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# Gröbner–Shirshov bases for metabelian Lie algebras ☆

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In this paper, we establish the Gröbner-Shirshov bases theory for metabelian Lie algebras. As applications, we find the Gröbner-Shirshov bases for partial commutative metabelian Lie algebras related to circuits, trees and some cubes.



## 1. Introduction

The class of metabelian Lie algebras is an important class of Lie algebras and attracts much attention. Let us mention the recent papers by E. Daniyarova, I. Kazatchkov, and V. Remeslennikov [4–6] on algebraic geometry of free metabelian Lie algebra, S. Findik and V. Drensky [7,8] on automorphisms of free metabelian Lie algebras, and V. Kurlin [9] on the Backer–Campbell–Hausdorff formula for free metabelian Lie algebras. Gröbner–Shirshov bases theory would be useful on this class of algebras. This theory was first considered by V.V. Talapov [10] in 1982. However, there are serious gaps in his paper. He missed several cases when he defined compositions. This means the theory was not established correctly. We refine his idea and complete the results.

It is well known that for many kinds of algebras, if $A_i = (X_i|S_i)$, $i = 1, 2$, are defined by generators and defining relations, where $S_1$ and $S_2$ are Gröbner–Shirshov bases respectively, then $S_1 \cup S_2$ is a

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Gröbner–Shirshov basis for the free product $A_1 * A_2 = (X_1 \cup X_2 | S_1 \cup S_2)$ of $A_1$ and $A_2$, for example, associative algebras, Lie algebras and for all classes with compositions of inclusion and intersection only (cf. [2,3]). We prove that it is not the case for metabelian Lie algebras, see Theorem 3.1, even in the case of $S_2 = \emptyset$. On the other hand, if $S_i \subset A_i^{(2)}$, then $S_1 \cup S_2$ is a Gröbner–Shirshov basis for the free metabelian Lie product $A_1 * A_2$, see Proposition 3.2.

Throughout this paper, all algebras will be considered over a field $\mathbf{k}$ of arbitrary characteristic. Suppose that $\mathcal{L}$ is a Lie algebra. Then $\mathcal{L}$ is called a metabelian Lie algebra if $\mathcal{L}^{(2)} = 0$, where $\mathcal{L}^{(0)} = \mathcal{L}$, $\mathcal{L}^{(n+1)} = [\mathcal{L}^{(n)}, \mathcal{L}^{(n)}]$. More precisely, the variety of metabelian Lie algebras is given by the identity

$$(x_1x_2)(x_3x_4) = 0.$$

## 2. Composition-Diamond lemma for metabelian Lie algebras

Let us begin with the construction of a free metabelian Lie algebra. Let $X$ be a set and $Lie(X)$ be the free Lie algebra generated by $X$. Then $\mathcal{L}_{(2)}(X) = Lie(X)/Lie(X)^{(2)}$ is the free metabelian Lie algebra generated by $X$. Any metabelian Lie algebra $\mathcal{ML}$ is a homomorphic image of a free metabelian Lie algebra generated by some $X$, that is, $\mathcal{ML}$ can be presented by generators $X$ and defining relations $S$: $\mathcal{ML} = \mathcal{L}_{(2)}(X|S)$.

We call a non-associative monomial on $X$ left-normed if it is of the form $(\cdots((ab)c)\cdots)d$. In the sequel, the brackets in the expression of left-normed monomials are omitted.

Let $X$ be well ordered. For an arbitrary set of indices $j_1, j_2, \ldots, j_m$, define an associative word

$$\langle a_{j_1} \cdots a_{j_m} \rangle = a_{i_1} \cdots a_{i_m},$$

where $a_{i_1} \leqslant \cdots \leqslant a_{i_m}$ and $i_1, i_2, \ldots, i_m$ is a permutation of the indices $j_1, j_2, \ldots, j_m$.

Let

$$R = \left\{ u = a_0a_1a_2 \cdots a_n \;\middle|\; a_i \in X \ (0 \leqslant i \leqslant n),\ a_0 > a_1 \leqslant \cdots \leqslant a_n,\ n \geqslant 1 \right\}$$

and $N = X \cup R$, where $u = a_0a_1a_2 \cdots a_n$ is left-normed.

Then $N$ forms a linear basis of the free metabelian Lie algebra $\mathcal{L}_{(2)}(X)$, i.e., $\mathcal{L}_{(2)}(X) = \mathbf{k}N$, see [1].

