Divisor Class Groups of Ladder Determinantal Varieties

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In this note, we compute the divisor class groups of certain ladder determinantal varieties. © 1996 Academic Press, Inc.

1. INTRODUCTION

For an $r \times s$ matrix $X$ of indeterminates over a field $K$, the ideal generated by the $t_i \times t_i$ minors in the first $s$ columns, for $i = 1, \ldots, n$, $1 \leq t_1 < t_2 < \cdots < t_n \leq \min(r, s)$, $1 \leq s_1 < s_2 < \cdots < s_n = s$, is a well-studied object. It is known that the ideal is prime and that the quotient ring is a normal, Cohen–Macaulay domain [EH, BV]. The divisor class group of this normal domain can also be found in [BV].

The restriction of the generators to certain subsets of $X$, called ladders, gives ladder determinantal ideals; the notion was introduced by Abhyankar [Ab]. Ladder determinantal ideals are prime [Na] and their quotient rings are Cohen–Macaulay [HT].

For ideals generated by minors of a fixed size, that is, for the case $n = 1$ above, Conca proved that ladder determinantal rings were normal and computed their divisor class groups [Co1, Co2].

In a recent note [MS1], we showed that the more general ladder determinantal rings (that is, for arbitrary $n$) are normal. In this paper, we compute the divisor class groups for the one-sided one-corner ladders.

As in all computations of divisor class groups of determinantal rings, Nagata’s Theorem (Theorem 2.2.2) provides the basic framework. It re-
duces the computation of the divisor class group to that of the minimal primes of finitely many “selected” elements of the ring. In the process, we obtain many new irreducible codimension 1 subvarieties of the ladder determinantal variety. One, in particular, is a new prime when we consider the full matrix too, and is the subject of [M S 2].

2. PRELIMINARIES

2.1.

Let \( X = (X_{ij}) \) be an \( r \times s \) matrix of independent indeterminates over an algebraically closed field \( K \), of arbitrary characteristic. Consider the ideal \( I \) in \( K[X] \) generated by the \( t_i \times t_j \) minors in the first \( s_i \) columns, for \( i = 1, \ldots, n \), \( 1 \leq t_1 < t_2 < \cdots < t_n \leq \min(r, s) \), \( 1 \leq s_1 < s_2 < \cdots < s_n = s \). We may, of course, assume that \( s_1 \geq t_1 > 1 \) and \( s_i - s_{i-1} > t_i - t_{i-1} \), for \( i = 2, \ldots, n \). Further, let \( t_0 = 1 \) and \( s_0 = 0 \). Denote this set of generators of \( I \) by \( G \). \( I \) can also be thought of as the ideal “cogenerated” by a fixed minor, say \( \alpha \), in \( K[X] \), that is, as the ideal generated by all minors in \( X \) which are not greater than or equal to \( \alpha \), in the usual order on minors [B V, H T].

Definitions. \( L \subseteq X \) is a ladder if, whenever \( X_{ij} \in L \) and \( X_{kl} \in L \), with \( i \leq k \) and \( j \leq l \), then \( X_{il} \in L \) and \( X_{kj} \in L \). Given a ladder \( L \subseteq X \), let \( G(L) = G \cap K[L] \) and denote the ideal in \( K[L] \) generated by \( G(L) \) by \( I(L) \).

We make the following assumptions on \( L \):

(1) every row of \( X \) and every column of \( X \) contains an element in \( L \),

(2) every indeterminate in \( L \) is involved in some minor of \( G(L) \), and

(3) \( L \) is connected, that is, it cannot be written as the disjoint union of two nonempty ladders \( L_1 \) and \( L_2 \), such that every minor in \( G(L) \) is either in \( L_1 \) or in \( L_2 \).

Note that the first two assumptions are not restrictive since they do not affect normality or the computation of the divisor class group.

Under the assumptions every ladder can be written as an intersection of one-sided one-corner (osoc) ladders of the form

\[
L_1 = \{(a, b) \in X | a > u \text{ or } b > v\}
\]

and

\[
L_2 = \{(a, b) \in X | a < x \text{ or } b < y\},
\]

for some \( 0 \leq u, x \leq r + 1 \) and \( 0 \leq v, y \leq s + 1 \). The point \((u, v)\) is the upper-left corner of the ladder \(L_2\), while \((x, y)\) is the lower-right corner of the ladder \(L_2\).
The subject of this paper is the computation of the divisor class groups of the rings $K[L_1]/I(L_1)$ and $K[L_2]/I(L_2)$.

2.2. The Divisor Class Group

The theory of fractionary ideals, divisorial ideals, and the divisor class group can be found in [Fo] or [Sa]. We shall follow the notation used by Conca [Co]. Throughout this section $R$ shall denote a Noetherian domain and $F$ denotes its field of fractions.

**Definitions.** A fractionary ideal of $R$ is a nonzero, finitely generated $R$-submodule of $F$. A fractionary ideal is called principal if it is generated by a single element of $F$. A divisorial ideal is a fractionary ideal which can be written as an intersection of principal fractionary ideals. One can characterize the divisorial ideals as those fractionary ideals $I$ which are reflexive, that is, which satisfy the condition $I = R_{v_F}(R_{v_F}I)$.

We introduce a binary operation “$\cdot$” on the set $\text{Div}(R)$ of divisorial ideals of $R$. For $I, J \in \text{Div}(R)$, let

$$I \cdot J = R_{v_F}(R_{v_F}I).$$

It is easy to see that $\text{Div}(R)$ is a commutative semigroup under “$\cdot$”, with $R$ as the identity.

Finally, we note that if $R$ is normal, $\text{Div}(R)$ is a group, called the group of divisors of $R$. The inverse of a divisorial ideal $I$ in $\text{Div}(R)$ is just $R_{v_F}I$.

Henceforth, we shall assume that $R$ is a Noetherian normal domain.

Let $M$ denote the set of maximal divisorial ideals contained in $R$. The elements of $M$ are precisely the height 1 primes of $R$. For $P \in M$, $R_P$ is a discrete valuation ring, since $R$ is normal. Denote by $v_P$ the valuation on $R_P$. For a fractionary ideal $I$, set $v_P(I) = \min(v_P(f) \mid f \in I \setminus \{0\})$. We describe the structure of $\text{Div}(R)$.

Let $G$ denote the group of all maps from $M$ to $\mathbb{Z}$ with finite support. Let $\text{div}(P)$ denote the map which sends $P$ to 1 and all other elements to zero. $G$ is therefore the free abelian group with basis $\{\text{div}(P) \mid P \in M\}$. For $f \in F \setminus \{0\}$ and $I$ a divisorial ideal, denote by $\text{div}(f)$ and $\text{div}(I)$ the elements $\sum v_P(f)\text{div}(P)$ and $\sum v_P(I)\text{div}(P)$ of $G$. We have [Fo]:

**2.2.1. Proposition.** The map $\text{div}: \text{Div}(R) \rightarrow G$ which sends the divisorial ideal $I$ to $\sum v_P(I)\text{div}(P)$ is an isomorphism of groups.

Let $\text{Prin}(R)$ denote the set of principal fractionary ideals of $R$. Clearly, $\text{Prin}(R)$ is a subgroup of $\text{Div}(R)$. The quotient group

$$\text{Cl}(R) = \text{Div}(R)/\text{Prin}(R)$$
is called the divisor class group of $R$. The residue class of a divisorial ideal $\text{div}(I)$ in $\text{Cl}(R)$ is denoted by $\text{cl}(I)$.

Let $A$ be an overring of $R$ and a normal domain too. Let $M_A$ and $M_R$ denote the sets of height 1 prime ideals of $A$ and $R$, respectively. Given $Q \in M_A$ and $P \in M_R$, denote the ramification index $e_Q(PA)$ of $P$ in $Q$ by $e(P, Q)$. The assignment $\text{div}(P) \mapsto \text{div}(Q)$ of $P$ in $Q$ defines a group homomorphism $g: \text{Div}(R) \to \text{Div}(A)$. For $g$ to induce a map at the divisor class group level, the following condition is sufficient:

\[ (\text{NBU}) \quad \text{for all } Q \in M_A, \quad \text{height}(Q \cap R) \leq 1. \]

In this case, we denote the induced map from $\text{Cl}(R)$ to $\text{Cl}(A)$ by $\bar{g}$.

If $A$ is flat over $R$, then (NBU) is satisfied. In this case $LA$ is a divisorial ideal of $A$ whenever $I$ is a divisorial ideal of $R$. Therefore the map $g$ takes $\text{div}(I)$ to $\text{div}(LA)$, and $\bar{g}$ takes $\text{cl}(I)$ to $\text{cl}(LA)$.

For a multiplicatively closed subset $S$ of $R$, $R_S$ is a flat extension of $R$. Further, $\bar{g}$ is surjective, since $g$ is. The kernel of $\bar{g}$ is described in the following theorem [Fo, Sa]:

\begin{align*}
2.2.2. \text{Theorem (Nagata).} & \quad \text{Let } R \text{ be a normal domain and let } S \text{ be a multiplicitively closed subset of } R. \text{ One has an exact sequence of groups:} \\
& \quad 0 \to \text{Ker}(\bar{g}) \to \text{Cl}(R) \xrightarrow{g} \text{Cl}(R_S) \to 0. \\
& \quad \text{The map } \bar{g} \text{ takes } \text{cl}(I) \text{ to } \text{cl}(IR_S), \text{ and } \text{Ker}(\bar{g}) \text{ is generated by the classes of the height 1 prime ideals } P \text{ of } R \text{ with } P \cap S \neq 0.
\end{align*}

As a corollary, we have [BV, Co1]:

\begin{align*}
2.2.3. \text{Corollary.} & \quad \text{Let } R \text{ be a normal domain and } B \text{ a factorial subring of } R. \text{ Assume that there exists } x \in B \text{ such that } B[x^{-1}] = R[x^{-1}]. \text{ Let } x = x_1 \cdots x_l \text{ be a factorization of } x \text{ in } B \text{ as a product of irreducible elements of } B. \text{ Denote by } P_1, \ldots, P_l \text{ the minimal prime ideals of } x \text{ in } R. \text{ Then } \text{Cl}(R) \text{ is generated by } \text{cl}(P_1), \ldots, \text{cl}(P_l). \text{ Furthermore, the syzygies between the given generators of } \text{Cl}(R) \text{ are linear combinations of the syzygies } \sum_{i=1}^{l} v_{P_i}(x_k) \text{cl}(P_i) = 0, k = 1, \ldots, l.
\end{align*}

2.3.

We shall require several versions of the Eagon–Hochster inversion trick [EH]. In its simplest form, it says:

\begin{align*}
2.3.1. \text{Lemma.} & \quad \text{Let } P = \{1, \ldots, t_1 - 1\} = Q, \ M = X_{P,Q} (a \ t_1 - 1 \times t_1 - 1 \text{ matrix}), \text{ and } u = \text{det } M. \text{ The rings} \\
& \quad \frac{K[X]}{I} \begin{bmatrix} 1 \\ u \end{bmatrix} \text{ and } \frac{K[X][V]}{I'} \begin{bmatrix} 1 \\ u \end{bmatrix}
\end{align*}
are isomorphic, where
\[
X' = \{ X_{ab} \in X \mid b \in Q \text{ or } b > s_1 \}
\]
\[
V = \{ X_{ab} \in X_1 \mid a \in P \text{ and } b \not\in Q \},
\]
and \( I' \) is the ideal in \( K[X'] \) generated by the \( t_j \times t_j \) minors in the first \( s_j - s_1 + t_1 - 1 \) columns of \( X' \), for \( j = 2, \ldots, n \).

**Notation.** By \( X_i \), we denote the submatrix of \( X \) consisting of its first \( s_i \) columns, for \( i = 1, \ldots, n \).

**Notation.** Let \( P \) be a subset of the rows (equivalently, a subset of \( \{1, \ldots, r\} \)) and let \( Q \) be a subset of the columns (equivalently, a subset of \( \{1, \ldots, s\} \)) of \( X \). We shall denote the submatrix formed by the rows of \( P \) and the columns of \( Q \) by \( X_{P,Q} \). Further, for \( T \subseteq X \), \( K[T] \) shall denote the polynomial ring over \( K \) in the variables \( X_{ij} \) such that \( (i, j) \in T \).

