Blocks with extraspecial defect groups of finite quasisimple groups

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Abstract

We classify all blocks of finite quasisimple groups with extraspecial defect groups.

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1. Introduction

In the modular representation theory of finite groups, a key theme is the determination of information about blocks from local information such as the defect groups and their normalizers. A number of important conjectures, such as those of Alperin, Dade and Broué, embody this theme. One approach to these conjectures is to use the classification of finite simple groups, and this is often done with some restriction on the defect groups. The class of extraspecial p-groups includes the smallest nonabelian p-groups and as such blocks with extraspecial defect groups are an obvious object of study. We determine the blocks of quasisimple groups with extraspecial defect groups, and it is hoped that this may prove useful in the application of the classification of finite simple groups.

In work on nilpotent and controlled blocks of finite quasisimple groups in [5] and [3], we notice that blocks with extraspecial defect groups play a special role. In particular it turns out that almost all controlled blocks with nonabelian defect groups of a quasisimple group have extraspecial or trivial intersection defect groups. We apply some techniques developed in the study of nilpotent and con-
trolled blocks of finite groups of Lie type to the classification of blocks of quasisimple groups whose defect groups are extraspecial.

In order to use inductive arguments, in treating the groups of Lie type we consider a broader class of $p$-groups: those whose derived subgroups have order 1 or $p$. We call such groups small derived subgroup groups, or SDS groups. The strategy is to prove strong results concerning classical groups (and certain related groups) in order to apply these results to the exceptional groups, whose blocks are not otherwise accessible for study. Our method for studying the blocks of the exceptional groups of Lie type is to consider the centralizer of an element in a defect group, which decomposes into classical groups and exceptional groups of smaller Lie rank.

Note that subgroups of SDS groups are themselves SDS groups, and if $P$ is an SDS group and $Z \leq Z(P)$, then $P/Z$ is also an SDS group. The principal examples we have in mind are extraspecial groups and abelian groups.

Let $G$ be a finite group and $p$ a prime. Although the classification concerns only blocks with respect to a field of characteristic $p$, we use methods from ordinary character theory, for example canonical characters, and so must use a $p$-modular system. Let $O$ be a local discrete valuation ring, complete with respect to the $p$-adic valuation, with field of fractions $K$ of characteristic zero and algebraically closed residue field $k = O/\mathfrak{O}$ of characteristic $p$. We assume that $O$ contains a primitive $|G|$th root of unity. Write $\text{Blk}(G)$ for the set of blocks of $O\text{G}$ and denote by $B_0(G)$ the principal block of $G$.

Let $N$ be a normal subgroup of $G$ and write $\text{Irr}(G)$ the set of irreducible $K$-characters of $G$. For $\theta \in \text{Irr}(N)$, we denote by $\text{Irr}(G | \theta)$ the subset of $\text{Irr}(G)$ consisting of characters covering $\theta$. We denote by $\text{Irr}(B)$ the set of irreducible characters belonging to $B$, $k(B) = |\text{Irr}(B)|$, and combine the above notations freely.

We use the convention $[x, y] = xyx^{-1}y^{-1}$.

In Section 2 we consider blocks of the symmetric and alternating groups and their covering groups. In Section 3 we list the extraspecial defect groups of the sporadic simple groups and their covers. In Section 4 we treat the classical groups. Here we prove stronger results than are necessary for the classification of their extraspecial defect groups, in order to provide the information necessary for the case of the exceptional groups of Lie type, which are considered in Section 5. Section 5 concludes with the exceptional covers of the alternating groups and groups of Lie type.

2. Symmetric and alternating groups

Let $G = \tilde{S}_n$, the double cover of $S_n$, where $n \geq 5$, and $\tilde{A}_n = K \leq G$. Let $B$ be a $p$-block of $K$ and $B_G$ a block of $G$ covering $B$. Suppose that $B_G$ has noncentral defect group $D_G$, so that $D := D_G \cap K$ is a non-central defect group for $B$. Then $D_G$ is isomorphic to a Sylow $p$-subgroup of $\tilde{S}_m$, where $m$ is the weight of $B_G$ (see [12]). Hence $D_GZ(G)/Z(G) \cong D_1 \times \cdots \times D_t$, where $D_i \cong \mathbb{Z}_p \wr \cdots \wr \mathbb{Z}_p$, each $D_i$ is nontrivial and at most $p - 1$ of the $D_i$ are isomorphic.

If $p$ is odd, then $D_GZ(G)/Z(G) \cong D_G = D$. It is immediate that $D$ is SDS if and only if $D_1 \cong \mathbb{Z}_p$ for each $i$, i.e., if $t = m < p$. Hence all SDS defect groups are abelian when $p$ is odd.

Suppose $p = 2$. Then $D_GZ(G)/Z(G)$ is SDS if and only if $m \leq 3$. Also, $DZ(G)/Z(G)$ is SDS if and only if $m \leq 3$. Note that since $[G : K] = 2$, $B_G$ is the unique block of $G$ covering $B$. The blocks of $G$ (resp. $K$) are in 1-1 correspondence with the blocks of $G/Z(G)$ (resp. $K/Z(G)$).

Now $B_C$ has an irreducible character $\chi$ labeled by a non-selfassociate partition, so by [21, 2.5.7] $\chi$ covers a $G$-stable irreducible character of $B$. Hence $B$ is $G$-stable, and $[D_G : D] = 2$.

We treat the cases $m = 1$, $m = 2$ and $m = 3$ in turn.

Suppose $m = 1$. Then $D_GZ(G)/Z(G) \cong \mathbb{Z}_2$ and $|D_G| = 4$, so $D_G$ is abelian.

Suppose $m = 2$. Then $D_GZ(G)/Z(G) \cong D_8$. This occurs precisely when $n = (r^2 + r + 8)/2$ for some $r$. In this case $DZ(G)/Z(G) \cong (\mathbb{Z}_2)^2$. Here $D_GZ(G)/Z(G)$, $DZ(G)/Z(G)$ and $D$ are SDS, but not $D_G$.

Suppose $m = 3$. Then $D_GZ(G)/Z(G) \cong D_8 \times \mathbb{Z}_2$. This occurs precisely when $n = (r^2 + r + 12)/2$ for some $r$. Since $p = 2$ and $m$ is odd, no non-faithful irreducible character of $B_G$ is labeled by a selfassociate partition. Hence every non-faithful irreducible character of $B_G$ covers a stable irreducible character of $B$, and so every non-faithful irreducible character of $B$ is $G$-stable. Since there is an irreducible character of $B_G$ of non-maximal defect, it follows that there is also a non-faithful irreducible character of $B$ of non-maximal defect (recall that the defect of an irreducible character $\theta$ of a finite
group $H$ is the non-negative integer $d$ such that $|H|/θ(1)_p = p^d$. Since Brauer’s height zero conjecture holds for the alternating groups (see [26, 4.8]) it follows that $DZ(G)/Z(G)$ is nonabelian and so isomorphic to $D_8$. Here $D_G Z(G)/Z(G)$, $DZ(G)/Z(G)$ and $D$ are SDS, but not $D_G$.

3. Sporadic groups

In Table 1 we list the blocks of quasisimple groups $G$ such that $G/Z(G)$ is a sporadic simple group, with defect groups $D$ such that $D := DZ(G)/Z(G)$ is extraspecial. In Table 1, $G$ is taken to be a full cover of the simple group. Except in the case $G/Z(G) ≅ Co_1$, all of the information may be extracted either from the library in [18] or from other sources which are listed in Table 1.

In [7] all of the blocks of $Co_1$ are given, as well as the 3-blocks for $2.Co_1$. Suppose $G ≅ 2.Co_1$. It remains to check the case $p = 5$. By [7] there is precisely one conjugacy class of radical extraspecial 5-subgroups of $G/Z(G)$ and this is self-centralizing, with normalizer $5_+^{1+2} : GL_2(5)$. Let $Q ≤ G$ such that $Q ≅ 5_+^{1+2}$. Since $N_G/Z(G)(Z(G)Q/Z(G)) = N_G(Q)/Z(G)$ it follows that $N_G(Q)$ is a double cover of $5_+^{1+2} : GL_2(5)$ and $C_G(Q) = Z(G) × Z(Q)$. Write $H = Z(G) × Q ≤ N_G(Q)$. Since $C_G(Q) ≤ H$, it follows that each of the two blocks of $H$ are covered by a unique block of $N_G(Q)$. Since each of the blocks of $H$ is $N_G(Q)$-stable, they must be covered by blocks of maximal defect. It follows that $N_G(Q)$ has no block with defect group $Q$, and so by Brauer’s first main theorem $G$ has no block with defect group $Q$. Hence the blocks with extraspecial defect groups of $G$ are as given in the table.

Using [18,7,4,23] we verify that the only blocks with extraspecial defect groups for groups with $G/Z(G)$ sporadic simple and $1 ≅ O_p(Z(G))$ occur for $G/Z(G) ≅ M_{12}$, $HS$, $J_2$ and $Ru$ for $p = 2$, in which case $|D| = 2^3$, and for $G/Z(G) ≅ Sz$, $O'N$ and $F_{14}'$ for $p = 3$, in which case $|D| = 3^3$.

We now turn to the case that $G$ is a quasisimple group such that $G/Z(G)$ is a sporadic simple group and $O_p(G) ≠ 1$, and $B$ is a block of $G$ with extraspecial defect group $D$. We assume again that $G$ is the full cover of $G/Z(G)$. Using the same references as in Table 1 we observe the following: when

<table>
<thead>
<tr>
<th>$G/Z(G)$</th>
<th>$D$ (faithful blocks)</th>
<th>$D$ (nonfaithful blocks)</th>
<th>Reference</th>
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<tr>
<td>$M_{11}$</td>
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<td>[18]</td>
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<tr>
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<td>$3_+^{1+2}$</td>
<td>[18]</td>
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<tr>
<td>$M_{22}$</td>
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<td>[18]</td>
</tr>
<tr>
<td>$M_{23}$</td>
<td>none</td>
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<td>[18]</td>
</tr>
<tr>
<td>$M_{24}$</td>
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<tr>
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<tr>
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<td>$3_+^{1+2}$</td>
<td>$3_+^{1+2}$</td>
<td>[18]</td>
</tr>
<tr>
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<td>[18]</td>
</tr>
<tr>
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<td>$3_+^{1+2}$, $3_+^{1+2}$, $11_+^{1+2}$</td>
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<td>[8]</td>
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<tr>
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<td>$5_+^{1+2}$</td>
<td>$5_+^{1+2}$</td>
<td>[18]</td>
</tr>
<tr>
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<td>$5_+^{1+2}$</td>
<td>$5_+^{1+2}$</td>
<td>[18]</td>
</tr>
<tr>
<td>$Suz$</td>
<td>$D_8$</td>
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<tr>
<td>$Ly$</td>
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<td></td>
<td>[27]</td>
</tr>
<tr>
<td>$He$</td>
<td>$D_8$, $3_+^{1+2}$, $7_+^{1+2}$</td>
<td></td>
<td>[18]</td>
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<tr>
<td>$Ru$</td>
<td>$3_+^{1+2}$, $5_+^{1+2}$</td>
<td>$3_+^{1+2}$, $5_+^{1+2}$</td>
<td>[18]</td>
</tr>
<tr>
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<td>$7_+^{1+2}$</td>
<td>$D_8$, $7_+^{1+2}$</td>
<td>[18]</td>
</tr>
<tr>
<td>$Co_3$</td>
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<tr>
<td>$Co_1$</td>
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<td>$3_+^{1+2}$</td>
<td>[7] and see below</td>
</tr>
<tr>
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<td>[18]</td>
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<tr>
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<tr>
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<td>[4]</td>
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<tr>
<td>$F_{14}'$</td>
<td>$D_8$, $7_+^{1+2}$</td>
<td>$D_8$, $7_+^{1+2}$</td>
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</tr>
<tr>
<td>$Th$</td>
<td>$5_+^{1+2}$</td>
<td></td>
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</tr>
<tr>
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<tr>
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<td>$D_8$</td>
<td></td>
<td>[9] and [23]</td>
</tr>
<tr>
<td>$F_1 = M$</td>
<td>$3_+^{1+2}$, $13_+^{1+2}$</td>
<td></td>
<td>[10] and [23]</td>
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</table>
Let $V$ be a linear, unitary, non-degenerate orthogonal or symplectic space over field $\mathbb{F}_q$, where $q$ is a power of $r$ for some prime $r \neq p$.

If $V$ is orthogonal, then there is a choice of equivalence classes of quadratic forms. Write $\eta(V)$ for the type of $V$ as defined in [17], so $\eta(V) = +$ or $-$. Let $\eta(V) = +$ if $V$ is linear and $\eta(V) = -$ if $V$ is unitary.

If $V$ is non-degenerate orthogonal or symplectic, then denote by $I$ the group of isometries on $V$ and let $I_0(V) = I(V) \cap SL(V)$. Then $I(V) = I_0(V) = \text{Sp}_{2n}(q)$ if $V$ is symplectic, $I(V) = (-1)^V \times I_0(V)$ with $I_0(V) = \text{SO}_{2n+1}(q)$ if $V$ is a $(2n + 1)$-dimensional orthogonal space and $I(V) = O^+(V) = O^+_{2n}(q)$ and $I_0(V) = \text{SO}_{2n}^-(q)$ if $V$ is a $2n$-dimensional orthogonal space with $\eta(V) = \eta$, where $\eta = +$ or $-$. If $V$ is a non-degenerate orthogonal or symplectic space, then denote by $J_0(V)$ the conformal isometries of $V$ with square determinant, and $D_0(V)$ the special Clifford group of an orthogonal space $V$ (cf. [17]).

We will follow the notation of [2,11,16]. Let $G = GL^0(V)$ or $I(V)$, and let $F_\varphi = F_\varphi(G)$ (resp. $F_\varphi^p$) be the set of polynomials serving as elementary divisors for all semisimple elements (resp. semisimple $p'$-elements) of $G$ (cf. [2, p. 6]). Let $d_\Gamma$ and $\delta_\Gamma$ be the degree and the reduced degree of $\Gamma \in F_\varphi$, and let $\epsilon_\Gamma$ be the sign given by [2, p. 6]), so that $d_\Gamma = \delta_\Gamma + \epsilon_\Gamma$ if $G = GL^0(V)$, $\epsilon_\Gamma = +$ when $G = GL(V)$ and $\epsilon_\Gamma = -$ or $0$ according as $\delta_\Gamma$ is odd or even when $G = U(V)$. For $\Gamma \in F_\varphi$, let $e_\Gamma$ be the multiplicative order of $\epsilon_\Gamma q^{k_r}$ modulo $p$ or 4 according as $p$ is odd or even, and let $m_\Gamma(s)$ be the multiplicity of $\Gamma$ in $s$. Thus $\epsilon_\Gamma \delta_\Gamma = ep^{e_\Gamma} \delta_\Gamma$ with $p \not| \delta_\Gamma$, where $e = e_{\epsilon_\Gamma} - 1$.

