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Generalized Jacobi operators in Krein spaces

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ABSTRACT

A special class of generalized Jacobi operators which are self-adjoint in Krein spaces is presented. A description of the resolvent set of such operators in terms of solutions of the corresponding recurrence relations is given. In particular, special attention is paid to the periodic generalized Jacobi operators. Finally, the spectral properties of generalized Jacobi operators are applied to prove convergence results for Padé approximants.

1. Introduction

Let μ be a positive Borel measure having infinite support $\operatorname{supp}\mu \subset [a,b] \subset \mathbb{R}$. To every such a measure μ there corresponds a linear functional defined on the linear space $\mathcal{P} = \operatorname{span}\{\lambda^k\colon\ k \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}\}$ by the formula

$$s_k = \mathfrak{S}\left(\lambda^k\right) := \int_a^b t^k \, d\mu(t), \quad k \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}.$$

The functional $\mathfrak{S}$ is positive definite on $\mathcal{P}$, that is, $\det(s_{i+j})_{i,j=0}^n > 0$ for all $n \in \mathbb{Z}_+$. Besides, the measure μ (or, equivalently, the functional $\mathfrak{S}$) generates the holomorphic function

$$\widehat{\mu}(\lambda) = \mathfrak{S}_t\left(\frac{1}{t-\lambda}\right) = \int_a^b \frac{d\mu(t)}{t-\lambda} = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_n}{\lambda^{n+1}} - \cdots \quad \left(|\lambda| > R\right), \tag{1.1}$$

where R is large enough. By using the Euclidean algorithm, P.L. Tchebyshev [31] expanded the function $\widehat{\mu}$ into the following continued fraction

$$\widehat{\mu}(\lambda) \sim -\cfrac{1}{\lambda - a_0 - \cfrac{b_0^2}{\lambda - a_1 - \cfrac{b_1^2}{\ddots}}} = -\frac{1}{\lambda - a_0} - \frac{b_0^2}{\lambda - a_1} - \frac{b_1^2}{\lambda - a_2} - \cdots, \tag{1.2}$$

where a_j are real numbers, b_j are positive numbers, and $\sup_{j\in\mathbb{Z}_+}\{|a_j| + b_j\} < \infty$. Such continued fractions are called J-fractions [21]. Note that the coefficients a_j and b_j are uniquely determined by the coefficients s_j of the Taylor series at infinity (see (1.1)).

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It is well known (see [5,21,28]), that the nth convergent $-Q_n(\lambda)/P_n(\lambda)$ of the continued fraction (1.2) is characterized by the following property

$$-\frac{Q_n(\lambda)}{P_n(\lambda)}=-\frac{s_0}{\lambda}-\frac{s_1}{\lambda^2}-\cdots-\frac{s_{2n-1}}{\lambda^{2n}}+O\left(\frac{1}{\lambda^{2n+1}}\right)\quad(\lambda\to\infty). \tag{1.3}$$

In other words, the rational function $-Q_n/P_n$ is the nth diagonal Padé approximant to $\widehat{\mu}$, that is,

$$\widehat{\mu}(\lambda)+\frac{Q_n(\lambda)}{P_n(\lambda)}=O\left(\frac{1}{\lambda^{2n+1}}\right)\quad(\lambda\to\infty).$$

Further, by using standard argumentation (see [21]), one can see that the polynomials P_n and Q_n are solutions of the three-term recurrence relations

$$b_{j-1}u_{j-1}+a_ju_j+b_ju_{j+1}=\lambda u_j\quad(j\in\mathbb{N}), \tag{1.4}$$

with initial conditions

$$P_0(\lambda)=1,\qquad P_1(\lambda)=\frac{p_0(\lambda)}{b_0},\qquad Q_0(\lambda)=0,\qquad Q_1(\lambda)=\frac{1}{b_0}. \tag{1.5}$$

On the other hand, to the recurrence relations (1.4) (or, equivalently, to the continued fraction (1.2)) there corresponds a linear bounded self-adjoint operator in the space $\ell^2_{[0,\infty)}$. More precisely, that operator is generated by the following tridiagonal matrix

$$H=\begin{pmatrix} a_0 & b_0 & & \\ b_0 & a_1 & b_1 & \\ & b_1 & a_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

and its m-function $m(\lambda)=((H-\lambda)^{-1}e,e)_{\ell^2_{[0,\infty)}}$, where $e=(1,0,\dots,0,\dots)^\top\in\ell^2_{[0,\infty)}$, coincides with $\widehat{\mu}(\lambda)$. So, one has an operator representation of $\widehat{\mu}$,

$$\widehat{\mu}(\lambda)=\left((H-\lambda)^{-1}e,e\right)_{\ell^2_{[0,\infty)}},\quad \lambda\in\mathbb{C}\setminus\mathbb{R}.$$

The relations between orthogonal polynomials, Padé approximants, and Jacobi operators are well known. These relations allow to use operator methods to the investigation of orthogonal polynomials and the Padé approximants (see [2,29]).

Note, that the above-mentioned results are also valid if the underlying functional $\mathfrak{S}$ is not positive and has the property

$$\det(s_{i+j})_{i,j=0}^n\neq 0,\quad s_j=\mathfrak{S}\left(\lambda^j\right) \tag{1.6}$$

for all $n\in\mathbb{Z}_+$ (see [1,6,7]). The functional $\mathfrak{S}$ having the property (1.6) is called *regular*.

In the present paper a similar relations for nonregular functionals are presented. In fact, the scheme proposed in [1,7] (see also [29]) for investigation of the convergence of Padé approximants is generalized here to the nonregular case. The paper is organized as follows. In Section 2, starting from a (not necessarily regular) functional on $\mathcal{P}$, three-term recurrence relations for associated polynomials are derived. In Section 3, a special class of generalized Jacobi matrices is presented. Weyl solutions of the three-term recurrence relations and Weyl functions are introduced in Section 4. In Section 5, following the scheme proposed in [1], the characterization of resolvent sets of generalized Jacobi operators is obtained. Section 6 is concerned with the case of periodic generalized Jacobi matrices. In Section 7, convergence results for Padé approximants are proved.

2. Associated polynomials

In this section, starting from a (not necessarily regular) functional on $\mathcal{P}$, three-term recurrence relations for associated polynomials are derived.

Let us consider a holomorphic in a neighborhood of infinity function φ such that

$$\varphi^\sharp(\lambda):=\overline{\varphi(\bar{\lambda})}=\varphi(\lambda).$$

So, φ has the Taylor expansion at infinity

$$\varphi(\lambda)=-\sum_{j=0}^{\infty}\frac{s_j}{\lambda^{j+1}},$$

where $s_j \in \mathbb{R}$. To every such a function one can associate a real linear functional on $\mathcal{P}$ defined by the formula

$$\mathfrak{S}(\lambda^k) := \frac{1}{2\pi i} \oint_{|\lambda|=R} \lambda^k \varphi(\lambda)\, d\lambda = s_k \in \mathbb{R}, \quad k \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\},$$

for sufficiently large R. Clearly, the functional $\mathfrak{S}$ is not necessarily regular, that is $\det(s_{i+j})_{i,j=0}^n = 0$ may vanish for some $n \in \mathbb{Z}_+$ (for instance, see [13]). In general case, the functional $\mathfrak{S}$ generates an indefinite inner product on $\mathcal{P}$ (see [3,32])

$$[f, g]_{\mathfrak{S}} := \mathfrak{S}\big(f(\lambda) g^\sharp(\lambda)\big) = \frac{1}{2\pi i} \oint_{|\lambda|=R} f(\lambda) g^\sharp(\lambda) \varphi(\lambda)\, d\lambda, \quad f, g \in \mathcal{P},$$

which is degenerate if and only if φ is rational. In what follows we suppose that φ is not rational. As in the regular case, one can associate to $\mathfrak{S}$ the following holomorphic function

$$\mathfrak{S}_z\left(\frac{1}{z-\lambda}\right) = \frac{1}{2\pi i} \oint_{|z|=R} \frac{\varphi(z)\, dz}{z-\lambda} = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_n}{\lambda^{n+1}} - \cdots \quad \big(|\lambda| > R\big). \tag{2.1}$$

Throughout this paper we suppose that the sequence $\mathbf{s} := \{s_j\}_{j=0}^\infty$ is *normalized*, i.e. the first nonvanishing moment has modulus 1. A number $n_j \in \mathbb{N}$ is called *a normal index* if $\det(s_{i+k})_{i,k=0}^{n_j-1} \neq 0$. Since φ is not rational, there exists an infinite number of normal indices (see [16, Section 16.10.2]). Let $n_1 < n_2 < \cdots < n_j < \cdots$ be a sequence of all normal indices. By the choice of n_1 one has $s_{n_1-1} \neq 0$. Let us set $\varepsilon_0 = s_{n_1-1}$ ($|\varepsilon_0| = 1$) and $\varphi_0 := \varphi$. The principal part of the Laurent expansion for $-\frac{1}{\varphi_0}$ is a polynomial of degree $k_0 := n_1$ with the leading coefficient ε_0. So, we have

$$-\frac{1}{\varphi_0(\lambda)} = \varepsilon_0 p_0(\lambda) + b_0^2 \varphi_1(\lambda), \quad b_0 > 0, \tag{2.2}$$

where p_0 is a monic polynomial of degree k_0 and φ_1 is holomorphic in a neighborhood of infinity. Furthermore, the function φ_1 satisfies the relation $\varphi_1^\sharp(\lambda) = \varphi_1(\lambda)$. Choose $b_0 > 0$ such that the sequence $\mathbf{s}^{(1)} = \{s_j^{(1)}\}_{j=0}^\infty$ defined by the following expansion

$$\varphi_1(\lambda) = -\frac{s_0^{(1)}}{\lambda} - \frac{s_1^{(1)}}{\lambda^2} - \cdots - \frac{s_{2(n-k_0)}^{(1)}}{\lambda^{2(n-k_0)+1}} - \cdots \tag{2.3}$$

at ∞ is normalized. This completes the first step of expanding the series (2.1) into a continued fraction (see [24,25]).

