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On Ky Fan's Inequality and Its Additive Analogue

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Let $x_i \in (0, 1/2]$ $(i = 1, \ldots, n)$ be real numbers. If A_n and G_n (respectively, A'_n and G'_n) denote the weighted arithmetic and geometric means of $x_1, \ldots, x_n$ (respectively, $1 - x_1, \ldots, 1 - x_n$), then

$$\frac{A'_n}{G'_n} \le 1 + 2(A'_n - G'_n) \le \frac{1 - G'_n + A'_n}{1 - A'_n + G'_n}$$

$$\le 1 + 2(A_n - G_n) \le \frac{1 - G_n + A_n}{1 - A_n + G_n} \le \frac{A_n}{G_n}, \qquad (*)$$

with equality holding if and only if $x_1 = \cdots = x_n$. The inequalities $(*)$ provide refinements of Ky Fan's inequality $A'_n/G'_n \le A_n/G_n$ and its additive analogue $A'_n - G'_n \le A_n - G_n$.

1. INTRODUCTION

In this paper we denote by A_n and G_n, respectively, A'_n and G'_n, the weighted arithmetic and geometric means of the real numbers $x_1, \ldots, x_n$, respectively, $1 - x_1, \ldots, 1 - x_n$, where $x_i \in (0, 1/2]$ $(i = 1, \ldots, n)$, that is,

$$A_n = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i, \qquad G_n = \prod_{i=1}^{n} x_i^{p_i/P_n},$$

respectively,

$$A'_n = \frac{1}{P_n} \sum_{i=1}^{n} p_i (1 - x_i), \qquad G'_n = \prod_{i=1}^{n} (1 - x_i)^{p_i/P_n},$$

where $p_1, \ldots, p_n$ are positive real numbers with $P_n = \sum_{i=1}^{n} p_i$.

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In 1961, the following "unpublished result" [4, p. 5] appeared in the well known book "Inequalities" by Beckenbach and Bellman [4, p. 5].

THEOREM A. *If* $x_i \in (0, 1/2]$ $(i = 1, \dots, n)$, *then*

$$A'_n/G'_n \le A_n/G_n, \tag{1.1}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Inequality (1.1), which is due to Ky Fan, was originally proved for unweighted means by forward and backward induction—a method used by Cauchy to present one of the first proofs for the arithmetic mean–geometric mean (A-G) inequality. It might be surprising to the reader that inequality (1.1) can be applied to give an elegant short proof for the A-G inequality; this was shown by Wang and Wang [6] in 1984.

Because of the strong resemblance between (1.1) and the A-G inequality, Wang in [5] calls (1.1) "a Ky Fan inequality of the complementary A-G type" [5, p. 502].

Ky Fan's inequality has evoked the interest of many mathematicians, and in numerous articles new proofs, extensions, refinements, and various related results of (1.1) have been published; see the survey paper [3] and the references therein.

In 1988, the author [1] proved an additive analogue of (1.1).

THEOREM B. *If* $x_i \in (0, 1/2]$ $(i = 1, \dots, n)$, *then*

$$A'_n - G'_n \le A_n - G_n, \tag{1.2}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

We remark that, just as for (1.1), inequality (1.2) was originally established for the special case $p_1 = \cdots = p_n$. Proofs of (1.1) and (1.2) for weighted means can be found in [3].

It is the aim of this paper to present new refinements of (1.1) and (1.2). In the next section we show that Ky Fan's inequality and its additive companion can be combined in one chain of inequalities. In particular, we obtain that inequality (1.2) is stronger than its multiplicative analogue (1.1).