We call elements of $N$ regular words on $X$ and those of $R$ regular $R$-words. Therefore, for any $f \in \mathcal{L}_{(2)}(X)$, $f$ has a unique presentation $f = f^{(1)} + f^{(0)}$, where $f^{(1)} \in \mathbf{k}R$ and $f^{(0)} \in \mathbf{k}X$. Moreover, the multiplication table of regular words is the following, $u \cdot v = 0$ if both $u, v \in R$, and

$$a_0a_1a_2 \cdots a_n \cdot b = \begin{cases} a_0\langle a_1a_2 \cdots a_nb \rangle & \text{if } a_1 \leqslant b, \\ a_0ba_1a_2 \cdots a_n - a_1b\langle a_0a_2 \cdots a_n \rangle & \text{if } a_1 > b. \end{cases}$$

If $u = a_0a_1 \cdots a_n \in R$, then the regular words $a_i$ $(0 \leqslant i \leqslant n)$ and $a_0\langle a_{i_1} \cdots a_{i_l} \rangle$ ($l \leqslant n$, $a_{i_1}, \ldots, a_{i_l}$ is a subsequence of the sequence $a_1, \ldots, a_n$) are called subwords of $u$. The words $a_i$ $(2 \leqslant i \leqslant n)$, and also $a_1$ if $a_0 > a_2$ are called strict subwords of $u$.

Define the length of regular words:

$$|a| = 1, \qquad |a_0a_1a_2 \cdots a_n| = n + 1,$$

where $a, a_0, \ldots, a_n \in X$. Now we order the set $N$ degree-lexicographically, i.e., for any $u, v \in N$,

$$u > v \quad \text{if } |u| > |v| \quad \text{or} \quad |u| = |v|, \quad u >_{lex} v.$$

Through out this paper, we will use this ordering.

The largest monomial occurring in $f \in \mathcal{L}_{(2)}(X)$ with nonzero coefficient is called the leading word of $f$ and is denoted by $\bar{f}$. Then we have $\overline{a_0a_1a_2 \cdots a_n \cdot b} = a_0\langle a_1a_2 \cdots a_nb \rangle$ and $|\overline{u \cdot b}| = |u| + 1$. For any $f \in \mathcal{L}_{(2)}(X)$, we call $f$ monic, (1)-monic and (0)-monic if the coefficients of $\bar{f}$, $\overline{f^{(1)}}$ and $\overline{f^{(0)}}$ are 1 respectively.

**Lemma 2.1.** *For any $u, v \in N$, if $u > v$ then*

$$(\forall b \in N) \quad u \cdot b \neq 0 \quad \Rightarrow \quad \overline{u \cdot b} > \overline{v \cdot b}.$$

**Proof.** The result is obvious if either $u, v \in X$ or $|u| > |v|$. Suppose that $u = a_0 a_1 a_2 \cdots a_n$, $v = a'_0 a'_1 a'_2 \cdots a'_n \in R$ and $b \in X$. If $a_0 > a'_0$ then we are done. If $a_0 = a'_0$, then $\langle a_1 a_2 \cdots a_n b\rangle > \langle a'_1 a'_2 \cdots a'_n b\rangle$ in $[X]$ since the deg-lex ordering on $[X]$ is monomial, where $[X]$ is the free commutative momoid generated by $X$. Now, the result follows. □

Let $S \subset \mathcal{L}_{(2)}(X)$. We denote $u_s = s v_1 v_2 \cdots v_n$, where $v_i \in N$, $s \in S$ and $n \geqslant 0$. We call $u_s$ an $s$-word (or $S$-word). It is clear that each element of the ideal $Id(S)$ of $\mathcal{L}_{(2)}(X)$ generated by $S$ is a linear combination of $S$-words.

**Definition 2.2.** Let $S \subset \mathcal{L}_{(2)}(X)$. Then the following two kinds of polynomials are called normal $S$-words:

(i) $s a_1 a_2 \cdots a_n$, where $a_i \in X$ $(1 \leqslant i \leqslant n)$, $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$, $s \in S$, $\bar{s} \neq a_1$ and $n \geqslant 0$;
(ii) $us$, where $u \in R$, $s \in S$ and $\bar{s} \neq u$.