**Proof of Lemma 2.3.1.** Consider the map
\[
\Phi : K[X_1] \left[ \frac{1}{u} \right] \rightarrow K[T_1] \left[ \frac{1}{u} \right]
\]
(where \( T_1 = \{ X_{ab} \in X_1 \mid a \in P \text{ or } b \in Q \} \)), defined as follows:

- \( \Phi \) is the identity on \( K[T_i] \) (therefore \( \Phi \) is surjective),
- for \( a \not\in P \) and \( b \not\in Q \), set
  \[
  R_a = X_{(a),Q} \quad \text{and} \quad C_b = X_{P,(b)},
  \]
both vectors of length \( t_1 - 1 \), and define
\[
\Phi(X_{ab}) = R_a M^{-1} C_b.
\]
Clearly, every column of \( X_1 \) is a linear combination of the columns \( Q \) of \( M \), after application of \( \Phi \). Therefore, \( \ker(\Phi) \) contains every \( t_1 \times t_j \) minor of \( X_1 \). Further, for \( j > 1 \), every \( t_j \times t_j \) minor of \( X \) is a linear combination of \( t_j \times t_j \) minors in the submatrix \( X' \). Therefore, we have an induced map
\[
\Phi : K[X] \left[ \frac{1}{I} \right] \rightarrow K[X'] \left[ \frac{1}{I'} \right],
\]
which is, in fact, an isomorphism, since \( I \) and \( I' \) are prime and the two rings have the same dimension (see, for instance, [BV]).
3. PRIMES OF HEIGHT 1 FOR THE LOWER-RIGHT CORNER LADDER

In this section we establish the primality of certain ideals which arise in the study of the divisor class group. The notation followed is that of Section 2.1. Henceforth, the ideal $I(L)$ shall simply be denoted by $I$ and the set $G(L)$ denoted by $G$.

Consider the osoc ladder $L$ with corner $C = (x, y)$ defined by

$$L = \{(a, b) \in X \mid a < x \text{ or } b < y\},$$

where $1 \leq x \leq r$ and $1 \leq y \leq s$. Let $p$ be such that $s_{p-1} < y \leq s_p$. Since a ring and its polynomial extension have isomorphic divisor class groups, we shall assume that $x$ and $y$ are such that every indeterminate in $L$ occurs in some non-trivial generator of $I$. Therefore

$$x - 1 \geq t_n \quad \text{and} \quad y - 1 - s_{p-1} > t_p - t_{p-1}.$$  

**Notation.** Let $L_s = L \cap X_s$. Consider the following submatrices of $L$:

$$Y = \{(a, b) \in L \mid b < y\} \quad \text{and} \quad Z = \{(a, b) \in L \mid a < x\}.$$  

Denote by $Y_i$ and $Z_i$, the submatrices $Y \cap L_s$ and $Z \cap L_s$, respectively, for $i = 1, \ldots, n$. Note that $Y_i = Y$, for $i \geq p$.

3.1. The Prime $\mathcal{D}_k$

Let $k$ be arbitrary with $1 \leq k \leq n$. Define

$$W_k = \{X_{ab} \in L \mid a \leq t_k - 1 \text{ and } b \leq s_k\}$$

and denote by $F_k$ the set of $t_k - 1 \times t_k - 1$ minors in $W_k$. Now consider the ideal over $I$ generated by the minors in $F_k$, that is, the ideal in $K[L]$ generated by $G \cup F_k$, which we denote by $\mathcal{D}_k$. We shall be concerned with the primality of $\sqrt{\mathcal{D}_k}$, the radical of $\mathcal{D}_k$, which we denote by $\mathcal{D}_k$. Since $\mathcal{D}_n$ is itself prime, being an ideal cogenerated by a fixed minor [BV], we shall assume $k < n$.

**Notation.** Denote by $W_i$ the region $L_i \cap W_k$, for $i = 1, \ldots, k - 1$. Further, let $R$ denote the ring $K[L]/\mathcal{D}_k$.

3.1.1. Lemma. Let $P = \{1, \ldots, t_k - 1\}$, $P_i = \{x - t_i + 1, \ldots, x - 1\}$, for $i = 1, \ldots, n$, $Q_1 = (1, \ldots, t_1 - 1)$, and $Q_i = Q_1 \cup \{s_1 + t_1 - t_2, \ldots, s_{i-1} + t_1 - t_{i-1}\}$, for $i = 2, \ldots, n$. Let $M = X_{p, q_1}$, $(a t_k - 1 \times t_k - 1 \text{ matrix of } Z)$, $v = \det M$, $M_i = X_{p, q_i}$, $(a t_i - 1 \times t_i - 1 \text{ matrix of } Z)$, $u_i = \det M_i$, for $i = 1, \ldots, n$, and $u = u_1 \cdot u_2 \cdots u_n$. Then $R[1/u]$ is a domain.

**Proof.** Let $Q_0 = \emptyset$. For $i = 1, \ldots, n$, let

$$L_i = \{X_{ab} \in L_i \mid b \in Q_{i-1} \text{ or } s_{i-1} < b \leq s_i\}.$$
and
\[ T_i = \{ X_{ab} \in L_i \mid a \in P, \text{ or } b \in Q \}. \]

To begin the proof, we imitate the map \( \Phi \) defined in Lemma 2.3.1, to obtain maps
\[ \Phi_j : K[L_j] \left[ \begin{array}{c} 1 \\ u_1 \\ \vdots \\ u_j \end{array} \right] \rightarrow K[T_j] \left[ \begin{array}{c} 1 \\ u_1 \\ \vdots \\ u_j \end{array} \right] \]
mapping the regions \( L_j \) onto the regions \( T_j \), via the inverses of \( M_j \), for \( j = 1, \ldots, k - 1 \). Now \( \text{ker}(\Phi_{k-1} \cdots \Phi_1) \) contains all \( t_j \times t_j \) minors of \( L_j \) (and therefore \( L_j \)), for \( j = 1, \ldots, k - 1 \).

Define
\[ W_j' = W_j \cap L_j, \quad \text{for } i = 1, \ldots, k. \]

We now map the submatrix \( L_k \). Consider
\[ \Phi_k : K[L_k] \left[ \begin{array}{c} 1 \\ u_1 \\ \vdots \\ u_k \end{array} \right] \rightarrow K[T_k] \left[ \begin{array}{c} 1 \\ u_1 \\ \vdots \\ u_k \end{array} \right] / \langle v \rangle \]
(where \( \langle v \rangle \) is the ideal generated by \( v \) in \( K[T_k] \), defined in the usual way:
- \( \Phi_k \) maps the variables in \( T_k \) canonically onto their images in \( K[T_k] / \langle v \rangle \) (and is therefore surjective).
- for \( a \not\in P_k \) and \( b \not\in Q_k \), set
\[ R_a = X_{(a),Q_k} \quad \text{and} \quad C_b = X_{P_k(b)} \]
both vectors of length \( t_k - 1 \), and define
\[ \Phi_k(X_{ab}) = R_a M_k^{-1} C_b \pmod{\langle v \rangle}. \]

Clearly, every column of \( X'_k \) is a linear combination of the columns \( Q_k \) of \( M_k \), after application of \( \Phi_k \cdots \Phi_1 \). We denote the map \( \Phi_k \cdots \Phi_1 \) by \( \Psi_k \).

Let
\[ A_k = \{ X_{ab} \in L_k \mid a \in P_k \}, \]
the submatrix of \( L_k \) consisting of its first \( t_k - 1 \) rows. Now \( \Psi_k(M) \) has rank \( t_k - 2 \), since we have quotiented by \( \langle v \rangle \). Therefore
\[ \Psi_k(W_k') = \Psi_k(M M_k^{-1} A_k) = \Psi_k(M) \Psi_k(M_k^{-1}) \Psi_k(A_k) \]
has rank at most \( t_k - 2 \). Therefore, \( \ker(\Phi_k \cdots \Phi_1) \) contains every \( t_i \times t_i \) minor of \( L_j \), for \( j = 1, \ldots, k \) and every \( t_k - 1 \times t_k - 1 \) minor of \( W_k \) (and therefore \( W'_k \)).

The region \( L \setminus L_k \) is covered by the maps \( \Phi_{k+1}, \ldots, \Phi_n \), via the inverses of \( M_{k+1}, \ldots, M_n \) in the usual way, to obtain a surjection onto

\[
\frac{K[T]}{\langle u \rangle}, \quad \text{where } T = \bigcup_{i=1}^{n} T_i.
\]

We denote \( \Phi_n \cdots \Phi_1 \) by \( \Phi \). Clearly \( \ker(\Phi) \) contains the ideal \( \mathcal{I}_k \). Further, \( \text{Im}(\Phi) \) is a domain, so it, in fact, contains \( \mathcal{E}_k \).

The ring \( R[1/u] \) is the ring of regular functions of the principal open set of \( \mathcal{E}_k \) in \( K[L] \), defined by the nonvanishing of the function \( u \), while \( \text{Im}(\Phi) \) is the ring of regular functions of the principal open set of \( \mathcal{E}_k \), again defined by the nonvanishing of \( u \). The map \( \Phi \) induces a map at the variety level, which is easily seen to be onto, and \( \Phi \) is injective.

By [HT], \( |T| \) is equal to the dimension of \( K[L]/\mathcal{E}_k \). Therefore, by the above lemma and the fact that \( v \), being the determinant of a matrix of indeterminates, generates a prime ideal, we have \( \dim K[L]/\mathcal{E}_k = \dim K[L]/I = 1 \).

We maintain the notation of the above lemma. Let \( V \) denote the variety of \( \mathcal{E}_k \). Let \( L \) be a point of \( V \) with nonzero, \( t_i - 1 \times t_i - 1 \) minors in \( Z_i \), say \( w_i \), for \( i = 1, \ldots, n \). Denote the matrices corresponding to the \( w_i \) by \( X_{p,q} \). Assume that \( Q_p \subset \{1, \ldots, y - 1\} \), that is, \( w_p \) is contained in \( Z_p \cap Y_p \).

Since the columns \( Q_i \) restricted to \( Z \) have full rank \((= t_i - 1)\), one can obtain \( t_i - t_{i-1} \) columns for \( Q_i \), say \( D_i \), which when added to the columns of \( Q_1 \cup D_2 \cup \cdots \cup D_{i-1} \) have rank \( t_i - 1 \), for \( i = 2, \ldots, n \). Therefore, we may select nonzero \( t_i - 1 \times t_i - 1 \) minors \( u_1, u_2, \ldots, u_n \), contained in \( Q_1, Q_1 \cup D_2, Q_1 \cup D_2 \cup D_3, \ldots, Q_1 \cup D_2 \cup \cdots \cup D_n \), respectively, and such that the rows of \( u_i \) contain the rows of \( u_{i-1} \), for \( i = 2, \ldots, n \) (that is, which satisfy the hypothesis of Lemma 3.1.1). Therefore, the point \( L \) is contained in an irreducible open subset of \( V \).

Let \( U \) denote the open subset of \( V \) consisting of points like \( L \). \( U \) is a finite union of irreducible open subsets whose intersection is nonempty (By Lemma 3.1.1, we need only establish the existence of a point in the variety of \( \langle v \rangle \) in \( K[T] \), with no \( t_i - 1 \) minor vanishing in the region \( T_i \), for \( i = 1, \ldots, n \). Since this variety is irreducible, such a point indeed exists.) Therefore \( U \) is itself irreducible. Let \( \overline{U} \) denote the closure of \( U \).

**Notation.** Let \( L \) be a point in \( K[L] \). By \( Z_{r+1} \), we shall denote the submatrix of rank \( x - 1 \), obtained by augmenting the region \( Z \) of \( L \) with \( x - 1 - \text{rank}(Z) \) linearly independent columns.
3.1.2 Theorem. \( \bar{U} = V \). Therefore the variety of \( \mathcal{S}_k \) is irreducible, that is, \( \mathcal{S}_k \) is prime.

Proof. We first tackle the region \( Y \). Let \( 1 \leq v \leq p \), and let \( L = (x_{ab}) \) be a point in \( V \) with \( \text{rank}(Y_i) = t_i - 1 \), for \( i = 1, \ldots, v - 1 \), and \( \text{rank}(Y_v) < t_v - 1 \). Let \( Z \) be a basis for the column space of \( Y_v \) be denoted by \( Z \). We may assume that \( C_1 \subseteq \cdots \subseteq C_v \subseteq C_p \). We shall show that \( L \) is in the closure of points with a higher rank for the region \( Y \).

(1) Let \( v < p \). Suppose that \( \text{rank}(Y_v) < \text{rank}(Y) \). Then \( w \) be the smallest integer greater than \( v \) such that \( \text{rank}(Y_v) < \text{rank}(Y_w) \). We seek a column not contained in \( \text{Span}(C_v) \), as follows:

(a) If \( w \leq k \) or \( v \geq k + 1 \) or \( \text{rank}(W) < t_k - 2 \), then we pick any column from \( C_w \). 

(b) If \( w > k, v < k + 1 \) and \( \text{rank}(W) = t_k - 2 \), then the column should be such that when restricted to its first \( t_k - 1 \) rows, it is in the column space of \( W \). Two cases:

   (b1) If \( \text{rank}(Y) - \text{rank}(Y_v) \geq 2 \), then let \( z \geq w \) be the least integer such that \( \text{rank}(Y_z) - \text{rank}(Y_v) \geq 2 \). Now since \( \text{rank}(W) \) is just one less than the maximum possible \( (W \) has \( t_k - 1 \) rows), a column satisfying our requirements can be obtained from \( \text{Span}(Y_z) \). 