2. THE MAIN RESULT

THEOREM. *If* $x_i \in (0, 1/2]$ $(i = 1, \dots, n)$, *then*

$$\begin{aligned} \frac{A'_n}{G'_n} &\le 1 + 2(A'_n - G'_n) \le \frac{1 - G'_n + A'_n}{1 - A'_n + G'_n} \\ &\le 1 + 2(A_n - G_n) \le \frac{1 - G_n + A_n}{1 - A_n + G_n} \le \frac{A_n}{G_n}. \end{aligned} \tag{2.1}$$

The sign of equality holds in each inequality of (2.1) *if and only if* $x_1 = \cdots = x_n$.

Proof. The left-hand and the right-hand inequalities of (2.1) can be written as $0 \le (2G'_n - 1)(A'_n - G'_n)$ and $0 \le (A_n - G_n)(A'_n - G_n)$, respectively. The validity of these inequalities follows immediately from $G_n \le A_n \le 1/2 \le G'_n \le A'_n$. The second and the fourth inequalities are equivalent to $0 \le (A'_n - G'_n)^2$ and $0 \le (A_n - G_n)^2$, respectively. The sign of equality holds in each of these inequalities if and only if $x_1 = \cdots = x_n$.

It remains to prove the third inequality of (2.1), which can be written as

$$A'_n - G'_n \le (A_n - G_n)(A_n + G'_n). \tag{2.2}$$

In what follows we establish an inequality which is even stronger than (2.2), namely,

$$A'_n - G'_n \le (A_n - G_n)(A'_n + G_n). \tag{2.3}$$

From inequality (1.2) we conclude that (2.3) implies (2.2).

Without loss of generality we may suppose that $P_n = 1$. Moreover, we suppose that the x_i's are not all equal, and we shall prove inequality (2.3) with "$<$" instead of "$\le$".

Let $n \ge 2$; we define the function

$$f\colon [0, 1/2]^n \to \mathbb{R}$$

by

$$f(x_1, \ldots, x_n) = G'_n - A'_n + (A_n - G_n)(A'_n + G_n),$$

and we denote the absolute minimum of f by $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$. Then we have to show that $\tilde{x}_1 = \cdots = \tilde{x}_n$. We assume, for a contradiction, that the $\tilde{x}_i$'s are not all equal.

First we suppose that $\tilde{x}$ is an interior point of $[0, 1/2]^n$. Then we obtain for $j = 1, \ldots, n$:

$$\tilde{x}_j(1 - \tilde{x}_j)\frac{1}{p_j}\frac{\partial f(x_1, \ldots, x_n)}{\partial x_j}\bigg|_{(x_1, \ldots, x_n) = (\tilde{x}_1, \ldots, \tilde{x}_n)}$$
$$= 2\tilde{x}_j(1 - \tilde{x}_j)(A'_n + G_n) - \tilde{x}_j G'_n - (1 - \tilde{x}_j)G_n(A'_n + 2G_n - A_n) = 0.$$

This means that the function

$$P(x) = 2x(1 - x)(A'_n + G_n) - xG'_n - (1 - x)G_n(A'_n + 2G_n - A_n)$$

has at least two distinct zeros on $(0, 1/2)$. From $2P(1/2) = A'_n - G'_n + 2G_n(A_n - G_n) > 0$ and $P(1) = -G'_n < 0$, we conclude that P has also a zero on $(1/2, 1)$. This contradicts the fact that P is a polynomial of degree 2.

Next we assume that $\tilde{x}$ is a boundary point of $[0, 1/2]^n$. We consider two cases.

Case 1. l components of $\tilde{x}$ are equal to 0 $(1 \le l \le n-1)$. Without loss of generality let

$$\tilde{x}_{k+1} = \cdots = \tilde{x}_n = 0$$
$$(1 \le k \le n-1) \qquad \text{and} \qquad 0 < \tilde{x}_i \le 1/2 \; (i = 1, \ldots, k).$$