By a simple observation, we have

$$\overline{s a_1 a_2 \cdots a_n} = \begin{cases} c_0 \langle c_1 \cdots c_k a_1 a_2 \cdots a_n\rangle & \text{if } \bar{s} = c_0 c_1 \cdots c_k, \\ c_0 a_1 a_2 \cdots a_n & \text{if } \bar{s} = c_0 > a_1, \\ a_1 c_0 a_2 \cdots a_n & \text{if } \bar{s} = c_0 < a_1, \end{cases}$$

and $\overline{us} = a_0 \langle a_1 \cdots a_k \overline{s^{(0)}}\rangle$, where $u = a_0 \langle a_1 \cdots a_k\rangle$. That is to say, if $u_s$ is a normal $s$-word, then $\overline{u_s}$ contains either $\bar{s}$ as a subword or $\overline{s^{(0)}}$ as a strict subword.

A regular word $u$ is called $S$-irreducible if for any $s \in S$, $u$ contains neither $\bar{s}$ as a subword nor $\overline{s^{(0)}}$ as a strict subword. Denote by $Irr(S)$ the set of all $S$-irreducible words. This means

$$Irr(S) = \{u \mid u \in N,\ u \neq \overline{v_s} \text{ for any normal } S\text{-word } v_s\}.$$

**Remark.** For any $s \in \mathcal{L}_{(2)}(X)$,

$$s a_1 a_2 \cdots a_n = s a_1 a_{j_2} \cdots a_{j_n},$$

where $\langle a_{j_2} \cdots a_{j_n}\rangle = a_2 \cdots a_n$.

**Lemma 2.3.** *Let $S \subset \mathcal{L}_{(2)}(X)$ and $Id(S)$ be the ideal of $\mathcal{L}_{(2)}(X)$ generated by $S$. Then for any $f \in Id(S)$, $f$ can be written as a linear combination of normal $S$-words.*

**Proof.** It is suffice to show that any $S$-word $u_s = s u_1 u_2 \cdots u_n$ is a linear combination of normal $S$-words, where $u_i \in N$, $1 \leqslant i \leqslant n$. We may assume that $s$ is monic. The proof will be proceeded by induction on $n$.

There is nothing to prove if $n = 0$.

Assume that $n = 1$. If $\bar{s} \neq u_1$, then either $s u_1$ or $u_1 s$ is normal. If $\bar{s} = u_1$, then $s = u_1 + \sum_{\bar{s} > v_j \in N} \alpha_j v_j$, $\alpha_j \in \mathbf{k}$ and

$$s u_1 = s\Big(s - \sum_{v_j < \bar{s}} \alpha_i v_j\Big) = -s \sum_{v_j < \bar{s}} \alpha_j v_j = -\sum_{v_j < \bar{s}} \alpha_j s v_j,$$

where for each $j$, either $v_j s$ or $s v_j$ is normal.

For $n \geqslant 2$, if $\exists u_i \in R$ $(i \geqslant 2)$, then $su_1u_2\cdots u_n = 0$; if $u_1 \in R$, then $(su_1)a_2\cdots a_n = s(u_1a_2\cdots a_n)$ which is the above case. So we may assume that $u_s = sa_1a_2\cdots a_n$ is normal and $u_{n+1} = a \in X$. Then

$$u_s \cdot u_{n+1} = sa_1a_2\cdots a_n \cdot a = sa_1\langle a_2\cdots a_na\rangle.$$

If $a \geqslant a_1$, then $sa_1\langle a_2\cdots a_na\rangle$ is normal. If $a < a_1$, then

$$\begin{aligned} u_s \cdot u_{n+1} &= sa_1aa_2\cdots a_n \\ &= saa_1a_2\cdots a_n - \big((a_1a)s\big)a_2\cdots a_n \\ &= saa_1a_2\cdots a_n - a_1aa_2\cdots a_n \cdot s. \end{aligned}$$

Clearly, by the previous proof, $a_1aa_2\cdots a_n \cdot s$ is normal. Now $saa_1a_2\cdots a_n$ is already normal provided that $\bar{s} \neq a$. If $\bar{s} = a$, then we substitute $a$ by $-\sum_{\bar{s}>v_j\in N}\alpha_jv_j$ where $s = a + \sum_{\bar{s}>v_j\in N}\alpha_jv_j$, and the result follows now. □

**Lemma 2.4.** *Let $u_s$ be a normal $S$-word and $w \in N$. If $\overline{u_s} < w$, then*

$$(\forall a \in X) \quad w \cdot a \neq 0 \quad \Rightarrow \quad \overline{u_s \cdot a} < \overline{w \cdot a}.$$

**Proof.** Suppose that $w = b_0b_1\cdots b_m$ where $m \geqslant 0$. Then

$$\overline{w \cdot a} = \begin{cases} b_0\langle b_1\cdots b_ma\rangle & \text{if } m > 0, \\ b_0a & \text{if } m = 0 \text{ and } b_0 > a, \\ ab_0 & \text{if } m = 0 \text{ and } b_0 < a. \end{cases}$$

