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On a Trigonometric Inequality of Vinogradov

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The sum $f(m, n) = \sum_{a=1}^{m-1} (|\sin(\pi an/m)|/\sin(\pi a/m))$ arises in bounding incomplete exponential sums. In this article we show that for positive integers m, n with $m > 1$, $f(m, n) < (4m/\pi^2)(\log m + \gamma + \frac{1}{8} - \log(\pi/2)) + (2/\pi)(2 - 1/\pi)$, where γ is Euler's constant. This improves earlier bounds for $f(m, n)$.

1. INTRODUCTION

Let m and n be positive integers with $m > 1$. The sum

$$f(m, n) = \sum_{a=1}^{m-1} \frac{|\sin(\pi an/m)|}{\sin(\pi a/m)}. \tag{1}$$

Vinogradov [1, Chap. 3, Example 11] obtained the estimate for $f(m, n)$,

$$f(m, n) < \begin{cases} m \log m - \dfrac{m}{3} \log\left(2\left[\dfrac{m}{6}\right] + 1\right) & \text{for } m \geqslant 6 \\ m \log m - \dfrac{m}{2} & \text{for } m \geqslant 12 \\ m \log m - m & \text{for } m \geqslant 60. \end{cases} \tag{2}$$

In 1987 Cochrane [2] proved that

$$f(m, n) < \frac{4m}{\pi^2} \log m + 0.38m + 0.608 + 0.116 \frac{d^2}{m}, \qquad \text{where } d = (m, n). \tag{3}$$

He also proved that

$$\frac{1}{m} \sum_{n=1}^{m} f(m, n) = \frac{4m}{\pi^2} \left(\log m + \gamma - \log \frac{\pi}{2}\right) + O(\log m \log \log m). \tag{4}$$

In this paper we prove the following.

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THEOREM 1. *For any positive integers m, n with $m > 1$ we have*

$$f(m, n) < \frac{4m}{\pi^2}\left(\log m + \gamma + \frac{1}{8} - \log\frac{\pi}{2}\right) + \frac{2}{\pi}\left(2 - \frac{1}{\pi}\right). \tag{5}$$

THEOREM 2. *For any positive integer $m > 1$ we have*

$$\frac{1}{m}\sum_{n=1}^{m} f(m, n) < \frac{4m}{\pi^2}\left(\log m + \gamma - \log\frac{\pi}{2}\right) + \frac{2}{\pi}\left(2 - \frac{1}{\pi}\right) - \frac{\pi}{6m}\left(1 - \frac{1}{m}\right). \tag{6}$$

2. SOME LEMMAS

LEMMA 1. *For any integer $m > 1$ we have*

$$\sum_{a=1}^{m-1} \frac{1}{\sin(\pi a/m)} < \frac{2m}{\pi}\left(\log m + \gamma - \log\frac{\pi}{2}\right) + 2 - \frac{1}{\pi}. \tag{7}$$

Proof. We have

$$\log\left(\tan\frac{x}{2}\right) - \log\frac{x}{2} = \sum_{k=1}^{\infty} \frac{C_k}{2k} x^{2k} \qquad (0 < x < \pi), \tag{8}$$

where C_k are positive constants such that for $0 < x < \pi$,

$$1/\sin x = \sum_{k=1}^{\infty} C_k x^{2k-1} \tag{9}$$

and $C_0 = 1$. Hence

$$\begin{aligned}\sum_{a=1}^{m-1} \frac{1}{\sin(\pi a/m)} &\leqslant 2\sum_{a=1}^{[m/2]} \frac{1}{\sin(\pi a/m)} = 2\sum_{k=0}^{\infty} C_k(\pi/m)^{2k-1} \sum_{a=1}^{[m/2]} a^{2k-1}\\ &= \frac{2m}{\pi}\sum_{a=1}^{[m/2]} \frac{1}{a} + 2\sum_{k=1}^{\infty} C_k(\pi/m)^{2k-1}\sum_{a=1}^{[m/2]} a^{2k-1}.\end{aligned} \tag{10}$$

If $m = 2p$ (even), then

$$\frac{2m}{\pi}\cdot\frac{1}{2[m/2]} = \frac{4p}{\pi}\cdot\frac{1}{2p} = \frac{2}{\pi},$$

and if $m = 2p + 1$ (odd), then

$$\frac{2m}{\pi}\cdot\frac{1}{2[m/2]} = \frac{4p+2}{\pi}\cdot\frac{1}{2[p+1/2]} = \frac{2}{\pi} + \frac{1}{p\pi} \leqslant \frac{3}{\pi}.$$