As was shown in [11], the set of the normal indices of $\mathbf{s}^{(1)}$ coincides with the following sequence

$$n_2 - k_0 < \cdots < n_j - k_0 < \cdots.$$

Now one can apply the above reasoning to the function φ_1 and so on. By recursion we obtain the following P-fraction

$$-\frac{\varepsilon_0}{p_0(\lambda) - \dfrac{\varepsilon_0\varepsilon_1 b_0^2}{p_1(\lambda) - \cdots - \dfrac{\varepsilon_{j-1}\varepsilon_j b_{j-1}^2}{p_j(\lambda) - \cdots}}}, \tag{2.4}$$

where $\varepsilon_j = \pm 1$, $b_j > 0$ and $p_j(\lambda) = \lambda^{k_j} + p_{k_j-1}^{(j)} \lambda^{k_j-1} + \cdots + p_1^{(j)} \lambda + p_0^{(j)}$ are real monic polynomials of degree k_j (see also [11,14]). Note, that $n_j = k_0 + k_1 + \cdots + k_{j-1}$.

It also should be mentioned that there exist holomorphic in a neighborhood of infinity functions for which the set $\{b_j, p_0^{(j)}, \ldots, p_{k_j-1}^{(j)}\colon j \in \mathbb{Z}_+\}$ of coefficients of the P-fraction is not necessarily bounded. In particular, the Cauchy transform of the signed measure constructed in [30] gives such an example.

The continued fraction (2.4) can be considered as a sequence of the linear-fractional transformations (see [21, Section 5.2])

$$T_j(\omega) := \frac{-\varepsilon_j}{p_j(\lambda) + \varepsilon_j b_j^2 \omega}$$

having the following matrix representation

$$\mathcal{W}_j(\lambda) = \begin{pmatrix} 0 & -\frac{\varepsilon_j}{b_j} \\ \varepsilon_j b_j & \frac{p_j(\lambda)}{b_j} \end{pmatrix}, \quad j \in \mathbb{Z}_+. \tag{2.5}$$

The superposition $T_0 \circ T_1 \circ \cdots \circ T_j$ of the linear-fractional transformations corresponds to the product of the matrices $\mathcal{W}_l(\lambda)$,

$$\mathcal{W}_{[0,j]}(\lambda) = \big(w_{ik}^{(j)}(\lambda)\big)_{i,k=1}^2 := \mathcal{W}_0(\lambda)\mathcal{W}_1(\lambda)\cdots\mathcal{W}_j(\lambda). \tag{2.6}$$

To give an explicit formula for $\mathcal{W}_{[0,j]}$ in terms of p_j, b_j, ε_j, define the polynomials $P_{j+1}(\lambda)$, $Q_{j+1}(\lambda)$ by the equalities

$$\begin{pmatrix} -Q_0 \\ P_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} -Q_{j+1}(\lambda) \\ P_{j+1}(\lambda) \end{pmatrix} := \mathcal{W}_{[0,j]}(\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad j \in \mathbb{Z}_+. \tag{2.7}$$

The relation $\mathcal{W}_{[0,j]}(\lambda) = \mathcal{W}_{[0,j-1]}(\lambda)\mathcal{W}_j(\lambda)$ (see (2.6)) yields

$$\mathcal{W}_{[0,j]}(\lambda) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathcal{W}_{[0,j-1]}(\lambda) \begin{pmatrix} 0 \\ \varepsilon_j b_j \end{pmatrix} = \begin{pmatrix} -\varepsilon_j b_j Q_j(\lambda) \\ \varepsilon_j b_j P_j(\lambda) \end{pmatrix}, \quad j \in \mathbb{N}. \tag{2.8}$$

So, the matrix $\mathcal{W}_{[0,j]}(\lambda)$ has the form

$$\mathcal{W}_{[0,j]}(\lambda) = \begin{pmatrix} -\varepsilon_j b_j Q_j(\lambda) & -Q_{j+1}(\lambda) \\ \varepsilon_j b_j P_j(\lambda) & P_{j+1}(\lambda) \end{pmatrix}, \quad j \in \mathbb{Z}_+. \tag{2.9}$$

Further, the equality

$$\begin{pmatrix} -Q_{j+1}(\lambda) \\ P_{j+1}(\lambda) \end{pmatrix} = \mathcal{W}_{[0,j-1]}(\lambda)\mathcal{W}_j(\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{b_j}\mathcal{W}_{[0,j-1]}(\lambda) \begin{pmatrix} -\varepsilon_j \\ p_j(\lambda) \end{pmatrix}, \quad j \in \mathbb{N},$$

shows that the polynomials $P_j(\lambda)$, $Q_j(\lambda)$ are solutions of the difference equation

$$\varepsilon_{j-1}\varepsilon_j b_{j-1} u_{j-1} - p_j(\lambda) u_j + b_j u_{j+1} = 0 \quad (j \in \mathbb{N}), \tag{2.10}$$

obeying the initial conditions

$$\begin{aligned} &P_0(\lambda) = 1, \qquad P_1(\lambda) = \frac{p_0(\lambda)}{b_0}, \\ &Q_0(\lambda) = 0, \qquad Q_1(\lambda) = \frac{\varepsilon_0}{b_0}. \end{aligned} \tag{2.11}$$

According to (2.9), the $(j+1)$th convergent of the continued fraction (2.4) is equal to

$$f_j := T_0 \circ T_1 \circ \cdots \circ T_j(0) = -Q_{j+1}(\lambda)/P_{j+1}(\lambda).$$

The relations (2.9), (2.6), and (2.5) imply the following statement.

Proposition 2.1. *(See [14].) The polynomials P_j, Q_j satisfy the following generalized Liouville–Ostrogradsky formula*

$$\varepsilon_j b_j \big(Q_{j+1}(\lambda)P_j(\lambda) - Q_j(\lambda)P_{j+1}(\lambda)\big) = 1 \quad (j \in \mathbb{Z}_+). \tag{2.12}$$

3. Generalized Jacobi matrices

The main goal of this section is to present a special class of generalized Jacobi matrices.

Let $p(\lambda) = p_n\lambda^n + \cdots + p_1\lambda + p_0$ be a monic scalar real polynomial of degree n, i.e. $p_n = 1$. Let us associate to the polynomial p its symmetrizator E_p and let the companion matrix C_p be given by

$$E_p = \begin{pmatrix} p_1 & \dots & p_n \\ \vdots & \iddots & \\ p_n & & \mathbf{0} \end{pmatrix}, \qquad C_p = \begin{pmatrix} 0 & \dots & 0 & -p_0 \\ 1 & & \mathbf{0} & -p_1 \\ & \ddots & & \vdots \\ \mathbf{0} & & 1 & -p_{n-1} \end{pmatrix}. \tag{3.1}$$

As is known, $\det(\lambda - C_p) = p(\lambda)$ and the spectrum $\sigma(C_p)$ of the companion matrix C_p is simple. The matrices E_p and C_p are related by (see [20])

$$C_p E_p = E_p C_p^{\top}. \tag{3.2}$$

So, $C_p E_p$ is a symmetric matrix.

Definition 3.1. (See [13,23].) Let p_j be real monic polynomials of degree k_j,

$$p_j(\lambda) = \lambda^{k_j} + p_{k_j-1}^{(j)}\lambda^{k_j-1} + \cdots + p_1^{(j)}\lambda + p_0^{(j)},$$

and let $\varepsilon_j = \pm 1$, $b_j > 0$ ($j \in \mathbb{N}$). The tridiagonal block matrix

$$H = \begin{pmatrix} A_0 & \widetilde{B}_0 & & \mathbf{0} \\ B_0 & A_1 & \widetilde{B}_1 & \\ & B_1 & A_2 & \ddots \\ \mathbf{0} & & \ddots & \ddots \end{pmatrix}, \tag{3.3}$$

where $A_j = C_{p_j}$ and $k_{j+1} \times k_j$ matrices B_j and $k_j \times k_{j+1}$ matrices $\widetilde{B}_j$ are given by

$$B_j = \begin{pmatrix} 0 & \dots & b_j \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \qquad \widetilde{B}_j = \begin{pmatrix} 0 & \dots & \widetilde{b}_j \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix} \quad (\widetilde{b}_j = \varepsilon_j \varepsilon_{j+1} b_j,\ j = 0, \dots, N-1), \tag{3.4}$$

will be called a *generalized Jacobi matrix* associated with the sequences of polynomials $\{\varepsilon_j p_j\}_{j=0}^{\infty}$ and numbers $\{b_j\}_{j=0}^{\infty}$.