If $u_s = sa_1a_2\cdots a_n$, then

$$\overline{u_s} = \begin{cases} c_0\langle c_1\cdots c_ka_1a_2\cdots a_n\rangle & \text{if } \bar{s} = c_0c_1\cdots c_k, \\ c_0a_1a_2\cdots a_n & \text{if } \bar{s} = c_0 > a_1, \\ a_1c_0a_2\cdots a_n & \text{if } \bar{s} = c_0 < a_1 \end{cases}$$

and

$$u_s \cdot a = \begin{cases} sa_1\langle a_2\cdots a_na\rangle & \text{if } a \geqslant a_1, \\ saa_1a_2\cdots a_n - a_1aa_2\cdots a_n \cdot s & \text{if } a < a_1. \end{cases}$$

Therefore,

$$\overline{u_s \cdot a} = \begin{cases} c_0\langle c_1\cdots c_ka_1a_2\cdots a_na\rangle & \text{if } \bar{s} = c_0c_1\cdots c_k, \\ c_0\langle a_1a_2\cdots a_na\rangle & \text{if } \bar{s} = c_0 > a_1, \\ a_1\langle c_0a_2\cdots a_na\rangle & \text{if } \bar{s} = c_0 < a_1. \end{cases}$$

If $u_s = a_0a_1\cdots a_n \cdot s$, then $\overline{u_s} = a_0\langle a_1\cdots a_n\overline{s^{(0)}}\rangle$ and $\overline{u_s \cdot a} = a_0\langle a_1\cdots a_n\overline{s^{(0)}}a\rangle$.
Since $\overline{u_s} < w$, in both cases we have $\overline{u_s \cdot a} < \overline{w \cdot a}$. □

**Definition 2.5.** Let $f$ and $g$ be momic polynomials of $\mathcal{L}_{(2)}(X)$ and $\alpha$ and $\beta$ be the coefficients of $\overline{f^{(0)}}$ and $\overline{g^{(0)}}$ respectively. We define seven different types of compositions as follows:

1. If $\bar{f} = a_0 a_1 \cdots a_n$, $\bar{g} = a_0 b_1 \cdots b_m$ $(n, m \geqslant 0)$ and $lcm(AB) \neq \langle a_1 \cdots a_n b_1 \cdots b_m \rangle$, where $lcm(AB)$ denotes the least common multiple in $[X]$ of associative words $a_1 \cdots a_n$ and $b_1 \cdots b_m$, then let $w = a_0 \langle lcm(AB) \rangle$. The composition of type I of $f$ and $g$ relative to $w$ is defined by

$$C_I(f, g)_w = f\left\langle \frac{lcm(AB)}{a_1 \cdots a_n} \right\rangle - g\left\langle \frac{lcm(AB)}{b_1 \cdots b_m} \right\rangle.$$

2. If $\bar{f} = \overline{f^{(1)}} = a_0 a_1 \cdots a_n$, $\overline{g^{(0)}} = a_i$ for some $i \geqslant 2$ or $\overline{g^{(0)}} = a_1$ and $a_0 > a_2$, then let $w = \bar{f}$. The composition of type II of $f$ and $g$ relative to $w$ is defined by

$$C_{II}(f, g)_w = f - \beta^{-1} a_0 a_1 \cdots \hat{a}_i \cdots a_n \cdot g,$$

where $a_0 a_1 \cdots \hat{a}_i \cdots a_n = a_0 a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$.
3. If $\bar{f} = \overline{f^{(1)}} = a_0 a_1 \cdots a_n$, $\bar{g} = \overline{g^{(0)}} = a_1$ and $a_0 \leqslant a_2$ or $n = 1$, then let $w = \bar{f}$. The composition of type III of $f$ and $g$ relative to $w$ is defined by

$$C_{III}(f, g)_{\bar{f}} = f + g a_0 a_2 \cdots a_n.$$

4. If $\bar{f} = \overline{f^{(1)}} = a_0 a_1 \cdots a_n$, $g^{(1)} \neq 0$, $\overline{g^{(0)}} = a_1$ and $a_0 \leqslant a_2$ or $n = 1$, then for any $a < a_0$ and $w = a_0 \langle a_1 \cdots a_n a \rangle$, the composition of type IV of $f$ and $g$ relative to $w$ is defined by

