-1} u_n$. Of course, $\|g\| = 1$ and g is non-negative. Note also that Remark 2.4 gives us that the supports of the u_n's are successive intervals of $\mathbb{N}$ whose union is $\mathbb{N}$. Since the g_n's are non-negative and each u_n is a positive block of (g_n), we can write

$$g = \sum_{n=1}^{\infty} c_n g_n, \quad \text{uniformly on } K, \text{ and } c_n > 0, \ \forall n \in \mathbb{N}.$$

Observe that $c_n \leqslant 1$ as $\|g_n\| \geqslant 1$, for all $n \in \mathbb{N}$. Moreover,

$$c_i < \epsilon_n, \quad \forall i \in \operatorname{supp} u_{n+1}, \ \forall n \in \mathbb{N}, \tag{2.5}$$

since $\|u_{n+1}\|_{c_0} < \|u_1\| \epsilon_n$, for all $n \in \mathbb{N}$ and $\|u\| \geqslant \|u_1\|$.

Now let $t_0 \in K$ be such that $g(t_0) = 1$ and choose a sequence of signs (σ_n) satisfying $\sigma_n f_n(t_0) = g_n(t_0)$ for all $n \in \mathbb{N}$. Define

$$a_n = \sigma_n c_n, \quad \text{and} \quad m_n = \max \operatorname{supp} u_n, \quad \forall n \in \mathbb{N}.$$

We now have, by the choices made, that $|a_n||f_n(t)| = c_n g_n(t)$, for all $t \in K$ and all $n \in \mathbb{N}$. It follows that the series $\sum_{n=1}^{\infty} |a_n||f_n|$ converges uniformly on K. Therefore, so does the series $\sum_{n=1}^{\infty} a_n f_n$ and if we let f denote its sum, then $f \in X$, as X is closed. Moreover, we have that for all $n \in \mathbb{N}$,

$$\begin{aligned} \left\| \sum_{i=1}^{m_n} |a_i||f_i| \right\| &= \left\| \sum_{i=1}^{m_n} c_i g_i \right\| \\ &= \|u\|^{-1} \left\| \sum_{i=1}^{n} u_i \right\| < (1+\delta_n)^{-1} \|u\|^{-1} \left\| \sum_{i=1}^{n+1} u_i \right\| \quad \big(\text{by } (2.4)\big) \\ &= (1+\delta_n)^{-1} \left\| \sum_{i=1}^{m_{n+1}} |a_i||f_i| \right\| \leqslant (1+\delta_n)^{-1} \|g\| = (1+\delta_n)^{-1}. \end{aligned}$$

Thus, (2) holds and also $\|f\| \leqslant 1$ and $|a_n| < 1$ for all $n \in \mathbb{N}$. Since $f(t_0) = g(t_0) = 1$ we obtain that $\|f\| = 1$ and so (1) holds as well. The validity of (3) follows by (2.5). This completes the proof of the first implication.

Conversely, suppose that X is a closed linear subspace of $C(K)$ satisfying the hypotheses of the theorem. Assume, to the contrary, that X contains a subspace isomorphic to c_0. We can now choose a normalized, basic sequence (f_n) in X which is equivalent to the usual c_0-basis. It follows that there is a constant $\tau \geqslant 1$ such that

$$\left\| \sum_{i=1}^{\infty} \lambda_i f_i \right\| \leqslant \tau \sup_i |\lambda_i|, \quad \forall (\lambda_i) \in c_0. \tag{2.6}$$

Let $\epsilon = \delta_1/(2\tau)$ and (δ_n) be a summable sequence in $(0, 1)$. Our hypotheses, applied to the sequence (f_n), yield some $f \in X$ with $\|f\| = 1$ and a scalar sequence (a_n) satisfying

$$f = \sum_{n=1}^{\infty} a_n f_n, \quad \text{and} \quad |a_1| \leqslant (1 + \delta_1)^{-1}, \quad |a_n| < \epsilon, \quad \forall n \geqslant 2. \tag{2.7}$$

But now,

$$\begin{aligned} 1 = \|f\| = \Big\|\sum_i a_i f_i\Big\| &\leqslant \|a_1 f_1\| + \Big\|\sum_{i>1} a_i f_i\Big\| \\ &\leqslant (1 + \delta_1)^{-1} + \tau \sup_{i>1} |a_i| \quad \big(\text{by (2.7) and (2.6)}\big) \\ &\leqslant (1 + \delta_1)^{-1} + \tau\epsilon \quad \big(\text{by (2.7)}\big) \\ &\leqslant (1 + \delta_1)^{-1} + \delta_1/2. \end{aligned}$$