Remark 3.2. The papers [13,14,23] are only concerned with the case of generalized Jacobi matrices which are finite rank perturbations of classical Jacobi matrices. In fact, the generalized Jacobi matrix in question is associated to the P-fraction (2.4) or, equivalently, to the sequence of matrices $\mathcal{W}_j$ having the form (2.5).

From now on, we suppose that

(A.1) there exists $N \in \mathbb{N}$: $\deg p_j \leqslant N$, $j \in \mathbb{Z}_+$,
(A.2) $\sup\{b_j, |p_k^{(j)}|:\ j \in \mathbb{Z}_+,\ k = 0, \dots, k_j - 1\} < +\infty$.

Let $\ell^2_{[0,\infty)}$ denote the Hilbert space of complex square summable sequences $(w_0, w_1, \dots)$ with the usual inner product. Setting

$$n_0 = 0, \qquad n_j = \sum_{i=0}^{j-1} k_i \quad (j \in \mathbb{N}), \tag{3.5}$$

define a standard basis in $\ell^2_{[0,\infty)}$ by the equalities

$$e_{j,k} = \{\delta_{l,n_j+k}\}_{l=0}^{\infty} \quad (j \in \mathbb{Z}_+;\ k = 0, \dots, k_j - 1), \qquad e := e_{0,0}.$$

Define the symmetric matrix G by the equality

$$G = \operatorname{diag}(G_0, G_1, \dots), \quad G_j = \varepsilon_j E_{p_j}^{-1}\ (j \in \mathbb{Z}_+). \tag{3.6}$$

Further, we may identify via usual matrix product the matrix G with an operator on the linear space $\mathfrak{C}_0$ of finite sequences of $\ell^2_{[0,\infty)}$. Its closure will be also denoted by G. In view of (A.1), (A.2), the operator G defined on $\ell^2_{[0,\infty)}$ is bounded and self-adjoint. Moreover, G^{-1} is a bounded linear operator in $\ell^2_{[0,\infty)}$.

Let $H_{[j,l]}$ ($G_{[j,l]}$) be a submatrix of H (G), corresponding to the basis vectors $\{e_{i,k}\}_{i=j,\dots,l}^{k=0,\dots,k_i-1}$ $(0 \leqslant j \leqslant l < +\infty)$. The matrix $H_{[j,l]}$ will be called a finite generalized Jacobi matrix.

Let $\mathfrak{H}_{[0,\infty)}$ be a space of elements of $\ell^2_{[0,\infty)}$ provided with the following indefinite inner product

$$[x, y] = (Gx, y)_{\ell^2_{[0,\infty)}} \quad \big(x, y \in \ell^2_{[0,\infty)}\big). \tag{3.7}$$

Let us recall [3] that a pair $(\mathfrak{H}, [\cdot,\cdot])$ consisting of a Hilbert space $\mathfrak{H}$ and a sesquilinear form $[\cdot,\cdot]$ on $\mathfrak{H} \times \mathfrak{H}$ is called *a space with indefinite inner product*. A space with indefinite metric $(\mathfrak{H}, [\cdot,\cdot])$ is called *a Krein space* if the indefinite scalar product $[\cdot,\cdot]$ can be represented as follows

$$[x, y] = (Jx, y)_{\mathfrak{H}}, \quad x, y \in \mathfrak{H},$$

where the linear operator J satisfies the following conditions

$$J = J^{-1} = J^*.$$

The operator J is called *the fundamental symmetry*. So, one can see that the space $\mathfrak{H}_{[0,\infty)}$ is the Krein space with the fundamental symmetry $J = \operatorname{sign} G$ (see [3] for details).

Proposition 3.3. *Under the assumptions* (A.1), (A.2), *the considered generalized Jacobi matrix defines a bounded self-adjoint operator H* (*a generalized Jacobi operator*) *in the Krein space $\mathfrak{H}_{[0,\infty)}$, that is,*

$$[Hx, y] = [x, Hy], \quad x, y \in \mathfrak{H}_{[0,\infty)}. \tag{3.8}$$

Proof. It is not hard to see that, according to (A.1), (A.2), the matrix in question generates a bounded operator in $\mathfrak{H}_{[0,\infty)}$. Relation (3.8) is implied by (3.2) (see [13] for details). □

Let us extend the system $\{P_j(\lambda)\}_{j=0}^{\infty}$, $\{Q_j(\lambda)\}_{j=0}^{\infty}$ by the equalities

$$P_{j,k}(\lambda) = \lambda^k P_j(\lambda), \qquad Q_{j,k}(\lambda) = \lambda^k Q_j(\lambda) \quad (j \in \mathbb{Z}_+;\ k = 0, \dots, k_j - 1). \tag{3.9}$$

Setting

$$\mathbf{P}_{[l,j]}(\lambda) = \big(P_{l,0}(\lambda), \dots, P_{l,k_l-1}(\lambda), \dots, P_{j,0}(\lambda), \dots, P_{j,k_j-1}(\lambda)\big),$$
$$\mathbf{Q}_{[l,j]}(\lambda) = \big(Q_{l,0}(\lambda), \dots, Q_{l,k_l-1}(\lambda), \dots, Q_{j,0}(\lambda), \dots, Q_{j,k_j-1}(\lambda)\big)$$

one can rewrite the system (2.10), (3.9) in the following manner

$$\mathbf{P}_{[0,j]}(\lambda)(\lambda - H_{[0,j]}) = \big(0, \dots, 0, b_j P_{j+1,0}(\lambda)\big) \quad (j \in \mathbb{Z}_+), \tag{3.10}$$
$$\mathbf{Q}_{[0,j]}(\lambda)(\lambda - H_{[0,j]}) = \big(\underbrace{0, \dots, 0, -\varepsilon_0}_{k_0}, 0, \dots, 0, b_j Q_{j+1,0}(\lambda)\big) \quad (j \in \mathbb{Z}_+). \tag{3.11}$$

Since $Q_{0,0}(\lambda) = \dots = Q_{0,k_0-1}(\lambda) \equiv 0$, the relation (3.11) reduces to

$$\mathbf{Q}_{[1,j]}(\lambda)(\lambda - H_{[1,j]}) = \big(0, \dots, 0, b_j Q_{j+1,0}(\lambda)\big) \quad (j \in \mathbb{N}). \tag{3.12}$$

It follows from (3.10) and (3.12) that the eigenvalues of $H_{[0,j]}$ and $H_{[1,j]}$ coincide with the roots of $P_{j+1}(\lambda)$ and $Q_{j+1}(\lambda)$, respectively.

Proposition 3.4. *(See [13].) The polynomials P_j and Q_j ($j \in \mathbb{N}$) can be found by the formulas*

$$P_j(\lambda) = (b_0 \dots b_{j-1})^{-1} \det(\lambda - H_{[0,j-1]}), \tag{3.13}$$
$$Q_j(\lambda) = \varepsilon_0 (b_0 \dots b_{j-1})^{-1} \det(\lambda - H_{[1,j-1]}). \tag{3.14}$$

The formulas (3.13) and (3.14) in the classical case can be found in [9, Section 7.1.2] and [4, Section 6.1]. The following statement is an easy consequence of the recurrence relations (2.10).

Proposition 3.5. *(See [13,14].) Let $j \in \mathbb{N}$. Then:*

(i) *The polynomials P_j and P_{j+1} have no common zeros.*
(ii) *The polynomials Q_j and Q_{j+1} have no common zeros.*
(iii) *The polynomials P_j and Q_j have no common zeros.*

Taking into account the equality $G_{[0,j]} H_{[0,j]} = H_{[0,j]}^{\top} G_{[0,j]}$ which is implied by (3.2) (see [13]) and setting

$$\pi_{[0,j]}(\lambda) = G_{[0,j]}^{-1} \mathbf{P}_{[0,j]}(\lambda)^{\top}, \qquad \xi_{[0,j]}(\lambda) = G_{[0,j]}^{-1} \mathbf{Q}_{[0,j]}(\lambda)^{\top},$$

one can rewrite (3.10), (3.11) in the form

$$(\lambda - H_{[0,j]}) \pi_{[0,j]}(\lambda) = \varepsilon_j b_j P_{j+1,0}(\lambda) e_{j,0} \quad (j \in \mathbb{Z}_+), \tag{3.15}$$
$$(\lambda - H_{[0,j]}) \xi_{[0,j]}(\lambda) + e_{0,0} = \varepsilon_j b_j Q_{j+1,0}(\lambda) e_{j,0} \quad (j \in \mathbb{Z}_+). \tag{3.16}$$

Further, let us set

$$\pi(\lambda) = G^{-1} \big(P_{0,0}(\lambda), \dots, P_{0,k_0-1}(\lambda), \dots\big)^{\top}, \qquad \xi(\lambda) = G^{-1} \big(Q_{0,0}(\lambda), \dots, Q_{0,k_0-1}(\lambda), \dots\big)^{\top}. \tag{3.17}$$

Now, it follows from (3.15)–(3.17) that the following formal equalities hold true

$$(\lambda - H)\pi(\lambda) = 0, \qquad (\lambda - H)\xi(\lambda) = -e_{0,0}. \tag{3.18}$$

The first equality in (3.18) allows us to characterize the point spectrum of H.