$$C_{IV}(f, g)_w = f a - \beta^{-1} a_0 a a_2 \cdots a_n \cdot g.$$

5. If $\bar{f} = \overline{f^{(1)}} = a_0 a_1 \cdots a_n$, $f^{(0)} \neq 0$, $g^{(1)} \neq 0$ and $\overline{g^{(0)}} = b \notin \{a_i\}_{i=1}^n$, then let $w = a_0 \langle a_1 \cdots a_n b \rangle$. The composition of type V of $f$ and $g$ relative to $w$ is defined by

$$C_V(f, g)_w = f b - \beta^{-1} a_0 a_1 \cdots a_n \cdot g.$$

6. If $\overline{f^{(0)}} = \overline{g^{(0)}} = a$ and $f^{(1)} \neq 0$, then for any $a_0 a_1 \in R$ and $w = a_0 \langle a_1 a \rangle$, the composition of type VI of $f$ and $g$ relative to $w$ is defined by

$$C_{VI}(f, g)_w = (a_0 a_1)\big(\alpha^{-1} f - \beta^{-1} g\big).$$

7. If $f^{(1)} \neq 0$, $g^{(1)} \neq 0$ and $\overline{f^{(0)}} = a > \overline{g^{(0)}} = b$, then for any $a_0 > a$ and $w = a_0 b a$, the composition of type VII of $f$ and $g$ relative to $w$ is defined by

$$C_{VII}(f, g)_w = \alpha^{-1}(a_0 b) f - \beta^{-1}(a_0 a) g.$$

Immediately, we have $\overline{C_\lambda(f, g)_w} < w$.

**Remark.** In the paper of V.V. Talapov [10], only the compositions of types I, II and III are defined.

**Definition 2.6.** Given a set $S$ of monic polynomials of $\mathcal{L}_{(2)}(X)$ and $w \in N$, a polynomial $f \in \mathcal{L}_{(2)}(X)$ is called trivial modulo $S$ and $w$, denoted by $f \equiv 0$ $mod(S, w)$, if $f$ is a linear combination of normal $S$-words whose leading words are less than $w$, i.e., $f = \sum_i \alpha_i u_{s_i}$, where $\alpha_i \in \mathbf{k}$, $u_{s_i}$ are normal $S$-words and $\overline{u_{s_i}} < w$. For any $f, g \in \mathcal{L}_{(2)}(X)$, we say $f \equiv g$ $mod(S, w)$ if $f - g \equiv 0$ $mod(S, w)$.

The set $S$ is a Gröbner–Shirshov basis in $\mathcal{L}_{(2)}(X)$ if $S$ is closed under compositions, which means every composition of any two elements of $S$ is trivial modulo $S$ and corresponding $w$, i.e., $(\forall f, g \in S)$ $C_\lambda(f, g)_w \equiv 0$ $mod(S, w)$.

**Lemma 2.7.** *If $sa_1a_2\cdots a_n$ is a normal s-word with leading word $w$, then for any $a_{i_1} < \bar{s}$,*

$$sa_1a_2\cdots a_n \equiv sa_{i_1}a_{i_2}\cdots a_{i_n} \quad mod(s, w),$$

*where $\langle a_{i_1}a_{i_2}\cdots a_{i_n}\rangle = a_1a_2\cdots a_n$.*

**Proof.** There is nothing to prove if $a_{i_1} = a_1$. Suppose that $a_{i_1} = a_j > a_1$ for some $j \geqslant 2$. Then we have

$$\begin{aligned} sa_1a_2\cdots a_n &= sa_1a_ja_2\cdots \hat{a}_j\cdots a_n \\ &= sa_ja_1a_2\cdots \hat{a}_j\cdots a_n + (a_ja_1)a_2\cdots \hat{a}_j\cdots a_n\cdot s. \end{aligned}$$

Since $a_{i_1} < \bar{s}$, it is easy to see that $\overline{(a_ja_1)a_2\cdots \hat{a}_j\cdots a_n\cdot s} < \overline{sa_1a_2\cdots a_n} = w$. The result follows. □

The following lemma plays a key role in this paper.

**Lemma 2.8.** *Let $S$ be a Gröbner–Shirshov basis in $\mathcal{L}_{(2)}(X)$. If $w = \overline{u_{s_1}} = \overline{u_{s_2}}$, where $s_1, s_2 \in S$ and $u_{s_1}, u_{s_2}$ are normal $S$-words, then for some $0 \neq \alpha \in \mathbf{k}$,*

$$u_{s_1} \equiv \alpha u_{s_2} \quad mod(S, w).$$

**Proof.** There are three main cases to consider.

Case 1. $u_{s_1} = s_1a_1a_2\cdots a_n$, $u_{s_2} = s_2b_1b_2\cdots b_m$.