Given a semisimple element $s \in G$, there is a unique orthogonal decomposition $V = \sum_{\Gamma \in F_\varphi} V_\Gamma(s)$, with $s = \prod_{\Gamma \in F_\varphi} s(\Gamma)$, where the $V_\Gamma(s)$ are nondegenerate subspaces of $V$ and $s(\Gamma) \in GL(V_\Gamma(s))$, $U(V_\Gamma(s))$ or $I(V_\Gamma(s))$ (depending on $G$) has minimal polynomial $(\Gamma)$. This is called the primary decomposition of $s$. Write $m_\Gamma(s)$ for the multiplicity of $\Gamma$ in $s(\Gamma)$. We have $C_G(s) = \prod_{\Gamma \in F_\varphi} C_{\Gamma}(s)$, where $C_{\Gamma}(s) = I(V_\Gamma(s))$ or $GL^r(m_{\Gamma}(s), q^{kr})$ as appropriate.

Suppose $G = GL^0_1(q) = GL^0(V)$, and let $B$ be a $p$-block of $G$ with a defect group $D$ and label $(s, \kappa)$.

Then
\[
V = V_0 \oplus V_+, \quad D = D_0 \times D_+, \quad s = s_0 \times s_+,
\]
where $V_0 = C_V(D)$, $V_+ = [D, V]$, $s_0 \in G_0 = GL^0(V_0)$ and $s_+ \in G_+ := GL^0(V_+)$. So if $p = 2$, then $V_0 = C_V(D) = 0$, $V_+ = V$, $D = D_+$ and $s = s_+$. We also denote $GL^0(V)$ by $G(V)$ and $SL^0(V)$ by $S(V)$.

For integers $c, m$ and the prime $p$, we write $p^c \not| m$ when $p^c \not| m$ and $p^{c+1} \not| m$, and we let $p^a(q^2 - 1)$ or $2a + 1 || (q^2 - 1)$ according as $p$ is odd or even.

**Proposition 4.1.** Let $K := SL^0(V) = SL^0(n, q) \leq G := GL^0(V)$, $Z \leq O_p(Z(K))$ and let $B_K \in B_k(K)$ have defect group $D_K$. Let $B_G \in B_k(G)$ be a weakly regular cover of $B_K$ and $D_G$ be a defect group of $B_G$, so that $D_K = D_G \cap K$. Suppose that $D_K/Z$ is an SDS group. Then $D_G$ is abelian if and only if $D_G/Z$ is abelian.

(a) If $D_K/Z$ is nonabelian, then one of the following holds.

(i) $p = 3$, $\mu = 1$, $3 \not| (q - \eta)$, $n = 3 \delta_\Gamma$ with $3 \not| \delta_\Gamma$ for some $\Gamma \in F_\varphi^p$, $D_G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $D_K/Z = D_K \cong 3_+^{1+2}$.

(ii) $p = 2$, $n = 2s$ with odd $s = \delta_\Gamma$ for some $\Gamma \in F_\varphi^p$, $D_G \cong SD_{2a+2}$ or $\mathbb{Z}_{2a} \times \mathbb{Z}_2$ according as $2 || (q - \eta)$ or $2^a || (q - \eta)$ and $a = 2$ or 3. We have $(Z, a) = (1, 2)$ or $(2, 3)$. In the former case $D_K = D_K/Z \cong Q_8$ and in the latter case $D_K \cong Q_{2^{a+1}}$, generalized quaternion of order $2^{a+1} = 16$, and $D_K/Z \cong D_8$. 

G/Z(G) \in \{M_{12}, J_2, Ru, HS\}$, we have $|Z(G)| = 2$ and there is a block $D$ with defect group $D \cong Q_8$; when $G/Z(G) \cong M_{12}$, we have $|Z(G)| = 12$ and there are blocks with defect group $D \cong 3_+^{1+2}$ covering each block of $Z(G)$; when $G/Z(G) \cong \text{Suz}$, we have $|Z(G)| = 6$ and there is a non-faithful block with defect group $D \cong 3_+^{1+2}$; when $G/Z(G) \cong O'N$, we have $|Z(G)| = 3$ and there is a block with defect group $D \cong 3_+^{1+4}$; these account for all such blocks. 

4. Classical groups
(b) If $D_K/Z$ is abelian but $D_G$ is nonabelian, then $Z = Z(D_K)$ and one of the following holds.

(i) $p = 3, 3|(q - \eta), n = 3\delta_\Gamma$ with $3\delta_\Gamma$ for some $\Gamma \in \mathcal{F}_q^P$, $D_G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_3$ and $D_K \cong \mathbb{Z}_3^{1+}$.

(ii) $p = 2, n = 2\delta$ with odd $\delta = \delta_\Gamma$ for some $\Gamma \in \mathcal{F}_q^P$, $D_G = SD_{2s}$. In addition, $D_K \cong \mathbb{Q}_8$.

Conversely, suppose $p = 2$ or $3$ and $3|(q - \eta)$ when $p = 3$. If $K = \text{SL}(p\delta_\Gamma, q)$ for some $\Gamma \in \mathcal{F}_q^P$ with $p \nmid \delta_\Gamma$, then there exists a block $B_K$ satisfying either (a) or (b).

**Proof.** Note that if $Q \leq D_K$, then $Q \cap (Q \cap Z) \cong Q/Z \leq D_K/Z$. So $Q/(Q \cap Z)$ is SDS.

Let $(s, k)$ be the label of $B := B_G$, and $V, D, s$ have the corresponding decomposition (4.1). Let $s_+ = \prod_\Gamma s(\Gamma)$ be a primary decomposition, so that $V_+ = \bigoplus_\Gamma V_\Gamma$ with $V_\Gamma$ the underlying space of $s(\Gamma)$. Thus

$$C_{G_+}(s_+) = \prod_\Gamma C_{G_\Gamma}, \quad C_{G_\Gamma} = \text{GL}_f(m_{G_\Gamma}, q^\delta_\Gamma) \quad (4.2)$$

with $m_{G_\Gamma} = m_\Gamma(s_+)$. We may suppose $D_+ \in \text{Syl}_p(C_{G_+}(s_+))$, that is, a Sylow subgroup of $C_{G_+}(s_+)$, so that

$$D_+ = \prod_\Gamma D_\Gamma, \quad D_\Gamma \in \text{Syl}_p(C_{G_\Gamma}) \quad (4.3)$$

So $D$ is a direct product of cyclic groups and wreath product $p$-groups. Denote by $X_{p^\alpha}$ a Sylow $p$-subgroup of the symmetric group $S_{p^\alpha}$, that is, $X_{p^\alpha} \in \text{Syl}_p(S_{p^\alpha})$. Here $X_{p^\alpha} = 1$ if $\alpha = 0$. Since $\text{SL}_f(m_{G_\Gamma}, q^\delta_\Gamma) \leq C_{G_\Gamma} \cap S(V_\Gamma)$ and since $D_K = D_G \cap K$, it follows that $D_K$ is abelian if and only if $m_\Gamma(s_+) < p$ for all $\Gamma$ if and only if $D_G$ is abelian.

Suppose $D_K/Z$ is SDS and $D_G$ is nonabelian. We have that

$$D_+ = D_1 \times D_2 \times \cdots \times D_m \quad (4.4)$$

with some $D_i$ nonabelian, where $D_i = P_i \times X_{p^{c_i}}$, $P_i \cong \mathbb{Z}_{p^{c_i}}$ with $c_i \geq 1$ or $P_i \in \{\mathbb{Z}_{2^{c_i}}, SD_{2^{c_i}+2}\}$ with $c_i \geq 2$ according as $p \geq 3$ or $p = 2$. Without loss of generality take $D_1$ to be nonabelian. Let $V_1$ be the underlying space of $D_1$, so that $X_{p^{c_1}} \cong \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \leq S(V_1)$. Let

$$M(D_1) = \langle [D_1, D_1], X_{p^{c_1}}, O_p(Z(S(V_1))) \rangle \quad (4.5)$$

so that $M(D_1) \leq S(V_1)$ and hence $M(D_1) \leq D_K$. Since $D_K/Z$ is SDS and since $[D_K, D_K]Z/Z \leq [D_K/Z, D_K/Z]$, it follows that $[D_K, D_K]Z/Z \leq \mathbb{Z}_p$ and in particular, $[D_K, D_K]$ is abelian of rank at most 2. Thus $[M(D_1), M(D_1)]Z_i/Z_i$ is cyclic of order 1 or $p$ for some $Z_i \leq O_p(Z(S(V_1)))$.

**Case 1.** Suppose $p$ is odd and $c_1 \geq 1$. Then $D_1 = Y_1 \times X_{p^{c_1}}$, where $Y_1 = (\mathbb{Z}_{p^{c_1}})^{p^{c_1}}$ is the base subgroup of $D_1$. Since $[X_{p^{c_1}}, X_{p^{c_1}}]Z_i/Z_i \cong [X_{p^{c_1}}, X_{p^{c_1}}] \leq \mathbb{Z}_p$, it follows that $c_1 = 1$ and $X_{p^{c_1}} \cong \mathbb{Z}_p$. Take $\sigma \in X_{p^{c_1}}$ such that $\sigma$ acts on $Y_1 = (\mathbb{Z}_{p^{c_1}})^{p^{c_1}}$ as the permutation $(12 \ldots p)$.

Suppose first that $p \geq 5$. Let $(w, w^{-1}, 1, \ldots, 1) \in [D_1, D_1]$ with $|w| = p^{c_1}$, so that

$$[(w, w^{-1}, 1, \ldots, 1), \sigma] = (w^{-1}, w^2, w^{-1}, 1, \ldots, 1) \in [M(D_1), M(D_1)]$$

Similarly, $[(1, w, w^{-1}, 1, \ldots, 1), \sigma] = (1, w^{-1}, w^2, w^{-1}, 1, \ldots, 1) \in [M(D_1), M(D_1)]$. Since $[M(D_1), M(D_1)]Z_i/Z_i$ is cyclic of order 1 or $p$, it follows that

$$(w^{-1}, w^2, w^{-1}, 1, \ldots, 1)^t = z(1, w^{-1}, w^2, w^{-1}, 1, \ldots, 1)$$

for some $z \in Z_i \leq O_p(Z(S(V_1)))$, where $1 \leq t \leq (p - 1)$, which is impossible.
Thus $p = 3$. Now $[(w, w^{-1}, 1), \sigma] = (w^{-1}, w^2, w^{-1})$, $[(1, w, w^{-1}), \sigma] = (w^{-1}, w^{-1}, w^2)$ and

$$[(w, 1, w^{-1}), \sigma] = (w^{-2}, w, w)$$

are elements of $[M(D_1), M(D_1)]$. But $[M(D_1), M(D_1)]Z_1/Z_1$ is cyclic of order 1 or $p$, so $c_1 = 1$ and $D_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. In particular, $D_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, and so $[D_1, D_1] = (\mathbb{Z}_3)^2$ and $D_1/Z(D_1) = 3^{1+2}$.

Suppose 3 $\not| (q - \eta)$, so that $O_3(Z(K)) = 1$ and $D_K$ is SDS and nonabelian. In particular, $[D_K, D_K] \cong \mathbb{Z}_3$. If $D_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $z_1 \in Z(D_1) \setminus \{1\}$, then $C_G(V_1)(z_1) \cong GL^*(38, q^2)$ for some $\delta \geq 1$.

Since $SL(2, q)$ contains an element of order 3, it follows that we may suppose $w \in SL(2, q)$ and hence $\det(w) = 1$. Thus $D_1 \leq S(V_1)$, and $D_1 \leq D_K$. But then $(\mathbb{Z}_3)^2 \equiv [D_1, D_1] \leq [D_K, D_K] \cong \mathbb{Z}_3$, a contradiction. Thus 3 $\not| (q - \eta)$. But then $3 \not| |Z(D_1)|$, so $a = 1$ and $3|(q - \eta)$.

Suppose $m \geq 2$. Since $D_2$ is either cyclic or $D_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, it follows that there exists $x \in D_2$ such that $\det(x) = w^{-1}$ and $x^2 = 1$, where $w \in O_3(F_q^*)$ with $|w| = 3$. Thus $\det((w, 1, 1) \times x) = 1$, and

$$[(w, 1, 1) \times x, \sigma] = (w^{-1}, w, 1) \in [D_K, D_K],$$

where $(w, 1, 1) \in Y_1$ and $\sigma \in X_p \leq D_1$. Similarly, $[(1, w, 1) \times x, \sigma] = (1, w^{-1}, w) \in [D_K, D_K]$. Let

$$Q_1 = \{(w, 1, 1) \times x, (1, w, 1) \times x, (1, 1, w) \times x, \sigma\}. \quad (4.6)$$

Then $Q_1 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, $Q_1 \leq D_1 \times (\langle x \rangle \times D_2)$, so that $Q_1 \leq S(V_1 + V_2)$ and $[Q_1, Q_1] = [D_1, D_1] \leq [D_K, D_K]$.

If $m \geq 3$, then $Z \cap Q_1 = 1$. So $(\mathbb{Z}_3)^2 \equiv [Q_1, Q_1]Z/Z = [Q_1Z/Z, Q_1Z/Z] \leq [D_K/Z, D_K/Z]$, contradicting our assumption that $D_K/Z$ is SDS. Hence $m = 2$.

Suppose $D_2$ is nonabelian, so that $D_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Take $x \in D_1$ with $\det(x) = w^{-1}$ and a similar proof to above shows that there exists a subgroup $Q_2 \cong \langle x \rangle \times D_2$ such that $Q_2 \leq S(V_1 + V_2)$ and $[Q_2, Q_2] = [D_2, D_2] \leq [D_K, D_K]$. This is impossible, since $[D_K, D_K]Z/Z \leq Z_p$ and $[D_2, D_2] \equiv [D_1, D_1] \equiv (\mathbb{Z}_3)^2$.

It follows that $D_2$ is cyclic of order $\mathbb{Z}_3^c$ with $c \geq 2$. Since $Z \leq O_3(Z(K)) \cong \mathbb{Z}_3$, it follows that $Z = 1$ or $O_3(Z(K))$. If $z \in Q_1 \cap Z$, then $z = a \gamma V_1$ and $z = z_1 \times z_2$, where $z_1 = a \gamma V_1 \in Z(D_1)$ and $z_2 = a \gamma V_2$. Thus $\det(z_1) = a^2 = 1$ and $\det(z_2) = 1$. Now $z_2 \in \langle x \rangle$ and $\det(x) = w^{-1}$, so $z_2 = x^t$ and $1 = \det(z_2) = \det(x^t) = w^{-t}$ and $3 \not| i$. Thus $z_2 = 1 \gamma V_2$ and $z_1 \gamma V_1$. In particular, $Q_1 \cong Q_1Z/Z \leq D_K/Z$, which is impossible. Hence we cannot have $m \geq 2$ after all.

If $m = 1$, then $D_K = D_G \cap K = 3^{1+2}$ (since $D_G \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and $D_K$ is a nonabelian SDS subgroup of $D_G$). Thus $Z = 1$ or $Z(D_K)$ according as $D_K/Z$ is nonabelian or abelian. In the former case (a)(i) holds and in the latter (b)(i).