Proposition 3.6. *$\lambda \in \sigma_p(H)$ if and only if $\pi(\lambda) \in \ell^2_{[0,\infty)}$.*

By the definition of $\pi(\lambda)$ and the assumptions (A.1), (A.2), we see that

$$\pi(\lambda) \in \ell^2_{[0,\infty)} \quad \Leftrightarrow \quad \sum_{j=0}^{\infty} \left|P_j(\lambda)\right|^2 < +\infty.$$

4. Weyl solutions and Weyl functions

If for some $\lambda \in \mathbb{C}$ there exists a solution $\{W_j(\lambda)\}_{j=0}^{\infty}$ of the recurrence relations (2.10) such that

$$\left\{W_j(\lambda)\right\}_{j=0}^{\infty} \in \ell^2_{[0,\infty)} \quad \text{and} \quad \left\{W_j(\lambda)\right\}_{j=0}^{\infty} \neq \left\{P_j(\lambda)\right\}_{j=0}^{\infty} \tag{4.1}$$

then we will say that there exists a Weyl solution $\{W_j(\lambda)\}_{j=0}^{\infty}$ of the recurrence relations (2.10) at the point λ. Since $\{P_j(\lambda)\}_{j=0}^{\infty}$ and $\{Q_j(\lambda)\}_{j=0}^{\infty}$ are linearly independent solutions of (2.10), the Weyl solution admits the following representation

$$W_j(\lambda) = Q_j(\lambda) + m(\lambda)P_j(\lambda), \tag{4.2}$$

where $m(\lambda)$ is a complex number. The following statement shows the relation between $m(\lambda)$ and the operator H.

Proposition 4.1. *Let $\lambda \in \rho(H)$ and let*

$$m(\lambda) = \left[(H-\lambda)^{-1}e, e\right], \quad e := e_{0,0}. \tag{4.3}$$

Then the family $\{W_j(\lambda)\}_{j=0}^{\infty}$ given by (4.2) *is the Weyl solution of* (2.10) *at λ. Moreover, $\{W_j(\lambda)\}_{j=0}^{\infty}$ has the form*

$$W_j(\lambda) = \left[(H-\lambda)^{-1}e, e_{j,0}\right], \quad j \in \mathbb{Z}_+. \tag{4.4}$$

Proof. For $\lambda \in \rho(H)$ the relation (3.18) implies that there exists a number $m(\lambda) \in \mathbb{C}$ such that

$$\xi(\lambda) + m(\lambda)\pi(\lambda) = (H-\lambda)^{-1}e \in \ell^2_{[0,\infty)}.$$

Since G is a bounded operator, we have

$$G\big(\xi(\lambda) + m(\lambda)\pi(\lambda)\big) \in \ell^2_{[0,\infty)}. \tag{4.5}$$

Using (3.17) yields

$$\left[(H-\lambda)^{-1}e, e_{j,0}\right] = \big(G\big(\xi(\lambda) + m(\lambda)\pi(\lambda)\big), e_{j,0}\big)_{\ell^2} = Q_j(\lambda) + m(\lambda)P_j(\lambda) =: W_j(\lambda) \quad \big(\neq P_j(\lambda)\big).$$

The latter relation means that $\{W_j(\lambda)\}_{j=0}^{\infty}$ is a solution of (2.10). Due to (4.5), we obtain that $\{W_j(\lambda)\}_{j=0}^{\infty} \in \ell^2_{[0,\infty)}$. Now, it follows from (2.11) that

$$W_0(\lambda) = m(\lambda) = \left[(H-\lambda)^{-1}e, e\right]. \qquad \square$$

Definition 4.2. The function m defined by (4.2) is called a Weyl function of the operator H.

Remark 4.3. It should be mentioned that a general treatment of the Weyl functions of classical Jacobi matrices in the framework of extension theory of nondensely defined symmetric operators was proposed in [26]. A general treatment of the Weyl functions of symmetric operators in Krein spaces in the framework of extension theory was presented in [15]. The function m defined by (4.3) on $\rho(H)$ is also called the m-function of H (see [13,18]).

Since H is bounded, m admits the representation

$$m(\lambda) = \left[(H-\lambda)^{-1}e, e\right] = -\sum_{j=0}^{\infty} \frac{s_j}{\lambda^{j+1}} \quad \big(|\lambda| > \|H\|\big), \tag{4.6}$$

where $s_i = [H^i e, e]$.

Analogously, one can define m-functions of shortened generalized Jacobi matrices.

Definition 4.4. The function

$$m_{[0,j]}(\lambda) = \left[(H_{[0,j]} - \lambda)^{-1}e, e\right] \tag{4.7}$$

is called the m-function of $H_{[0,j]}$.

Making use of the structure of $H_{[0,j]}$, we obtain that (see [13])

$$m_{[0,j]}(\lambda) = -\varepsilon_0 \frac{\det(\lambda - H_{[1,j]})}{\det(\lambda - H_{[0,j]})}. \tag{4.8}$$

According to (3.13), (3.14), the formula (4.8) can be rewritten as follows

$$m_{[0,j]}(\lambda) = -\frac{Q_{j+1}(\lambda)}{P_{j+1}(\lambda)}. \tag{4.9}$$

It follows from (2.10) (see [13] for details) that the m-function $m_{[0,j]}(\lambda)$ and the m-function $m_{[1,j]}(\lambda)$ of $H_{[1,j]}$ are related by the equality

$$m_{[0,j]}(\lambda) = \frac{-\varepsilon_0}{p_0(\lambda) + \varepsilon_0 b_0^2 m_{[1,j]}(\lambda)}, \quad j \in \mathbb{N}. \tag{4.10}$$

An analogous statement for infinite generalized Jacobi matrices is an essential ingredient in the proof of the following result.

Theorem 4.5. *Under the assumptions* (A.1), (A.2), *the generalized Jacobi matrix H is uniquely determined by its Weyl function m.*

Proof. By using the Frobenius formula, one can see that the Weyl function m of H and the Weyl function

$$m_{[1,\infty)}(\lambda) = \left[(H_{[1,\infty)} - \lambda)^{-1}e_{1,0}, e_{1,0}\right]$$

of $H_{[1,\infty)}$ are related by the equality

$$m(\lambda) = \frac{-\varepsilon_0}{p_0(\lambda) + \varepsilon_0 b_0^2 m_{[1,\infty)}(\lambda)}, \quad |\lambda| > \|H\| \geqslant \|H_{[1,\infty)}\| \tag{4.11}$$

(a more detailed reasoning can be found in [13]). Further, consecutive applications of the relation (4.11) leads to the P-fraction (2.4). So, one can uniquely recover the generalized Jacobi matrix H. □

Remark 4.6. In the definite case, formulas (4.8) and (4.9) are easy consequences of the theory developed in [26]. In this case, it is well known that the Weyl function determines the classical Jacobi matrix uniquely (for instance, see [18,26]). It is worth to mention that this result as well as Borg type uniqueness result was recently extended to the case of normal matrices (see [27]). Besides, a canonical form of a normal matrix (an analog of Jacobi matrix for self-adjoint matrices) was also introduced there. Some inverse problems for finite generalized Jacobi matrices were considered in [12,13].

5. The resolvent set of *H*

Here, following the scheme proposed in [1], the characterization of resolvent sets $\rho(H)$ of generalized Jacobi operators is obtained. We begin with an auxiliary lemma which gives a criterion of the density of $\operatorname{ran}(H - \lambda)$ in $\ell^2_{[0,\infty)}$.

Lemma 5.1. *For $\lambda \in \mathbb{C}$ the equation*

$$(H - \lambda I)x = e_{j,k}$$

has a solution $x = x(j,k) \in \ell^2_{[0,\infty)}$ for all $j, k \in \mathbb{Z}_+$ iff there exists a Weyl solution at λ.