(1.1) If $\bar{s}_1 = \overline{s_1^{(1)}} = c_0c_1\ldots c_k$ and $\bar{s}_2 = \overline{s_2^{(1)}} = d_0d_1\ldots d_l$, then $c_0 = d_0$ and

$$w = c_0\langle c_1\cdots c_ka_1a_2\cdots a_n\rangle = d_0\langle d_1\cdots d_lb_1b_2\cdots b_m\rangle = c_0\big\langle lcm(CD)T\big\rangle,$$

where $T \in [X]$ such that $\langle c_1\cdots c_ka_1a_2\cdots a_n\rangle = \langle d_1\cdots d_lb_1b_2\cdots b_m\rangle = \langle lcm(CD)T\rangle$. Thus, By Lemmas 2.7 and 2.4 we have

$$\begin{aligned} s_1a_1a_2\cdots a_n - s_2b_1b_2\cdots b_m &= s_1\left\langle \frac{lcm(CD)}{c_1\cdots c_k}T\right\rangle - s_2\left\langle \frac{lcm(CD)}{d_1\cdots d_l}T\right\rangle \\ &\equiv \left(s_1\left\langle \frac{lcm(CD)}{c_1\cdots c_k}\right\rangle - s_2\left\langle \frac{lcm(CD)}{d_1\cdots d_l}\right\rangle\right)\langle T\rangle \\ &\equiv C_I(s_1, s_2)_{w'}\langle T\rangle \\ &\equiv 0 \quad mod(S, w), \end{aligned}$$

where $w' = c_0\langle lcm(CD)\rangle$ and $w = \overline{w'\langle T\rangle}$.

(1.2) If $\bar{s}_1 = \overline{s_1^{(1)}} = c_0c_1\ldots c_k$ and $\bar{s}_2 = \overline{s_2^{(0)}} = d$, then there are two subcases to be discussed.

(1.21) If $d > b_1$ then

$$w = c_0\langle c_1\cdots c_ka_1a_2\cdots a_n\rangle = db_1b_2\cdots b_m,$$

which implies $c_0 = d$ and $\langle c_1\cdots c_ka_1a_2\cdots a_n\rangle = b_1b_2\cdots b_m$.

Hence,

$$\begin{aligned} s_1a_1a_2\cdots a_n - s_2b_1b_2\cdots b_m &\equiv s_1a_1a_2\cdots a_n - (s_2c_1\cdots c_k)a_1a_2\cdots a_n \\ &\equiv (s_1 - s_2c_1\cdots c_k)a_1a_2\cdots a_n \\ &\equiv C_I(s_1, s_2)_{\bar{s}_1}a_1a_2\cdots a_n \\ &\equiv 0 \quad mod(S, w). \end{aligned}$$

(1.22) If $d < b_1$ then $a_1 \geqslant c_1$. In fact, if $a_1 < c_1$ $(< c_0)$, then $w = c_0a_1\langle c_1\cdots c_ka_2\cdots a_n\rangle = b_1db_2\cdots b_m$, which implies $c_0 = b_1$, $a_1 = d$ and $\langle c_1\cdots c_ka_2\cdots a_n\rangle = b_2\cdots b_m$. This is impossible because $c_1 < c_0 = b_1 \leqslant b_i$ $(2 \leqslant i \leqslant m)$. Thus we have $a_1 \geqslant c_1$ and

$$w = c_0c_1\langle c_2\cdots c_ka_1a_2\cdots a_n\rangle = b_1db_2\cdots b_m,$$

which implies $c_0 = b_1$, $c_1 = d$ and $\langle c_2\cdots c_ka_1a_2\cdots a_n\rangle = b_2\cdots b_m$.

By noting that $c_0 = b_1 \leqslant b_i = c_2$ for some $2 \leqslant i \leqslant m$, we have

$$\begin{aligned} s_1a_1a_2\cdots a_n + s_2b_1b_2\cdots b_m &= s_1a_1a_2\cdots a_n + (s_2c_0c_2\cdots c_k)a_1a_2\cdots a_n \\ &= (s_1 + s_2c_0c_2\cdots c_k)a_1a_2\cdots a_n \\ &\equiv C_{III}(s_1, s_2)_{\bar{s}_1}a_1a_2\cdots a_n \\ &\equiv 0 \quad mod(S, w). \end{aligned}$$

(1.3) If $\bar{s}_1 = \overline{s_1^{(0)}} = c$ and $\bar{s}_2 = \overline{s_2^{(0)}} = d$, then we have $n = m$. Thus, we may assume that $n = m \geqslant 1$. There are two subcases to consider.