**Case 2.** Suppose $p = 2$. Then $D_1 = Y_1 \times X_2$, where $Y_1 = (P_1)^{2^n}$ is the base subgroup of $D_1$ with $P_1 \cong \mathbb{Z}_2^{c_1}$ or $SD_2^{c_1+2}$, $c_1 \geq 2$. Since $[M(D_1), M(D_1)]$ is abelian and $X_2^{2^n} \leq S(V_1)$, it follows that $\alpha_1 \leq 2$.

Suppose $\alpha_1 = 2$. Take $\sigma_j \in X_2$, $\alpha_j \leq 2$. Since $[M(D_1), M(D_1)]$ is abelian and $X_2^{2^n} \leq S(V_1)$, it follows that $\alpha_1 \leq 2$.

Suppose $\alpha_1 = 2$. Take $\sigma_j \in X_2^{2^n}$ such that $\sigma_j = (123)$ and $\sigma_3 = (123)(24)$ acting on $Y_1$. Thus $(w, 1, w^{-1}) \in [D_1, D_1]$ for any $w \in P_1$.

$$[(w, 1, w^{-1}, 1), \sigma] = (w^{-1}, w, 1, 1) \in [M(D_1), M(D_1)] \leq [D_K, D_K]$$

and $[(w, 1, w^{-1}, 1), \sigma_2] = (1, w, 1, w^{-1}) \in [M(D_1), M(D_1)] \leq [D_K, D_K]$. Thus

$$(w^{-1}, w, 1, 1) = z(1, 1, w, w^{-1})$$

for some $z \in Z(S(V_1))$. This is impossible, since we can choose $w \in P_1$ such that $|w| = 4$. Thus $\alpha_1 \leq 1$ and so $D_1 \cong P_1 \times \mathbb{Z}_2$ or $P_1$.

Suppose $P_1 = SD_2^{c_1+2} \leq G(U_1)$ and $\alpha_1 = 1$, where $U_1 \subseteq U_1 = V_1$. Then $P_1 \cap S(U_1)$ contains a generalized quaternion group $Q_1$ and for each $w \in Q_1$, $(w, w^{-1}) \in [D_K, D_K]$. But $[D_K, D_K]$ is abelian, so this is a contradiction. Thus $\alpha_1 = 0$ and $D_1 = SD_2^{c_1+2} = P_1$. So the generalized quaternion group $Q_2^{c_1+1}$ of order $2^{c_1+1}$ is a subgroup of $D_1 \cap S(V_1)$ and so $Q_2^{c_1+1} \leq D_K$. Since $D_1 = SD_2^{c_1+1}$ is a Sylow
2-subgroup of $\GL(\mathbb{F}_L, 2, q^{\delta r})$ for some $\Gamma$ and $Z(G(V_i))) \leq \GL(\mathbb{F}_L, 2, q^{\delta r})$, it follows that $2\|q^{\delta r} - \epsilon_r\|$ and so $2\|q - \eta\|$. In particular, $O_2(Z(S(V_i))) \cong \mathbb{Z}_2$ and $[Q_{2q^{\varepsilon_1}}, Q_{2q^{\varepsilon_2}}] \cong \mathbb{Z}_{2^{q^{\varepsilon_1} - 1}} \leq [D, D]$. Now $[D, D]Z/Z \leq \mathbb{Z}_2$ and $c_i \geq 0 \geq 2$, so $c_i - 1 = 1$ or $2$ and $c_i = 2$ or $3$.

Suppose $D_i \cong \mathbb{Z}_{2^{q^{\varepsilon_1}}}$, $\mathbb{Z}_2 \leq G(V_i)$, so that $2^{\|q - \eta\|}(q^{\delta r} - \epsilon_r)$ with $c_i \geq 2$. Suppose $\delta r$ is odd, so that $c_i = a$. Since $(q - \eta) \mid (q^{\delta r} - \epsilon_r)$ and $\delta r$ is odd, it follows that $q^{\delta r} \equiv \eta^{\delta r} = \eta \equiv \epsilon_r \mod q - \eta$, and $\eta - \epsilon_r = -q$ for some integer $t$. Thus $\eta = \epsilon_r$ except when $\eta = 1, q = 3$ and $\epsilon_r = -1$. But if $\eta = 1$, then $G$ is general linear and so $\epsilon_r = 1$ for all $\Gamma$. Thus $\eta = \epsilon_r$, $2^{\|q - \eta\|}$ and $Q_{2q^{\varepsilon_1}} = D_1 \cap S(V_i) = Z(G(V_i)) \cong \mathbb{Z}_{q^{\varepsilon_2} - q^{\varepsilon_1}}$. But $O_2(Z(S(V_i))) \cong \mathbb{Z}_{\gcd(2^{\varepsilon_1}, q^{\varepsilon_1} - q^{\varepsilon_2})} \cong \mathbb{Z}_2$, so $[Q_{2q^{\varepsilon_1}}, Q_{2q^{\varepsilon_2}}] \cong \mathbb{Z}_2$ or $[Q_{2q^{\varepsilon_1}}, Q_{2q^{\varepsilon_2}}] / \mathbb{Z}_2 \cong \mathbb{Z}_2$. It follows that $a = 2$ or $3$ and $c_i = 2$ or $3$.

Suppose $\delta r$ is even, so that $c_i = a + \alpha r = a + 1 \geq 3$. Let $\delta r = \delta r/2$, so that $SL(2, q^{\delta r}) \leq S(U_i)$. Since $SL(2, q^{\delta r})$ contains an element $w$ of order $2^i$, it follows that $\det(w) = 1$ and $D_i \cong \mathbb{Z}_2 \leq S(V_i)$. Thus $D_i \leq D_K$ and $[D_i, D_i] \cong \mathbb{Z}_2 \leq [D_K, D_K]$. If $2\|q - \eta\|$, then $O_2(Z(S(V_i))) \cong \mathbb{Z}_2$ and $[D_i, D_i] / O_2(Z(S(V_i))) \cong \mathbb{Z}_{2^{q^{\varepsilon_1} - 1}} \neq \mathbb{Z}_2$. This is impossible, since $[D_K, D_K]Z/Z \leq \mathbb{Z}_2$. If $2\|q - \eta\|$, then $[D_i, D_i] = \langle w, w^{-1} \rangle \leq Y_i$ with $|w| = 2^{\varepsilon_1}$ and $O_2(Z(S(V_i))) = \langle (w^{2^{\varepsilon_1} - q}, w^{2^{\varepsilon_1} - q}) \rangle \leq Y_i$. Since $[D_K, D_K]Z/Z \leq \mathbb{Z}_2$ and $[D_K, D_K]Z/Z \leq Z(D_K/Z)$, it follows that

\[(w, w^{-1})^\sigma = (w^{-1}, w) = z(w, w^{-1})\]

for some $z \in O_2(Z(S(V_i)))$, where $D_i = \langle Y_i, \sigma \rangle$ and $\sigma$ acts on $Y_i$ are the 2-cycle (12). If $z = (\alpha, \alpha) \in Y_i$, then $w^{-1} = \alpha w$ and $w = \alpha w^{-1}$, and so $w^{-2} = \alpha = w^2$. Thus $w^1 = 1$, which is impossible as $|w| = 2^{\varepsilon_1} \geq 8$. Thus $\delta r$ is not even.

So if $D_i$ is nonabelian, then $D_i = SD_{2^{\varepsilon_1} - 2}$ or $Z_{2^{\varepsilon_1}} \leq S(V_i)$. This implies that $2\|q - \eta\|$ and $D_i \cong \mathbb{Z}_{2^{\varepsilon_1}} \leq \mathbb{Z}_2$. Then take $x_i = \langle w_1 \rangle \in Y_i$ with $|w_1| = 2^{\varepsilon_1}$ and $\sigma_i | D_i$ acts as 2-cycle on the two factors of the base group $Y_i$, then $\sigma_i \in S(V_i)$, $x_1 \times x_2^{-1} \in S(V_i + V_2)$,

\[\left[x_1 \times x_2^{-1}, \sigma_1\right] = \left[x_1, \sigma_1\right] = \left< w_1^{-1}, w^{-1}\right> \in [D, D] \cap S(V_i)\]

and $[x_1^{-1} \times x_2, \sigma_2] = (w_2, w_2^{-1}) \in [D, D] \cap S(V_2)$. Thus

\[H := \left< \left< w_1, w_1^{-1}\right>, \left< w_2, w_2^{-1}\right> \right> \leq [D, D, D]. \quad (4.7)\]

Since $O_2(Z(S(V_1 + V_2))) \cong \mathbb{Z}_2$, it follows that $|H/O_2(Z(S(V_1 + V_2)))| \geq 2^{\varepsilon_1}$. But $[D_K, D_K]Z/Z \leq \mathbb{Z}_2$, this is impossible. So each $D_i$ is abelian when $i \geq 2$.

Suppose $2\|q - \eta\|$, and $D_1 = SD_{2^{\varepsilon_1}}$ and $D_2 = SD_{2^{\varepsilon_2}}$. The proof is similar to the case when $2\|q - \eta\|$. Take $x_i \in D_i$ such that $|x_i| = 2^{\varepsilon_1} - 1$ and $\det(x_i) = \det(x_2)$, and take $\sigma_i \in D_i \cap S(V_i)$ such that $x_i = x_2^{-1} - 1$. Then

\[H := \left< \left[x_1 \times x_2, \sigma_1\right] \times \left[x_1 \times x_2, \sigma_2\right] \right> \leq [D, D, D]\]

and $|H| = 2^{2\varepsilon_1}$. But $O_2(Z(S(V_1 + V_2))) \cong \mathbb{Z}_2$, so $H/O_2(Z(S(V_1 + V_2)))$ has order $2^{2\varepsilon_1} - 1$. This is impossible, since $[D_K, D_K]Z/Z \leq \mathbb{Z}_2$. Thus each $D_i$ is abelian when $i \geq 2$.

Suppose $m \geq 2$. A proof similar to that of Case 1 show there exists an element $x \in D_2$ and subgroup $Q \leq (D_1 \times x) \cap S(V_1 + V_2)$ such that $x^{2^m} = 1$, $\det(x) = \delta_2 \times (\delta_2 + D_2)$, and $Q \cong \mathbb{Z}_2$, where $2^m = O_2(Z_{q^{\varepsilon_1} - q})$. If $m \geq 3$, then $Q \cap Z \geq 1$, $Q = Z \cap Z = Q \leq D_K/Z$, which is impossible. So $m = 2$.

If $2\|q - \eta\|$, then $D_1 = SD_{2^{\varepsilon_1}}$. If $1 \neq z \in Q \cap Z$, then $z = z_1 \times z_2 := \alpha \times \varepsilon_1 \times \varepsilon_2$ for some scalar $\alpha$, where $z_1 \in G(V_i)$. Since $z_1 \in O_2(Z(S(G(V_i)))) = (-1)\times 1$ and $\dim V_1$ is even, it follows that $z_1 = -1$, $\det(z_2) = 1$, and so $\det(z_2) = 1$ and $z_2 = -1$. But $z_2 \in (x)$, so $z_2 = x', 1 = \det(z_2) = \det(x')$ and $i$ is even. Thus $z_2 = x^i = 1$, and $Q = Z \cap Z \leq D_K/Z$, which is impossible.

Hence $m = 1$. Then $D_G \cong \mathbb{Z}_{2^m} \times \mathbb{Z}_2$ or $SD_{2^{\varepsilon_1}}$ according as $2\|q - \eta\|$ or $2\|q - \eta\|$. Thus $D_K = D_G \cap S(V) \cong O_{2^{\varepsilon_1}}$. If $a = 2$, then $D_K \cong \mathbb{Q}_6$ and so $Z = 1$. If $a = 3$, then $Z \cong \mathbb{Z}_2$ and $D_K/Z \cong \mathbb{D}_8$. Thus (a)(ii) holds.
Suppose $2^a \| (q - \eta)$, so that $D_1 \cong \mathbb{Z}_{2^a} \times \mathbb{Z}_2$ and $\vert Z \vert \leq 2^a$. If $z \in Q \cap Z$ with $\vert z \vert = 2^k > 1$, then $z = z_1 \times z_2 := \alpha V_1 \times \alpha V_2$ for some scalar $\alpha$ with $\vert \alpha \vert = 2^k$, where $z_1 \in G(V_1)$. Then $1 = \det(z) = \det(z_1) \det(z_2)$ and $\det(z_2) = \alpha^{2n_1}$, where $\dim V_1 = 2n_1$ with odd $n_1$. Since $z_2 \in \langle x \rangle$, it follows that $z_2 = x^i$ for some $i$. But then

$$\alpha^{2n_1} = \det(z_2)^{-1} = \det(x)^{-i} = w^{-i}$$

and $\vert w^i \vert = \vert \alpha^{2n_1} \vert = \vert \alpha^2 \vert = 2^{k-1}$. Thus $\vert x^i \vert = \vert w^i \vert = 2^{k-1}$, $\vert z_2 \vert = \vert x^i \vert = 2^{k-1}$ and so $\vert z_1 \vert = \vert z_2 \vert = \vert \alpha \vert = 2^{k-1}$. This is a contradiction, since $\vert z \vert = \vert \alpha \vert = 2^k$. It follows that $m = 1$ and (a), (b) hold.

Conversely, suppose $p = 2$ or $3$ with $3 \| (q - \eta)$ when $p = 3$. Let $K = \text{SL}^g(p \delta^r, q) = \text{SL}^g(V)$ for some $\Gamma \in Z^g_0$ with $p \nmid \delta^r$ and $G = \text{GL}^g(V)$. Take $s \in G$ such that $m_r(s) = p$ and so $C_G(s) = \text{GL}^{g,r}(p, q^{\delta^r})$. Let $B_G = \mathcal{E}_p(G, (s))$ and $B_K$ a block of $K$ covered by $B_G$. Then $D(B_K)$ and $D(B_G)$ satisfy (a) and (b). □

Let $V$ be a non-degenerate orthogonal or symplectic space, $G = I_0(V)$ and let $G^*$ be the dual group of $G$. Note that

$$\text{Sp}_{2n}(q)^* = \text{SO}_{2n+1}(q), \quad \text{SO}_{2n+1}(q)^* = \text{Sp}_{2n}(q), \quad \text{SO}_{2n}^g(q)^* = \text{SO}_{2n}^g(q).$$

If $B$ is a block of $I_0(V)$, then there exists a semisimple $p'$-element $s \in I_0(V)^*$ such that

$$B \subseteq \mathcal{E}_p(I_0(V), (s)).$$

Let $(D, b_D)$ be a Sylow $B$-subgroup of $I_0(V)$. Then $V$ and $D$ have corresponding decompositions

$$V = V_0 \perp V_+, \quad D = D_0 \times D_+.$$  \hspace{1cm} (4.8)

If $p$ is odd, then $V_0 = C_V(D), V_+ = [V, D], D_0 = \{1_{V_0}\}$ and $D_+ \leq I_0(V_+)$. If $p = 2$, then by [1, (5.1)], $D_0 \leq I_0(V_0)$ is an elementary abelian $2$-subgroup and $D_+ \leq I_0(V_+)$. Let $G_0 := I_0(V_0), G_+ := I_0(V_+), C_+ := C_{I_0(V_+)}(D_+)$ and let $V^*$ be the underlying space of $I_0(V)^*$.