Proof. (1) First, let us consider the case where $j = k = 0$. It follows from (3.18) that the equation $(H - \lambda)x = e_{0,0}$ has a solution belonging to $\ell^2_{[0,\infty)}$ if and only if there exists a number $m(\lambda) \in \mathbb{C}$ such that $\xi(\lambda) + m(\lambda)\pi(\lambda) \in \ell^2_{[0,\infty)}$. Furthermore, in this case we have

$$x = x(0,0) = \xi(\lambda) + m(\lambda)\pi(\lambda). \tag{5.1}$$

(2) Next, let $k = 0$ and $j \in \mathbb{N}$. According to (3.15) and (3.16), we see that

$$(H - \lambda)\xi_{[0,j]}(\lambda) = e_{0,0} - \varepsilon_j b_j Q_{j+1,0}(\lambda)e_{j,0} + \varepsilon_j b_j Q_{j,0}(\lambda)e_{j+1,0}, \tag{5.2}$$

$$(H - \lambda)\pi_{[0,j]}(\lambda) = -\varepsilon_j b_j P_{j+1,0}(\lambda)e_{j,0} + \varepsilon_j b_j P_{j,0}(\lambda)e_{j+1,0}. \tag{5.3}$$

Adding (5.2) multiplied by $-P_j$ and (5.3) multiplied by Q_j, one obtains

$$(H-\lambda)\big[-P_j(\lambda)\xi_{[0,j]}(\lambda)+Q_j(\lambda)\pi_{[0,j]}(\lambda)\big]=-P_j(\lambda)e_{0,0}-\varepsilon_j b_j\big(P_{j+1}(\lambda)Q_j(\lambda)-Q_{j+1}(\lambda)P_j(\lambda)\big)e_{j,0}. \tag{5.4}$$

Due to (2.12), (5.4) can be rewritten as follows

$$(H-\lambda)\big[-P_j(\lambda)\xi_{[0,j]}(\lambda)+Q_j(\lambda)\pi_{[0,j]}(\lambda)\big]+P_j(\lambda)e_{0,0}=e_{j,0}.$$

The latter relation shows that the equation $(H-\lambda)x=e_{j,0}$ has a solution $x=x(j,0)\in\ell^2_{[0,\infty)}$ if and only if $e_{0,0}\in\operatorname{ran}(H-\lambda)$. Moreover, the solution admits the representation

$$x=x(j,0)=-P_j(\lambda)\xi_{[0,j]}(\lambda)+Q_j(\lambda)\pi_{[0,j]}(\lambda)+P_j(\lambda)\big[\xi(\lambda)+m(\lambda)\pi(\lambda)\big]. \tag{5.5}$$

(3) Finally, assume that $k\neq 0$. Observe that

$$(H-\lambda)e_{j,0}=e_{j,1}-\lambda e_{j,0},\quad \ldots,\quad (H-\lambda)e_{j,k_j-2}=e_{j,k_j-1}-\lambda e_{j,k_j-2}. \tag{5.6}$$

The chain of equalities (5.6) implies that the equation $(H-\lambda)x=e_{j,k}$ has a solution $x=x(j,k)\in\ell^2_{[0,\infty)}$ if and only if $e_{j,0}\in\operatorname{ran}(H-\lambda)$ (or, equivalently, $e_{0,0}\in\operatorname{ran}(H-\lambda)$). Besides, the solution $x=x(j,k)$ can be expressed in the following manner

$$x=x(j,k)=e_{j,k-1}+\lambda e_{j,k-2}+\cdots+\lambda^k\big(-P_j(\lambda)\xi_{[0,j]}(\lambda)+Q_j(\lambda)\pi_{[0,j]}(\lambda)+P_j(\lambda)\big[\xi(\lambda)+m(\lambda)\pi(\lambda)\big]\big). \qquad \square$$

Remark 5.2. In fact, Lemma 5.1 gives a way to express the formal inverse operator R_λ to $H-\lambda$. Moreover, it is not so hard to see that the Weyl solution at λ exists if and only if $e\in\operatorname{ran}(H-\lambda)$.

Now we are ready to prove the main result of the present paper.

Theorem 5.3. *Under the assumptions* (A.1), (A.2), $\lambda\in\rho(H)$ *if and only if there exist a Weyl solution* $\{W_j(\lambda)\}_{j=0}^\infty$ *at* λ *and numbers* $q\in(0,1)$, $C>0$ *such that*

$$\big|P_i(\lambda)W_j(\lambda)\big|\leqslant Cq^{n_j-n_i},\quad i\leqslant j. \tag{5.7}$$

Proof. Let us prove the sufficiency. Let $\mathcal{H}_k:=\overline{\operatorname{span}}\{e_{j,k}\mid j\in\mathbb{Z}_+\}$ and let $\mathcal{P}_{\mathcal{H}_k}$ be the orthogonal projector onto $\mathcal{H}_k$ in $\ell^2_{[0,\infty)}$. We start with proving the boundedness of the operator $R_\lambda\mathcal{P}_{\mathcal{H}_k}$ for any $k\in\{0,\ldots,N-1\}$. First, it is convenient to consider the operator $GR_\lambda\mathcal{P}_{\mathcal{H}_0}$. Taking into account (5.5), we obtain

$$(GR_\lambda\mathcal{P}_{\mathcal{H}_0}e_{j,0})_{i,k}=(GR_\lambda e_{j,0})_{i,k}:=(GR_\lambda e_{j,0},e_{i,k})=\begin{cases}\lambda^kP_i(\lambda)(Q_j(\lambda)+m(\lambda)P_j(\lambda)), & i\leqslant j,\\ \lambda^kP_j(\lambda)(Q_i(\lambda)+m(\lambda)P_i(\lambda)), & i>j.\end{cases}$$

It is clear that one can represent the operator $GR_\lambda\mathcal{P}_{\mathcal{H}_0}$ as the sum of upper and lower triangular operators: $GR_\lambda\mathcal{P}_{\mathcal{H}_0}=R_\lambda^{(1)}+R_\lambda^{(2)}$. To be more precise, we choose $R_\lambda^{(1)}$ in the following way

$$R_\lambda^{(1)}e_{j,k}=0\quad\text{for } k\neq 0,\qquad R_\lambda^{(1)}e_{j,0}=y_j^{(1)},\quad\text{where } \big(y_j^{(1)}\big)_{i,k}=\begin{cases}\lambda^kP_i(\lambda)(Q_j(\lambda)+m(\lambda)P_j(\lambda)), & i\leqslant j,\\ 0, & i>j.\end{cases}$$

Now, one can prove that $R_\lambda^{(1)}$ is bounded. Indeed, setting

$$x=\sum_{j=0}^{l}\sum_{k=0}^{k_j-1}x_{j,k}e_{j,k}\quad\text{and}\quad y=\sum_{j=0}^{l}\sum_{k=0}^{k_j-1}y_{j,k}e_{j,k}$$

we get the following relation

$$\big(R_\lambda^{(1)}x,y\big)_{\ell^2}=\sum_{j=0}^{l}x_{j,0}\left(R_\lambda^{(1)}e_{j,0},\sum_{i=0}^{l}\sum_{k=0}^{k_i-1}y_{i,k}e_{i,k}\right)=\sum_{j=0}^{l}x_{j,0}\sum_{i=0}^{j}\sum_{k=0}^{k_i-1}\lambda^kP_i(\lambda)\big(Q_j(\lambda)+m(\lambda)P_j(\lambda)\big)\overline{y}_{i,k}.$$

Thus, (5.7) yields

$$\big|\big(R_\lambda^{(1)}x,y\big)_{\ell^2}\big|\leqslant C\max\big\{1,|\lambda|^{N-1}\big\}\sum_{j=0}^{l}|x_{j,0}|\sum_{i=0}^{j}\sum_{k=0}^{k_i-1}q^{n_j-n_i}|y_{i,k}|.$$

Notice that $n_j-n_i=\sum_{l=i+1}^{j}k_l\geqslant\sum_{l=i+1}^{j}1=j-i$ and, therefore, we have

$$\left|\left(R_\lambda^{(1)}x, y\right)_{\ell^2}\right| \leqslant C\max\left\{1, |\lambda|^{N-1}\right\}\sum_{j=0}^{l}\sum_{i=0}^{j}\sum_{k=0}^{k_i-1} q^{j-i}|x_{j,0}||y_{i,k}| \leqslant C\max\left\{1, |\lambda|^{N-1}\right\}\sum_{s=0}^{l} q^s \sum_{j=s}^{l}|x_{j,0}|\sum_{k=0}^{k_{j-s}-1}|y_{j-s,k}|$$

$$\leqslant C\sqrt{N}\max\left\{1, |\lambda|^{N-1}\right\}\sum_{s=0}^{l} q^i\|x\|_{\ell^2}\|y\|_{\ell^2} = \widetilde{C}\frac{1-q^{l+1}}{1-q}\|x\|_{\ell^2}\|y\|_{\ell^2}.$$

Hence, $R_\lambda^{(1)}$ is a bounded operator. Similarly, one can prove the boundedness of $R_\lambda^{(2)}$. So, we have proved that $GR_\lambda\mathcal{P}_{\mathcal{H}_0}$ is bounded. Since G^{-1} is bounded, $R_\lambda\mathcal{P}_{\mathcal{H}_0}$ is also bounded.