   (b2) If \( \text{rank}(Y) - \text{rank}(Y_v) = 1 \), then let \( z \geq w \) be the least integer such that \( \text{rank}(Z) - \text{rank}(Z_v) \geq 2 \). It exists as \( \text{rank}(Z_v) \geq x - 1 - t_v + 1 \). In this case the column can be obtained from \( \text{Span}(Z) \) 

We denote the chosen column by \( d \). If it was obtained from \( Y_v \), it has \( r \) entries; otherwise it has \( x - 1 \) entries, in which case we set \( d_s = \cdots = d_r = 0 \).

Let \( c \) be a column in \( Y_v \) (that is, \( (s_{v-1} + 1, \ldots, s_v) \) which does not belong to \( C_v \). Therefore \( c \) can be expressed as a linear combination of the columns of \( C_v \). Let \( \alpha \) be an indeterminate. Consider the ladder \( P = (p_{ab}) \), with entries in \( K[\alpha] \), defined as follows:

- \( p_{ac} = x_{ac} + \alpha \cdot d_a \), for all \( a = 1, \ldots, r \).
- \( p_{ab} = x_{ab} \), otherwise.

For \( \alpha \in K \setminus \{0\} \), the points \( P \) are such that the ranks for the region \( Y \) are 1 more than that for the point \( L \). Therefore \( L \) is in the closure of the set of these points. We claim that the points \( P \) all lie in \( V \). First, since adding column \( d \) to any earlier column \( c \) will increase \( \text{rank}(W) \) by at most 1, \( d \) is so chosen that for all the points \( P \), \( \text{rank}(W) \leq t_k - 2 \). Next, \( \text{rank}(Y_i) \leq t_i - 1 \), for all \( i = 1, \ldots, p \). In case (a),
that is clear, and in case (b), note that \( L \) is such that the \( Y_c \) (resp. \( Z_c \), depending on \( \text{rank}(Y) = \text{rank}(Y_c) \)) is the first region such that its rank is 2 more than that of \( Y_c \). Therefore the regions \( Y_{c+1}, \ldots, Y_{c-1} \) (resp. \( Y_{c+1}, \ldots, Y_p, Z_p, \ldots, Z_{c-1} \)) have ranks precisely one more than \( Y_c \), and so none of them has its maximum permissible rank. Since in the points \( P \), the ranks go up by just 1, our claim holds.

If \( \text{rank}(Y_c) = \text{rank}(Y) \), the above arguments hold verbatim, except that the indices \( w \) and \( z \) above are obtained from the \( Z \) region (therefore, \( p \leq w \leq z \)). We may therefore assume that \( \text{rank}(Y_i) = t_i - 1 \), for \( i = 1, \ldots, p \). Denote the region \( Y_p \cap Z_p = Y \cap Z \) by \( D \).

(2) Let \( p = p \). If \( \text{rank}(Y) < \text{rank}(Z) \), we take \( d \) to be any column in \( \text{Span}(Z_p) \setminus \text{Span}(D) \). Otherwise, we seek the index \( w \) (and possibly \( z \)) from the region \( Z \), just as we did in (1) above. Repeating the arguments of (1), we may assume that \( \text{rank}(Y_i) = t_i - 1 \), for \( i = 1, \ldots, p \). We are now in a position to handle points with \( \text{rank}(Y_i \cap Z) < t_i - 1 \), for \( v \leq p \).

(3) Let \( v < p \). Let \( \text{rank}(Z_i) = t_i - 1 \), for \( i = 1, \ldots, p - 1 \), and \( \text{rank}(Z_{v+1}) < t_{v+1} - 1 \). Let \( R_i \) and \( C_{p_i} \) be the rows and columns of a nonzero minor in \( Z_i \), of maximum size (that is, of size rank\((Z_i)\)).

Since \( \text{rank}(Y_c) > \text{rank}(Z_i) \), there exists a column in \( Y_i \), say \( \lambda \), such that the columns in \( C_{p_i} \cup \{ \lambda \} \) are linearly independent in \( Y \). Therefore there exists a row \( c \geq x \), such that the (\( \text{rank}(Z_i) + 1 \))-sized minor, with row indices \( R_i \cup \{ c \} \) and column indices \( C_{p_i} \cup \{ \lambda \} \), is nonzero. Further, let \( d \) be a column chosen from \( \text{Span}(Z_i) \setminus \text{Span}(Z_{v+1}) \), just as it was in (1), and consider the ladder \( P = (p_{ab}) \), with entries in \( K[\alpha] \), defined as follows:

\[
\begin{align*}
p_{ab} &= x_{ab} + \alpha \cdot c_b \cdot d_a \quad \text{for all } (a, b) \text{ such that } 1 \leq a \leq x - 1 \text{ and } 1 \leq b \leq y - 1. \\
p_{ab} &= x_{ab}, \quad \text{otherwise.}
\end{align*}
\]

It is easily seen that for all \( \alpha \in K \setminus \{0\} \), the points \( P \) are such that the columns in \( C_{p_i} \cup \{ \lambda \} \) are linearly independent in the region \( Z_{v+1} \) itself, and so \( \text{rank}(Z_i) \) is 1 more than that for the point \( L \). Note that since we have added a multiple of column \( d \) to the columns of \( D \), the ranks of the regions \( Z_{v+1} \), for \( i > v \), are still bounded by \( t_i - 1 \). The ranks of \( Y_i \), for \( i = 1, \ldots, p \), do not change either, since we have added multiples of a row of \( Y_i \) to the rows of \( D \). Therefore, these points belong to \( V \). We conclude that \( L \) is in their closure.

(4) Let \( v = p \). Identical to (3), except that \( Z_p \) should be replaced by \( D \). Therefore we may assume that \( \text{rank}(Z_i) = t_i - 1 \), for all \( i = 1, \ldots, p - 1 \), and that \( \text{rank}(Z_p) = t_p - 1 = \text{rank}(D) \).

(5) Let \( v > p \). Let \( L \) be such that \( \text{rank}(Z_i) = t_i - 1 \), for \( i = 1, \ldots, v - 1 \), and let \( \text{rank}(Z_{v+1}) = t_{v+1} - 1 \). As before, we seek regions \( Z_{v+1} \) (and perhaps \( Z_p \)), with \( v < w \leq z \leq n + 1 \), which will provide us with a column \( d \).
satisfying the various conditions described in (1). Therefore assume that 
\( \text{rank}(Z_i) = t_i - 1 \), for all \( i = 1, \ldots, n \).

(6) Finally, if the point \( L \in V \) is such that \( \text{rank}(W) < t_k - 2 \), one takes a row, say \( d \), in \( \text{Span}(Z_k) \setminus \text{Span}(W) \), and a row in \( W \), say \( c \), which is not among the basis rows of \( W \), and considers the points \( P = (p_{ab}) \), with entries in \( K[\alpha] \), defined as follows:

- \( p_{cb} = x_{cb} + \alpha \cdot d_b \) for all \( 1 \leq b \leq s \).
- \( p_{ab} = x_{ab} \) otherwise.

As always, \( L \) is in the closure of the points \( P \), for all \( \alpha \in K \). 

3.2. The Prime \( \mathcal{R}_k \)

Define \( \mathcal{J}_k \) to be the ideal over \( I \) generated by all \( t_k - 1 \times t_k - 1 \) minors in the first \( s_{k-1} + t_k - t_{k-1} \) columns of \( L \), for all \( k = 1, \ldots, n \), and let \( \mathcal{R}_k \) be its radical. Since \( \mathcal{J}_k \) is an ideal cogenerated by a fixed minor (it is a determinantal ideal of the same type as \( I \)), it is prime, and so \( \mathcal{J}_k = \mathcal{R}_k \). By [HT], one sees that its height is one more than that of \( I \).

3.3. The Prime \( \mathcal{R}_C \)

Recall that the corner \( C = (x, y) \) is such that \( s_{p-1} < y \leq s_p \). Define

\[ D = \{ X_{ab} \in L \mid a < x \text{ and } b < y \}, \]

that is, \( D = Y \cap Z \). Denote by \( F \) the set of \( t_p - 1 \times t_p - 1 \) minors in \( D \). Consider the ideal \( \mathcal{J}_C \) in \( K[L] \), generated by \( G \cup F \). Further, denote \( \sqrt[\mathcal{J}_C] \) by \( \mathcal{R}_C \) and the ring \( K[L]/\mathcal{R}_C \) by \( R \).

**Notation.** Denote by \( D_i \) the region \( L_i \cap D \), for \( i = 1, \ldots, p \).

3.3.1. **Lemma.** Let \( P_i = \{ x - t_i + 1, \ldots, x - 1 \} \), for \( i = 1, \ldots, p - 1 \), \( Q_1 = \{ 1, \ldots, t_1 - 1 \} \), and \( Q_i = Q_{i-1} \cup \{ s_1 + 1, \ldots, s_1 + t_2 - t_1, s_2 + 1, \ldots, s_2 + t_3 - t_2, \ldots, s_{i-1} + 1, \ldots, s_{i-1} + t_i - t_{i-1} \} \), for \( i = 2, \ldots, p - 1 \). Further, let \( P = \{ x - t_p + 2, \ldots, x - 1 \} \), \( Q = Q_{p-1} \cup \{ s_{p-1} + 1, \ldots, s_{p-1} + t_p - t_{p-1} - 1 \} \), \( P'_p = P \cup \{ x - t_p + 1 \} \), \( Q'_p = Q \cup \{ y \} \), \( P_p = P \cup \{ x \} \), \( Q_p = Q \cup \{ s_p + 1, \ldots, s_p + t_{p+1} - t_p, \ldots, s_{p-1} + 1, \ldots, s_{p-1} + t_i - t_{i-1} \} \), and \( P_i = \{ x - t_i + 1, \ldots, x - 1 \} \) and \( Q_i = Q_i \cup \{ y \} \), for \( i = p + 1, \ldots, n \), \( P_i = \{ x - t_i + 1, \ldots, x - 1 \} \) and \( Q_i = Q_i \cup \{ y \} \), for \( i = p + 1, \ldots, n \), \( P_i = \{ x - t_i + 1, \ldots, x - 1 \} \) and \( Q_i = Q_i \cup \{ y \} \), for \( i = p + 1, \ldots, n \). Let \( M = X_{P, Q} \) (a \( t_p - 2 \times t_p - 2 \) matrix of \( D \)), \( M'_i = X_{P_i, Q_i} \) (a \( t_i - 1 \times t_i - 1 \) matrix of \( Z_i \)), for \( i = 1, \ldots, n \), and \( M_p = X_{P'_p, Q'_p} \) (a \( t_{p+1} - 1 \times t_{p+1} - 1 \) matrix of \( Y \)). Let \( u = \text{det} \ M \), \( u_i = \text{det} \ M'_i \), \( v_p = \text{det} \ M_p \), and \( w = u_1 \cdots u_{p-1} \cdot u \cdots v \cdot u_p \cdots u_n \). Then \( R[1/w] \) is a domain.

**Proof.** Let \( L'_1 = L_1 \) and, for \( j = 2, \ldots, n \), let

\[ L'_j = \{ X_{ab} \in L_j \mid b \in Q_{j-1} \text{ or } s_{j-1} < b \leq s_j \}. \]
Also, for \( j = 1, \ldots, p - 1 \), let
\[
T_j := \{ X_{ab} \in L_j \mid a \in P_j \text{ or } b \in Q_j \}.
\]

As always, the proof begins by imitating the map \( \Phi \) defined in Lemma 2.3.1 to obtain maps
\[
\Phi_j : K[L_j] \left[ \begin{array}{c}
1 \\
u_1 & \cdots & u_j
\end{array} \right] \rightarrow K[T_j] \left[ \begin{array}{c}
1 \\
u_1 & \cdots & u_j
\end{array} \right]
\]
mapping the regions \( L_j \) onto the regions \( T_j \), via the inverses of \( M_j \), for \( j = 1, \ldots, p - 1 \). Now \( \ker(\Phi_{p-1} \cdots \Phi_1) \) contains all \( t_j \times t_j \) minors of \( L_j \) (and therefore \( L_j \)), for \( j = 1, \ldots, p - 1 \).

We now map the submatrix \( D' = L'_p \cap D \). Define
\[
S = \{ X_{ab} \in D' \mid a \in P \text{ or } b \in Q \}.
\]

Consider
\[
\Psi : K[D'] \left[ \begin{array}{c}
1 \\
u_1 & \cdots & u_{p-1} \cdot u
\end{array} \right] \rightarrow K[S] \left[ \begin{array}{c}
1 \\
u_1 & \cdots & u_{p-1} \cdot u
\end{array} \right]
\]
defined in the usual way, that is, the variables in \( S \) are mapped identically onto themselves, while the rest of the variables in \( D' \) are mapped via \( M^{-1} \), to obtain the desired linear relations. Therefore \( \ker(\Psi \cdot \Phi_{p-1} \cdots \Phi_1) \) contains every \( t_j \times t_j \) minor in \( L_j \) for \( j = 1, \ldots, p - 1 \) and every \( t_{p-1} \times t_{p-1} \) minor of \( D' \) (and therefore of \( D \), since the columns \( Q \) of \( M \) contain the columns \( Q_{p-1} \) of \( M_{p-1} \)).