Using (2.1) in [2] we have

$$\frac{2m}{\pi}\sum_{a=1}^{[m/2]}\frac{1}{a}<\frac{2m}{\pi}\left(\log\,[m/2]+\gamma+\frac{1}{2[m/2]}\right)\leqslant\frac{2m}{\pi}\left(\log m+\gamma-\log 2\right)+\frac{3}{\pi}. \tag{11}$$

Now

$$\sum_{a=1}^{[m/2]}a^{2k-1}<\int_0^{m/2}t^{2k-1}\,dt+\left(\frac{m}{2}\right)^{2k-1}=\frac{1}{2k}\left(\frac{m}{2}\right)^{2k}+\left(\frac{m}{2}\right)^{2k-1}\qquad (k=1,2,...). \tag{12}$$

Hence, by (8), (9), and (12) we have

$$\begin{aligned}2\sum_{k=1}^{\infty}C_k\left(\frac{\pi}{m}\right)^{2k-1}\sum_{a=1}^{[m/2]}a^{2k-1}&<\frac{2m}{\pi}\sum_{k=1}^{\infty}\frac{C_k}{2k}\left(\frac{\pi}{2}\right)^{2k}+2\sum_{k=1}^{\infty}C_k\left(\frac{\pi}{2}\right)^{2k-1}\\&=\frac{2m}{\pi}\left(\log\left(\tan\frac{\pi}{4}\right)-\log\frac{\pi}{4}\right)\\&\quad+2\left(\frac{1}{\sin(\pi/2)}-\frac{2}{\pi}\right)\\&=-\frac{2m}{\pi}\log\frac{\pi}{4}+2-\frac{4}{\pi}.\end{aligned} \tag{13}$$

Combining (10), (11), and (13) yields the result of the lemma.

LEMMA 2. *For any positive integers m, n, k with $m>3$ we have*

$$\sum_{a=1}^{m-1}\frac{\cos(2knax/m)}{\sin(ax/m)}>-\frac{m}{4\pi}. \tag{14}$$

Proof. (1) Define

$$a_r=\begin{cases}\dfrac{2}{\sin(\pi/m)}-\dfrac{1}{\sin(2\pi/m)}, & r=0,\\[2mm] \dfrac{1}{\sin(r\pi/m)}, & r=1,2,...,m-1,\\[2mm] 2a_{r-1}-a_{r-2}, & r=m,m+1,...,2m.\end{cases} \tag{15}$$

Now we show

$$a_{m+p}=\frac{p+2}{\sin(\pi/m)}-\frac{p+1}{\sin(2\pi/m)}\qquad (p=0,1,...,m). \tag{16}$$

In fact, if $p=0$, by (15)

$$\begin{aligned} a_m &= 2a_{m-1} - a_{m-2} \\ &= \frac{2}{\sin((m-1)\,\pi/m)} - \frac{1}{\sin((m-2)\,\pi/m)} \\ &= \frac{2}{\sin(\pi/m)} - \frac{1}{\sin(2\pi/m)}. \end{aligned}$$

Formula (16) is true for $p=0$. Assume that Eq. (16) is true for $p=s$; i.e.,

$$a_{m+s} = \frac{s+2}{\sin(\pi/m)} - \frac{s+1}{\sin(2\pi/m)}.$$

Thus, by (15), we have

$$\begin{aligned} a_{m+s+1} &= 2a_{m+s} - a_{m+s-1} \\ &= 2\left(\frac{s+2}{\sin(\pi/m)} - \frac{s+1}{\sin(2\pi/m)}\right) - \left(\frac{s+1}{\sin(\pi/m)} - \frac{s}{\sin(2\pi/m)}\right) \\ &= \frac{s+3}{\sin(\pi/m)} - \frac{s+2}{\sin(2\pi/m)}. \end{aligned}$$

Hence, (16) is true for $p=s+1$. So (16) follows.

(2) Suppose first that $0 < 2kn\pi/m < 2\pi$. By (15) we have

$$s = \sum_{a=1}^{m-1} \frac{\cos(2kna\pi/m)}{\sin(a\pi/m)} = \sum_{r=1}^{2m} a_r \cos\frac{2knr\pi}{m} - \sum_{r=m}^{2m} a_r \cos\frac{2knr\pi}{m} = s_1 - s_2. \qquad (17)$$

We infer by partial summation

$$\begin{aligned} \frac{a_0}{2} + \sum_{r=1}^{2m} a_r \cos\frac{2knr\pi}{m} &= \sum_{r=0}^{2m-1} (a_r - a_{r+1})\, D_r + a_{2m} D_{2m} \\ &= \sum_{r=0}^{2m-2} (a_r - 2a_{r+1} + a_{r+2})\, B_r \\ &\quad + (a_{2m-1} - a_{2m})\, B_{2m-1} + a_{2m} D_{2m}. \end{aligned} \qquad (18)$$