Further, let $k \in \{1, \dots, N-1\}$. From (5.6) one can deduce

$$R_\lambda e_{j,k} = e_{j,k-1} + \lambda R_\lambda e_{j,k-1} = V_k e_{j,k} + \lambda R_\lambda V_k e_{j,k},$$

where $V_k : e_{j,k} \mapsto e_{j,k-1}$ is an isometric operator from $\mathcal{H}_k$ to $\mathcal{H}_{k-1}$. If $h \in \mathcal{H}_k$ then $R_\lambda h = V_k h + \lambda R_\lambda V_k h$. So, the boundedness of V_1 and $R_\lambda\mathcal{P}_{\mathcal{H}_0}$ implies that $R_\lambda\mathcal{P}_{\mathcal{H}_1}$ is bounded. Analogously, $R_\lambda\mathcal{P}_{\mathcal{H}_k}$ is bounded for $k \in \{2, \dots, N-1\}$. This implies that $R_\lambda = \sum_{i=0}^{N-1} R_\lambda\mathcal{P}_{\mathcal{H}_i}$ is a bounded operator. The latter means that the domain of R_λ is $\ell^2_{[0,\infty)}$. Since λ is not an eigenvalue, we have $\ker(H-\lambda) = \{0\}$ and $\operatorname{ran}(H-\lambda) = \ell^2_{[0,\infty)}$. Now, applying the Banach theorem on inverse operators we obtain that $\lambda \in \rho(H)$.

The necessity of (5.7) follows from [10] and the relation

$$(R_\lambda e_{j,0})_{i,k_i-1} = \begin{cases} P_i(\lambda)(Q_j(\lambda) + m(\lambda)P_j(\lambda)), & i \leqslant j, \\ P_j(\lambda)(Q_i(\lambda) + m(\lambda)P_i(\lambda)), & i > j, \end{cases}$$

which directly follows from (5.5). □

Remark 5.4. In the case of nonsymmetric tridiagonal operators Theorem 5.3 was proved in [1]. In fact, we have extended the scheme proposed in [1] to the case of generalized Jacobi matrices. A similar result for banded matrices with nonvanishing extreme diagonals in terms of the corresponding vector polynomials was obtained in [8].

6. The Floquet theory

In the present section, by using Theorem 5.3, we give a description of spectra of periodic generalized Jacobi operators.

Definition 6.1. Let $s \in \mathbb{N}$. A generalized Jacobi matrix satisfying the properties

$$A_{js+k} = A_k, \qquad B_{js+k} = B_k, \qquad \varepsilon_{js+k} = \varepsilon_k, \quad j \in \mathbb{Z}_+,\ k \in \{0, \dots, s-1\},$$

will be called an s-periodic generalized Jacobi matrix. The corresponding generalized Jacobi operator in $\mathcal{H}_{[0,\infty)}$ will be also called an s-periodic generalized Jacobi operator.

Evidently, any s-periodic generalized Jacobi matrix satisfies the assumptions (A.1), (A.2) and we have

$$\mathcal{W}_{js+k}(\lambda) = \mathcal{W}_k(\lambda), \quad j \in \mathbb{Z}_+,\ k \in \{0, \dots, s-1\}. \tag{6.1}$$

The main tool for analysis of periodic generalized Jacobi operators is the following matrix

$$T(\lambda) := \mathcal{W}_{[0,s-1]}(\lambda) = \begin{pmatrix} -\varepsilon_{s-1}b_{s-1}Q_{s-1}(\lambda) & -Q_s(\lambda) \\ \varepsilon_{s-1}b_{s-1}P_{s-1}(\lambda) & P_s(\lambda) \end{pmatrix}.$$

The matrix $T(\lambda)$ is called *the monodromy matrix*. Using (6.1), we get the following relation

$$\mathcal{W}_{[0,js+k-1]}(\lambda) = T^j(\lambda)\mathcal{W}_{[0,k-1]}(\lambda), \quad j \in \mathbb{Z}_+,\ k \in \{1, \dots, s\}. \tag{6.2}$$

Let $w_1 = w_1(\lambda)$ and $w_2 = w_2(\lambda)$ be the roots of the characteristic equation $\det(T(\lambda) - w) = 0$. Introduce the following notations

$$E := \left\{\lambda \in \mathbb{C}:\ \left|w_1(\lambda)\right| = \left|w_2(\lambda)\right|\right\}, \qquad E_p := \left\{\lambda \in \mathbb{C}:\ P_{s-1}(\lambda) = 0,\ \left|b_{s-1}Q_{s-1}(\lambda)\right| > \left|P_s(\lambda)\right|\right\}.$$

Now, we are ready to give a description of spectra of periodic generalized Jacobi operators.

Theorem 6.2. *The spectrum of an s-periodic generalized Jacobi operator has the form*

$$\sigma(H) = E \cup E_p, \qquad \sigma_p(H) = E_p.$$

Proof. Since $\det T(\lambda) \equiv 1$, we have that

$$w_1(\lambda) w_2(\lambda) = 1.$$

Step 1. First, let us prove that

$$\{\lambda \in \mathbb{C}:\ P_{s-1}(\lambda) \neq 0,\ |w_1(\lambda)| \neq |w_2(\lambda)|\} \subset \rho(H).$$

To be definite, assume that $|w_1(\lambda)| > |w_2(\lambda)|$. In this case, we see that

$$T(\lambda) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} w_1(\lambda) & 0 \\ 0 & w_2(\lambda) \end{pmatrix} \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}, \qquad \det \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = 1. \tag{6.3}$$

Now, (6.2) can be rewritten in the form

$$\mathcal{W}_{[0,js+k-1]}(\lambda) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} w_1^j & 0 \\ 0 & w_2^j \end{pmatrix} \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix} \mathcal{W}_{[0,k-1]}(\lambda). \tag{6.4}$$

Further, (6.4) is reduced to the following relation

$$\mathcal{W}_{[0,js+k-1]}(\lambda) = \begin{pmatrix} x_1x_4w_1^j - x_2x_3w_2^j & -x_1x_2w_1^j + x_1x_2w_2^j \\ x_3x_4w_1^j - x_4x_3w_2^j & -x_3x_2w_1^j + x_1x_4w_2^j \end{pmatrix} \mathcal{W}_{[0,k-1]}(\lambda). \tag{6.5}$$

Multiplying (6.5) by the vector $(1\ 0)^\top$ we obtain

$$\begin{pmatrix} -\varepsilon_{k-1}b_{k-1}Q_{js+k-1}(\lambda) \\ \varepsilon_{k-1}b_{k-1}P_{js+k-1}(\lambda) \end{pmatrix} = \begin{pmatrix} x_1x_4w_1^j - x_2x_3w_2^j & -x_1x_2w_1^j + x_1x_2w_2^j \\ x_3x_4w_1^j - x_4x_3w_2^j & -x_3x_2w_1^j + x_1x_4w_2^j \end{pmatrix} \begin{pmatrix} -\varepsilon_{k-1}b_{k-1}Q_{k-1}(\lambda) \\ \varepsilon_{k-1}b_{k-1}P_{k-1}(\lambda) \end{pmatrix}.$$

Thus, for the polynomials $Q_.$ one has

$$Q_{js+k-1}(\lambda) = -w_1^j\big(x_1x_4Q_{k-1}(\lambda) + x_1x_2P_{k-1}(\lambda)\big) + w_2^j\big(x_2x_3Q_{k-1}(\lambda) + x_1x_2P_{k-1}(\lambda)\big). \tag{6.6}$$

Similarly, for the polynomials $P_.$ we have

$$P_{js+k-1}(\lambda) = -w_1^j\big(x_3x_4Q_{k-1}(\lambda) + x_3x_2P_{k-1}(\lambda)\big) + w_2^j\big(x_4x_3Q_{k-1}(\lambda) + x_1x_4P_{k-1}(\lambda)\big). \tag{6.7}$$

Note that $x_3 \neq 0$. Indeed, if $x_3 = 0$ then according to (6.3) we would have that $P_{s-1}(\lambda) = 0$. Now, formulas (6.6) and (6.7) yield

$$Q_{js+k-1}(\lambda) - \frac{x_1}{x_3}P_{js+k-1} = -w_2^j\left(\frac{x_1}{x_3}P_{k-1}(\lambda) + Q_{k-1}(\lambda)\right). \tag{6.8}$$

For brevity, define $C_k(\lambda) := -(\frac{x_1}{x_3}P_{k-1}(\lambda) + Q_{k-1}(\lambda))$. Then the equality (6.8) can be rewritten in the following way

$$Q_{js+k-1}(\lambda) - \frac{x_1}{x_3}P_{js+k-1} = (\widetilde{w}_2)^{js+k-1}C_k(\lambda), \quad \widetilde{w}_2 = w_2^{1/s},\ k \in \{1,\dots,s\}. \tag{6.9}$$

Since $|w_2| < 1$, it follows from (6.9) that $\xi(\lambda) - \frac{x_1}{x_3}\pi(\lambda) \in \ell^2_{[0,\infty)}$. A linear independence of $\pi(\lambda)$ and $\xi(\lambda)$, and (6.7), (6.6) imply $\pi(\lambda) \notin \ell^2_{[0,\infty)}$. Now we are ready to verify the condition (5.7) of Theorem 5.3. Let us assume that $i < j$. Then

$$\left|P_i(\lambda)\left[Q_j(\lambda) - \frac{x_1}{x_3}P_j(\lambda)\right]\right| = \left|\big(-\widetilde{w}_1^i f_1^{(k_1)}(\lambda) + \widetilde{w}_2^i f_2^{(k_1)}(\lambda)\big)\widetilde{w}_2^j C_{k_2}(\lambda)\right|, \quad \widetilde{w}_1 = \widetilde{w}_2^{-1},\ k_1, k_2 \in \{1,\dots,s\},$$

where $k_1 \equiv i \pmod s$ and $k_2 \equiv j \pmod s$. Since $(n_j - n_i)/N = (\sum_{l=i+1}^j k_l)/N \leqslant (\sum_{l=i+1}^j N)/N = j - i$ we have that $(n_j - n_i)/N \leqslant j - i$. Thus, one obtains

$$\left|P_i(\lambda)\left[Q_j(\lambda) - \frac{x_1}{x_3}P_j(\lambda)\right]\right| \leqslant |\widetilde{w}_2|^{j-i}C(\lambda) \leqslant q^{n_j-n_i}C(\lambda),$$

where $C(\lambda) = \sup_{k,k_2\in\{1,\dots,s\}}\{(|f_1^{(k_1)}(\lambda)| + |f_2^{(k_2)}(\lambda)|)|C_{k_2}(\lambda)|\}$ and $q = \widetilde{w}_2 < 1$.