It follows that $\delta_1 \geqslant 1$ contradicting our initial choices. The proof of the theorem is now complete. □

Remark 2.5. The proof of Theorem 1.10 shows that, given a summable sequence (δ_n) in $(0, 1)$, X contains no isomorph of c_0 if, and only if, for every sequence (f_n) in X with $\inf_n \|f_n\| \geqslant 1$ and every sequence (ϵ_n) in $(0, 1)$ there exist $f \in X$ with $\|f\| = 1$, a sequence of non-zero scalars (a_n) in $(-1, 1)$ and an infinite sequence of integers $1 = m_1 < m_1 < \cdots$ fulfilling the following properties:

(1) $f = \sum_{n=1}^{\infty} a_n f_n$, uniformly on K.
(2) $\|\sum_{i=1}^{m_n} a_i f_i\| \leqslant (1 + \delta_n)^{-1}$ for all $n \in \mathbb{N}$.
(3) $|a_i| < \epsilon_n$ for every $i \in (m_n, m_{n+1}]$ and all $n \in \mathbb{N}$.

3. The isometric version of Rosenthal's problem

In this section we give the proof of Theorem 1.2. The latter will follow after establishing the next lemma.

Lemma 3.1. *Let K be compact Hausdorff and let $\mathcal{M} \subset C(K)^*$ be norm-separable. Suppose that (f_n) is a weakly null sequence in $C(K)$ and let (ϵ_n) be a sequence of positive scalars. Then, given $N \in [\mathbb{N}]$ there exists $M \in [N]$ such that for every $L \in [M]$, $L = (l_n)$, and all $\mu \in \mathcal{M}$ we have that*

$$|\mu|\Big(\limsup_n \big[|f_{l_{2n}}| \geqslant \epsilon_{l_{2n-1}}\big]\Big) = 0.$$

Proof. Fix some $\nu \in C(K)^*$, a scalar $a > 0$ and some $P \in [\mathbb{N}]$. We define

$$\mathcal{D} = \big\{(l_1, l_2) \in [P]^2 \colon |\nu|\big(\big[|f_{l_2}| \geqslant \epsilon_{l_1}\big]\big) \geqslant a\big\}.$$

Ramsey's theorem yields some $Q \in [P]$ such that either $[Q]^2 \subset \mathcal{D}$, or $[Q]^2 \cap \mathcal{D} = \emptyset$. If the former, let $m = \min Q$. Then for all $n \in Q$, $n > m$, we have that $(m,n) \in \mathcal{D}$ and so $|\nu|([|f_n| \geqslant \epsilon_m]) \geqslant a$. It follows from this that

$$|\nu|\big(|f_n|\big) \geqslant \int_{[|f_n| \geqslant \epsilon_m]} |f_n|\,d|\nu| \geqslant \epsilon_m |\nu|\big(\big[|f_n| \geqslant \epsilon_m\big]\big) \geqslant \epsilon_m a > 0,$$

for all $n \in Q$ with $n > m$. But this contradicts our assumption that (f_n) is weakly null in $C(K)$.

Therefore, $[Q]^2 \cap \mathcal{D} = \emptyset$ and so $|\nu|([|f_{l_2}| \geqslant \epsilon_{l_1}]) < a$, for all $l_1 < l_2$ in Q. By applying the preceding argument, successively, for $a = 1/2^n$, $n \in \mathbb{N}$, we obtain a nested sequence $Q_1 \supset Q_2 \supset \cdots$ of infinite subsets of P such that

$$|\nu|\big(\big[|f_{l_2}| \geqslant \epsilon_{l_1}\big]\big) < 1/2^n, \quad \forall l_1 < l_2 \in Q_n,\ \forall n \in \mathbb{N}.$$

Choose an infinite sequence of integers $q_1 < q_2 < \cdots$ with $q_n \in Q_n$ for all $n \in \mathbb{N}$. It follows that for all $L \in [\{q_n\colon n \in \mathbb{N}\}]$, $L = (l_n)$, one has that

$$\sum_{n=1}^{\infty} |\nu|\big(\big[|f_{l_{2n}}| \geqslant \epsilon_{l_{2n-1}}\big]\big) < \sum_{n=1}^{\infty} 1/2^n = 1.$$

This is so because $l_{2n-1} < l_{2n}$ belong to Q_n for all $n \in \mathbb{N}$.