The rest of the \( Y \) region, that is, the region \( Y \setminus (D \cup Y_{p-1}) \) shall now be mapped by \( \Phi'_p \), via the inverse of \( M'_p \). The details are: let \( Y' \) denote the submatrix of \( Y \) consisting of its last \( r - x + t_p - 1 \) rows, that is, the submatrix of \( Y \) whose first \( t_p - 1 \) rows are the rows \( P'_p \) of \( M'_p \), and define the free region
\[
T'_p = \{ X_{ab} \in Y' \mid a \in P'_p \text{ or } b \in Q'_p \}.
\]

\( \Phi'_p \) maps \( T'_p \) onto itself, and the rest of the variables in \( Y' \setminus Y_{p-1} \) in the usual way, via the inverse of \( M'_p \), to obtain the desired linear relations among the columns of \( Y' \).

The region above the ladder, that is, the region \( Z \setminus D \), remains to be mapped. Since \( M \subset M_p \subset M_{p+1} \subset \cdots \subset M_n \), the description of the maps \( \Phi_{p+1}, \ldots, \Phi_n \) for the regions \( Z_{p+1} \setminus D, Z_{p+2} \setminus Z_{p+1}, \ldots, Z_n \setminus Z_{n-1} \), respectively, are mere repetitions of those for \( \Phi_2, \ldots, \Phi_{p-1} \). Denoting \( L_j \cap Z \)
by $Z_j$, the required free regions are

$$T_j = \{X_{ab} \in Z_j | a \in P_j \text{ or } b \in Q_j\},$$

for $j = p, \ldots, n$.

Denote by $\Phi$ the composite map $\Phi_n \cdots \Phi_p \cdot \Phi' \cdot \Psi \cdot \Phi_{p-1} \cdots \Phi_1$ and let

$$T = S \cup T'_p \cup \bigcup_{i=1}^{n} T_i,$$

denote the free region. We have a surjection

$$\Phi: K[L] \left[ \frac{1}{w} \right] \to K[T] \left[ \frac{1}{w} \right].$$

Clearly $\ker(\Phi)$ contains the ideal $I_C$, and since $\text{Im}(\Phi)$ is a domain, it also contains $R_C$.

The ring $R[1/w]$ is the ring of regular functions of the principal open set of the variety of $R_C$ in $K[I]$ defined by the nonvanishing of the function $w$, while $\text{Im}(\Phi)$ is the ring of regular functions of the principal open set of the affine space $K[I]$, again defined by the nonvanishing of $w$. The map $\Phi$ induces a map at the variety level, which is easily seen to be onto, and $\Phi$ is an isomorphism.

Now $|T|$ is 1 less than the dimension of $K[L]/I$, therefore $\dim K[L]/R_C = \dim K[L]/I - 1$.

We maintain the notation of the above lemma. Let $V$ denote the variety of $R_C$. Let $L$ be a point of $V$ with nonzero $t_i - 1 \times t_i - 1$ minors in $Z_i$, say $w_i$, for $i = 1, \ldots, p - 1$, a nonzero $t_p - 2 \times t_p - 2$ minor in $D$, say $w$, nonzero $t_p - 1 \times t_p - 1$ minors $w'_p$ and $w_p$, in $Y_p$ and $Z_p$, respectively, and nonzero $t_i - 1 \times t_i - 1$ minors in $Z_i$, say $w_i$, for $i = p + 1, \ldots, n$. One can easily show (as we did in Section 3.1) that $L$ satisfies the hypothesis of Lemma 3.3.1, and is therefore contained in an irreducible open subset of $V$.

Let $U$ denote the open subset of $V$ consisting of points like $L$. $U$ is a finite union of irreducible open subsets whose intersection is nonempty (by Lemma 3.3.1) and the fact that the affine space $K[I]$ is irreducible. Therefore $U$ is itself irreducible. Let $\overline{U}$ denote the closure of $U$.

3.3.2. **Theorem.** $\overline{U} = V$. Therefore $R_C$ is prime.

**Proof.** Let $L = (x_{ab})$ be a point in $V \setminus U$.

(1) The region up to $Y_{p-1}$ is handled exactly as in (1)(a) of the proof of Theorem 3.1.2. Therefore assume that rank($Y_i$) = $t_i - 1$, for $i = 1, \ldots, p - 1$. 

(2) If \( \text{rank}(Y) \leq t_p - 2 \), we again mimic the procedure of (1a) of the proof of Theorem 3.1.2 to raise its rank, that is, locate the first column (call it \( d \)) in \( Z \setminus D \) which is not in \( \text{Span}(D) \) and add \( \alpha \cdot d \) to a nonbasis column of \( D \). Therefore assume that \( \text{rank}(Y) \geq t_p - 2 \).

(3) Now, using the methods of (3) of the proof of Theorem 3.1.2, we can assume that \( \text{rank}(Z_i) = t_i - 1 \), for all \( i = 1, \ldots, p - 1 \), and that \( \text{rank}(D) = t_p - 2 \).

If \( \text{rank}(Z_p) = t_p - 2 \) (= \( \text{rank}(D) \)), increasing its rank is easy: we merely add multiples of a suitably obtained column in \( \text{Span}(Z) \setminus \text{Span}(Z_p) \) to a column in \( Z_p \setminus D \). The ranks of \( Z_i \), for \( i > p \), are also increased in the same manner.

(4) It remains to be shown that if \( \text{rank}(Y) = t_p - 2 \) (see (2) above), then it is in the closure of points of \( V \) with \( \text{rank}(Y) = t_p - 1 \).

We seek a vector in \( K^{t_p - 1} \) that does not belong to the row span of \( Y \), but its restriction to its first \( s_{p-1} \) entries belongs to the row span of \( Y_{p-1} \).

Let \( R_{p-1} \) denote a set of \( t_{p-1} - 1 \) linearly independent rows in \( Y_{p-1} \) and let \( e_i \), for \( i = 1, \ldots, y-1 \), denote the standard basis for the row space of \( Y \). We claim that at least one of the vectors in \( \{e_{y-1}^1, \ldots, e_{y-1}^y\} \) does not belong to the row span of \( Y \). If that were not so, the vectors corresponding to the rows in \( R_{p-1} \) and the vectors \( e_{y-1}^1, \ldots, e_{y-1}^y \) would be linearly independent in \( Y \). But that would imply \( \text{rank}(Y) > t_p - 2 \), since the assumption made on the ladder was that \( y - 1 - s_{p-1} > t_p - t_p - 1 \).

We consider any one such vector, and denote it by \( d \). Let \( c \geq x \) be any row in \( Y \setminus D \). Consider the points \( P = (p_{ab}) \), with entries in \( K[\alpha] \), defined as follows:

- \( p_{cb} = x_{cb} + \alpha \cdot d_b \) for all \( 1 \leq b \leq y - 1 \).
- \( p_{ab} = x_{ab} \), otherwise.

The points \( P \), for \( \alpha \neq 0 \), have \( \text{rank}(Y) = t_p - 1 \) and are in \( V \). As always, \( L \) is in their closure.

4. PRIMES OF HEIGHT 1 FOR THE UPPER-LEFT CORNER LADDER

Consider the osoc ladder \( L \) with corner \( C = (x,y) \) defined by

\[
L = \{(a,b) \in X \mid a > x \text{ or } b > y\},
\]

where \( 1 \leq x \leq r \) and \( 1 \leq y \leq s \). Let \( p \) be such that \( s_{p-1} \leq y < s_p \). As before, we may assume that \( x \) and \( y \) are such that every indeterminate in \( L \) occurs in some nontrivial generator of \( I \). Therefore

\[
r - x \geq t_p \quad \text{and} \quad y \leq s - t_n.
\]
Assumption 1. We assume that the corner is sufficiently far from column $s_p$, that is, $s_p - y \geq t_p$. The other case is handled similarly.

Assumption 2. We assume that $r - x \geq t_p$. The results are identical in the other case.

Notation. Recall that $L_i = L \cap X_i$. Consider the following submatrices of $L$:

$$Y = \{(a, b) \in L \mid a > x\} \quad \text{and} \quad Z = \{(a, b) \in L \mid b > y\}.$$ 

Denote by $Y_i$ and $Z_i$ the submatrices $Y \cap L_i$ and $Z \cap L_i$, respectively, for $i = 1, \ldots, n$. Note that $Z_i = \emptyset$, for $i < p$. Further, define

$$D = \{(a, b) \in L \mid a > x \text{ and } b > y\},$$

that is, $D = Y \cap Z$. Also, let $D_i = D \cap L_i$, for $i = p, \ldots, n$.

4.1. The Prime $\mathcal{E}_k$

Let $k$ be arbitrary with $1 \leq k \leq n$. Define

$$W_k = \{x_{ab} \in L \mid x + 1 \leq a \leq x + t_k - 1 \text{ and } b \leq s_k\}$$

and denote by $F_k$ the set of $t_k - 1 \times t_k - 1$ minors in $W_k$. Denote by $\mathcal{J}_k$ the ideal in $K[L]$ generated by $G \cup F_k$ and denote its radical by $\mathcal{E}_k$. Further, let $R$ denote the ring $K[L]/\mathcal{E}_k$.

Notation. We denote $W_k$ by $W$ and denote the regions $L_i \cap W$ by $W_i$, for $i = 1, \ldots, n - k - 1$.

The proof of the primality of $\mathcal{E}_k$ resembles that of its namesake in Section 3, so we shall keep the details to a minimum.

Assumption. We assume that $k \geq p$, since the proof for $k < p$ follows the same line of argument and is somewhat simpler.

4.1.1. Lemma. Let $P = \{x + 1, \ldots, x + t_k - 1\}$, $P_i = \{r - t_i + 2, \ldots, r\}$, for $i = 1, \ldots, n$, $Q_1 = \{1, \ldots, t_1 - 1\}$, $Q_i = Q_1 \cup \{s_1 + 1, \ldots, s_1 + t_1 - 1, s_2 + 1, \ldots, s_2 + t_2 - t_1, \ldots, s_i + t_i - t_{i-1}, 1, \ldots, s_i + t_i - t_{i-1} + 1\}$, for $i = 2, \ldots, p - 1$, $Q_p = \{y + 1, \ldots, y + t_p - 1\}$, and $Q_i = Q_p \cup \{s_p + 1, \ldots, s_p + t_{p+1} - t_p, \ldots, s_{i-1} + 1, \ldots, s_{i-1} + t_i - t_{i-1}\}$, for $i = p + 1, \ldots, n$. Let $M = X_{P, Q_i}$ (a $t_k - 1 \times t_k - 1$ matrix of $D$), $v = \det M$, $M_i = X_{P, Q_i}$ (a $t_i - 1 \times t_i - 1$ matrix of $Y_i$), $u_i = \det M_i$, for $i = 1, \ldots, n$, and $u = u_1 \cdot u_2 \cdots u_n$. Then $R[1/u]$ is a domain.

Proof. Let $L'_1 = L_1$ and, for $i = 2, \ldots, n$, let

$$L_i = \{X_{ab} \in L_i \mid b \in Q_{i-1} \text{ or } s_{i-1} < b \leq s_i\}.$$
By repeated applications of Lemma 2.3.1, we map the region up to column $s_{p-1}$ and denote the corresponding maps by $\Phi_1, \ldots, \Phi_{p-1}$. Define, for $i = 1, \ldots, p - 1$,

$$V_i = \{ X_{ab} \in L'_i | a \in P_i \text{ and } b \notin Q_i \}. $$

We now map the regions $L'_p, \ldots, L'_{k-1}$, via the inverses of $M_p, \ldots, M_{k-1}$, in the usual way, and denote the maps by $\Phi_p, \ldots, \Phi_{k-1}$. Note that $Q_{p-1} \not\subset Q_p$, but $Q_i \subset Q_{i+1}$ for $i \geq p$. Therefore define, for $i = p, \ldots, n$, the free regions

$$T_i = \{ X_{ab} \in L_i | a \in P_i \text{ or } b \in Q_i \}. $$

Note that under $\Phi_p$, the columns in $Q_{p-1} \setminus Q_p$ are no longer free. Therefore define, for $i = p, \ldots, n$,

$$\Phi_i: K[L'_i] \left[ \frac{1}{u_1 \cdots u_p} \right] \longrightarrow K \left[ \bigcup_{i=1}^{p-1} V_i \cup T_p \right] \left[ \frac{1}{u_1 \cdots u_p} \right]. $$

We now map the subladder $L'_k$. Consider

$$\Phi_k: K[L'_k] \left[ \frac{1}{u_1 \cdots u_k} \right] \longrightarrow K[T_k] \left[ \frac{1}{u_1 \cdots u_k} \right] $$

defined exactly as in Lemma 3.1.1. That the kernel contains all $t_k - 1 \times t_k - 1$ minors in $W_k$ also follows exactly as it did in Lemma 3.1.1.