We define the function

$$g\colon [0, 1/2]^k \to \mathbb{R}$$

by

$$g(x_1, \ldots, x_k) = f(x_1, \ldots, x_k, 0, \ldots, 0).$$

Further, we denote by $\tilde{A}'_n$ and $\tilde{G}'_n$ the weighted arithmetic and geometric means of the n numbers $1 - x_1, \ldots, 1 - x_k, 1, \ldots, 1$. Since $0 < x_j \le 1/2$ implies $2 \ge 1/(1 - x_j)$ and $\tilde{A}'_n > \tilde{G}'_n$, we get for $j = 1, \ldots, k$:

$$\frac{1}{p_j} \frac{\partial g(x_1, \ldots, x_k)}{\partial x_j} = 2\tilde{A}'_n - \tilde{G}'_n/(1 - x_j) > 0.$$

This means that the function $g(x_1, \ldots, x_k)$ is strictly increasing with respect to each component, which leads to

$$f(\tilde{x}_1, \ldots, \tilde{x}_n) = g(\tilde{x}_1, \ldots, \tilde{x}_k) > g(0, \ldots, 0) = f(0, \ldots, 0).$$

Case 2. No component of $\tilde{x}$ is equal to 0. Then l components of $\tilde{x}$ are equal to $1/2$ $(1 \le l \le n-1)$. We may assume that

$$\tilde{x}_{k+1} = \cdots = \tilde{x}_n = 1/2$$
$$(1 \le k \le n-1) \qquad \text{and} \qquad 0 < \tilde{x}_i < 1/2 \; (i = 1, \ldots, k).$$

This implies that the function

$$h\colon [0, 1/2]^k \to \mathbb{R},$$
$$h(x_1, \ldots, x_k) = f(x_1, \ldots, x_k, 1/2, \ldots, 1/2),$$

attains its absolute minimum at the interior point $(\tilde{x}_1, \ldots, \tilde{x}_k)$. If we denote by $\overline{A}_n$ and $\overline{G}_n$ (respectively, $\overline{A}'_n$ and $\overline{G}'_n$) the weighted arithmetic and geometric means of the n numbers $\tilde{x}_1, \ldots, \tilde{x}_k, 1/2, \ldots, 1/2$ (respec-

tively, $1-\tilde{x}_1,\dots,1-\tilde{x}_k,1/2,\dots,1/2)$, then we get

$$\frac{1}{p_j}\frac{\partial h(x_1,\dots,x_k)}{\partial x_j}\bigg|_{(x_1,\dots,x_k)=(\tilde{x}_1,\dots,\tilde{x}_k)}$$
$$= 1-\overline{G}_n'/\left(1-\tilde{x}_j\right)+\left(1-\overline{G}_n/\tilde{x}_j\right)\left(\overline{A}_n'+2\overline{G}_n-\overline{A}_n\right)=0$$

for $j=1,\dots,k$. Hence, the quadratic polynomial

$$Q(x)=x(1-x)-x\overline{G}_n'+(1-x)\left(x-\overline{G}_n\right)\left(\overline{A}_n'+2\overline{G}_n-\overline{A}_n\right)$$

has k zeros on $(0,1/2)$. From $Q(1/2)=\overline{G}_n(\overline{A}_n-\overline{G}_n)+(\overline{A}_n'-\overline{G}_n')/2>0$ and $Q(1)=-\overline{G}_n'<0$, we conclude that $\tilde{x}_1=\cdots=\tilde{x}_k=x_0$, say.

Next we prove that the function

$$H(x)=h(x,\dots,x)$$

attains only positive values on $(0,1/2)$. A simple calculation yields

$$\frac{1}{\alpha}H'(x)=1-\left[2-(2x)\right]^{\alpha-1}$$
$$+\left[1-(2x)^{\alpha-1}\right]\left[\alpha(1-(2x))+(2x)^{\alpha}\right], \tag{2.4}$$

with $\alpha=P_k\in(0,1)$. Setting $t=2x$ and denoting the right-hand side of (2.4) by $F(t)$, we get

$$F''(t)/(\alpha-1)=2(1-2\alpha)t^{2\alpha-3}+\alpha(\alpha+1)t^{\alpha-2}+\alpha(2-\alpha)t^{\alpha-3}$$
$$+(2-\alpha)(2-t)^{\alpha-3}. \tag{2.5}$$

We shall prove that $F''(t)<0$ for all $t\in(0,1)$. This implies $F'(t)>F'(1)=0$ and $F(t)<F(1)=0$ for $t\in(0,1)$. Then we get $H'(x)<0$, and, finally, $H(x)>H(1/2)=0$ for all $x\in(0,1/2)$.