Let $z \in D$ be a primitive element. If $p$ is odd, then $z \in Z(D)$ with $\vert z \vert = p$ (cf. [17, p. 176]). If $p = 2$, then $z$ is given by the proof (3) of [1, Remark 2.2.9], so $\vert z \vert = 4, z \in K$ and $[V, D_+] = [V, z] = V_+$. Thus

$$z = z_0 \times z_+, \quad L := C_G(z) = L_0 \times L_+, \quad L_0 := G_0, \quad L_+ := \text{GL}^r(m, q^e),$$

where $z_0 = 1_{V_0}, z_+ \leq D_+$ and $\dim V_+ = 2em$. Then $L$ is a regular subgroup of $G$ and we may suppose $s \in L^* \leq G^*$. In particular,

$$V^* = U_0 \perp U_+ \quad \text{and} \quad s = s_0 \times s_+,$$  \hspace{1cm} (4.10)

where $U_0 = V_0^*, s_0 \in L_0^* = I_0(U_0), U_+$ is the underlying space of $L_+^*$ and $s_+ \in L_+^* \leq I_0(U_+)$. Let $p = 2$ and let $s = \prod_{\Gamma} s(\Gamma)$ be the primary decomposition of $s$ in $I_0(V^*)$, and let $U_\Gamma$ be the underlying vector space of $C_\Gamma$. Then $C_{I_0(V^*)}(s) = \prod_{\Gamma} C_\Gamma$ and

$$C_\Gamma = \text{GL}^{g, r}(m_\Gamma(s), q^{\delta^r}) \quad \text{or} \quad I_0(U_\Gamma)$$

according as $\Gamma \neq X - 1$ or $\Gamma = X - 1$. In particular, $C_{I_0(V^*)}(s)$ is a regular subgroup of $G^*$.

**Proposition 4.2.** Let $K := \Omega^g_2(q) := \Omega^g_2(V) \leq G = \text{SO}^g(V) \leq J := I_0(V), Z \leq O_p(Z(K)), B_K \in \text{Blk}(K), B_C \in \text{Blk}(G)$ covering $B_K$ and $B_J \in \text{Blk}(J)$ covering $B_C$. Let $D_G, D_K$ and $D_J$ be defect groups for $B_C, B_K$ and $B_J$ respectively. Then $D_K$ is abelian if and only if $D_G$ is abelian if and only if $D(B_J)$ is abelian.
Self-dual, we have and only if $H$.

Hence in order to show that $D_K/Z$ is SDS for some $Z \leq O_2(Z(K))$ and $D_G$ is nonabelian. Then $p = 2$ and $(G, D_G, D_K, a, Z, D_K/Z)$ is listed in Table 2, where $\delta := \delta_1 = \delta'$ or $2\delta'$ with odd $\delta'$ for some $\Gamma \in \mathcal{F}_K^Q \setminus \{ \Gamma \}$. In the last three cases, $B_K$ is the principal block $B_0(K)$ of $D_K/Z$.

Conversely, if $G = SO^0(V)$ is given in Table 2 and $K = \Omega^0(V) \leq G$, then there exist blocks $B_K \in \text{Blk}(K)$ and $B_C \in \text{Blk}(G)$ covering $B_K$ with defect groups $D_K$ and $D_C$ respectively as given in Table 2.

**Proof.** Let $B = B_G$ and $D = D_G$, so that we may suppose $D_K = D \cap K$ and $D = D_J \cap K$. Since $G$ is self-dual, we have $V = V^\ast$, $U_0 = V_0$, $U_+ = V_+$ in (4.10) and $U_\Gamma = V_\Gamma$ in (4.11).

Write $K_0 = \Omega(J_0)$, $K_+ = \Omega(J_+)$, $K_\Gamma = \Omega(V_\Gamma)$ and $M_+ := SL^\Gamma(m, q^\delta) \leq L_+ \cap K_+$. so that $K_0 \times M_+ \leq C_K(z) \leq L_0 \times L_+$, $C_K(z) = \{ K_0 \times C_{K_+}(z), t_0 \times t_+ \}$ and $[L_+, C_{K_+}(z)] \leq 2$, where $t_0 \in L_0 \setminus K_0$ and $t_+ \in L_+$. 

**Case 1.** Suppose $p$ is odd. Since $|G : K| = 2$, it follows that $D = D_K$. Let $(z, B_2)$ be a major subsection of $K_+$. Then $B_2$ covers a block $B_0 \times B_+$ of $K_0 \times C_{K_+}(z_+)$ with $B_0 \in \text{Blk}(K_0)$ and $B_+ \in \text{Blk}(C_{K_+}(z_+))$ such that $D$ is a defect group for both $B_2$ and $B_0 \times B_+$.

By [14, Lemma 4.1], there exists a $B$-subgroup $(z, B_1)$ such that $B_1$ covers $B_2$. Thus $(z, B_1)$ is a major subsection of $K_+$. Since $I_0 = L_0 \times L_+$ with $L_0 = G_0 = \Omega(V_0)$, it follows that $B_1 = B_{L_0} \times B_{L_+}$ with $B_{L_0} \in \text{Blk}(L_0)$ and $B_{L_+} \in \text{Blk}(L_+)$. But $B_1$ covers $B_2$ and $B_2$ covers $B_0 \times B_+$, so $B_{L_0}$ covers $B_0$ and $B_{L_+}$ covers $B_+$. In particular, $D_+$ is a defect group for $B_+$. Thus $D_+ = \text{Syl}_p(C_{L_+}(s_+))$. But $Z(K) = 1$ or $Z_2$ and $p$ is odd, so $D_+ / Z \leq D_+$ and by Proposition 4.1, $D_+$ cannot be nonabelian and SDS. 

By [17, (1A)], $C_{J_1}(z) = (L, \tau)$, where $\tau = t_0 \times t_+$ with $[\tau_+, L_+] = 1$ and $J = \langle G, \tau \rangle$. Since $D_K = D_+ \langle 1_{L_0} \rangle \times D_+$, $D_+ \leq L_+$, it follows that $[\tau_+, t_J = 1]$. But $D_J / D_+ \leq \langle \tau \rangle$, it follows that $D_K$ is abelian if and only if $D_+$ is abelian, if and only if $D$ is abelian.

Since $D \cong D_+$, it follows that if $p$ is odd, then we cannot have a situation where $D_K/Z$ is SDS and $D_G$ is nonabelian.

**Case 2.** Suppose $p = 2$, so that $B = E_2(G, s)$ and $D \in \text{Syl}_2(C_{C}(s))$. Follow the notation of (4.11), and let $B_\Gamma = E_2(C_\Gamma, (s, (\Gamma)))$ and $D_\Gamma := D(B_\Gamma)$. Thus $D = \prod \Gamma D_\Gamma$, $D_\Gamma \in \text{Syl}_2(C_\Gamma)$ and $D_K = D \cap K$.

Note that if $D_K$ is abelian, then $m_\Gamma(s) \leq 1$ for each $\Gamma \neq X - 1$ and so $D_\Gamma$ is abelian for $\Gamma \neq X - 1$. Hence in order to show that $D_K$ is abelian if and only if $D$ is abelian, it suffices to consider $\Gamma = X - 1$.

Write $2n_{X-1} = \dim V_{X-1}$ and $n_{X-1} = \eta(V_{X-1})$. Thus $C_{X-1} = \Omega(V_{X-1})$. Let $Q_{X-1} \in \text{Syl}_2(I(V_{X-1}))$, so that $D_{X-1} = Q_{X-1} \cap I_0(V_{X-1})$ and $\eta_{X-1} = Q_0 \times Q_1 \times \cdots \times Q_m$. 

$$Q_{X-1} = Q_0 \times Q_1 \times \cdots \times Q_m, \quad (4.12)$$

where $Q_0 = 1$ or $Z_2 \times Z_2$ according as $q^{X-1} \equiv \eta_{X-1} \pmod{4}$ or $-\eta_{X-1} \pmod{4}$, and $Q_i = D_{2^{2i+1}} \times X_{2^{2i}}$ with $X_{2^{2i}} \in \text{Syl}_2(S_{2^{2i}})$. In addition, $Q_0 \cap I_0(V(0)) = \langle -1, V(0) \rangle$ and $-1_{V(0)} \not\in \Omega(V(0))$.
when \( Q_0 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Thus if \( D_K \) is abelian, then \( m_{X-1}(s) \leq 2 \) or 4 according as \( q^{n-1} \equiv \eta^{n-1} \) (mod 4) or \(-\eta^{n-1} \) (mod 4). So \( D_K \) is abelian if and only if \( D \) is abelian.

It follows by [17, (1A)] that \( C_f(s) = (C_G(s), \tau) \) with \( \tau \in J \setminus K \) and \( (\tau, C_G(s)) = 1 \). Thus \( D_J \in \text{Syl}_2(C_J(s)) \) and \( D_J \) is abelian. Conversely, if \( D_J \) is abelian, then \( D_J \) and \( D_K \) are abelian, since \( D = D_J \cap G \) and \( D_K = D_J \cap K \). It follows that \( D_J \) is abelian if and only if \( D \) is abelian if and only if \( D_K \) is abelian, proving the first part of the proposition.

Now suppose that \( D_K \) is nonabelian and that \( D_K/z \) is SDS for some \( Z \leq Z(K) \). Then there exists \( \Gamma \in \mathcal{F} \) such that \( m_{\Gamma}(s) \geq 2 \) or 4 according as \( \Gamma \neq X - 1 \) or \( \Gamma = X - 1 \). Since \( Z = 1 \) or \( \mathbb{Z}_2 \), it follows that \( Z \leq D_K \). If \( K \neq D_J \), then \( Z \cap K = 1 \), \( D_J \cap K \cong 1 \) and in particular \( D_J \cap K \) is SDS. Hence by Proposition 4.1 \( D_J \cong \mathbb{Z}_{2^c} \) or \( SD_{2+2} \), where \( c = 2 \) or 3 and \( 2^{c+1} || q^{2^c} - 1 \). In addition, \( D_J : D_J \cap K = 2 \), since \( [C_J : C_K(s_J)] = 2 \). In particular, \( m_{\Gamma}(s) = 2 \) and \( \eta(V_\Gamma) = (\epsilon q^{\delta})^m(s) = +1 \). But dim(\( V_\Gamma \)) = 4\( q \), so \(-1 \) is odd, 2 || \( q^{\delta} - \epsilon \). Hence \( D_J \cap K = 1 \). It follows that

\[
-1 \in J = D_J \setminus K.
\]

and we set \( u_\Delta = -1 \).

If \( D_J \cong \mathbb{Z}_{2^c} \) for some \( c \geq 2 \), then \( C_\Delta = \text{GL}^\Delta(1, q^{\delta}) \) and \( 2^c || (q^{\delta} - \epsilon) \). In this case, \( -1 \in K \) and there exists \( u_\Delta = D_J \setminus K \).

Now suppose that in addition we have \( m_{X-1}(s) \geq 4 \). Follow the notation of (4.12). Let \( V(I) \) be the underlying space of \( Q_i \) and \( V(+) = \oplus_{i \geq 1} V(i) \), so that \( V_{X-1} = V(0) \perp V(+) \). If \( z_{X-1} \) is a primary element of \( Q_{X-1} \) as given in [1, Remark 2.29 (2)], then \( |z_{X-1}| = 4 \), \( z_{X-1} = l_0(V_{X-1}) \), \( z_{X-1} = D_{X-1} \) and

\[
z_{X-1} = 1_{V(0)} \times z_+(X - 1), \quad C_{l_0(V_{X-1} - 1)}(z_{X-1}) = l_0(V(0)) \times L(+) \quad L(+) = \text{GL}^\epsilon(W_{X-1}, q),
\]

where \( 2w_{X-1} = \dim(V_{X-1}) - \dim(V(0)) \) and \( z_+(X - 1) \in l_0(V(+)) \). If \( a \geq 2 \), then \( z_{X-1} = K_{X-1} \). If \( a \geq 3 \), then \( z_{X-1} = K_{X-1} = z_{X-1} \) is a square of some 2-element of \( l_0(V_{X-1}) \) and there exists \( u_{X-1} = D_{X-1} \setminus K_{X-1} \).

Recall that \( D_J \cong \mathbb{Z}_{2^c} \cap \mathbb{Z}_2 \) or \( SD_{2+2} \), where \( c = 2 \) or 3 and \( 2^c || (q^{\delta} - \epsilon) \). In each case we define an element \( u_f \) of \( D_J \setminus K \) as follows.

If \( D_J \cong \mathbb{Z}_{2^c} \cap \mathbb{Z}_2 \leq C_J = \text{GL}^\epsilon(2, q^{\delta}) \), then take \( u_f = \text{diag}(w, 1) \in C_J \) such that \( w \) is an element of order \( 2^c \) in \( \mathbb{F}_{q^{\delta}}^* \), so that \(-1 \in J = D_J \setminus K \). If \( D_J = SD_{2+2} \leq C_J = \text{GL}^\epsilon(2, q^{\delta}) \), then \( SD_{2+2} = \langle x, y \mid x^{2^c+1} = y^4, y^{-1}xy = x^2 \rangle \) with \( \det(y) = 1 \). Thus we take \( u_f = x \) and so \( u_f = D_J \setminus K \).
Suppose $D_\Delta \neq 1$ for some $\Delta \neq \Gamma$. If $\Delta \neq X - 1$ and $m_\Delta(s) \geq 2$, then define $u_\Delta \in D_\Delta \setminus K_\Delta$ as for $\Gamma$. Otherwise, define $u_\Delta$ as above. Define
\[
P := \langle D_\Gamma \cap K_\Gamma, u_\Gamma \times u_\Delta \rangle.
\]
Then $P \leq (D_\Gamma \times \langle u_\Delta \rangle) \cap \Omega(V_\Gamma + V_\Delta) \leq D_K$. Since $[P, P] = [D_\Gamma, D_\Gamma] \leq K_\Gamma$ and since $K_\Gamma \cap Z = 1$, it follows that $[D_\Gamma, D_\Gamma] \cong [P, P]/Z \leq [D_K/Z, D_K/Z]$. This is impossible, since $[D_K/Z, D_K/Z] \leq \mathbb{Z}_2$ and $[D_\Gamma, D_\Gamma] \cong \mathbb{Z}_2^2$. Thus $D_\Delta = 1$ for all $\Delta \neq \Gamma$.

**Case 2.2** Suppose $D = D_\Gamma \in \{\mathbb{Z}_2^k : \mathbb{Z}_2, SD_{2^2+1}\}$ with $3 \leq c \geq a \geq 2$ and let $\delta = \delta_\Gamma$, so that $C = C_\Gamma = \text{GL}^\delta(2, q^{\delta}) \supseteq M_\Gamma = \text{SL}(2, q^{\delta})$. Since $K = [G, G]$, it follows that
\[
Q_2^{c+1} = D \cap M_\Gamma \leq D \cap K = D_K.
\]
But $C/M_\Gamma$ is cyclic and $D/Q_2^{c+1} \leq C/M_\Gamma$, so $D_K$ is the unique subgroup of $D$ such that $Q_2^{c+1} \leq D_K$ and $[D : D_K] = 2$. Since $[C : M_\Gamma Z(C)] = 2$, it follows that $D_K \in \text{Syl}_2(M_\Gamma Z(C))$, $D_K \cong Q_2^{c+1} \leq \mathbb{Z}_2^c$ or $Q_2^{c+1}$ according as $D \cong \mathbb{Z}_2^2 \cap \mathbb{Z}_2$ or $SD_{2^2+2}$.