The region $L \setminus L_k$ is covered by the maps $\Phi_{k+1}, \ldots, \Phi_n$, via the inverses of $M_{k+1}, \ldots, M_n$ in the usual way, to obtain a surjection onto

$$K[\langle T \rangle] \left[ \frac{1}{u} \right], \quad \text{where } T = \bigcup_{i=1}^{p-1} V_i \cup \bigcup_{i=p}^{n} T_i. $$

We denote $\Phi_n \cdots \Phi_1$ by $\Phi$. Clearly, $\ker(\Phi)$ contains the ideal $\mathcal{J}_k$ and, since $\text{Im}(\Phi)$ is a domain, the ideal $\mathcal{E}_k$ too.

The ring $K[1/u]$ is the ring of regular functions of the principal open set of the variety of $\mathcal{E}_k$ in $K[L]$ defined by the nonvanishing of the function $u$, while $\text{Im}(\Phi)$ is the ring of regular functions of the principal open set of the variety of $\langle u \rangle$ in $K[T]$, again defined by the nonvanishing of $u$. The map $\Phi$ induces a map at the variety level, which is easily seen to be onto, and $\Phi$ is an isomorphism.

Again, by [HT] and the above lemma, $\dim K[L]/\mathcal{E}_k = \dim K[L]/I - 1$.

Let $V$ denote the variety of $\mathcal{E}_k$. Let $U \subset V$ consist of points $L$ with nonzero $t_i - 1 \times t_i - 1$ minors in $Y_i$, for $i = 1, \ldots, p - 1$, and nonzero $t_i - 1 \times t_i - 1$ minors in $D_i$ for $i = p, \ldots, n$. The arguments of Section 3 carry over, and $U$ is irreducible. Let $\overline{U}$ denote the closure of $U$. 
Notation. Let \( L \) be a point in \( K^{(L)} \). By \( Y_{n+1} \), we shall denote the submatrix of rank \( r - x \), obtained by augmenting the region \( Y \) of \( L \) with \( r - x - \text{rank}(Y) \) linearly independent columns. Similarly, by \( Z_{n+1} \), we shall denote the matrix obtained by augmenting the region \( Z \) of \( L \) with linearly independent columns so as to achieve a rank of \( \min(r, s - y) \). Recall that \( t_n \leq \min(r, s - y) \).

4.1.2. Theorem. \( \bar{U} = V \). Therefore \( \mathcal{E}_k \) is prime.

Proof. Let \( L \) be a point in \( V \setminus U \).

(1) The region up to \( Y_{p-1} \) is handled exactly as in (1) of the proof of Theorem 3.1.2, and we may assume that \( \text{rank}(Y) = t_i - 1 \), for all \( i = 1, \ldots, p - 1 \).

We shall now show that if \( L \) is such that \( \text{rank}(D_p) < \text{rank}(Y) \), then it is in the closure of points of \( V \) with a higher rank for \( D_p \), while the ranks of the other regions are not less than the ranks of the corresponding regions for \( L \).

(2) Let \( \text{rank}(D_p) < \text{rank}(Y_p) \). There are several cases to consider.

(a) Suppose \( \text{rank}(Z_i) < t_i - 1 \), for all \( i = p, \ldots, n \). Consider a column, say \( d \), in \( \text{Span}(Y_p) \setminus \text{Span}(D_p) \) and a column in \( D_p \), say \( c \), not among the basis columns of \( D_p \). The points obtained by adding nonzero multiples of \( d \) to \( c \) all have \( \text{rank}(D_p) \) greater than that for the point \( L \). These points are all in \( V \) as the ranks of the regions \( Z_i \), for \( i = p, \ldots, n \), can increase by at most 1. Of course, the ranks of \( Y_i \), for \( i = 1, \ldots, n \), remain unaffected.

As for \( \text{rank}(W) \), since we have assumed \( k \geq p \), it too remains unaltered.

As always, \( L \) is in the closure of these points.

(b) Let \( w \leq n \) be the least index such that \( \text{rank}(Z_w) = t_w - 1 \). Two cases:

(b1) If \( \text{rank}(D_p) = \text{rank}(Z_w) \), let \( C_w \) be a set of basic columns of \( D_p \). This set is also a basis for the region \( Z_w \). Since \( \text{rank}(D_p) = t_w - 1 \), which is the maximum possible, we have \( \text{Span}(Y_w) = \text{Span}(D_p) \). Therefore any column of \( Y_w \), in particular, a column of \( Y_p \) which does not belong to \( \text{Span}(D_p) \), can be expressed as a linear combination of the columns in \( C_w \). We denote the column in \( \text{Span}(Z_w) \) obtained by this linear combination by \( d \). That is, \( d \) restricted to its last \( r - x \) rows is equal to the said column of \( Y_p \setminus \text{Span}(D_p) \). Let \( c \) be a column in \( D_p \), not among its basic columns. Consider the ladder \( P = (p_{ab}) \), with entries in \( K[\alpha] \), defined as follows:

- \( p_{ac} = x_{ac} + \alpha \cdot d_a \), for all \( a = 1, \ldots, r \).
- \( p_{ab} = x_{ab} \), otherwise.

For \( \alpha \in K \setminus \{0\} \), the points \( P \) are such that the ranks for the region \( D_p \) are 1 more than that for the point \( L \). Further, ranks of the regions \( Y_i \), for
For all 
\[
\alpha \in K \setminus \{0\}, \text{ the points } P \text{ are such that } \text{rank}(D_i) = 1 \text{ more than for } L. \text{ Since we have added a multiple of row } d \text{ to the rows of } D, \text{ the ranks of the regions } Z_i, \text{ for } i = p, \ldots, n, \text{ are unaltered. Further, since we have added a column of } Y_j \text{ to columns of } D, \text{ rank}(W) \text{ and the ranks of } Y_j, \text{ for } i = p, \ldots, n, \text{ remain unaffected (the ranks of } D_i, \text{ for } i = p + 1, \ldots, n, \text{ could, of course, increase). Therefore, these points belong to } V. \text{ We conclude that } L \text{ is in their closure.}
\]

(b2.2) Suppose \( v > p \). We shall show that \( L \) is in the closure of points with \( \text{rank}(D_p) < \text{rank}(Z_p) \), so that they satisfy the conditions of (b2.1).

Let \( C_i \) be a set of basic columns for \( D_p \) that includes a basis \( C_{p-1} \) for \( D_{p-1} \). Note that the columns of \( C_{p-1} \), when considered as columns of \( Z_i \), form a basis for \( Z_{p-1} \). Since \( \text{rank}(Z_{p-1}) > \text{rank}(D_{p-1}) \), there exists a column in \( Z_{p-1} \) which does not belong to \( \text{Span}(C_{p-1}) \). Now this column restricted to its last \( r - x \) rows is in the linear span of the columns in \( C_{p-1} \) similarly restricted. Therefore, we can obtain a column in \( \text{Span}(Z_{p-1}) \setminus \text{Span}(C_{p-1}) \), which has zeros for its last \( r - x \) entries. Call this column \( d \). Adding nonzero multiples of \( d \) to a nonbasic column of \( Z_{p-1} \) increases the ranks of the regions \( Z_{p-1}, \ldots, Z_{p-1} \), while leaving the ranks of all the other regions unchanged. As always, \( L \) is in their closure.

Therefore we may assume that \( \text{rank}(Y_{p-1}) = t_i - 1 \), for \( i = 1, \ldots, p - 1 \), and that \( \text{rank}(D_{p-1}) = \text{rank}(Y_{p-1}) \).

Let \( R_p \) and \( C_p \) be sets of basic rows and columns of \( D_p \). Therefore every column of \( Y_{p-1} \) is contained in \( \text{Span}(C_p) \). We shall now show that \( L \) is in the closure of points with a higher rank for the region \( D_j \), for some \( j \geq p \). These points must however be such that the columns \( C_{p,j} \) of \( D_j \) are the same as that for \( L \), lest the ranks of the \( Y_i \) regions, for \( i < p \), be affected.
So let \( w \geq p \) be the least integer such that \( \text{rank}(D_w) < \text{rank}(Z_w) \). We consider a row in \( \text{Span}(Z_w) \setminus \text{Span}(D_w) \), and zero out the entries corresponding to the columns in \( C_p \) (by adding an appropriate linear combination of the rows in \( R_p \)). The row thus obtained continues to belong to \( \text{Span}(Z_w) \setminus \text{Span}(D_w) \). Adding nonzero multiples of it to a nonbasic row of \( D_w \), not among the rows of \( W \), gives points in \( V \) with a higher rank for the regions \( D_w \) and \( Y_w \), while the ranks of the other regions do not decrease.

If however \( \text{rank}(D_i) = \text{rank}(Z_i) \) for all \( i = p, \ldots, n \), let \( w \geq p \) be the least integer such that \( \text{rank}(D_w) < t_w - 1 \). In this case we obtain a row from \( K^{(i-1)} \setminus \text{Span}(D_w) \). The rest of the argument is the same as that in the previous paragraph.

### 4.2. The Prime \( \mathcal{P}_k \)

For \( k = 1, \ldots, p - 1 \), we define the ideal \( \mathcal{J}_k = \mathcal{P}_k \) exactly as we did in Section 3.2, that is, the ideal over \( I \) generated by the \( t_k - 1 \times t_k - 1 \) minors in the first \( s_{k-1} + t_k - t_{k-1} \) columns of \( L \). This ideal too, is a height 1 prime in the ring \( K[L]/I \). Further, define

\[
O_p = \{ X_{ab} \in L \mid y + 1 \leq b \leq y + t_p - 1 \}
\]

and, for \( p < i \leq n \), let

\[
O_i = \{ X_{ab} \in L \mid y + 1 \leq b \leq s_{i-1} + t_i - t_{i-1} \}.
\]

Let \( k \) be arbitrary with \( p \leq k \leq n \), and denote by \( E_k \) the set of \( t_k - 1 \times t_k - 1 \) minors in \( O_k \). Denote by \( \mathcal{J}_k \) the ideal in \( K[L] \) generated by \( G \cup E_k \) and denote its the radical by \( \mathcal{P}_k \). Further, let \( R \) denote the ring \( K[L]/\mathcal{P}_k \).

#### 4.2.1. Lemma

Let \( P_i = (r - t_i + 2, \ldots, r) \), for \( i = 1, \ldots, n \), \( Q_1 = \{ 1, \ldots, t_1 - 1 \} \), \( Q_i = Q_1 \cup \{ s_1 + 1, \ldots, s_1 + t_2 - t_1, \ldots, s_{i-1} + 1, \ldots, s_{i-1} + t_i - t_{i-1} \} \), for \( i = 2, \ldots, p - 1 \), \( Q_p = \{ s_p - t_p + 2, \ldots, s_p \} \), \( Q_i = Q_p \cup \{ s_{p+1} - t_{p+1} + t_p + 1, \ldots, s_p + t_i - t_{i-1} + 1, \ldots, s_i \} \), for \( i = p + 1, \ldots, n \).

If \( k = p \), let \( Q = \{ y + 1, \ldots, y + t_p - 1 \} \) and if \( k > p \), let \( Q = Q_{k-1} \cup \{ s_{k-1} + 1, \ldots, s_{k-1} + t_k - t_{k-1} \} \). Let \( M = X_{P_i, Q} \) (a \( t_k - 1 \times t_k - 1 \) matrix of \( D \)), \( v = \det M, M_i = X_{P_i, Q} \) (a \( t_i - 1 \times t_i - 1 \) matrix of \( Y \)), \( u_i = \det M_i \), for \( i = 1, \ldots, n \), and \( u = u_1 \cdot u_2 \cdot \ldots \cdot u_n \). Then \( R[1/u] \) is a domain.

**Proof.** The maps \( \Phi_1, \ldots, \Phi_n \) are identical to those defined in Lemma 4.1.1. Since we quotient by \( v \), one shows as in Lemma 4.1.1, that the kernel of \( \Phi_k \) contains all \( t_k - 1 \times t_k - 1 \) minors in the region \( O_k \). We omit the details.

A gain, by [HT] and the above lemma, \( \dim K[L]/\mathcal{P}_k = \dim K[L]/I - 1 \).
Let $V$ denote the variety of $\mathcal{R}_k$. Let $U \subset V$ consist of points $L$ with nonzero $t_i - 1 \times t_i - 1$ minors in $Y_i$, for $i = 1, \ldots, p - 1$, and nonzero $t_i - 2 \times t_i - 2$ minors in $D_i$, for $i = p, \ldots, n$. The arguments of Section 3 carry over, and $U$ is irreducible. Let $\overline{U}$ denote the closure of $U$.

4.2.2. Theorem. $\overline{U} = V$. Therefore $\mathcal{R}_k$ is prime.

Proof. Similar to that of Theorem 4.1.2, except that in this case $\text{rank}(O_k)$ must not exceed $t_k - 2$. Now this bound is easily maintained, since $Z_k$ has more columns than $O_k$. Therefore we can always obtain a column in $Z_k \setminus O_k$ for increasing the rank of $Z_k$ to $t_k - 1$. $\Box$

4.3. The Prime $\mathcal{R}_C$

Let $F$ denote the set of $t_i - 1 \times t_i - 1$ minors in $D_i$, for $i = p, \ldots, n$. Consider the ideal $\mathcal{R}_C$ in $K[L]$, generated by $G \cup F$. We denote $\sqrt{\mathcal{R}_C}$ by $\mathcal{R}_C$ and denote the ring $K[L]/\mathcal{R}_C$ by $R$.