Step 2. Let us show that

$$\{\lambda \in \mathbb{C}:\ P_{s-1}(\lambda) = 0,\ |b_{s-1}Q_{s-1}(\lambda)| > |P_s(\lambda)|\} \subset \sigma_p(H).$$

In this case the monodromy matrix can be represented as follows

$$T(\lambda) = \begin{pmatrix} -\varepsilon_{s-1}b_{s-1}Q_{s-1}(\lambda) & -Q_s(\lambda) \\ 0 & P_s(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_1(\lambda) & 0 \\ 0 & w_2(\lambda) \end{pmatrix} \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix},$$

where $w_1(\lambda) = -\varepsilon_{s-1} b_{s-1} Q_{s-1}(\lambda)$ and $w_2(\lambda) = P_s(\lambda)$. Further, we have that

$$\begin{pmatrix} -\varepsilon_{k-1} b_{k-1} Q_{js+k-1}(\lambda) \\ \varepsilon_{k-1} b_{k-1} P_{js+k-1}(\lambda) \end{pmatrix} = \begin{pmatrix} w_1^j & w_1^j x_1 - w_2^j x_1 \\ 0 & w_2^j \end{pmatrix} \begin{pmatrix} -\varepsilon_{k-1} b_{k-1} Q_{k-1}(\lambda) \\ \varepsilon_{k-1} b_{k-1} P_{k-1}(\lambda) \end{pmatrix},$$

that is, the following formulas hold true

$$P_{js+k-1}(\lambda) = w_2^j P_{k-1}(\lambda),$$
$$Q_{js+k-1}(\lambda) = -w_1^j Q_{k-1}(\lambda) + \left(w_1^j - w_2^j\right) x_1 P_{k-1}(\lambda).$$

Since $|w_2(\lambda)| < |w_1(\lambda)|$, $G\pi(\lambda) \in \ell^2_{[0,\infty)}$ and, therefore, $\pi(\lambda) \in \ell^2_{[0,\infty)}$. So, we have proved that $E_p \subset \sigma(H)$.

Step 3. By the same reasoning as in Step 1 it can be shown that

$$\{\lambda \in \mathbb{C}:\ P_{s-1}(\lambda) = 0,\ |b_{s-1} Q_{s-1}(\lambda)| > |P_s(\lambda)|\} \subset \rho(H).$$

Step 4. We complete this proof by proving the following inclusion

$$\{\lambda \in \mathbb{C}:\ |w_1(\lambda)| = |w_2(\lambda)|\} \subset \sigma(H).$$

Since $w_1 w_2 = 1$ we have that $|w_1(\lambda)| = |w_2(\lambda)| = 1$. If T is diagonalizable then, due to (6.6), (6.7), there exist numbers $\alpha_k^{(1)}(\lambda), \alpha_k^{(2)}(\lambda), \beta_k^{(1)}(\lambda), \beta_k^{(2)}(\lambda) \in \mathbb{C}$ such that

$$Q_{js+k-1}(\lambda) = w_2^{-j} \alpha_k^{(1)} + w_2^j \alpha_k^{(2)}, \quad j \in \mathbb{Z}_+,\ k \in \{1, \dots, s\}, \tag{6.10}$$
$$P_{js+k-1}(\lambda) = w_2^{-j} \beta_k^{(1)} + w_2^j \beta_k^{(2)}, \quad j \in \mathbb{Z}_+,\ k \in \{1, \dots, s\}. \tag{6.11}$$

Since the sequences $Q_.$ and $P_.$ are linearly independent, the sequence $\{Q_j + mP_j\}_{j=0}^\infty$ is nonzero for any $m \in \mathbb{C}$. Also, observe that the sequence $\{|w^j \alpha + \beta|\}_{j=1}^\infty$ ($\alpha \neq 0$, $|w| = 1$) converges only for $w = 1$. Taking into account these observations one can conclude that the series

$$\sum_{j=0}^\infty |Q_j(\lambda) + mP_j(\lambda)|^2 = \sum_{i=0}^\infty \sum_{k=0}^{s-1} \left|\alpha_k^{(1)} + m\beta_k^{(1)} + w_2^{2i}\left(\alpha_k^{(2)} + m\beta_k^{(2)}\right)\right|^2$$

does not converge for any number $m \in \mathbb{C}$. So, we have that $\xi(\lambda) + m\pi(\lambda) \notin \ell^2_{[0,\infty)}$ for any $m \in \mathbb{C}$. Similarly, one can conclude that $\pi(\lambda) \notin \ell^2_{[0,\infty)}$.

If T is similar to a Jordan block then $w_1 = w_2$ and $w_1 w_2 = 1$. So, $w_1 = \pm 1$. First, let us consider the case $w_1 = 1$. The monodromy matrix takes the form

$$T(\lambda) = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_4 & -x_2 \\ -x_3 & x_1 \end{pmatrix}.$$

Further, (6.2) yields

$$\mathcal{W}_{[0,js+k-1]}(\lambda) = \begin{pmatrix} x_2 x_4 - x_1 x_3 j - x_2 x_3 & -x_2^2 + x_1^2 j - x_1 x_2 \\ -x_3^2 j & -x_3 x_2 + x_3 x_1 j + x_4 x_1 \end{pmatrix} \mathcal{W}_{[0,k-1]}(\lambda).$$

Thus for any $m \in \mathbb{C}$ the vectors $\xi(\lambda) + m\pi(\lambda)$ and $\pi(\lambda)$ do not belong to $\ell^2_{[0,\infty)}$. Analogously, one can consider the case $w_1 = -1$. $\square$

Remark 6.3. A description of spectra of classical Jacobi operators was obtained in [17].

Let us remind that w_1 and w_2 are the roots of the equation

$$w^2 - \left(P_s(\lambda) - \varepsilon_{s-1} b_s Q_{s-1}(\lambda)\right) w + 1 = 0. \tag{6.12}$$

Due to (6.12), one has

$$w_1 + w_2 = P_s(\lambda) - \varepsilon_{s-1} b_s Q_{s-1}(\lambda), \tag{6.13}$$
$$w_1 w_2 = 1. \tag{6.14}$$

Remark 6.4. (See [7].) Formulas (6.13), (6.14) allow us to give another description of E,

$$E = \{\lambda \in \mathbb{C}:\ \left(P_s(\lambda) - \varepsilon_{s-1} b_s Q_{s-1}(\lambda)\right) \in [-2, 2]\}. \tag{6.15}$$

Example 6.5. Let us consider the following difference equation

$$\frac{1}{2}u_{j+1} - \lambda^2 u_j - \frac{1}{2}u_{j-1} = 0 \quad (j \in \mathbb{N}). \tag{6.16}$$

Clearly, the three-term recurrence relations (6.16) generate the following 1-periodic generalized Jacobi matrix

$$H = \begin{pmatrix} A & B & \mathbf{0} \\ B & A & \ddots \\ \mathbf{0} & \ddots & \ddots \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

In this case, by easy calculations, we have

$$P_0(\lambda) = 1, \qquad P_1(\lambda) = 2\lambda^2, \qquad Q_0(\lambda) = 0, \qquad Q_1(\lambda) = 2.$$

Since $P_0(\lambda) = 1 \neq 0$, it follows from Theorem 6.2 that $\sigma_p(H) = \emptyset$. Next, according to Theorem 6.2 and (6.15), we have

$$\sigma(H) = \{\lambda \in \mathbb{C}:\ 2\lambda^2 \in [-2, 2]\} = [-1, 1] \cup [-i, i].$$

By the same reasoning as in [7], one can prove the following statement on the structure of the set E.

Proposition 6.6. *(See [7].) The compact set E has no interior points. The open set $D := \mathbb{C} \setminus E$ is connected. The functions w_1 and w_2 are single-valued in D.*

7. Padé approximants

Our goal in this section is to prove convergence results for Padé approximants.