If $\delta = \delta'$ is odd, then $c = a$ and $Z = 1$ or $Z(K)$. This implies the first 6 cases in Table 2. Suppose $\delta$ is even, so that $\alpha := \alpha_\Gamma \geq 1$ and $c = a + \alpha \geq a + 1 \geq 3$. Thus $c = 3$, $\alpha = 1$ and $\delta = 2\delta'$ for some odd $\delta'$. In particular, $\delta = 2^j \delta'$. Thus $(D, D_K) = (\mathbb{Z}_2^2 : \mathbb{Z}_2, Q_2^{c+1} \cap \mathbb{Z}_2^c)$ or $(SD_{2^2+2}, Q_2^{c+1})$ according as $\alpha_\Gamma = +$ or $-$. In this case $Z = Z(G) \cong \mathbb{Z}_2^2$ and the cases 7 and 8 in Table 2 hold.

**Case 2.3** Suppose that $m_{X-1}(s) \geq 4$ and that $D_\Gamma$ is abelian for all $\Gamma \neq X - 1$ and $D_{X-1}$ is non-abelian. Recall that $(D_{X-1} \cap K_{X-1})/(Z \cap K_{X-1})$ is SDS and $D_{X-1} \cap K_{X-1}$ is a Sylow 2-subgroup of $K_{X-1}$. Note that $w_{X-1} \geq 2$.

Let $M(\pm) = \text{SL}^\delta(w_{X-1}, q) \leq \text{S}L^\delta(w_{X-1}^c, q)$, so that $M(\pm) \leq \Omega(V(\pm))$. Since $Z \cap \Omega(V(\pm)) \leq Z(M(\pm))$, it follows that $D_{X-1} \cap M(\pm)/Z(M(\pm))$ is SDS. By Proposition 4.1 $D_{X-1} \cap L(\pm) = SD_{2^{a+2}}$ or $\mathbb{Z}_2^a \cap \mathbb{Z}_2^c$ according as $2 \parallel (q - \epsilon)$ or $2 \parallel (q + \epsilon)$. Since $4 \parallel (q - \epsilon)$, it follows $D_{X-1} \cap L(\pm) \cong \mathbb{Z}_2^c \cap \mathbb{Z}_2^c$ with $a = 2$ or $3$. In particular, $w_{X-1} = 2$.

A similar proof to that of Case 2.1 shows that $D_\Gamma = 1$ for any $\Gamma \neq X - 1$ and $D_0 = 1$. So dim $V = 4 = m_{X-1}(s)$ and $K_{X-1} = K = \Omega^4_{\pm}(q)$. If $\eta = -$, then $K = K_{X-1} = \Omega^4_{\pm}(q) = P \text{SL}_2(q^2)$, so that $D_K \cong Q_2^{a+1}(Z/SL_2(q^2)) = D_2^{a+1}$ and $Z = 1$. Thus $a = 2$ and $D_K/Z = D_K = D_2^2$ and $D_G = D_2^2$.

If $\eta = +$, then $K = K_{X-1} = SL_2(q) \times SL_2(q)$ and $D_K \cong Q_{2^a+1} \cap Q_2^{2a+1}$. If $a \geq 3$, then $D_K/Z(K)$ is nonabelian and not SDS. Thus $a = 2$, $D_K = D_2^2 \times D_2$, $D_G = D_2^4 \times D_2$, and $Z = 1$ or $Z(K) \cong \mathbb{Z}_2^2$.

Conversely, suppose $G = SO_6^\delta(\mathbb{F}_q)$ is as given in Table 2 and $K = \Omega(V) \leq G$. If dim $V = 4$, then let $B = B_0(K)$, $B_G = B_0(G)$ and so the defect groups $D_G$ of $B_G$ and $D_K$ of $B_K$ are as given in Table 2. If dim $V > 4$, then take a semisimple $2'$-element $s \in G$ such that $m_{\Gamma}(s) = 2$ and let $B_G = E_2^\delta(G, s)$.

Thus $C = C_G(s) = \text{GL}^\delta(2, q^{\delta})$ and $D_G \in \text{Syl}_2(C)$ for some $D_G = D(B_G)$. If $B_K$ is the block of $K$ covered by $B_G$, then we may suppose $D_G = D_K \cap K$ for some $D_K = D(B_K)$. By Case 2.2, the defect groups $D_G$ and $D_K$ are as given in Table 2.

**Proposition 4.3.** Let $K := \Omega_{2^n+1}(q) := \Omega(V) \text{ or } \text{Sp}_{2n}(q) = \text{Sp}(V)$, $G = SO_{2n+1}(q) = \text{SO}(V)$ or $\text{Sp}_{2n}(q)$, and $J = \text{SO}(V)$ or $J_0(V)$, so that $K \leq G \leq J$. Let $B_K \in \text{Bk}(K)$, $B_G \in \text{Bk}(G)$ covering $B_K$ and $B_J \in \text{Bk}(J)$ covering $B_G$. Let $D_K$, $D_G$, and $D_J$ be blocks of $B_K$, $B_G$, and $B_J$ respectively. Then $D_K$ is abelian if and only if $D_G$ is abelian if and only if $D_J$ is abelian.

Suppose $D_G$ is nonabelian and $D_K/Z$ is SDS for some $Z \leq O_2^\delta(Z(K))$. Then $p = 2$ and $(G, D_G, D_K, a, Z, D_K/Z)$ is as listed in Table 3, where $D_K = D_G \cap K$ and $\delta = \delta_\Gamma$ for some $\Gamma \in \mathcal{F}_2^\delta \setminus \{X - 1\}$ with odd $\delta$.

Conversely, if $G = SO_{2n+1}(q) \text{ or } \text{Sp}_{2n}(q)$ is as given in Table 3 and $K = \Omega_{2n+1}(q) \text{ or } \text{Sp}_{2n}(q)$ such that $K \leq G$, then there exist blocks $B_K \in \text{Bk}(K)$ and $B_G \in \text{Bk}(G)$ covering $B_K$ with defect groups $D_K$ and $D_G$ as given in Table 3.

**Proof.** For $p$ odd, the proof is similar to the proof of Case 1 of Proposition 4.2 and for reasons of space we omit it.
Table 3
Defect 2-groups of SO(2n + 1, q) and Sp(2n, q) with quotient a SDS group.

<table>
<thead>
<tr>
<th>G</th>
<th>D_G</th>
<th>a</th>
<th>D_K</th>
<th>Z</th>
<th>D_K/Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sp(2, q)</td>
<td>Q_{2a+1}</td>
<td>2</td>
<td>Q_{2a+1}</td>
<td>1</td>
<td>Q_2</td>
</tr>
<tr>
<td>Sp(2, q)</td>
<td>Q_{2a+1}</td>
<td>2</td>
<td>Q_{2a+1}</td>
<td>Z_2</td>
<td>Z_2 × Z_2</td>
</tr>
<tr>
<td>Sp(2, q)</td>
<td>Q_{2a+1}</td>
<td>3</td>
<td>Q_{2a+1}</td>
<td>Z_2</td>
<td>D_8</td>
</tr>
<tr>
<td>SO(3, q)</td>
<td>D_{2a+1}</td>
<td>1</td>
<td>D_8</td>
<td>1</td>
<td>D_8</td>
</tr>
<tr>
<td>SO(4 + 1, q)</td>
<td>Z_2 × Z_2</td>
<td>2</td>
<td>Q_{2a+1} o Z_4</td>
<td>1</td>
<td>Q_8 o Z_4</td>
</tr>
<tr>
<td>SO(4 + 1, q)</td>
<td>SD_{2a+2}</td>
<td>2</td>
<td>Q_{2a+1}</td>
<td>1</td>
<td>Q_8</td>
</tr>
</tbody>
</table>

Table 4
Defect 2-groups of Spin(V) with Z_c ≠ Z and quotient SDS.

<table>
<thead>
<tr>
<th>K</th>
<th>a</th>
<th>D_K</th>
<th>Z</th>
<th>D_K/Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spin^+(4δ', q)</td>
<td>3</td>
<td>Q_{2a+1} o Z_2 × Z_2</td>
<td>Z_2</td>
<td>D_8 × Z_2 × Z_2</td>
</tr>
<tr>
<td>Spin^-(4δ', q)</td>
<td>2</td>
<td>Q_{2a+1} o Z_2 × Z_2</td>
<td>Z_2</td>
<td>D_8 × Z_2 × Z_2</td>
</tr>
<tr>
<td>Spin^+(4δ', q)</td>
<td>2</td>
<td>Q_8 o Z_2 × Z_2 × Z_2</td>
<td>1</td>
<td>Q_8 o Z_8</td>
</tr>
<tr>
<td>Spin(4 + 1, q)</td>
<td>2</td>
<td>Q_8 o Z_2 × Z_2 × Z_2</td>
<td>1</td>
<td>Q_8 o Z_8</td>
</tr>
<tr>
<td>Spin(4, q)</td>
<td>2</td>
<td>Q_8 o Z_2 × Z_2 × Z_2</td>
<td>Z_2</td>
<td>Z_2 × Z_2 × Q_8</td>
</tr>
<tr>
<td>Spin(3, q)</td>
<td>2</td>
<td>Q_8 o Z_2 × Z_2 × Z_2</td>
<td>1</td>
<td>Q_8</td>
</tr>
</tbody>
</table>

Suppose p = 2, so that B_G = E_2(G, (s)). Let s^a be a dual element of s in l_0(V) given by [2, (4A)]. Then D_C ∈ Syl_2(C_0^1(s^a)). A proof similar to the Case 2 of proof of Proposition 4.2 with s replaced by s^a shows that Proposition 4.3 holds. Note that \Omega_3(q) \cong PSL_2(q) and Z(\Omega_2n+1(V)) = 1.

**Proposition 4.4.** Let K := Spin^+(V) < J such that J/K is abelian, C_J(K) ≤ Z(J) and J/Z(J) ≅ SO(V) or J_0(V)/Z(J_0(V)) according as dim V is odd or even. Let B_K ∈ Blk(K) and let B_J be a block of J covering B_K. Let D_K and D_J be defect groups for B_K and B_J respectively, chosen with D_K = K ∩ D_J. Let Z_c ≤ Z(K) such that K_c := K/Z_c = Ω^+(V), so that |Z_c| = gcd(2, q – η). Then D_K is abelian if and only if D_J is abelian. In addition, if p = 2 and D_K is abelian and J/KZ(J) = J_0(V)/KZ(J_0(V)), then D_J/D_K is isomorphic to the outer diagonal automorphism group Out.diag(K) of K.

Suppose D_K is nonabelian and D_K/Z is SDS for some Z ≤ O_p(Z(K)). Then p = 2 and Z can be taken to be any subgroup of Z(K). If Z_c ≤ Z, then D_K/Z is given by Tables 2 or 3. If Z_c ≠ Z, then D_K/Z is given by Table 4, where \delta = \delta_1 = \delta_2 = 2\delta and \delta_3 = \delta_4 with odd \delta_3 for some \Gamma ∈ \mathcal{F}_q^2 and \beta is the integer such that 2^\beta \parallel (q^4 – 1). Thus \beta = 1 or a ≥ 1. In the last two cases of the Table 4, B = B_0(K).

Conversely, if K = Spin(V) and K/K/Z is as given in one of Tables 2, 3 and 4 for some Z ≤ O_p(Z(K)), then there exists a block B_K ∈ Blk(K) such that D_K is a defect group for B_K and is as given in Tables 2, 3 and 4.

**Proof.** Write B = B_K and D = D_K, and Z_+ ≤ Z(D_0(V)) such that G = D_0(V)/Z_+ = SO(V), so that Z_c = Z_+ ∩ K and Z_+ ≅ Z_q-1.

**Case 1.** Suppose p is odd. We may suppose D ≅ DZ_c/Z_c ≤ K_c. Thus D is of defect type in K_c, where p-subgroup Q of K_c is of defect type if Q is a Sylow p-subgroup of a centralizer C_{K_c}(t) of a semisimple p'-element t. So D is of defect type of G, and D has a primary element z ∈ Z(D) (see [17, Section 5]). Thus we have the corresponding decompositions V = V_0 ⊕ V_+. D = D_0 × D_+, z = z_0 × z_+ ∈ C_{K_c}(z) = L_0 × L_+, given by (4.8) and (4.9). In particular, D_0 = (1V_0), Z ∈ Z(D) and |Z| = p. Let t = t_0 × t_+ with t_0 ∈ L_0 and t_+ ∈ L_+, and let B_{t_0} = E_p(L_+, (t_+)). Then D_+ ∈ Syl_p(C_{L_+}(t_+)) and D(B_{t_0}) = D_+.

Suppose first that D is nonabelian and D/Z is SDS for some Z ≤ O_p(Z(K)). Then Z = 1 and D/Z = D is nonabelian, and so D is SDS. But then by Proposition 4.1 (a) and (b), D_+ cannot be SDS, and hence D cannot be SDS after all, a contradiction.

Suppose D is abelian. Since D = D_J ∩ K for some D_J and J/KZ(J) is a 2-group, it follows that D_J ≅ KZ(J) and D_J = DO_p(Z(J)), which is abelian. Conversely, if D_J is abelian, then D is abelian as D = D_J ∩ K.
that is, we identify $G$ with $G^*$. If dim $V = 2n + 1$, then identify $s$ with its dual $s^* \in G$. Then a defect group $D_G$ for $B_C$ satisfies $D_G \in \text{Syl}_2(C_G(s))$. Thus $G = \text{SO}(V)$, and since $D_G = \langle \bar{G} \rangle$, for some semisimple $2'$-element $s \in G^*$. If $\dim V = 2n$, then identify $G$ with $G^*$.

It follows that $CD$ and $D_J$ are abelian. Then $s \in E_2^s$ or $E_2^s$ is abelian, and hence $\mathcal{V}$ is odd or even for some $\eta \in \mathcal{X}$. If $\dim V = 2n + 1$, then identify $s$ with its dual $s^* \in G$. Then a defect group $D_G$ for $B_C$ satisfies $D_G \in \text{Syl}_2(C_G(s))$. Thus $G = \text{SO}(V)$, and since $D_G = \langle \bar{G} \rangle$, for some semisimple $2'$-element $s \in G^*$. If $\dim V = 2n$, then identify $G$ with $G^*$.

Write $D_c := D/\mathcal{Z}$, a defect group for $B_C$. Suppose $D_c$ is abelian. Then $D_c$ is abelian and so $C_G(s^*)$ is a maximal torus of $G$. Suppose $D_G = \langle \bar{G} \rangle$, and $D$ is abelian. In the notation of (4.11), $C_G(s^*) = \prod_r C_r$, where $C_r = \text{GL}_r(1, q^r)$ or $I_0(V_{X-1})$. Since $D_G \in \text{Syl}_2(C_G(s^*))$ and $D_c$ is abelian, it follows that $I_0(V_{X-1}) = 1$ or $\text{SO}^q(2, q) = \text{GL}^q(1, q)$. We may suppose $C_G = \text{GL}_r^q(1, q^r) \circ \eta_0(V_{X-1})$ for all $r$. By [17, (2E)], $C_D(s^*)$ is a central product of groups $L_r$, where $L_r$ is a central extension of $C_r$ by $Z_r$. If $q_r = O_2(L_r)$, then $D$ is a central product of $Q_r \cap K_r$, where $K_r = \text{Spin}(V_r)$.