4.3.1. Lemma. Let $P_i = \{x + 1, \ldots, x + t_i - 1\}$, for $i = 1, \ldots, p - 1$, $Q_1 = \{1, \ldots, t_1 - 1\}$, $Q_i = Q_{i-1} \cup \{s_1 + 1, \ldots, s_1 + t_1, \ldots, s_i + 1, \ldots, s_i + t_i - 1\}$, for $i = 2, \ldots, p - 1$, $P_p = \{x + 1, \ldots, x + t_p - 2\}$, and $Q_p = \{y + 1, \ldots, y + t_p - 2\}$. Further, let $P' = P_p \cup \{x + t_p - 1\}$, $Q' = Q_p \cup \{y\}$, $P_i = \{x + 1, \ldots, x + t_i - 2\}$, and $Q_i = Q_{i-1} \cup \{s_{i-1} + 1, \ldots, s_{i-1} + t_i - 1\}$, for $i = p + 1, \ldots, n$.

Also, let $A_i = P_i \cup \{x\} \text{ and } B_i = Q_i \cup \{y + t_p - 1\}$, for $i = p, \ldots, n$. Finally, let $M_i = X_{P_i, Q_i} (a t_i - 1 \times t_i - 1 \text{ matrix of } Y_i)$, $u_i = \det M_i$, for $i = 1, \ldots, p - 1$, $M'_i = X_{P'_i, Q'_i} (a t_i - 2 \times t_i - 2 \text{ matrix of } D_i)$, $v_i = \det M'_i$, for $i = p, \ldots, n$, $M''_p = X_{P''_p, Q''_p} (a t_p - 1 \times t_p - 1 \text{ minor of } Y_p)$, $w_p = \det M''_p$, $N_i = X_{A_i, B_i} (a t_i - 1 \times t_i - 1 \text{ matrix of } Z_i)$, $w_i = \det N_i$, for $i = p, \ldots, n$, and $u = u_1 \cdots u_p \cdot v_p \cdots v_n \cdot w_p \cdots w_n$. Then $R[1/u]$ is a domain.

Proof. The maps for the region $Y_{p-1}$ (that is, $\Phi_1, \ldots, \Phi_{p-1}$) are identical to those of Lemmas 4.1.1 and 4.2.1. The regions $D_i$ are mapped via the inverses of $M_i$ and the regions $Z_i \setminus D_i$ are mapped via the inverses of $N_i$, for $i = p, \ldots, n$. Last, the variables in $(Y_p \setminus Y_{p-1}) \cup Q_{p-1}$ which are not in $D_p$ are mapped via the inverse of $M''_p$. The image is a localization of a polynomial ring over $K$. Its Krull dimension (equal to the cardinality of the free region) is easily computed to be 1 less than that of the ring $K[L]/I$. The induced map at the variety level can again be shown to be onto. We omit the details. $\Box$

Let $V$ denote the variety of $\mathcal{R}_C$. Let $U \subset V$ consist of points $L$ with nonzero $t_i - 1 \times t_i - 1$ minors in $Y_i$, for $i = 1, \ldots, p$, nonzero $t_i - 2 \times t_i - 2$ minors in $D_i$, for $i = p, \ldots, n$, and nonzero $t_i - 1 \times t_i - 1$ minors in $Z_i$, for $i = p, \ldots, n$. The arguments of Section 3 carry over and $U$ is irreducible. Let $\overline{U}$ denote the closure of $U$. 
4.3.2. Theorem. $\overline{U} = V$. Therefore $\mathcal{R}_C$ is prime.

Proof. Let $L$ be a point in $V \setminus U$.

(1) The region up to $Y_{p-1}$ is handled exactly as in (1) of Lemma 4.1.1, and we may assume that $\text{rank}(Y_i) = t_i - 1$, for all $i = 1, \ldots, p - 1$. We shall now show that whenever a region $D_i$ of $L$ has rank less than $t_i - 2$, then it is in the closure of points with a higher rank for that region, and with the regions $Y_i$, $Z_i$, and $D_i$ having the same rank as the corresponding regions for $L$, for all $i < i$.

(2) Let $q \geq n$ be the least integer such that $\text{rank}(D_q) < t_q - 2$. Now $L$ satisfies one of the following conditions:

(a) $\text{rank}(Z_q) < t_q - 1$ and $\text{rank}(D_q) < t_q - 2$, for all $i = q, \ldots, n$.

(b) There exist $\gamma \leq n$ and $\delta$ such that $\text{rank}(D_q) = t_q - 2$, $\text{rank}(D_q) < t_q - 2$, for all $q \leq i < \gamma$, $\text{rank}(Z_q) = t_q - 1$, $\text{rank}(Z_q) < t_q - 1$, for all $q \leq i < \delta$, and $\gamma \leq \delta$.

(c) There exist $\gamma$ and $\delta$ such that $\text{rank}(D_q) = t_q - 2$, $\text{rank}(D_q) < t_q - 2$, for all $q \leq i < \gamma$, $\text{rank}(Z_q) = t_q - 1$, $\text{rank}(Z_q) < t_q - 1$, for all $q \leq i < \delta$, and $\delta < \gamma$.

If (a) is satisfied, we obtain a column in $\text{Span}(Y_{n+1}) \setminus \text{Span}(D_q)$ as follows:

(a1) If $\text{rank}(D_q) < \text{rank}(Y_q)$, then pick any column in $\text{Span}(Y_q) \setminus \text{Span}(D_q)$.

(a2) If $\text{rank}(D_q) = \text{rank}(Y_q)$ (therefore $\text{rank}(D_q) = \text{rank}(D_q)$, for all $i = q + 1, \ldots, n$) and $\text{rank}(D_q) < \text{rank}(D_q)$, then let $u$ be the smallest index such that $\text{rank}(D_q) < \text{rank}(D_q)$, and pick any column from $\text{Span}(D_u) \setminus \text{Span}(D_q)$.

(a3) If neither (a1) nor (a2) is satisfied, pick any column from $\text{Span}(Y_{n+1}) \setminus \text{Span}(D_q)$.

Adding nonzero multiples of the chosen column to a nonbasic column of $D_q$ gives points with the region $D_q$ having a rank 1 greater than that for $L$. These points are in $V$ as the ranks of $D_i$ and $Z_i$, for all $i = q, \ldots, n$, can increase by at most 1.

If (b) is satisfied, let $w \in \{q + 1, \ldots, n\}$ be the least integer such that $\text{rank}(D_w) = t_w - 2$. By assumption, $\text{rank}(Z_i) < t_i - 1$, for all $i = q, \ldots, w - 1$. There are two cases.

(b1) If $\text{rank}(Y_i) < t_i - 1$, for all $i = q, \ldots, w - 1$, our task is easy. Consider a column, say $d$, in $D_w \setminus \text{Span}(D_q)$ and a column, say $c$, in $D_q$, not among its basic columns. We add nonzero multiples of column $d$ of $Z_w$ to column $c$ of $Z_q$, that is, we consider points $P = (p_{ab})$, with entries in
$K[\alpha]$, defined as follows:

- $p_{ac} = x_{ac} + \alpha \cdot d_{ac}$, for all $a = 1, \ldots, r$.
- $p_{ab} = x_{ab}$, otherwise.

For $\alpha \in K \setminus \{0\}$, the points $P$ are such that the ranks for the region $D_q$ are one more than that for the point $L$. The ranks of the regions $Y_i$, $Z_i$, and $D_i$, for $i = q, \ldots, w - 1$, can increase by at most 1, so the points $P$ are in $V$. A gain, $L$ is in their closure.

(b2) Let $v$ be the smallest integer in $\{q, \ldots, w - 1\}$ such that $\text{rank}(Y_v) = t_v - 1$. Since $\text{rank}(D_v) = t_v - 2$ and $\text{rank}(Y_v) = t_v - 2$ or $t_v - 1$, the rank–nullity theorem implies that the rank of $\text{Span}(Y_v) \cap \text{Span}(D_v)$ is at least $t_v - 2$. Therefore we can obtain a column from this intersection which does not belong to $\text{Span}(D_v)$. A fortiori, it does not belong to $\text{Span}(D_v)$. Call it $d'$. It can be expressed as a linear combination of the basic columns of $D_w$. Extend this linear combination to the whole of $Z$ and denote the column in $\text{Span}(Z_w)$ obtained by this linear combination by $d$. That is, $d$ restricted to its last $r - x$ rows is equal to $d'$. Let $c'$ be a column in $D_q \setminus D_{q-1}$, not among its basic columns. Column $c'$ considered as a column of $Z$ shall be denoted by $c$. Adding nonzero multiples of $d$ to column $c$ gives points with a higher rank for the regions $D_i$, for all $i = q, \ldots, v - 1$. Since $d' \in \text{Span}(Y_v)$, rank($Y_v$) could increase, for $i = 1, \ldots, v - 1$. Further, the ranks of the regions $Z_i$ and $D_i$, for $i = q, \ldots, w - 1$, can increase by at most 1. The points are therefore in $V$.

Finally, if (c) is satisfied, let $w$ be the least integer $i \in \{q, \ldots, n\}$ such that $\text{rank}(Z_i) = t_i - 1$. By assumption, $\text{rank}(D_i) < t_i - 2$, for all $i = q, \ldots, w$. In fact, $\text{rank}(D_i) < t_i - 2$, for all $i = q, \ldots, n$. For if that isn’t so, then there exists $v > w$ such that $\text{rank}(D_v) = t_v - 2$. Now there exist two linearly independent rows in $Z_w$ which do not belong to the row span of $D_w$. These two rows obviously continue to be linearly independent of the row span of $D_v$, when extended to the region $Z_w$. Since $\text{rank}(D_v) = t_v - 2$, we get $\text{rank}(Z_i) \geq t_i$. But $L$ is a point in $V$.

If either $\text{rank}(D_q) = \text{rank}(Z_q)$ or $\text{rank}(Z_{q-1}) = \text{rank}(Z_q)$ (if $p < q$), we consider a column in $\text{Span}(Z_w) \setminus \text{Span}(Z_q)$, with zeros for its last $r - x$ entries (such a column exists as $\text{rank}(D_v) < \text{rank}(Z_w)$), and add nonzero multiples of it to a nonbasic column of $Z_q$ to obtain points with a higher rank for the regions $Z_i$, for $i = q, \ldots, w - 1$, while leaving the ranks of the $Y_i$ and $D_i$ regions unchanged, for all $i = 1, \ldots, n$.

Now let $d'$ be a row in the row span of $Z_q$, which does not belong to the row span of $D_q$, and, if $p < q$, with zeros for its first $s_{q-1} - y$ entries, that is, in the region $Z_{q-1}$. A gain, such a row is available since $\text{rank}(Z_q)$ is greater than both $\text{rank}(D_q)$ and $\text{rank}(Z_{q-1})$. 

(c1) If \( \text{rank}(Y_i) < t_i - 1 \), for all \( q \leq i \leq n \), we choose a row in \( D_{q^t} \), not among its basic rows, and denote it by \( c \). Adding nonzero multiples of the whole of row \( d \) (that is, all \( s - y \) columns) to row \( c \) gives points \( P \) with a higher rank for the region \( D_q \). The ranks of the regions \( Z_i \) remain unaffected, while the ranks of the regions \( Y_i \) and \( D_i \) can increase by at most 1 for all \( i = q, \ldots, n \). These points are therefore in \( V \), and \( L \) is in their closure.

(c2) Let \( v \) be the least integer in \( \{q, \ldots, n\} \) such that \( \text{rank}(Y_v) = t_v - 1 \). Since \( \text{rank}(D_v) < \text{rank}(Y_v) \), there exists a column in \( Y_v \setminus D_v \) (= \( Y \setminus D \)), which does not belong to \( \text{Span}(D_v) \). Call it \( c \) and consider the ladder \( P = (p_{ab}) \), with entries in \( K[\alpha] \), defined as follows:

- \( p_{ab} = x_{ab} + \alpha \cdot c_{a} \cdot d_{b} \) for all \( (a, b) \) such that \( x + 1 \leq a \leq r \) and \( y + 1 \leq b \leq s \).
- \( p_{ab} = x_{ab} \), otherwise.

For all \( \alpha \in K \setminus \{0\} \), the points \( P \) are such that \( \text{rank}(D_v) \) is 1 more than for \( L \). Since we have added a multiple of row \( d \) to the rows of \( D \), the ranks of the regions \( Z_i \), for \( i = p, \ldots, n \) are unaltered. The ranks of the regions \( D_i \), for \( i = q, \ldots, n \), can increase by at most 1, while the ranks of the \( Y_i \) regions, for all \( i = 1, \ldots, n \), remain unchanged, since column \( c \) was obtained from \( Y \setminus D \). These points are therefore in \( V \), and \( L \) is in their closure.