(1.31) If either $c > a_1$, $d > b_1$ or $c < a_1$, $d < b_1$, then

$$w = ca_1\cdots a_n = db_1\cdots b_n$$

or

$$w = a_1ca_2\cdots a_n = b_1db_2\cdots b_n,$$

which implies $c = d$, $a_i = b_i$ for any $i$.

It is easy to see that

$$\begin{aligned} s_1a_1a_2\cdots a_n - s_2b_1b_2\cdots b_n &= (s_1 - s_2)a_1\cdots a_n \\ &= C_I(s_1, s_2)_{\bar{s}_1}a_1\cdots a_n \\ &\equiv 0 \quad mod(S, w). \end{aligned}$$

(1.32) If $c > a_1$ but $d < b_1$, then

$$w = ca_1\cdots a_n = b_1db_2\cdots b_n,$$

which implies $c = b_1$, $d = a_1$, $a_i = b_i$ for any $i > 1$.

Obviously,

$$\begin{aligned} s_1a_1a_2\cdots a_n+s_2b_1b_2\cdots b_n &= (s_1\bar{s}_2-\bar{s}_1s_2)a_2\cdots a_n \\ &= \bigl(s_1(\bar{s}_2-s_2)-(\bar{s}_1-s_1)s_2\bigr)a_2\cdots a_n \\ &\equiv 0 \quad mod(S,w). \end{aligned}$$

Case 2. $u_{s_1}=s_1a_1a_2\cdots a_n$, $u_{s_2}=b_0b_1b_2\cdots b_m\cdot s_2$. We may assume that $s_2$ is (0)-monic and $\overline{s_2^{(0)}}=d$. Then $w=b_0\langle b_1\cdots b_md\rangle$.

(2.1) If $\bar{s}_1=\overline{s_1^{(1)}}=c_0c_1\ldots c_k$, then $c_0=b_0$ and

$$w=c_0\langle c_1\cdots c_ka_1a_2\cdots a_n\rangle=b_0\langle b_1\cdots b_md\rangle.$$

(2.11) If $d\notin\{c_i\}_{i=1}^k$, then there exists an $a_i$ $(1\leqslant i\leqslant n)$ such that $d=a_i$. Thus,

$$\begin{aligned} s_1a_1a_2\cdots a_n-b_0b_1b_2\cdots b_m\cdot s_2 &\equiv (s_1a_i)a_1a_2\cdots\hat{a}_i\cdots a_n-(c_0c_1\cdots c_k\cdot s_2)a_1a_2\cdots\hat{a}_i\cdots a_n \\ &\equiv \bigl(s_1\overline{s_2^{(0)}}-\bar{s}_1s_2\bigr)a_1a_2\cdots\hat{a}_i\cdots a_n \quad mod(S,w). \end{aligned}$$

If $s_2^{(1)}=0$, then

$$\begin{aligned} \bigl(s_1\overline{s_2^{(0)}}-\bar{s}_1s_2\bigr)a_1a_2\cdots\hat{a}_i\cdots a_n &= (s_1\bar{s}_2-\bar{s}_1s_2)a_1a_2\cdots\hat{a}_i\cdots a_n \\ &\equiv 0 \quad mod(S,w). \end{aligned}$$

If $s_1^{(0)}=0$, i.e., $s_1=s_1^{(1)}=\bar{s}_1+r_1^{(1)}$, then let $s_2^{(0)}=\overline{s_2^{(0)}}+r_2^{(0)}$ and we have

$$\begin{aligned} s_1\overline{s_2^{(0)}}-\bar{s}_1s_2 &= \bigl(\bar{s}_1+r_1^{(1)}\bigr)\overline{s_2^{(0)}}-\bar{s}_1s_2^{(0)} \\ &= r_1^{(1)}\overline{s_2^{(0)}}-\bar{s}_1r_2^{(0)} \\ &= r_1^{(1)}\overline{s_2^{(0)}}-\bar{s}_1r_2^{(0)}+r_1^{(1)}r_2^{(0)}-r_1^{(1)}r_2^{(0)} \\ &= r_1^{(1)}s_2^{(0)}-s_1r_2^{(0)} \\ &= r_1^{(1)}s_2-s_1r_2^{(0)}, \end{aligned}$$

which implies $(s_1\overline{s_2^{(0)}}-\bar{s}_1s_2)a_1a_2\cdots\hat{a}_i\cdots a_n\equiv 0\ mod(S,w)$ immediately.