Let $\tau_r$ induce an outer diagonal automorphism of order 2 on $\text{Spin}(V_r)$. The centralizers $\text{CSpin}(V_r)(\tau_r)$ and $\text{CSpin}(V_r)(\tau_r)$ are given by [19, Table 4.5.2]. It follows that $C_D(s^*) = \text{GL}_r^q(\delta_r, q) \circ \eta_0(V_{X-1}) \circ \eta_0(V_{X-1})$ or $\text{SL}_r^q(\delta_r, q) \circ \eta_0(V_{X-1})$ or $\eta_0(V_{X-1}) \circ \eta_0(V_{X-1}) \times 2$ according as $\delta_r$ is odd or even. In particular,

$$\eta_0(V_{X-1}) \circ \eta_0(V_{X-1}) \times 2$$

and hence $D_J$ is abelian, since $D_J \leq \langle D_{O_p}(Z(J)), \tau \rangle$, where $\tau = \prod_r \tau_r$. In addition, $\tau \in T_J(B)$ and we may suppose $\tau \in D_J$, and so $D_J = D \cong \text{Outdiag}(K)$.

Suppose $D_c$ is nonabelian. Since $Z(K)$ is a 2-group, it follows that $Z(K) \leq Z(D)$. But $D/Z(K) \cong (D/Z)(Z(K)/Z)$, so $D/Z(K)$ is SDS. Suppose $D/Z(K) \cong D_c/Z(K_c)$ and since $D_c$ is nonabelian if and only if $D_c$ is abelian, it follows by Propositions 4.2 and 4.3 that dim $V = 2d_r + 1$ or $4d_r$ according as $\dim V$ is odd or even for some $r \neq q$. Moreover, $D/Z(K) = D_c/Z(K_c)$ and $D/Z_c = D_c$ are given in Tables 2 and 3.

Suppose $Z \neq Z_c$ and $Z \neq Z(K)$, so that $Z$ is cyclic of order 1 or 2. Suppose $\Gamma = X - 1$, so that by Tables 2 and 3, $s^* = 1_v$ and $B_C$ is the principal block. If dim $V$ is odd, then $\Omega_2(1, q) = \text{PSL}_2(q)$, $K = \text{SL}_2(q)$, and $\Omega_3(1, q) = \text{PSL}_2(q^2)$ if and only if $D_c$ is nonabelian, it follows by Propositions 4.2 and 4.3 that dim $V = 2d_r + 1$ or $4d_r$ according as $\dim V$ is odd or even for some $r \neq q$. Moreover, $D/Z(K) = D_c/Z(K_c)$ and $D/Z_c = D_c$ are given in Tables 2 and 3.

If dim $V = 2d_r + 1$ or $4d_r$ according as $\dim V$ is odd or even for some $r \neq q$. Moreover, $D/Z(K) = D_c/Z(K_c)$ and $D/Z_c = D_c$ are given in Tables 2 and 3.

Suppose $\Gamma = X - 1$ and so $\delta := \delta_r = \delta'$ or $2\delta'$ with odd $\delta'$, and $C_G(s^*) = \text{GL}_r^q(2, q^\delta)$. Since $Z(\text{GL}_r^q(2, q^\delta)) \leq K$ and $[K : K_c] = 2$, it follows that $C_{K_c}(s^*) = \text{GL}_r^q(2, q^\delta) \circ \eta_0(V_{X-1}) \circ \eta_0(V_{X-1}) \times 2$.

By [17, (2E)], $C_D(s^*)$ is a central extension of $C_G(s^*)$ by $Z_c$. If dim $V = 2n + 1$, then let $t$ be an element of $K$ inducing the central involution of $Z(\text{GL}_r^q(2, q^\delta))$. By [19, Table 4.5.2], $C_K(t) = \text{Spin}_2^+(q)$ and so

$$C_K(s^*) = C_K(t) \circ (\text{GL}_r^q(2, q^\delta) \circ (q^\delta - \delta_r)) \times 2.$$
If \( \dim V \) is odd, then \( Z = 1 \) and hence \( a = 2 \). If \( \dim V \) is even, then \( Z = 1 \) or \( Z = Z(SL^{\xi}(2, q^\lambda)) \cong \mathbb{Z}_2 \). In the former case \( c = a = 2 \) and in the later case \( c = 3 \) and \( D/Z = D_8 \times \mathbb{Z}_{5\beta-1} \times \mathbb{Z}_2 \). If \( c = 3 \), then \( a = 3 = c \) and \( \delta = \delta' \), or \( a = 2 \) and \( \delta = 2\delta' \). Thus Table 4 holds.

Conversely, suppose \( K = \text{Spin}(V) \) and \( K \) or \( K/Z \) is given in Tables 2, 3 and 4 for some \( Z \leq Z(K) \). If \( \dim V \leq 4 \), then take \( B_K = B_0(K) \). If \( \dim V > 4 \), then take a \( 2' \)-element \( s \in K \) such that \( m_I(sZ_c) = 2 \) and let \( B_K = E_2(K, (s)) \). Then \( D_K \) and \( D_K/Z \) are as given in Table 2, 3 or 4. \( \square \)

We now specialize to the case of extraspecial groups.

**Theorem 4.5.** Let \( K \) be a finite quasi-simple group of classical type over a field \( \mathbb{F}_q \), \( B \in \text{Blk}(K) \) and \( \eta = \pm \). Let \( D \) be a defect group for \( B \). If \( p \nmid q \), then \( D/Z \) is an extraspecial group for some \( Z \leq O_p(Z(K)) \) if and only if \( K = K_u/Z_0 \) and \( D = p^{1+2} \in \text{Syl}_p(K) \) and \( Z = 1 \), where \( K_u = SL^n(3, p) \) and \( Z_0 \) is any subgroup of \( Z(K_u) \).

Suppose \( p \mid q \). Then \( D/Z \) is an extraspecial group for some \( Z \leq O_p(Z(K)) \) if and only if either \( K, a, D, Z \) is given by Table 5 or \( p = 2 \), \( V \) is orthogonal, \( K = \text{Spin}(V) \), \( Z_c \leq Z(K) \) with \( K/Z_c = \Omega(V) \) and \( (K/Z_c, a, D/Z_c, Z/Z_c) \) is given by Table 5. Here \( \delta = \delta_I = \delta' \) with \( p \nmid \delta' \) or \( p = 2 \) and \( \delta = 2\delta' \) with odd \( \delta' \) for some \( I \in \mathcal{F}_q^p \). In addition, if \( K = \Omega^+(4\delta' + 1, q) \), then \( 2\|\langle q^{\delta'} - \epsilon_I \rangle \).

**Proof.** If \( p \nmid q \), then the result follows by Propositions 4.1, 4.2, 4.3 and 4.4.

Suppose \( p \mid q \), so that \( D \) is a Sylow \( p \)-subgroup of \( K \). In particular, \( |D| = p^{1+2\gamma} \) for some \( \gamma \neq 0 \). Note that \( Z(K_u) \) is a \( p' \)-group, so we may take \( K = K_u/Z \) for any \( Z \leq Z(K_u) \).

Now blocks of positive defect of \( SL^n(4, q) \) have non-cyclic derived subgroups and so are not SDS. Hence \( SL^n(4, q) \) cannot be a subgroup of \( K \). In particular, the Lie rank of \( K \) is at most 3. A Sylow \( p \)-subgroup of \( SL_2(q) \) is abelian, so \( K \cong SL_2(q) \). If \( K = SL^n(3, q) \), then \( D \) is special with derived subgroup isomorphic to \( \mathbb{F}_q^\times \). Thus \( q = p \) and \( D \cong 3^{1+2} \).

If \( K = D_6^m(q) \), then by [19, Table 2.2], \( |D| = q^{m(m-1)} \) and so \( D \) cannot be extraspecial. If \( K = C_m(q) \) or \( B_m(q) \), then \( |D| = q^{2m^2} \) and so \( m = 3 \). Since \( P \text{Sp}_6(q) \cong \Omega_5(q) \) (with odd \( q \)), it follows that we may suppose \( K = \text{Sp}_6(q) \) or \( \text{SO}_7(q) \). In both cases the derived subgroup of \( D \) is non-cyclic, and hence \( D \) cannot be extraspecial. \( \square \)

5. Exceptional groups

We will follow the notation of [19].

**Theorem 5.1.** Let \( K \) be a finite quasisimple group of exceptional type over a field \( \mathbb{F}_q \) and \( B \in \text{Blk}(K) \). Let \( D \) be a defect group of \( B \). If \( p \nmid q \), then \( D \) is not extraspecial. Suppose \( p \mid q \). Then \( D/Z \) is extraspecial for some \( Z \leq O_p(Z(K)) \) if and only if \( p = 3 \) and

\[
(K, a, Z, B, D) = (G_2(q), 2, 1, B_0(G_2(q)), 3^{1+2})
\]

or \((2F_2(2^{m+1}), 2, 1, B_0(2F_2(2^{m+1})), 3^{1+2})\).
Proof. If $p \mid q$, then $D \in SYLP(K)$ and $O_p(Z(K)) = 1$. By [19, Table 2.2], $|D| = q^N$ for some even $N$ or $N = 63$ according as $K \neq E_7(q)$ or $E_8(q)$. In the former case $D$ cannot be extraspecial. In the latter case, $K$ contains a subgroup $SL^2_4(q)$ and so $D$ cannot be extraspecial in this case either.

Suppose $p \mid q$. For the proof we need to consider not only $K$ but certain overgroups of $K$. Let $K < H$ such that $C_H(K) \leq Z(H)$, $H/K$ is cyclic and $H$ induces inner-diagonal automorphisms on $K$. Let $B_H$ be a block of $H$ covering $B$, and let $D_H$ be a defect group of $B_H$ with $D = K \cap D_H$.

Let $K_u$ be the universal group, so that $K = K_u/Z_0$ for some $Z_0 \leq Z(K_u)$. If $Z(K) \neq \Omega_1(Z(D))$, then take $z \in Z(D) \setminus Z(K)$ with $|z| = p$. If $Z(K) = \Omega_1(Z(D))$, then take $z \in D$ such that $|z| = p^2$ and $zK \in Z(D/Z(K))$. Let $(z, B_z)$ be a $B$-subsection, and suppose it is major when $z \in Z(D)$. In any case, let $D_z$ be a defect group for $B_z$ with $D_z \leq D$. Then $B_z$ is a block of $C := C_K(z)$, and $D$ is a defect group for $B_z$ when $(z, B_z)$ is major. By [19, Theorem 4.2.2],

$$C = O^{r'}(C)T, \quad O^{r'}(C) = L_1 \circ L_2 \circ \cdots \circ L_\ell$$

where each $L_i \in Lie(r)$, and $T$ is an abelian $r'$-group inducing inner-diagonal automorphisms on each $L_i$. In general, $z \notin O^{r'}(C)$, so we perform the following modifications. If $Z(C) \leq O^{r'}(C)$, then let $\ell = k$ and $L = O^{r'}(C)$. If $Z(C) \not\leq O^{r'}(C)$, then let $k = \ell + 1$ and $L_k = Z(C)$. Thus

$$C = LT \quad \text{with} \quad L := L_1 \circ L_2 \circ \cdots \circ L_k, \quad (5.1)$$

$z \in Z(C) \leq L$ and $L \leq C$. Let $B_1$ be a block of $L$ covered by $B_2$ and $\chi \in \text{Irr}(B_1)$. Note that $B_1$ has defect group $D_1 = D_2 \cap L$. Then $\chi = \chi_1 \circ \cdots \circ \chi_k$ for some $\chi_i \in \text{Irr}(L_i)$, so that $\chi_i \in \text{Irr}(B_1)$ for some $B_i \in \text{Blk}(L_i)$ and we write $B_L = B_1 \circ B_2 \circ \cdots \circ B_k$.

Since $L = (L_1 \times L_2 \times \cdots \times L_k)/A$ for a central subgroup $A \leq (L_1 \times L_2 \times \cdots \times L_k)$, it follows by [24, Theorems 5.8.8 and 5.8.10] that

$$D_1 = (D(B_1) \times D(B_2) \times \cdots \times D(B_k))/A$$

where $D(B_i)$ is some defect group of $B_i$ and $D(B_i)$ is isomorphic to a subgroup of $D_1$.

Each element $t \in T$ has the form $t_1t_2 \cdots t_kt'$, where $t'$ centralizes $L$ and $t_1$ induces an inner-diagonal automorphism on $L_1$ and $[L_1, t_j] = 1$ for $i \neq j$. Let

$$J_i := \langle L_i, t_i : t = t_1t_2 \cdots t_kt' \in T \rangle$$

and $T' = (t' : t = t_1t_2 \cdots t_kt' \in T)$. Then $LT \triangleleft J := J_1 \circ J_2 \circ \cdots \circ J_k \circ T'$ and $T'$ is abelian. Let $B_J$ be a block of $J$ covering $B_2$, so that $B_J$ covers $B_L$. Thus

$$B_J = B_{J_1} \circ B_{J_2} \circ \cdots \circ B_{J_k} \circ B_{T'}, \quad (5.2)$$

where $B_{J_i} \in \text{Blk}(J_i)$ covering $B_i$ and $B_{T'} \in \text{Blk}(T')$. Let $D(B_{J_i})$ be a defect group of $B_{J_i}$ with $D(B_i) \leq D(B_{J_i})$.

Our strategy is as follows. If $D$ is SDS, then so are $D_1$ and $D(B_i)$ for each $i$. We treat the exceptional groups case-by-case, progressing from low Lie rank to high (for inductive purposes we must consider the inner-diagonal versions of the groups). In each case we treat the subcases that each $D(B_i)$ is abelian, and that some $B_i$ has nonabelian defect groups. When $L_i$ is classical, this situation has been fully explored in Section 4. When $L_i$ is exceptional, we may use the previously treated exceptional groups of lower Lie rank.

Before treating the exceptional groups case-by-case, we gather together some information.

Suppose $p = 2$, $\ell \geq 2$ and $D(B_i)$ is nonabelian for some $i$. Then by [19, Table 4.5.2], $\ell = 2$, $Z(C) \leq L_1 \circ L_2$, $k = \ell$, $L = L_1 \circ L_2$ and the possible $(K, C)$ are given in Table 6, where $\eta = -$ or $+$. Here $C = (L_1 \circ L_2, (2, 2))$ means that $C = [L_1 \circ L_2, x]$ such that $x$ induces inner-diagonal automorphism of order 2 on each $L_i$.  