Therefore we may assume that \( L \) is such that \( \text{rank}(Y_i) = t_i - 1 \), for all \( i = 1, \ldots, p - 1 \), and that \( \text{rank}(D_i) = t_i - 2 \), for all \( i = p, \ldots, n \). We now show that it is in the closure of points of \( V \) with the maximum permissible rank for the regions \( Z_i \) and \( Y_i \), for all \( i = p, \ldots, n \). We first handle the region \( Z \).

(3) If \( \text{rank}(Z_i) = \text{rank}(D_i) \), for all \( i = p, \ldots, n \), then obtain a vector from \( K^{s-y} \) which is not in the row span of \( Z \) and add nonzero multiples of it to a nonbasic row of \( Z \setminus D \), thereby increasing the rank of \( Z \), while leaving the regions \( Y_i \) and \( D_i \) unaltered, for any \( i \).

So let \( w \leq n \) be the least integer such that \( \text{rank}(D_w) < \text{rank}(Z_w) \). Since \( \text{rank}(Z_{w-1}) = \text{rank}(D_{w-1}) \), \( \text{rank}(D_{w+1}) < \text{rank}(D_w) \), and \( \text{rank}(D_w) < \) \( \text{rank}(Z_w) \), there exists a column in \( \text{Span}(Z_w) \setminus \text{Span}(Z_{w-1}) \), which has zeros for its last \( r - x \) rows (that is, when restricted to \( D \), it is the zero column). Adding nonzero multiples of this column to a nonbasic column of \( Z_p \) gives points \( P \) with a higher rank for the region \( Z_i \), for all \( i = p, \ldots, w - 1 \). The regions \( Y_i \) and \( D_i \) clearly remain unaltered, for all \( i \). Therefore these points are in \( V \), and, of course, \( L \) is in their closure.

Therefore assume that \( L \) is such that \( \text{rank}(D_i) = t_i - 2 \) and \( \text{rank}(Z_i) = t_i - 1 \), for all \( i = p, \ldots, n \). It remains to show that if there exists a \( j \in \{p, \ldots, n\} \) such that \( \text{rank}(Y_j) < t_j - 1 \), then it is in the closure of points...
of $V$ with a higher rank for that region, while no region has a lower rank than the corresponding region for $L$.

(4) If $L$ is such that rank($D_i$) = rank($Y_i$) for all $i = p, \ldots, n$, then simply obtain a column from $K^{(r-\epsilon)}$, which does not belong to the column span of $Y_i$ and add nonzero multiples of it to a nonbasic column in $(Y_p \setminus \{p+1\}) \setminus D_p$ (that is, to a column in $(s_p + 1, \ldots, y)$), thereby obtaining points with rank($Y_i$) = $t_i - 1$, for all $i = p, \ldots, n$, while all other regions are left unaffected.

Let $w \in \{p, \ldots, n\}$ be such that rank($Y_w$) = $t_w - 1$. Therefore, there exists a column in $Y_w \setminus D_w$ (that is, $Y \setminus D$) which does not belong to the column span of $D_w$. A fortiori, it does not belong to the column span of $D_i$, for all $i = p, \ldots, w - 1$ and so rank($Y_i$) = $t_i - 1$, for all $i = p, \ldots, w - 1$. Therefore, there exists an index $v \geq p$ such that rank($Y_v$) = rank($D_v$) = $t_v - 2$ and rank($Y_i$) = $t_i - 1$, for all $i = p, \ldots, v - 1$. Of course, this implies that rank($Y_i$) = $t_i - 2$, for all $i = v + 1, \ldots, n$.

Let $C_i$ be a set of basic columns for $D_i$, for $i = p, \ldots, n$. As always, assume that $C_p \subset \cdots \subset C_n$. Further, let $c$ be a column in $Y_{v-1} \setminus D_{v-1}$, not belonging to span($D_{v-1}$). Therefore $C_v \cup \{c\}$ is a basis for $Y_i$, for all $i = p, \ldots, v - 1$. Now since $c \in $ Span($C_v$), $c$ can be expressed as a linear combination of the columns of $C_v$. Since $c \notin $ Span($C_{v-1}$), there exists a column in $C_v \setminus C_{v-1}$, which has a nonzero coefficient in the linear expression for $c$. Call this column $k$.

Now every column in $D_v \setminus D_{v-1}$ ($D_{v-1}$ = $\emptyset$, of course) can be uniquely expressed as a linear combination of the columns in $C_v$, for all $i = v, \ldots, n$. For any such column $b$, let $b_k$ denote the coefficient of column $k$ in this linear expression.

Finally, let $d'$ be a column in $K^{(r-\epsilon)} \setminus $ Span($Y$). Prefix $d'$ with $x$ zeros to obtain a column in $K'$, with its first $x$ entries being zero. Call this column $d$ and consider the ladder $P = (p_{ab})$, with entries in $K[\alpha]$, defined as follows:

- $p_{ab} = x_{ab}$, for all $b \leq s_{v-1}$.
- $p_{ab} = x_{ab} + b_k \cdot \alpha \cdot d_{s_v}$, for all $b > s_{v-1}$.

For all $\alpha \in K \setminus \{0\}$, the points $P$ are such that rank($Y_i$) is 1 more than for $L$, for all $i = v, \ldots, n$, since column $d$ does not belong to span($Y$). Since we have maintained the linear dependence relations for the points $P$, just as they were in $L$, of every nonbasic column of $D$, the columns in $C_i$ continue to be bases for the regions $D_i$, for all $i = p, \ldots, n$. For the same reason, the ranks of $Z_i$ are also unaltered. These points are therefore in $V$, and $L$ is in their closure. \(\blacksquare\)
5. THE DIVISOR CLASS GROUP FOR THE LOWER-RIGHT CORNER LADDER

We shall be applying the corollary to Nagata's theorem (Corollary 2.2.3) to the lower-right corner ladder of Section 3. Henceforth, \( R \) shall denote the ring \( K[L]/I \).

5.1. Lemma. Let \( P_i = \{1, \ldots, t_i - 1\} \), for \( i = 1, \ldots, n \), \( Q_i = \{1, \ldots, t_i - 1\} \) and \( Q_i = Q_1 \cup \{s_1 + 1, \ldots, s_1 + t_2 - t_1, \ldots, s_{i-1} + 1, \ldots, s_{i-1} + t_i - t_{i-1}\} \), for \( i = 2, \ldots, n \). Let \( M_i = X_{P_i, Q_i} \) (a \( t_i \times t_i - 1 \) matrix of \( Z \)), \( u_i = \det M_i \), for \( i = 1, \ldots, n \), and \( u = u_1 \cdot u_2 \cdots u_n \). Then \( R[1/u] \) is isomorphic to a localization of a polynomial ring over \( K \) and is therefore regular.

Proof. The isomorphism has been defined several times in the previous sections and we omit it.

For future use, we describe the polynomial ring obtained in the above isomorphism in more detail. Let \( Q_0 = \emptyset \). For \( i = 1, \ldots, n \), let
\[
L_i = \{ X_{ab} \in L_i \mid b \in Q_{i-1} \text{ or } s_{i-1} < b \leq s_i \}
\]
and
\[
T_i = \{ X_{ab} \in L_i \mid a \in P_i \text{ or } b \in Q_i \}.
\]
Then \( R[1/u] \) is isomorphic to \( K[T][1/u] \), where \( T = \bigcup_{i=1}^n T_i \). Of course, \( u_i \), being the determinant of a matrix of indeterminates, is irreducible in \( K[T] \), for all \( i = 1, \ldots, n \). The corollary to Nagata's theorem is therefore applicable.

The notation of the above lemma shall be retained for the entire section.

5.2. Lemma. Let \( A \) be a matrix with entries in \( K \), such that \( \text{rank}(A) = t \). Let \( M_1 = A_{P_1, Q_1} \) and \( M_2 = A_{P_2, Q_2} \) be two square submatrices of size \( t \), with nonzero determinants. Then \( N_1 = A_{P_1, Q_2} \) and \( N_2 = A_{P_2, Q_1} \) also has nonzero determinants.

Proof. Since \( \text{rank}(A) = t \), the columns in \( Q_1 \), as well as in \( Q_2 \), are bases for the column space of \( A \). Therefore the columns of \( N_1 \) can be expressed as linear combinations of the columns of \( M_1 \), and conversely. Since \( \det(M_1) \) is nonzero, so is \( \det(N_1) \). Similarly for \( M_2 \) and \( N_2 \).

5.3. Lemma. (a) Let \( 1 \leq i \leq p - 1 \). Then
\[
\mathcal{I}_1 \cdot \mathcal{I}_2 \cdots \mathcal{I}_i \cdot \mathcal{J}_i \subseteq \sqrt{I + u_i}.
\]
(b) Let \( p \leq i \leq n \). Then
\[
\mathcal{I}_1 \cdot \mathcal{I}_2 \cdots \mathcal{I}_i \cdot \mathcal{J}_i \cdot \mathcal{K}_i \subseteq \sqrt{I + u_i}.
\]
Proof. (a) We shall show that the generators of \( \mathcal{I}_1 \cdot \mathcal{I}_2 \cdots \mathcal{I}_i \mathcal{I}_j \) vanish whenever \( I + u_i \) does. Assume the contrary. Therefore, there is a point in the variety of \( I \) where \( u_i \) vanishes, but there exist nonvanishing \( t_j - 1 \times t_j - 1 \) minors \( v_j \) in \( \mathcal{I}_j \), for \( j = 1, \ldots, i \), and a nonvanishing \( t_i - 1 \times t_i - 1 \) minor \( w_i \) in \( \mathcal{I}_j \).

Since we are in the variety of \( I \), the columns of \( v_i \) are a basis for the column space of the sublatter \( L_i \). Further, by the definition of the ideals \( \mathcal{I}_j \), the columns of \( v_j \) are among the first \( s_j - 1 + t_j - t_j - 1 \) columns of \( L_i \) for \( j = 2, \ldots, i \). But since every \( t_j - 1 \times t_j - 1 \) minor in the first \( s_j - 1 \) columns is zero, the columns of \( v_j \) must include the columns \( s_j - 1 + 1, \ldots, s_j - 1 + t_j - t_j - 1 \), and therefore contain precisely \( t_j - 1 - 1 \) columns from among the first \( s_j - 1 \) columns of \( L_i \). Since the column indices of \( v_i \) are \( 1, \ldots, t_i - 1 \), it is easy to see that the columns \( 1, \ldots, t_i - 1, s_i + 1, \ldots, s_i + t_2 - t_2, \ldots, s_i - 1 + 1, \ldots, s_i - 1 + t_i - t_i - 1 \) form a basis for the column space of the submatrix \( L_i \). Therefore they contain a square submatrix of size \( t_i - 1 \) with nonzero determinant. Since \( w_i \) is also a \( (t_i - 1) \)-sized minor in \( L_i \), with row indices \( 1, \ldots, t_i - 1 \), by Lemma 5.2 we conclude that \( u_i \neq 0 \).

(b) As in the proof of (a), assume there is a point in the variety of \( I \) where \( u_i \) vanishes but there exist nonvanishing \( t_j - 1 \times t_j - 1 \) minors \( v_j \) in \( \mathcal{I}_j \), for \( j = 1, \ldots, i \), a nonvanishing \( t_i - 1 \times t_i - 1 \) minor \( x_p \) in \( \mathcal{I}_p \), and a nonvanishing \( t_i - 1 \times t_i - 1 \) minor \( w_i \) in \( \mathcal{I}_j \).

Recall that \( Y \) and \( Z \) denote the submatrices of \( L \) consisting of its first \( y - 1 \) columns and first \( x - 1 \) rows, respectively. Since the assumption on the corner way \( y - 1 - s_{p-1} > t_p - t_{p-1} \), the minors \( v_1, \ldots, v_p \) are entirely contained in \( Y \), and by the definition of \( \mathcal{I}_p \), the minor \( x_p \) is contained in \( Y \cap Z \), in particular, its rows are contained in \( Z \). Reasoning exactly as we did in the proof of (a) (with Lemma 5.2 applied to the submatrix \( Y \)), we conclude that there exists a nonzero \( (t_i - 1) \)-sized minor, say \( v'_p \), with column indices \( 1, \ldots, t_i - 1, s_i + 1, \ldots, s_i + t_2 - t_2, \ldots, s_i - 1 + 1, \ldots, s_i - 1 + t_i - t_i - 1 \) and row indices in \( Z \).

Finally, since \( v'_p, v_{p+1}, \ldots, v_i \) and \( w_i \) are all contained in the submatrix \( Z \), we can yet again reason as in (a), and the proof is complete. \( \blacksquare \)

An immediate consequence is the following theorem.

Notation. Denote the images in \( K[L]/I \) of the primes \( \mathcal{P}_i, \mathcal{E}_i \), and \( \mathcal{R}_C \) by \( \alpha_i, \beta_i \), and \( \gamma_i \), respectively.

5.4. Theorem. For \( i = 1, \ldots, p - 1 \), the minimal prime ideals of \( u_i \) in \( K[L]/I \) are \( \alpha_2, \ldots, \alpha_i \) and \( \beta_i \), while for \( i = p, \ldots, n \), the minimal prime ideals of \( u_i \) are \( \alpha_1, \ldots, \alpha_i, \beta_i \), and \( \gamma_i \).
5.5. Lemma. \( v_{\alpha_i}(u_i) = 1 \), for all \( i = 1, \ldots, n \).