To establish $F''(t)<0$ for $t\in(0,1)$ we consider two cases.

Case 1. $0<\alpha<1/2$. Then each term on the right-hand side of (2.5) is positive, so that we obtain $F''(t)/(\alpha-1)>0$.

Case 2. $1/2\le\alpha<1$. Then we get

$$F''(t)t^{3-\alpha}/(\alpha-1)>2(1-2\alpha)t^{\alpha}+\alpha(\alpha+1)t+\alpha(2-\alpha)$$
$$=K(t),$$

say. The function K thus attains its absolute minimum at $t_0 = [2(2\alpha - 1)/(\alpha + 1)]^{1/(1-\alpha)}$. Since $\alpha \in [1/2, 1)$, we get

$$\left(\frac{\alpha(2-\alpha)}{1-\alpha^2}\right)^{1-\alpha} \geq 1 > \frac{2(2\alpha - 1)}{\alpha + 1},$$

which leads to $K(t_0) > 0$. This implies $F''(t)/(\alpha - 1) > 0$ for $t \in (0, 1)$.

Thus, we have proved that

$$\begin{aligned} f(\tilde{x}_1, \ldots, \tilde{x}_n) &= f(\tilde{x}_1, \ldots, \tilde{x}_k, 1/2, \ldots, 1/2) \\ &= h(\tilde{x}_1, \ldots, \tilde{x}_k) = h(x_0, \ldots, x_0) > 0 = f(x_0, \ldots, x_0). \end{aligned}$$

This contradicts the assumption that f attains its absolute minimum at $(\tilde{x}_1, \ldots, \tilde{x}_n)$. The proof of the theorem is complete.

3. CONCLUDING REMARKS

In the final part of this paper we want to draw attention to an open problem concerning a refinement of inequality (1.2). We denote by H_n and H'_n the weighted harmonic means of $x_1, \ldots, x_n$ and $1 - x_1, \ldots, 1 - x_n$, that is,

$$H_n = P_n \Big/ \sum_{i=1}^{n} p_i/x_i \qquad \text{and} \qquad H'_n = P_n \Big/ \sum_{i=1}^{n} p_i/(1 - x_i),$$

where $p_1, \ldots, p_n$ are positive real numbers with $P_n = \sum_{i=1}^{n} p_i$. Recently, the author [2] proved a counterpart of (1.2) involving arithmetic and harmonic means.

THEOREM C. *If $x_i \in (0, 1/2]$ $(i = 1, \ldots, n)$ and $p_1 = \cdots = p_n$, then*

$$A'_n - H'_n \leq A_n - H_n, \tag{3.1}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

We remark that the differences $G'_n - H'_n$ and $G_n - H_n$ cannot be compared in general; see [2]. It has been conjectured in [2] that the following inequality holds.

CONJECTURE. If $x_i \in (0, 1/2]$ $(i = 1, \ldots, n)$ and $p_1 = \cdots = p_n$, then

$$\frac{1}{n}\left[(A_n - H_n) - (A'_n - H'_n)\right] \leq (A_n - G_n) - (A'_n - G'_n), \tag{3.2}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

From (3.1) it follows that the validity of (3.2) would provide a new sharpening of inequality (1.2) for unweighted means. Until now, neither a proof nor a disproof of (3.2) is known. We conclude by asking: If (3.2) is true, does there also exist a weighted version?

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