Suppose $p = 3$ and $L_1$ is classical. Suppose also that $D(B_1)$ is nonabelian and SDS. By Propositions 4.1, 4.2, 4.3 and 4.4, $L_1 = SL^+(3d_1, q_1)/Z_1$ and $D(B_1) = 3^{1+2}$, where $Z_1 \leq SL^+(3d_1, q_1)$, gcd$(6, d_1) = 1$ and $3||q_1 - e_1$. By [19, Table 4.7.3A], $(q_1, e_1) = (q, e)$ or $(q^2, 1)$ and $(K, C)$ are given in Table 7, where $L_e := SL^+(q)$. 

**Case 1.** Suppose, moreover that $K := 2^2 B_2(2^{2m+1})$, $2^2 G_2(3^{2m+1})$, $2^4 F_4(2^{2m+1})$, $G_2(q)$, $3^3 D_4(q)$, $F_4(q)$ or $E_6^-(q)$ with $q = e \pmod{3}$. Note that $Z(K) = 1$, so $Z = 1$, and that $p \nmid |G : H|$, so $D_H = D$. Then $D$ is nonabelian and SDS if and only if $p = 3$ and

$$\left(K, a, Z, B, D(B) = (G_2(q), 2, 1, B_0(G_2(q)), 3^{1+2})\right)$$

or $(2^2 F_2(2^{2m+1}), 2, 1, B_0(2^2 F_2(2^{2m+1})), 3^{1+2})$.

Suppose that $D$ is nonabelian and SDS. Since $Z(K) = 1$, it follows that $z \in Z(D)$ (so $(z, B_2)$ is a major subsection) and $z$ induces an inner automorphism on $K$. Note that each $L_1$ is a classical group (possibly $L_k$ abelian). Suppose each $D(B_j)$ is abelian. By Propositions 4.1, 4.2, 4.3 and 4.4, each defect group $D(B_j)$ of each $B_j$ is abelian and so is a defect group $D(B_j)$ of $B_j$, since $D(B_j)$ is isomorphic to a quotient group of $D(B_j) \times \cdots \times D(B_j) \times D(B_T)$. But $D_2 \cong D(B_1) \cap C$ and $(z, B_2)$ is major, so $D = D_2$ is abelian. Suppose that $D(B_i)$ is nonabelian for some $i$. By Propositions 4.1, 4.2, 4.3 and 4.4 again, $p = 2$ or 3. We treat these two cases in turn.

**Case 1.1.** Suppose $p = 2$. Note that $K \neq 2^2 B_2(2^{2m+1})$, $2^4 F_4(2^{2m+1})$, and we may suppose $K \neq 2 G_2(3^{2m+1})$ since a Sylow 2-subgroup of $2 G_2(3^{2m+1})$ is elementary abelian of order 8.

Suppose $\ell \geq 2$. Then $\ell = 2$, $Z(C) \leq L_1 \circ L_2$, $k = \ell$, $L = L_1 \circ L_2$ and the possible $(K, C)$ are given in Table 6.

Let $L_1 := SL_2(q) \leq J_1 := (L_1, x_1) \leq G_1 = GL^2_2(q)$, $L_2 := SL_2(q^2) \circ SL_2(q)$, $Sp_6(q)$ or $SL^+\epsilon(q)$, $J_2 = (L_2, x_2) \leq G_2$ with $G_2 = GL_2^2(q^3)$, $GL_2(q)$, $CSp_6(q)$, $GL_6^\epsilon(q)$, where $\delta$ is the sign such that $2\| (q - \delta)$ and $x = x_1 \times x_2 \in C \setminus L$ such that $\frac{1}{x}$ induces outer-diagonal automorphism of order 2 on $L_1$. Then $C < J := J_1 \circ J_2$. If $B_j \in Blk(J)$ covering $B_2$, then $B_j$ covering $B_1$, $B_j = B_{J_2} \circ B_{J_2}$ for some $B_{J_i} \in Blk(J_i)$ covering $B_i$, $D(B_j) = D(B_{J_2}) \circ D(B_{J_2})$ and $D_2 = D(B_{J_2}) \cap C$.

Let $B_{C_i}$ be a block of $G_i$ covering $B_1$. Then $G_{C_i} = E_2(G_i, (s_i))$ for some semisimple 2'-element $s_i \in G_i$. Identify $G_i$ with its dual $G_i^*$ except when $G_i = CSp_6(q)$, in which case identify $s_i$ with its dual $s_i^* \in G_i$. Thus $D(B_{C_i}) \in SyL_2(C_{C_i}(s_i))$ and in particular, $|D(B_{C_i})| = |D(B_{C_i})| \geq 2$ and hence $|D(B_j)| = |D_{B_j}| = 2$ as $D(\frac{1}{x}) = D(B_{C_i}) \cap J_j$.

Since $D$ is SDS, it follows that each $D(B_i)$ is also SDS. We have treated the case that both $D(B_1)$ and $D(B_2)$ are abelian. We now treat the cases $D(B_1)$ nonabelian and $D(B_2)$ nonabelian.
Suppose that $D(B_1)$ is nonabelian. Since $|D(B_{j_1}) : D(B_{j_1})| = 2$ for $i = 1$ and 2 and since $D_2 = D(B_j) \cap C$, it follows that $D_2$ contains a subgroup which is isomorphic to $D(B_{j_1})$. So $D(B_{j_1})$ is nonabelian and SDS. But by Proposition 4.1 (b), $D(B_{j_1}) \cong S_2$ or $S_3$, a contradiction.

Suppose that $D(B_2)$ is nonabelian. Then by Proposition 4.3, $K \neq F_4(q)$. A proof similar to above shows that $D_2$ contains a subgroup which is isomorphic to the nonabelian group $D(B_{j_2})$, and by Proposition 4.1, $D(B_{j_2}) = \not \cong S_2$ or $S_3$, a contradiction. It follows that $D$ cannot be SDS.

Suppose $\ell = 1$. By [19, Table 4.7.1], $L_1 = \text{Spin}_9(q) \text{ or } \text{Spin}_{10}^{10}(q)$ and $C = L_1/2$ or $L_1.(\text{gcd}(4, q + \epsilon))$ according as $K = F_4(q)$ or $E_6^7(q)$. Writing $Z_1 = \Omega_1(Z(L_1))$, we have $C/Z_1 = \text{SO}_9(q)$ or $\text{SO}_7(q).\text{gcd}(4, q + \epsilon)/2$. By Proposition 4.4, $D = D_2$ is not SDS.

**Case 1.2.** Suppose $p = 3$ and $D(B_1)$ is nonabelian. By the discussion above $L_1 = \text{SL}_3(3d_1, q_1)/Z_1$ and $D(B_1) = 3^1_{1+2}$, where $Z_1 \equiv \mathbb{Z} / (\text{SL}_3(3d_1, q_1), \text{gcd}(6, d_1) = 1$ and $3|(q_1 - \epsilon_1)$. We have $(q_1, \epsilon_1) = (q, \epsilon)$ or $(q^2, 1)$ and $(K, C)$ are given in Table 7, where $L_\epsilon := \text{SL}_2(q)$.

If $K = G_2(q) = 2^F_4(2^{m+1})$, then $\ell = 1$ and $L = C$ and $B_2 = B_L = B_0(L)$, so $B = B_0(K)$ with $D(B) = 3^1_{1+2}$.

Let $K = 3D_4(q)$, so that $C \cong \mathbb{Z}_3^{(q_2+\epsilon_2+1)} \times H_\epsilon$, where $H_\epsilon = (L_\epsilon, x)$ with $x$ inducing outer-diagonal automorphism of order 3 on $L_\epsilon$. Thus $H_\epsilon \leq G_\epsilon = \text{GL}_2(q)$ and $D_2 \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \cong \text{Syl}_3(G_\epsilon)$. Since $Z_1 = 1$, it follows that $D = D_2$ is not SDS.

Suppose $K = E_6^6(q)$ or $F_4(q)$, so that $C = \langle L_1 \circ L_2, x \rangle$, where $L_1 = L_\epsilon$ or $\text{SL}_2(q^2)$, and $x = x_1 \times x_2$ such that each $x_i$ induces outer-diagonal automorphism of order 3 on $L_i$. Let $J_i = \langle L_i, x_i \rangle$, $J = J_1 \times J_2$ and $B_{j_1} \subseteq \text{Bil}(J_i)$ covering $B_i$. Let $G_i = \text{GL}_2(q^2)$ or $\text{GL}_2(q^2)$ such that $J_1 \subseteq G_1$, and let $B_{j_1}$ be a 3-block of $G_1$ covering $B_{j_1}$. Then $D(B_{j_1}) = D(B_G C_i) \cap J_1$ and $D(B_{j_1}) \subseteq \text{Syl}_3(G_i(s))$ for some semisimple 3-element $s_i$ of $G_i$. It follows that $D(B_{j_1}) \neq D(B_{j_1}) \cap (Z(G_i))$ and so $D(B_{j_1}) \neq D(B_{j_1}) = D(B_{j_1}) \cap (Z(G_i(s_i)))$. If $D(B_{j_1})$ is nonabelian, then $D(B_{j_1}) = 3^1_{1+2}$, it follows that $D(B_{j_1}) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ and hence $D = (D(B_{j_1}) \cap D(B_{j_2})) \cap C$ contains a subgroup which is isomorphic to $D(B_{j_1})$, which is impossible.

**Case 2.** Suppose $3 | (q - \epsilon)$ and $K_0 := 3.E_6^6(q) \not \subseteq H := 3.E_6^6(q).3$ Then $D$ cannot be nonabelian and SDS. In addition, $D$ is abelian if and only if $D_H$ is abelian, and $|D_H : D| = 3$ when $p = 3$.

Write $m^* := \text{gcd}(m, q - \epsilon)$.

**Case 2.1.** Suppose $p = 2$. We have $z \in Z(D)$ with $|z| = 2$. By [19, Table 4.5.2], $C := C_K(z) = \langle \text{Spin}_{10}^1(q) \circ (q - \epsilon), t \rangle$ or given in Table 6, where $t = 4^*$. By [19, Table 4.5.1], $C_H(z) = \langle \text{Spin}_{10}^1(q) \circ (q - \epsilon), t_H \rangle$ or $\langle \text{SL}_2(q) \circ \text{SL}_2(q), x_H \rangle$, where $t_H = 4^* \times 3$ and $x_H = 2^6 : 6$. A proof similar to that of Case 1.1 shows that $D = D_Z$ cannot be both nonabelian and SDS. Similarly, since $D = D_Z$, it follows by the first part of the proof Case 1 that each $D(B_{j_1})$ is abelian if and only if $D$ is abelian if and only if $D_H$ is abelian (as $D = D_H$).

**Case 2.2.** Suppose $p$ is odd. Since $z$ is parabolic or equal-rank type and $z$ induces an inner automorphism on $K$, it follows that each $L_i$ is classical.

Suppose, moreover that each $D(B_{j_1})$ is abelian. Then as in Case 1, $D_Z$ is abelian. If $p \geq 5$, then $D_H = D_z = D$ and so both $D_H$ and $D$ are abelian.

Suppose $p = 3$. If $z \in Z(D)$, then $(z, B_{j_2})$ is major and so $D = D_Z$ is abelian. Let $(z, B_{H_2})$ be a $B_{H_2}$-subgroup such that $B_{H_2}$ covers $B_2$. Then $B_{H_2}$ is a block of $C_H(z)$. By [19, Table 4.7.3A],

$$C_H(z) = \langle \text{SL}_2(q) \times \text{SL}_2(q) \circ \text{SL}_2(q), 3 : 3, 1 : 1, 3 : 3 \rangle, \quad \text{SL}_2(q) \circ (q - \epsilon), (3 \times 2^*).$$

Then $\text{Spin}_{10}^1(q) \circ (q - \epsilon), ((q - \epsilon) \times (q - \epsilon)), (2^* \times 2^* \times 3), \text{Spin}_{10}^1(q) \circ (q - \epsilon), (3 \times 2^* \times 2^*)$ (when $q \equiv \epsilon \not \equiv (\text{mod } 9)$) with $2^* = 1$ or $2^*$ according as $\epsilon = - \text{ or } +$, or $\langle \text{SL}_2(q) \times \text{SL}_2(q) ) \circ (q - \epsilon), (2^* \times 3 \rangle$ (when $q \equiv \epsilon \not \equiv (\text{mod } 9)$).

Suppose $C_H(z) = \langle \text{SL}_2(q) \times \text{SL}_2(q) \circ \text{SL}_2(q), t, x \rangle$, so that $L = \langle \text{SL}_2(q) \times \text{SL}_2(q) \circ \text{SL}_2(q), T = (t) \subseteq K$ with $t$ induces $3 : 3 : 3$ on $L$, and $x \in H \setminus K$ induces $1 : 3 : 3$ on $L$. Let $L_i = \text{SL}_2(q) \subseteq G_i := \text{GL}_2(q), t = t_i t_2 t_3, x = x_i x_2 x_3$ with $t_i, x_i$ act on $L_i$ and centralizes $L_j$ when $i \neq j$. In addition, let $J_i = \langle L_i, t_i, x_i \rangle$, so that $J_i \subseteq G_i$. Let $S \equiv \mathbb{Z}_{q - \epsilon} \times \mathbb{Z}_{q - \epsilon} \subseteq \text{SL}_2(q)$ be a maximal torus, and $S \times S \not \subseteq S \leq L$. Since $C_{G_i}(S) = \mathbb{Z}_{q - \epsilon} \times \mathbb{Z}_{q - \epsilon} \times \mathbb{Z}_{q - \epsilon}$ is a maximal torus, it follows that $A := C_{J}(S \times S \circ 3 S)$ is abelian such that
$A \cap K \simeq \mathbb{Z}^6_{q^2}$ is a maximal torus of $K$ and $A/(A \cap K) \cong \mathbb{Z}_3$. In particular, we may suppose $t, x \in A$ and $CH(z) = LA$ with abelian $A$ and $L \subset CH(z)$.

Similarly, if $CH(z) = ((SL_3^6(q) \times (q - \epsilon)) \delta^*, \text{Spin}^6_8(q) \circ (q - \epsilon)).(2^* \times 6^*)$, then $A \leq CH(z)$ and so $CH(z) = LA$ with abelian $A$ and $L \subset CH(z)$, and $A$ induces inner-diagonal automorphisms on each $L_i$.

A proof similar to that in Case 1 with $LT$ replaced by $LA$ and some modifications shows that if each $D(B_i)$ is abelian, then $D(B_H)$ is abelian. Moreover, a proof similar to that of Case 1 shows that $|D(B_H) : D| = 3$. Thus if $z \in Z(D)$, then $D = D_2$ and so $D(B_H) = D_H$, since $|D(B_H) : D| = 3$ and $H/K \cong \mathbb{Z}_3$. It follows that if each $D(B_i)$ is abelian and $z \in Z(D)$, then $D_H$ is abelian with $|D_H : D| = 3$.

Conversely, if $D_H$ is abelian, then $D = D_H \cap K$ is also abelian.

**Case 2.3.** Suppose $p$ is odd and $D(B_i)$ is nonabelian for some $i$, so that as in Case 1, $p = 3$ and $(K, C)$ is given in Table 7. A proof similar to Case 1 shows that $D_2$ is not SDS. Since $D_2 \leq D$, it follows that $D$ is not SDS.