We next consider a countable, norm-dense subset $\{\mu_k\colon k \in \mathbb{N}\}$ of $\mathcal{M}$. Our previous argument yields some $M_1 \in [N]$ such that

$$\sum_{n=1}^{\infty} |\mu_1|\big(\big[|f_{l_{2n}}| \geqslant \epsilon_{l_{2n-1}}\big]\big) < \infty, \quad \forall L \in [M_1],\ L = (l_n).$$

By the same reasoning, we may choose a nested sequence $M_1 \supset M_2 \supset \cdots$ of infinite subsets of N so that

$$\sum_{n=1}^{\infty} |\mu_k|\big(\big[|f_{l_{2n}}| \geqslant \epsilon_{l_{2n-1}}\big]\big) < \infty, \quad \forall L \in [M_k],\ L = (l_n),\ \forall k \in \mathbb{N}.$$

We now choose an infinite sequence of integers $m_1 < m_2 < \cdots$ with $m_k \in M_k$ for all $k \in \mathbb{N}$, and set $M = (m_k)$. One checks that for every $L \in [M]$, $L = (l_n)$, and every $k \in \mathbb{N}$ we have that

$$\sum_{n=k}^{\infty} |\mu_k|\big(\big[|f_{l_{2n}}| \geqslant \epsilon_{l_{2n-1}}\big]\big) < \infty, \quad \text{as } \{l_n\colon n \geqslant 2k-1\} \subset M_k.$$

Hence,

$$\sum_{n=1}^{\infty} |\mu_k|\big(\big[|f_{l_{2n}}| \geqslant \epsilon_{l_{2n-1}}\big]\big) < \infty, \quad \forall L \in [M],\ L = (l_n),\ \forall k \in \mathbb{N}.$$

We conclude that

$$|\mu_k|\Big(\limsup_n\big[|f_{l_{2n}}| \geqslant \epsilon_{l_{2n-1}}\big]\Big) = 0, \quad \forall L \in [M],\ L = (l_n),\ \forall k \in \mathbb{N}.$$

The assertion of the lemma now follows by the norm-density of the sequence (μ_k) in $\mathcal{M}$. □

Proof of Theorem 1.2. If X^* is not separable, then by a result of R. Haydon [12] we obtain that $\mathcal{M}$ is not separable in norm and so the assertion of the theorem holds in this case.

We next assume that X^* is separable. In this case, we employ Rosenthal's theorem [20] to find a normalized weakly null sequence (h_n) in X. Suppose $\mathcal{M}$ is norm-separable. Let (δ_n) be a summable sequence of scalars in $(0, 1)$. Then choose another such sequence (ϵ_n) satisfying $2\sum_{i>n} \epsilon_i < \delta_n$, for all $n \in \mathbb{N}$. Apply Lemma 3.1 to the weakly null sequence (h_n), the norm-separable subset $\mathcal{M}$ of $C(K)^*$ and the scalar sequence (ϵ_n), to obtain $L \in [\mathbb{N}]$, $L = (l_n)$, so that

$$|\mu|\Big(\limsup_n\big[|h_{l_{2n}}| \geqslant \epsilon_{l_{2n-1}}\big]\Big) = 0, \quad \forall \mu \in \mathcal{M}. \tag{3.1}$$

Put $f_n = h_{l_{2n}}$ for all $n \in \mathbb{N}$. Now, (f_n) is a normalized sequence in X and we may apply Corollary 1.11 to this sequence and the scalar sequence (δ_n) to obtain $f \in X$ and a sequence of scalars (a_n) in $(-1, 1)$ so that

(1) $f = \sum_{n=1}^{\infty} a_n f_n$, in the $C(K)$-norm, and $\|f\| = 1$.
(2) $\|\sum_{i=1}^{n} |a_i||f_i|\| \leqslant (1 + \delta_n)^{-1}$, for all $n \in \mathbb{N}$.

Since $\mathcal{M}$ is w^*-compact and norms X isometrically, there is a $\mu \in \mathcal{M}$ such that $|\int_K f\, d\mu| = 1$. It follows now, as $\|f\| = 1$ and $\|\mu\| \leqslant 1$, that $\int_K |f|\, d|\mu| = 1$ and so

$$|\mu|\big(\big[|f| = 1\big]\big) = 1.$$

We claim however, that $[|f| = 1] \subset \limsup_n[|h_{l_{2n}}| \geqslant \epsilon_{l_{2n-1}}]$ and so we contradict (3.1). To prove the claim, we assume it does not hold and derive a contradiction as follows: Choose some $t \in K$ such that $|f(t)| = 1$, yet $t \notin \limsup_n[|h_{l_{2n}}| \geqslant \epsilon_{l_{2n-1}}]$. We next choose some $n \in \mathbb{N}$ such that

$$\big|h_{l_{2i}}(t)\big| < \epsilon_{l_{2i-1}}, \quad \forall i > n. \tag{3.2}$$