Here $\theta = 2kn\pi/m$, so that $0 < \theta < 2\pi$,

$$D_0 = \frac{1}{2}, \qquad D_R = \frac{1}{2} + \sum_{k=1}^{R} \cos k\theta = \frac{\sin(R+1/2)\,\theta}{2\sin(\theta/2)}, \qquad (19)$$

$$B_0 = \frac{1}{2}, \qquad B_R = \sum_{k=0}^{R} D_k = \frac{1}{2}\left(\frac{\sin(R+1)(\theta/2)}{\sin(\theta/2)}\right)^2. \qquad (20)$$

From (16), (19), and (20),

$$\left.\begin{aligned} a_{2m-1}-a_{2m} &= \left(\frac{m+1}{\sin(\pi/m)}-\frac{m}{\sin(2\pi/m)}\right)-\left(\frac{m+2}{\sin(\pi/m)}-\frac{m+1}{\sin(2\pi/m)}\right) \\ &= \frac{-1}{\sin(\pi/m)}+\frac{1}{\sin(2\pi/m)}, \\ D_{2m} &= \frac{\sin(2m+1/2)(2kn\pi/m)}{2\sin(kn\pi/m)}=\frac{1}{2}, \\ B_{2m-1} &= \frac{1}{2}\left(\frac{\sin(2m-1+1)(kn\pi/m)}{\sin(kn\pi/m)}\right)^2=0. \end{aligned}\right\} \qquad (21)$$

Let $\varphi(x)=(1/\sin(\pi x/m))$ for $0<x<m$. We have

$$\varphi''(x)=\frac{(\pi/m)^2}{\sin^3(\pi x/m)}(\sin^2(\pi x/m)+2\cos^2(\pi x/m))>0.$$

Hence

$$\frac{1}{\sin(r\pi/m)}-\frac{2}{\sin((r+1)\pi/m)}+\frac{1}{\sin((r+2)\pi/m)}>0 \qquad (r=1, 2, ..., m-3). \qquad (22)$$

By (15) and (22), we get

$$a_r-2a_{r+1}+a_{r+2}\geqslant 0 \qquad (r=0, 1, 2, ..., 2m-2). \qquad (23)$$

Hence, by (18), (21), and (23), we have

$$s_1\geqslant \tfrac{1}{2}(a_{2m}-a_0)=\tfrac{1}{2}(a_2+a_{2m}-2a_1). \qquad (24)$$

Similarly,

$$\begin{aligned} \frac{a_{m-1}}{2}+\sum_{r=m}^{2m} a_r\cos\frac{2knr\pi}{m} &= \frac{a_{m-1}}{2}+\sum_{r=0}^{m} a_{m+r}\cos\frac{2kn(m+r)\pi}{m} \\ &= \frac{a_{m-1}}{2}+\sum_{r=0}^{m} a_{m+r}\cos\frac{2knr\pi}{m} \\ &= \sum_{r=-1}^{m-2}(a_{m+r}-2a_{m+r+1}+a_{m+r+2})B_r \\ &\quad +(a_{2m-1}-a_{2m})B_{m-1}+a_{2m}D_m. \end{aligned} \qquad (25)$$

By (15), (19), and (20), we get

$$a_{m+r}-2a_{m+r+1}+a_{m+r+2}=0 \qquad (r=-1,0,1,\ldots,m-2) \tag{26}$$

$$\begin{aligned} D_m &= \frac{\sin(m+1/2)(2kn\pi/m)}{2\sin(kn\pi/m)}=\frac{1}{2}, \\ B_{m-1} &= \frac{1}{2}\left(\frac{\sin(m-1+1)(kn\pi/m)}{\sin(kn\pi/m)}\right)^2=0. \end{aligned} \tag{27}$$

Thus, by (25), (26), and (27), we have

$$s_2=\tfrac{1}{2}(a_{2m}-a_{m-1}). \tag{28}$$

From (9), (17), (24), and (28), we get

$$\begin{aligned} s &= \sum_{a=1}^{m-1}\frac{\cos(2kn\pi a/m)}{\sin(a\pi/m)} \geqslant \frac{1}{2}(a_{m-1}+a_2-2a_1) \\ &= \frac{1}{2}\left(\frac{1}{\sin((m-1)\pi/m)}+\frac{1}{\sin(2\pi/m)}-\frac{2}{\sin(\pi/m)}\right) \\ &= \frac{1}{2}\left(\frac{1}{\sin(2\pi/m)}-\frac{1}{\sin(\pi/m)}\right) \\ &= \frac{1}{2}\sum_{k=0}^{\infty} C_k\left(\frac{\pi}{m}\right)^{2k-1}(2^{2k-1}-1) \\ &= -\frac{m}{4\pi}+\sum_{k=1}^{\infty} C_k\left(\frac{\pi}{m}\right)^{2k-1}(2^{2k-1}-1) \\ &> -\frac{m}{4\pi}. \end{aligned} \tag{29}$$