Definition 7.1. (See [28].) The $[L/M]$ Padé approximant to the function $\varphi(\lambda) = -\sum_{j=0}^{+\infty} \frac{s_j}{\lambda^{j+1}}$ is defined as a ratio

$$f^{[L/M]}(\lambda) = \frac{A^{[L/M]}(\frac{1}{\lambda})}{B^{[L/M]}(\frac{1}{\lambda})}$$

of two polynomials $A^{[L/M]}$, $B^{[L/M]}$ of formal degree L and M, respectively, such that $B^{[L/M]}(0) \neq 0$ and

$$\sum_{j=0}^{+\infty} \frac{s_j}{\lambda^{j+1}} + f^{[L/M]}(\lambda) = O\left(\lambda^{-(L+M+1)}\right) \quad (\lambda \to \infty).$$

In the case $L = M = n$, the $[n/n]$ Padé approximant is also called *the nth diagonal Padé approximant*.

Let us consider the Weyl function $m(\lambda) = [(H - \lambda)^{-1}e, e]$, where the corresponding matrix H satisfies the assumptions (A.1), (A.2). The representation (4.7) of $m_{[0,j-1]}$ yields

$$m_{[0,j-1]}(\lambda) = -\frac{Q_j(\lambda)}{P_j(\lambda)} = -\frac{s_0}{\lambda} - \frac{s_1}{\lambda^2} - \cdots - \frac{s_{2n_j-2}}{\lambda^{2n_j-1}} + O\left(\frac{1}{\lambda^{2n_j}}\right) \quad (\lambda \to \infty), \tag{7.1}$$

where $s_i = [H^i e, e]$ $(i = 0, \ldots, 2n_j - 2)$. Moreover, it was shown in [13] that $m_{[0,j-1]}(\lambda)$ has the following asymptotic expansion

$$m_{[0,j-1]}(\lambda) = -\sum_{i=0}^{2n_j-2+k_j} \frac{s_i}{\lambda^{i+1}} + O\left(\frac{1}{\lambda^{2n_j+k_j}}\right) \quad (\lambda \to \infty), \tag{7.2}$$

where $s_i = [H^i e, e]$ $(i = 0, \ldots, 2n_j - 2 + k_j)$. The latter means that the rational function

$$f^{[n_j/n_j]}(\lambda) = m_{[0,j-1]}(\lambda) = \frac{A^{[n_j/n_j]}(1/\lambda)}{B^{[n_j/n_j]}(1/\lambda)} = \frac{-\frac{1}{\lambda^{n_j}} Q_j(\lambda)}{\frac{1}{\lambda^{n_j}} P_j(\lambda)} \quad (j = 1, 2, \ldots) \tag{7.3}$$

is the $[n_j/n_j]$ Padé approximant to m. Besides, it follows from the Padé theorem (see [5, Theorem 1.4.3]), that for L and M satisfying

$$L \geqslant n_j, \qquad M \geqslant n_j, \qquad L + M \leqslant 2n_j + k_j - 1$$

the $[L/M]$ Padé approximants coincide with $f^{[n_j/n_j]}$, and for L, M satisfying

$$L \leqslant n_j + k_j - 1, \qquad M \leqslant n_j + k_j - 1, \qquad L + M \geqslant 2n_j + k_j$$

the $[L/M]$ Padé approximants do not exist (for details see [13]).

In what follows we need the following definition.

Definition 7.2. The set $\Theta(H) := \{(Hy, y)_{\ell^2}\colon \ \|y\| = 1\} \subset \mathbb{C}$ is called a numerical range of the operator H.

Clearly, the numerical range of a bounded operator is a bounded set. By the Hausdorff theorem we have that $\sigma(H) \subset \overline{\Theta(H)}$ (see [22]).

Theorem 7.3. *Let $m(\lambda) = [(H - \lambda)^{-1}e, e]$ be the Weyl function of H satisfying (A.1), (A.2). Then there exists a subsequence of diagonal Padé approximants $f^{[n_j/n_j]}$ to m, which converges to m locally uniformly in $\mathbb{C} \setminus \overline{\Theta(H)}$.*

Proof. First, note that $\Theta(H_{[0,n]}) \subset \Theta(H)$, and, therefore, $\overline{\Theta(H_{[0,n]})} \subset \overline{\Theta(H)}$. As a consequence, we have that if $\lambda \in \mathbb{C} \setminus \overline{\Theta(H)}$ then $\lambda \in \rho(H_{[0,n]})$ for all $n \in \mathbb{Z}_+$. Let $\lambda \in \mathbb{C} \setminus \overline{\Theta(H)}$. Then $\operatorname{ran}(H - \lambda) = \ell^2_{[0,\infty)}$ and for any finite vector ϕ we have

$$(H_{[0,j]} - \lambda)^{-1}\phi \to (H - \lambda)^{-1}\phi, \quad j \to +\infty.$$

Thus, one can obtain

$$f^{[n_j/n_j]}(\lambda) = m_{[0,j-1]}(\lambda) = \left[(H_{[0,j-1]} - \lambda)^{-1}e, e\right] = \left((H_{[0,j-1]} - \lambda)^{-1}e, Ge\right)_{\ell^2} \to \left((H - \lambda)^{-1}e, Ge\right)_{\ell^2} = m(\lambda). \tag{7.4}$$

According to [22, Theorem 3.2, p. 336] one has

$$\left\|(H_{[0,j]} - \lambda)^{-1}\right\| \leqslant \frac{1}{\operatorname{dist}(\lambda, \Theta(H_{[0,n]}))} \leqslant \frac{1}{\operatorname{dist}(\lambda, \Theta(H))}.$$

Hence, the family $f^{[n_j/n_j]} = m_{[0,j]}$ is uniformly bounded on compact sets in $\mathbb{C} \setminus \overline{\Theta(H)}$. Actually, the following estimates hold true

$$\left|m_{[0,j]}(\lambda)\right| = \left|\left((H_{[0,j]} - \lambda)^{-1}e, Ge\right)_{\ell^2}\right| \leqslant \left\|(H_{[0,j]} - \lambda)^{-1}e\right\|_{\ell^2} \|Ge\|_{\ell^2} \leqslant \frac{\|Ge\|_{\ell^2}}{\operatorname{dist}(\lambda, \Theta(H))}.$$

So, the family $f^{[n_j/n_j]}$ is uniformly bounded and, therefore, $f^{[n_j/n_j]}$ is precompact. Now, to complete the proof it is sufficient to apply (7.4) and the Vitali theorem. □

Remark 7.4. In the proof of Theorem 7.3 we used the method proposed in [7] for complex Jacobi matrices. Note that Theorem 7.3 is a generalization of [19, Theorem 1]. More precisely, we do not suppose existence of all diagonal Padé approximants and all poles of the existed diagonal Padé approximants belong to the convex set $\overline{\Theta(H)}$.

Further, following the scheme proposed in [1], we find out a behavior of the associated polynomials at the points of the resolvent set.

Proposition 7.5. *For any $\lambda \in \rho(H)$ the following inequality holds true*

$$\limsup_{j \to +\infty} \left|P_j(\lambda)\right|^{1/j} > 1. \tag{7.5}$$

Proof. In fact, the proof is in line with that in [1]. However, we give the proof here for the convenience of the readers. Using (2.12) and (4.2), one can see

$$P_j(\lambda)\left(\varepsilon_j b_j W_{j+1}(\lambda)\right) - P_{j+1}(\lambda)\left(\varepsilon_j b_j W_j(\lambda)\right) = 1. \tag{7.6}$$

Due to (A.1), (A.2), and (5.7), one has

$$\left|\varepsilon_j b_j W_j(\lambda)\right| \leqslant C_1 {q_1}^j, \quad 0 < q_1 < 1. \tag{7.7}$$

It follows from (7.6) and (7.7) that the sequence $P_j(\lambda)$ cannot be majorized by a geometric sequence p^j with $p < 1/q_1$, that is, for any $p < 1/q_1$ and any positive constant C_2 the inequality

$$\left|P_j(\lambda)\right| \leqslant C_2 p^j$$

is not satisfied for an infinite number of indices j. We thus have

$$\left|P_{j_k}(\lambda)\right| \geqslant C_2 p^{j_k}.$$

If we choose p such that $1/q_1 > p > 1$ we obtain the required result. □

From these statements one can deduce the following result on convergence of diagonal Padé approximants.

Theorem 7.6. *Under the assumptions* (A.1), (A.2), *for any* $\lambda \in \rho(H)$ *there exists a subsequence of diagonal Padé approximants to* $m(\lambda) = [(H-\lambda)^{-1}e, e]$, *which converges to* $m(\lambda)$ *at* λ.

Proof. From (7.5) we get that there exists a subsequence j_k such that

$$\left|P_{j_k}(\lambda)\right| \geqslant C_2 p^{j_k}, \quad p > 1. \tag{7.8}$$

Further, Theorem 5.3 and (7.8) imply the relation

$$\left|m(\lambda) + \frac{Q_{j_k}(\lambda)}{P_{j_k}(\lambda)}\right| = \left|\frac{W_{j_k}(\lambda)}{P_{j_k}(\lambda)}\right| \leqslant C_3 \left(\frac{q}{p}\right)^{j_k},$$

which completes the proof. □

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