Proof. Let \( s_0 = 0, t_0 = 1, C_j = Q_j \), for \( j = 1, \ldots, i - 1, C_i = C_{i-1} \cup \{ s_j - t_i + t_{i-1} + 1, \ldots, s_j \} \), and \( C_j = C_i \cup \{ s_j + 1, \ldots, s_j + t_{j+1} - t_i, \ldots, s_j - 1 + 1, \ldots, s_j - t_i - t_{i-1} \} \), for \( j = i + 1, \ldots, n \). Let \( N_j = X_{p, C_j} \) (a \( t_i - 1 \times t_i - 1 \) matrix of \( Z_j \)) and \( v_i = \det N_j \) for \( j = 1, \ldots, n \) (note that \( v_i = u_i \), for \( j = 1, \ldots, i - 1 \)). Finally, let \( v = v_1 \cdots v_n \). Then exactly as in Lemma 5.1, \( R[1/v] \) is isomorphic to a localization of a polynomial ring over \( K \) and, just as in the remark following Lemma 5.1, we can obtain the set of variables of this polynomial ring. We denote this set by \( S \).

Now none of the minors \( v_j \) belongs to \( \alpha_i \). Further, the variables occurring in \( u_j \), that is, the variables in the submatrix \( X_{p, C_j} \), all belong to \( S \). Therefore, under the isomorphism, the minor \( u_j \), being a determinant of indeterminates, is an irreducible polynomial and generates a prime ideal. This proves that \( u_i \) is a uniformising parameter for \( R[1/v] \).

5.6. Theorem. The divisor class group of \( R \) is free of rank \( n + 1 \) and is generated by \( \text{cl}(\beta_1), \ldots, \text{cl}(\beta_n) \) and \( \text{cl}(\gamma_i) \).

Proof. By Corollary 2.2.3, \( \text{cl}(\alpha_1), \ldots, \text{cl}(\alpha_n), \text{cl}(\beta_1), \ldots, \text{cl}(\beta_n), \) and \( \text{cl}(\gamma_i) \) generate \( \text{Div}(R) \). Now by the second part of Corollary 2.2.3 and Theorem 5.4, since \( v_{\alpha_i}(u_i) = 1 \) (Lemma 5.5), \( \text{cl}(\alpha_i) \) can be generated by \( \text{cl}(\beta_1), \ldots, \text{cl}(\beta_n) \) and \( \text{cl}(\gamma_i) \). Finally, one easily sees that there is no nontrivial relation among \( \text{cl}(\beta_1), \ldots, \text{cl}(\beta_n) \) and \( \text{cl}(\gamma_i) \).

6. The Divisor Class Group for the Upper-Left Corner Ladder

We apply the corollary to Nagata's theorem to the upper-left corner ladder of Section 4. \( R \) again denotes the ring \( K[L]/I \).

6.1. Lemma. Let \( P_i = \{ x + 1, \ldots, x + t_i - 1 \} \), for \( i = 1, \ldots, n \), \( Q_1 = \{ 1, \ldots, t_1 - 1 \} \), \( Q_i = Q_1 \cup \{ s_1 + 1, \ldots, s_1 + t_2 - t_1, \ldots, s_{i-1} + 1, \ldots, s_{i-1} + t_i - t_{i-1} \} \), for \( i = 2, \ldots, p - 1 \), \( Q_p = \{ y + 1, \ldots, y + t_p - 1 \} \), and \( Q_i = Q_1 \cup \{ s_p + 1, \ldots, s_p + t_{p+1} - t_p, \ldots, s_{i-1} + 1, \ldots, s_{i-1} + t_i - t_{i-1} \} \), for \( i = p + 1, \ldots, n \). Let \( M_i = X_{p, Q_i} \) (a \( t_i - 1 \times t_i - 1 \) matrix of \( Y_i \)), \( u_i = \det M_i \) for \( i = 1, \ldots, n \), and \( u = u_1 \cdots u_n \). Then \( R[1/u] \) is isomorphic to a localization of a polynomial ring over \( K \) and is therefore regular.

Proof. Omitted.

We describe the polynomial ring obtained in the above isomorphism in more detail. Let \( Q_0 = \emptyset \). For \( i = 1, \ldots, n \), let

\[
L_i = \{ X_{ab} \in L_i | b \in Q_{i-1} \text{ or } s_{i-1} < b \leq s_i \}.
\]
For $i = 1, \ldots, p - 1$, let 
\[ T_i = \{ X_{ab} \in L_i \mid a \in P_i \text{ and } b \notin Q_i \}, \]
and for $i = p, \ldots, n$, let 
\[ T_i = \{ X_{ab} \in L_i \mid a \in P_i \text{ or } b \in Q_i \}. \]

Then $R[1/u_i]$ is isomorphic to $K[T \| 1/u_i]$, where $T = \bigcup_{i=1}^{n} T_i$. As in Section 5, $u_i$, being the determinant of a matrix of indeterminates, is irreducible in $K[T]$, for all $i = 1, \ldots, n$.

The notation of the above lemma shall be retained for the entire section.

6.2. Lemma. (a) Let $1 \leq i \leq p - 1$. Then
\[ \mathcal{I}_1 \cdot \mathcal{I}_2 \cdots \mathcal{I}_i \subseteq \sqrt{I + u_i}. \]
(b) Let $p \leq i \leq n$. Then
\[ \mathcal{I}_p \cdot \mathcal{I}_{p+1} \cdots \mathcal{I}_i \subseteq \sqrt{I + u_i}. \]

Proof. The proof of (a) is identical to that of its counterpart, Lemma 5.3(a).

(b) As in the proof of (a), assume there is a point in the variety of $I$ where $u_i$ vanishes, but there exist nonvanishing $t_j - 1 \times t_j - 1$ minors $v_j$ in $\mathcal{J}_j$, for $j = p, \ldots, i$, a nonvanishing $t_q - 1 \times t_q - 1$ minor $x_q$ in $\mathcal{H}_C$, for some $q \in \{ p, \ldots, n \}$, and a nonvanishing $t_i - 1 \times t_i - 1$ minor $w_i$ in $\mathcal{J}_i$.

Recall that $Y$ and $Z$ denote the submatrices of $L$ consisting of its last $r - x$ rows and last $s - y$ columns, respectively. Applying the reasoning of the proof of Lemma 5.3(a) to the minors $v_p, \ldots, v_i$, which are all contained in the submatrix $Z$, we conclude that there exists a nonzero $(t_i - 1)$-sized minor, say $v'_i$, with column indices $y + 1, \ldots, y + t_p - 1, s_p + 1, \ldots, s_p + t_p - 1, s_{p+1} - 1, \ldots, s_{p+1} - 1, \ldots, s_{i-1} - 1, s_{i-1} - 1, t_i - t_i - 1$ (that is, the column indices $Q_i$ of $u_i$). Therefore these columns are a basis for the column space of the submatrix $Z$. The minor $x_q$ is contained in $D_q (= Y_q \cap Z_q)$. Therefore its $t_q - 1$ columns, denoted by $C_q$, are a basis for the submatrix $Z_q$, and the same columns restricted to their last $r - x$ rows (that is, to $Y$) are a basis for the column space of $Y_q$. Finally, since $w_i$ is contained in $Y_i$, its $t_i - 1$ columns, denoted by $B_i$, are a basis for the column space of $Y_i$.

Let $q < i$. Since the columns in $C_q$ are linearly independent in $Y_q$, we can find $t_i - t_q$ columns in $B_q$, say $E_q$, which when added to $C_q$, also form a basis for $Y_q$. Note that the columns in $E_q$ are necessarily from $Y_q \setminus Y_q$, since $C_q$ is a basis for $Y_q$. Therefore the $t_i - 1$ columns $C_q \cup E_q$ contain a nonzero $(t_i - 1)$-sided minor. This minor is obviously in $Y_i$, and so applying
Lemma 5.2 to it and \( w_i \), gives us a nonzero \((t_i - 1)\)-sized minor with row indices \( P_i \) (the rows of \( w_i \)) and column indices \( C_q \cup E_i \). Now applying Lemma 5.2 to this minor and \( v_i \), we conclude that \( u_i \) is nonzero.

Let \( i \leq q \). Since the columns \( Q_i \) of \( v_i \) are linearly independent in \( Z \), we can find \( t_q - t_i \) columns in \( C_q \), say \( E_{q} \), which when added to \( Q_i \), continue to be a linearly independent set, and, in fact form a basis for \( Z_q \). Therefore they contain a nonzero \((t_q - 1)\)-sized minor. Since \( w_{q} \) is in \( Y \cap Z \), by Lemma 5.2 (applied to the region \( Z \)), we obtain a nonzero \((t_q - 1)\)-sized minor with column indices \( Q_i \cup E_{q} \) and row indices in \( Y \). Now the columns in \( Q_i \) restricted to this minor are still linearly independent, so we can find a \((t_i - 1)\)-sized nonzero minor among them. Therefore we have a \((t_i - 1)\)-sized nonzero minor in \( Y \), with column indices \( Q_i \). Applying Lemma 5.2 to this minor and \( w_i \), we conclude that \( u_i \) is nonzero.

A gain denoting the images in \( K[L]/I \) of the primes \( P, Q, \) and \( R_C \) by \( \alpha, \beta, \) and \( \gamma \) respectively, we have

6.3. **Theorem.** For \( i = 1, \ldots, p - 1 \), the minimal prime ideals of \( u_i \) in \( R \) are \( \alpha_1, \ldots, \alpha_i \) and \( \beta_i \), while for \( i = p, \ldots, n \), the minimal prime ideals of \( u_i \) are \( \alpha_p, \ldots, \alpha_i, \beta_i, \) and \( \gamma_i \).

6.4. **Lemma.** \( v_{a_i}(u_i) = 1 \), for all \( i = 1, \ldots, n \).

**Proof.** Let \( s_0 = 0, t_0 = 1, \) and \( C_j = Q_j \), for \( j = 1, \ldots, i - 1 \).

Suppose \( i \leq p - 1 \). Then let \( C_j = C_{j-1} \cup (s_j - t_j + t_{j-1} + 1, \ldots, s_j) \) and \( C_j = C_j \cup (s_j + 1, \ldots, s_j + t_j - 1, \ldots, s_j) \), for \( j = i + 1, \ldots, p - 1 \). Further, let \( C_j = Q_j \), for \( j = p, \ldots, n \). Let \( N_j = X_{p, Q_j} \) (a \( t_j - 1 \times t_j - 1 \) matrix of \( Z_j \)) and \( v_j = \text{det} N_j \), for \( j = 1, \ldots, n \), and let \( v = v_1 \cdots v_n \). Then, as always, \( R[1/v] \) is isomorphic to a localization of a polynomial ring over \( K \). We denote the set of free variables by \( S \).

Note that none of the minors \( v_j \) belongs to \( \alpha_i \) and that the variables occurring in \( u_j \) that is, the variables in the submatrix \( X_{p, Q_j} \) all belong to \( S \). Therefore, as in Lemma 5.5, \( v_{a_i}(u_i) = 1 \).

Suppose \( i = p \). Then let \( C_j = \{s_j - t_j + 2, \ldots, s_j\} \) and \( C_j = C_j \cup \{s_j + 1, \ldots, s_j + t_{j-1} + 1, \ldots, s_{j-1} + t_j - t_{j-1}\} \), for \( j = p + 1, \ldots, n \). Let \( N_j = X_{p, C_j} \) (a \( t_j - 1 \times t_j - 1 \) matrix of \( Z_j \)) and \( v_j = \text{det} N_j \), for \( j = 1, \ldots, n \), and let \( v = v_1 \cdots v_n \). Then, as always, \( R[1/v] \) is isomorphic to a localization of a polynomial ring over \( K \). We denote the set of free variables by \( S \).

Now \( v_j \notin \alpha_p \), since by Assumption 1 of Section 4 we have \( s_p - y \geq t_p \). Of course, the other \( v_j \) do not belong to \( \alpha_p \) either. A gain, the variables in \( X_{p, Q_j} \) all belong to \( S \), that is, \( u_p \) generates a prime ideal, and we have \( v_{a_i}(u_i) = 1 \).
Suppose $i > p$. In this case let $C_i = C_{i-1} \cup \{s_i - t_i + t_{i-1} + 1, \ldots, s_i\}$ and $C_j = C_i \cup \{s_j + 1, \ldots, s_i + t_{i+1} - t_i, \ldots, s_j + t_{i-1} + 1, \ldots, s_{j-1} + t_j - t_{j-1}\}$, for $j = i + 1, \ldots, n$. The rest is mere repetition.

6.5. **Theorem.** The divisor class group of $R$ is free of rank $n + 1$ and is generated by $\text{cl}(\beta_1), \ldots, \text{cl}(\beta_n)$ and $\text{cl}(\gamma_i)$.

**Proof.** Identical to that of Theorem 5.6.

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