Again, if $2kn\pi/m=2\pi$, then

$$\sum_{a=1}^{m-1}\frac{\cos(2kna\pi/m)}{\sin(a\pi/m)}>0. \tag{30}$$

If $2kn\pi/m>2\pi$, then there exist positive integers k', n' such that $0<2k'n'\pi/m\leqslant 2\pi$. We have

$$\sum_{a=1}^{m-1}\frac{\cos(2kna\pi/m)}{\sin(a\pi/m)}=\sum_{a=1}^{m-1}\frac{\cos(2k'n'a\pi/m)}{\sin(a\pi/m)}>-\frac{m}{4\pi}. \tag{31}$$

Obviously, our lemma follows from (29), (30), and (31).

3. Proof of Theorem 1

Using the Fourier series

$$|\sin\theta| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos 2k\theta}{4k^2-1},$$

we have

$$\begin{aligned} f(m,n) &= \sum_{a=1}^{m-1}\frac{|\sin(\pi an/m)|}{\sin(\pi a/m)} \\ &= \frac{2}{\pi}\sum_{a=1}^{m-1}\frac{1}{\sin(\pi a/m)} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\sum_{a=1}^{m-1}\frac{\cos(2kna\pi/m)}{\sin(a\pi/m)}. \end{aligned} \tag{32}$$

By Lemma 1,

$$\frac{2}{\pi}\sum_{a=1}^{m-1}\frac{1}{\sin(a\pi/m)} < \frac{4m}{\pi^2}(\log m + r - \log(\pi/2)) + \frac{2}{\pi}(2 - 1/\pi). \tag{33}$$

By Lemma 2,

$$-\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\sum_{a=1}^{m-1}\frac{\cos(2kna\pi/m)}{\sin(\pi a/m)} < \frac{m}{2\pi^2}. \tag{34}$$

Combining (32), (33), and (34), we have

$$f(m,n) < \frac{4m}{\pi^2}\left(\log m + \gamma + \frac{1}{8} - \log(\pi/2)\right) + \frac{2}{\pi}\left(2 - \frac{1}{\pi}\right) \qquad (m > 3).$$

Suppose now that $m = 2, 3$; then we have

$$\begin{aligned} f(2,n) &= \frac{|\sin(n\pi/2)|}{\sin(\pi/2)} \\ &\leqslant 1 < \frac{8}{\pi^2}\left(\log 2 + \gamma + \frac{1}{8} - \log\frac{\pi}{2}\right) + \frac{2}{\pi}\left(2 - \frac{1}{\pi}\right), \\ f(3,n) &= \frac{|\sin(n\pi/3)|}{\sin(\pi/3)} + \frac{|\sin(2n\pi/m)|}{\sin(2\pi/3)} \\ &\leqslant \frac{2}{\sin(\pi/3)} = \frac{4}{\sqrt{3}} \\ &< \frac{12}{\pi^2}\left(\log 3 + \gamma + \frac{1}{8} - \log\frac{\pi}{2}\right) + \frac{2}{\pi}\left(2 - \frac{1}{\pi}\right), \end{aligned}$$

completing the proof of the theorem.

4. Proof of Theorem 2

Lemma 3 [2]. *For any integers m, n with $m > 1$, we have*

$$\sum_{a=1}^{m} \left|\sin \frac{\pi a n}{m}\right| < \frac{2m}{\pi} - \frac{\pi d^2}{6m}, \tag{35}$$

where $d = (m, n)$.

Proof of Theorem 2. By Lemma 1, (35), and $(a, m)^2/\sin(\pi a/m) \geqslant 1$, we have

$$\begin{aligned} \sum_{n=1}^{m} f(m, n) &= \sum_{a=1}^{m} \frac{1}{\sin(\pi a/m)} \sum_{n=1}^{m} |\sin(\pi a n/m)| \\ &< \frac{2m}{\pi} \sum_{a=1}^{m-1} \frac{1}{\sin(\pi a/m)} - \frac{\pi}{6m} \sum_{a=1}^{m-1} \frac{(a, m)^2}{\sin(\pi a/m)} \\ &\leqslant \frac{2m}{\pi} \sum_{a=1}^{m-1} \frac{1}{\sin(\pi a/m)} - \frac{\pi}{6m} \sum_{a=1}^{m-1} 1 \\ &< \frac{4m^2}{\pi^2} (\log m + \gamma - \log(\pi/2)) + \frac{2m}{\pi}\left(2 - \frac{1}{\pi}\right) - \frac{\pi}{6m}(m-1). \end{aligned}$$

So the theorem follows.

References

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2. T. Cochrane, On a trigonmetric inequality of Vinogradov, *J. Number Theory* **27** (1987), 9–16.