We now have the estimates

$$\begin{aligned} 1 = \big|f(t)\big| &= \left|\sum_{i=1}^{\infty} a_i f_i(t)\right| = \left|\sum_{i=1}^{\infty} a_i h_{l_{2i}}(t)\right| \\ &\leqslant \sum_{i=1}^{n} |a_i|\big|h_{l_{2i}}(t)\big| + \sum_{i>n} |a_i|\big|h_{l_{2i}}(t)\big| \\ &\leqslant (1+\delta_n)^{-1} + \sum_{i>n} \epsilon_{l_{2i-1}} \quad \big(\text{by (2) and (3.2)}\big) \\ &\leqslant (1+\delta_n)^{-1} + \sum_{i>n} \epsilon_i < (1+\delta_n)^{-1} + (1/2)\delta_n. \end{aligned}$$

So, it must be the case that $\delta_n > 1$ contradicting our choice of the sequence (δ_n). Therefore, our claim holds and we arrived at a contradiction by assuming $\mathcal{M}$ was separable in norm. This completes the proof of the theorem. □

References

[1] D. Alspach, Y. Benyamini, $C(K)$ quotients of separable $\mathcal{L}_\infty$ spaces, Israel J. Math. 32 (1979) 145–160.
[2] G. Androulakis, A subsequence characterization of sequences spanning isomorphically polyhedral Banach spaces, Studia Math. 127 (1998) 65–80.
[3] Y. Benyamini, An extension theorem for separable Banach spaces, Israel J. Math. 29 (1978) 24–30.
[4] C. Bessaga, A. Pelczynski, On bases and unconditional convergence of series in Banach spaces, Studia Math. 17 (1958) 151–164.
[5] C. Bessaga, A. Pelczynski, Spaces of continuous functions IV, Studia Math. 19 (1960) 53–62.
[6] C. Bessaga, A. Pelczynski, On extreme points in separable conjugate spaces, Israel J. Math. 4 (1966) 262–264.
[7] J. Bourgain, A result on operators on $C[0,1]$, J. Operator Theory 3 (2) (1980) 275–289.
[8] J. Elton, Extremely weakly unconditionally convergent series, Israel J. Math. 40 (1981) 255–258.
[9] V. Fonf, A property of Lindenstrauss–Phelps spaces, Funct. Anal. Appl. 13 (1) (1979) 66–67.
[10] V. Fonf, Weakly extremal properties of Banach spaces, Math. Notes 45 (5–6) (1989) 488–494.
[11] P. Hajek, Smooth norms that depend locally on finitely many coordinates, Proc. Amer. Math. Soc. 123 (12) (1995) 3817–3821.
[12] R. Haydon, An extreme point criterion for separability of a dual Banach space and a new proof of a theorem of Corson, Quart. J. Math. Oxford 27 (1976) 379–385.
[13] J. Lindenstrauss, L. Tzafriri, Classical Banach spaces I, Springer-Verlag, New York, 1977.
[14] J. Lindenstrauss, D.E. Wulbert, On the classification of the Banach spaces whose duals are L_1 spaces, J. Funct. Anal. 4 (1969) 332–349.
[15] A. Miljutin, Isomorphism of the spaces of continuous functions over compact sets of the cardinality of the continuum, Teor. Funkciĭ Funkcional. Anal. i Priložen. Vyp. 2 (1966) 150–156 (in Russian).
[16] A. Pelczynski, Projections in certain Banach spaces, Studia Math. 19 (1960) 209–228.
[17] A. Pelczynski, On $C(S)$-subspaces of separable Banach spaces, Studia Math. 31 (1968) 513–522.
[18] A. Pelczynski, Z. Semadeni, Spaces of continuous functions III. Spaces $C(\Omega)$ for Ω without perfect subsets, Studia Math. 18 (1959) 211–222.
[19] H.P. Rosenthal, On factors of $C[0,1]$ with non-separable dual, Israel J. Math. 13 (1972) 361–378.
[20] H.P. Rosenthal, A characterization of Banach spaces containing ℓ_1, Proc. Nat. Acad. Sci. (USA) 71 (1974) 2411–2413.
[21] H.P. Rosenthal, A characterization of Banach spaces containing c_0, J. Amer. Math. Soc. 7 (3) (1994) 707–748.
[22] H.P. Rosenthal, The Banach spaces $C(K)$, in: W.B. Johnson, J. Lindenstrauss (Eds.), Handbook of the Geometry of Banach Spaces, vol. 2, North-Holland, 2003, pp. 1547–1602.