**Case 2.4.** Suppose $p$ is odd and $z \notin Z(D)$, so that $p = 3$ and $|z| = 9$. If $D(B_i)$ is nonabelian, then by Case 2.3, $D$ is not SDS.

We may suppose each $D(B_i)$ is abelian, $z \in D$ with $|z| = 9$ and $zZ(K) \in Z(D/Z(K))$. By [19, Table 4.7A], $9|z - \epsilon)$ and $C_H(z) = \text{Spin}^6_8(q) \circ (q - \epsilon).(|6^* \times 2^*)$ or $(SL_2(q) \times SL_3^6(q)) \circ (q - \epsilon).6^*$.

In this case, $C_H(z)/Z(K)$ is also given by [19, Table 4.7A], and we have $C_K(z/Z(K)) = C_K(z)/Z(K)$. Thus $D/Z(K) \leq C_K(z)/Z(K)$ and $D \leq C_K(z)$. In particular, $z \in Z(D)$ and we may suppose $(z, B_2)$ is major. Hence $D = D_2$ is also abelian.

**Case 2.5.** Suppose $D = Z(K) \cong \mathbb{Z}_3$, so that $D_1/D = 1$ or $3$ and $D_H$ is abelian. We claim that $|D_H : D| = 3$. Let $\theta$ be the canonical character of $B$, so that $D \leq \text{Ker}(\chi)$ and so $\theta \in \text{Irr}(K_a)$, where $K_a = K_a/Z(K_a)$. Let $K^* = \text{the dual group of } K$, so that $K^* = K_a.3 = \text{Innd}iag(K_a)$.

Let $\chi \in \text{Irr}(K^*)$ covering $\theta$, and let $(s, \mu)$ be the label of $\chi$. Thus $s$ is a semisimple character of $K$ and $\mu$ is a unipotent character of $C_K(s)$. So $\chi(1) = |K:C_K(s)||\mu(1)|$. But $\chi(1) = t = \theta(1)$ for some $t \in [1, 3]$ and $\theta(1) \in |K_a|$, so

$$3|\mu(1) = t|C_K(s)|_3$$

and $|\mu(1)| = |C_K(s)|_3$ or $|C_K(s)|/3$.

If $s = 1$, then $\mu$ is a unipotent character of $K$. By [13, pp. 480, 481], $|\mu(1)| \neq |K|_3$ and $|K|_3/3$. Thus $\mu \neq 1$. The centralizer $C_K(s)$ and its order are given by [15]. So $O^r(C_K(s))$ is a central product of classical groups of type $A^m_2$, $A^2_1$, $D^m_2$ (with $m \geq 4$). It follows by the hook-length formula [16, (1.15)] and [25, (22)] that $|\mu(1)| = |O^r(C_K(s))| = 1$ and so $C_K(s)$ is a maximal torus of $K$. In addition, $|C_K(s)|_3 = 3$ and so $O_3(C_K(s)) = Z(K)$. Thus $|K:C_K(s)| = |K_a|_3$ and $\theta(1) = \chi(1)$. Since $K^*/K_a \cong \mathbb{Z}_3$, it follows that $\chi(1) = \theta(1)$ and hence $\chi \mid K_a = \theta$. In particular, $T_H(\theta) = T_H(B) = H$. But $|T_H(B) : D_H| \equiv 1 \pmod{3}$ (replacing $D_H$ by a conjugate if necessary), so $|D_H : D| = 3$.

**Case 3.** Suppose that $K/Z(K) \cong E_3(q)$ with $q$ even. Note that there are no outer diagonal automorphisms of $G$. If $D$ is SDS, then $D$ is abelian.

Suppose that $D$ is SDS. Then $p$ is odd, and since $Z(K) = 1$ we have $z \in Z(D)$ (and $z, B_2$ is a major subsection). Each $L_i$ is either classical or an exceptional group treated in Case 1 or 2.

Suppose $D(B_i)$ is nonabelian for some $i$, and recall that $D(B_i)$ is SDS. By Propositions 4.1, 4.2, 4.3 and 4.4 and Cases 1 and 2, $p = 3, D(B_i) = 3^{1+2}$ and $(K, C)$ is given in Table 7. A proof similar to Case 1.2 shows that $D = D_2$ cannot be SDS after all. It follows that each $D(B_i)$ is abelian. If $L_i$ is classical, then apply Propositions 4.1, 4.2, 4.3 and 4.4. If $L_i$ is exceptional, then apply the results given in cases 1 and 2. Thus $D = D_2$ is abelian.

**Case 4.** Let $q$ be odd, $K = K_a = 2.E_7(q) \leq H := 2.E_7(q).2$. Then $D$ is abelian if and only if $D_H$ is abelian, and moreover, $D_H : D = 2$ when $2 = 2$. If $D/Z$ is nonabelian and SDS, then $p = 2$ and $D$ is given by Table 8 and in addition, $|D_H : D| = 2$, where $\beta = 1$ or $a$. In particular, $D/Z$ is not isomorphic to an extraspecial group for any $Z \leq Z(K)$.

As before, write $m^* := \gcd(m, q - \epsilon)$. 

Case 4.1. Suppose $p = 2$. Since $z$ induces an inner automorphism on $K$, it follows by [19, Table 4.5.2] that $C_K(z) \cong (\text{SL}_2(q) \circ |_{Z_0} \text{Spin}_1(2,q), t)$ with $t = 2 \cdot s$, $(\text{SL}_2(q) \circ (q \cdot x))$ with $x = (8^*/4) : 1$, or $(\text{SL}_2(q) \circ (q \cdot e), w)$ with $w = (8^* : 1)$, where the sign $e = \pm$ is chosen so that $4 \mid (q - e)$. Further, by [19, Tables 4.5.1 and 4.5.2]

$$C_H(z) \cong \langle \text{SL}_2(q) \circ \text{Spin}_1(2,q), t, t_H \rangle, \text{ with } t = 2 \cdot 2, t_H = 1 : 2,$$

(5.3)

$$\langle \text{SL}_2(q) \circ \text{Spin}_1(2,q), t, t_H \rangle, \text{ with } t = 2 \cdot 2, t_H = 1 : 2,$$

(5.3)

or $(3^* \cdot E_8(q))(q \cdot e), w, \tilde{w}$ with $\tilde{w} = (8^* : 1)$ and $\tilde{v} = v = \gamma \cdot i$. Here $\gamma$ and $i$ are graph and inverse automorphisms, respectively. Since $D \unlhd K$ and $D/Z \leq C_H(z)/Z_0$, it follows that $D/Z \leq C_H(z)/Z_0$, $z \in Z(D)$ and so $(z, B_2)$ is always major, as required.

If each $D(B_i)$ is abelian, then a proof similar to that of Case 1 shows that $D = D_2$ is abelian.

Let $(z, B_2)$ be a $B_2$-subgroup such that $B_2$ covers $B_2$, so that $B_2$ covers $B_1$. Let $D(B_2)$ be a defect group for $B_2$ with $D(B_2) \leq D_H$. Suppose each $D(B_i)$ is abelian. A proof similar to that in Case 1 shows that $D(B_2)$ is abelian. In addition, by Propositions 4.1 and 4.4, $|D(B_2) : D_2| = 2$. Since $D = D_2$ and $H : K = 2$, it follows that $D_H = D_H$, and so $D_H$ is abelian when $D_2$ is abelian, it follows that $D = D_2$ is abelian and if only if $D$ is abelian.

Suppose $D_2$ is nonabelian and $D_2/Z$ is SDS for some $Z \subseteq Z(K)$.

We have that $\ell \leq 2$. Suppose $\ell = 1$, so that $L = L_1 = \text{SL}_2(q) \circ (E_8(q))$. Thus $D(B_1)$ is nonabelian and $D(B_1)Z$ is SDS. By Proposition 4.1 and Case 2, this is impossible.

Hence $\ell = 2$, so that $L = L_2$. Let $L_2 = \text{SL}_2(q)$ and $L_2 = \text{Spin}_1(2,q)$. If $D(B_2)$ is abelian, then by Proposition 4.4, $|D(B_1) : D(B_2)| = 2$. Since $D_2$ is nonabelian and $D_2/Z$ is SDS, it follows that $D(B_1)$ is nonabelian and $D(B_1)Z/Z$ is SDS. So $D(B_1) \in \text{Spin}_2(L, L_1)$. Let $B_1 = B_0(L_1)$. Then $D(B_1) \in \text{Spin}_2(L, L_1)$.

Thus there exists a subgroup $Q \leq D_2$ such that $Q \cong D(B_1) \cong SD_{2q_2+2}$, which is impossible. Thus $D(B_2)$ is nonabelian.

By Proposition 4.4 and its proof (cf. [15]),

$$D(B_1) \circ D(B_2) \cong \langle \text{SL}_2(q) \circ \text{SL}_2(q^2) \circ (q^3 - \eta) \rangle. (2 : 1) \leq C_K(z)$$

(5.4)

and $D(B_2) \in \text{Spin}_2(2q_2(q^3 - \eta))$, where $\eta = \pm 1$. Note that $Z(K) = Z(L)$ is a subgroup of $Z(L_2)$ generated by an element $z_L$ and $L_2/Z_2 \cong \Omega_2(2q_2)$. If $D(B_1)$ is nonabelian, then as shown above $D_2$ contains a subgroup $Q \cong D(B_1) \cong SD_{2q_2+2}$, which is impossible. Thus $D(B_1)$ is abelian and hence $D(B_2)$ is abelian. If $G_1 = \text{GL}_{q_2}(2)$, then $D(B_2) \in \text{Spin}_2(C_{G_1}(t))$ for some $2^2$-element $t \in G_1$. Thus $D(B_2) \cong Z_{2q_2+1}$ or $Z_2 \times Z_2$ and $D = D(B_1) \circ D(B_2)$. Thus $D$ is given by Table 8.

By (5.3) and (5.4),

$$D(B_2) = D_1 \circ D_2 \cong \langle \text{SL}_2(q) \circ \text{SL}_2(q^2) \circ (q^3 - \eta) \rangle, (2 : 1, 1 : 2) \leq C_H(z),$$

(5.5)
where $D_1$ is an abelian subgroup of $SL_2(q)$.2 and $D_2$ is a defect group of $(SL_2(q^2) \circ (q^2 - \eta)).2$ containing $D(B_2)$. Thus $D_2$ is a Sylow 2-subgroup of $(SL_2(q^2) \circ (q^2 - \eta)).2$. In particular we have shown that $|D_H : D_1| = 2$. Note that $(SL_2(q^2) \circ (q^2 - \eta)).2$ is isomorphic to $GL_2^q(q^2)$ and so $D_2 \cong SD_{2e+2}$ or $Z_{2e} \rtimes Z_2$ according as $2\| (q - \eta)$ or $2^2 \| (q - \eta)$.

**Case 4.2.** Suppose $p$ is odd, so $z \in Z(D)$. Either $L_1$ is classical, or $L_1$ is an exceptional group treated in Case 1 or 2. Suppose each $D(B_i)$ is abelian. If $L_1$ is classical, then apply Propositions 4.1, 4.2, 4.3 and 4.4. If $L_1$ is exceptional, then apply the results given in Cases 1 and 2. Thus $D = D_2$ is abelian. Since $D_H = D = D_2$, it follows that $D_H$ is abelian. Suppose $D(B_1)$ is nonabelian, so that $D(B_1) \cong 3_+^{1+2}$ and $(K, C)$ is given by Table 7. A proof similar to Case 1.2 with some obvious modifications shows that $D = D_2$ cannot be SDS.

**Case 5.** Suppose $K := E_8(q)$, so that $Z(K) = 1$. Then $K$ has no block with extraspecial defect group. Since $Z(K) = 1$, we may choose $(z, B_2)$ to be a major subsection of $B$. If each $D(B_i)$ is abelian, then a proof similar to that in Case 1 shows that $D = D_2$ is abelian. If $D(B_1)$ is nonabelian and $p$ is odd, then $D(B_1) \cong 3_+^{1+2}$ and $(K, C)$ is given by Table 7. A proof similar to Case 1.2 with some obvious modifications shows that $D = D_2$ cannot be SDS.

Suppose $D(B_1)$ is nonabelian for some $i$ and $p = 2$. Then $C_K(z) \cong \text{Spin}_{16}(q)/\langle z \rangle$ or is given by Table 6. In the former case by Proposition 4.4, $D = D_2$ is not isomorphic to an extraspecial group. In the latter case $C_K(z) \cong (SL_2(q) \circ E_7(q_4))(2)$. Thus $L_1 \sim L_2$ with $L_1 \cong SL_2(q)$ and $L_2 \cong E_7(q_4)$. If $D(B_1)$ is abelian, then $D$ contains a subgroup isomorphic to $D(B_{j_1}) \leq \text{Sy}(L_1, 2)$. But $31^6$ is impossible, Thus $D(B_1)$ is abelian and $D(B_{j_1}) \cong Z_2 \times Z_2$ or $Z_{2n+1}$. In addition, $D(B_2)$ is nonabelian and given by Table 8. Thus $D = (D(B_1) \circ D(B_2)).2$ and as shown in the proof of Case 4.1, $D$ contains a subgroup isomorphic to $SD_{2n+2}$ or $Z_{2n} \rtimes Z_2$, so $D$ is not isomorphic to an extraspecial group. □

To complete the treatment, it remains to consider the simple groups with exceptional covers, namely the prefect groups $G$ where $G/Z(G) \cong A_6$, $A_7$, $PSL_3(2)$, $PSL_3(4)$, $PSU(2)$, $PSU(4)$, $PSU(6)$, $Sz(8)$, $Sp_6(2)$, $O_7(3)$, $O_8^+(2)$, $G_2(3)$, $G_2(4)$, $F_4(2)$ and $E_6(2)$. We take $G$ to be the full cover of $G/Z(G)$.

In none of these cases do we have any new examples of extraspecial groups of $G/O_2(Z)$ (every extraspecial defect group for a faithful block is already a defect group for a non-faithful block). We may use [18] to verify this in all but the cases $F_4(2)$ and $E_6(2)$ for $p = 3$. For the blocks of the double cover of $F_4(2)$ for $p = 3$, we refer to [20]. For $G/Z(G) \cong E_6(2)$ and $p = 3$, we use that fact that for every $p$-subgroup $Q$ we have $C_G(Q)/Z(G) = C_G(Z_G(Q)/Z(G))$, since $Q$ and $Z(G)$ have coprime order, and use an analysis similar to that given in the proof of Theorem 5.1. In each case, every extraspecial defect group for a faithful block is already a defect group for a non-faithful block.

However, we do have new examples of blocks of such groups $G$ with extraspecial defect groups. Using the same references as above: if $G/Z(G) \cong A_6$ or $A_7$, then $|Z(G)| = 6$ and there are blocks with defect group $D \cong 3_+^{1+2}$ covering each block of $Z(G)$; if $G/Z(G) \cong G_2(4)$, then $|Z(G)| = 2$ and there are blocks with defect group $D \cong 3_+^{1+2}$ covering each block of $Z(G)$; if $G/Z(G) \cong PSL_3(4)$, then $|Z(G)| = 12$ and there are blocks with defect group $D \cong 3_+^{1+2}$ covering each block of $Z(G)$; all other blocks of these groups with extraspecial defect groups are already accounted for.

**References**