If $s_2^{(1)}\neq 0$ and $s_1^{(0)}\neq 0$, then

$$\begin{aligned} \bigl(s_1\overline{s_2^{(0)}}-\bar{s}_1s_2\bigr)a_1a_2\cdots\hat{a}_i\cdots a_n &\equiv C_V(s_1,s_2)_{w'}a_1a_2\cdots\hat{a}_i\cdots a_n \\ &\equiv 0 \quad mod(S,w), \end{aligned}$$

where $w'=c_0\langle c_1\cdots c_kd\rangle$ and $w=\overline{w'a_1a_2\cdots\hat{a}_i\cdots a_n}$.

(2.12) If $d = c_i$ for some $i \geqslant 2$, or $d = c_1$ and $c_0 > c_2$, then

$$\begin{aligned} s_1a_1a_2\cdots a_n - b_0b_1b_2\cdots b_m\cdot s_2 &\equiv s_1a_1a_2\cdots a_n - (c_0c_1\cdots\hat{c}_i\cdots c_k\cdot s_2)a_1a_2\cdots a_n \\ &\equiv (s_1 - c_0c_1\cdots\hat{c}_i\cdots c_k\cdot s_2)a_1a_2\cdots a_n \\ &\equiv C_{II}(s_1,s_2)_{\bar{s}_1}a_1a_2\cdots a_n \\ &\equiv 0 \quad mod(S,w), \end{aligned}$$

where $c_i = d$.

(2.13) If $d = c_1$ and $c_0 \leqslant c_2$, then by the form of $w$, we have $b_0b_1\cdots b_m = c_0\langle c_2\cdots c_ka_1\cdots a_n\rangle \in R$, which implies $c_2 \geqslant c_0 > a_1$. Thus,

$$\begin{aligned} s_1a_1a_2\cdots a_n - b_0b_1b_2\cdots b_m\cdot s_2 &= s_1a_1a_2\cdots a_n - c_0a_1\langle c_2\cdots c_ka_2\cdots a_n\rangle\cdot s_2 \\ &= (s_1a_1 - c_0a_1c_2\cdots c_k\cdot s_2)a_2\cdots a_n \\ &= C_{IV}(s_1,s_2)_{w'}a_2\cdots a_n \\ &\equiv 0 \quad mod(S,w), \end{aligned}$$

where $w' = c_0\langle c_1\cdots c_ka_1\rangle$ and $w = w'a_2\cdots a_n$.

(2.2) If $\bar{s}_1 = \overline{s_1^{(0)}} = c$ and $\overline{s_2^{(0)}} = d$, then $n = m+1 \geqslant 2$ since $w = b_0\langle b_1\cdots b_md\rangle$ and $m \geqslant 1$.

(2.21) If $c > a_1$, then $w = ca_1\cdots a_n = b_0\langle b_1\cdots b_md\rangle$, which implies $b_0 = c$.

(2.211) If $d \geqslant b_1$, then $a_1 = b_1$, $a_2\cdots a_n = \langle b_2\cdots b_md\rangle$ and

$$\begin{aligned} s_1a_1a_2\cdots a_n - b_0b_1b_2\cdots b_m\cdot s_2 &= (s_1b_1d)b_2\cdots b_m - \big((b_0b_1)\cdot s_2\big)b_2\cdots b_m \\ &= \big(s_1b_1d - (b_0b_1)\cdot s_2\big)b_2\cdots b_m \\ &= \big(s_1b_1\overline{s_2^{(0)}} - (\bar{s}_1b_1)\cdot s_2^{(0)}\big)b_2\cdots b_m \\ &= \big(s_1b_1\big(\overline{s_2^{(0)}} - s_2^{(0)}\big) - \big((\bar{s}_1 - s_1)b_1\big)\cdot s_2^{(0)}\big)b_2\cdots b_m \\ &= (s_1b_1)\big\langle r_2^{(0)}b_2\cdots b_m\big\rangle - (r_1b_1)b_2\cdots b_m\cdot s_2 \\ &\equiv 0 \quad mod(S,w), \end{aligned}$$

where $s_2^{(0)} = \overline{s_2^{(0)}} + r_2^{(0)}$ and $s_1 = \bar{s}_1 + r_1$.

(2.212) If $d < b_1$, then $w = cdb_1\cdots b_m$. Suppose that $s_1 = c + \sum_{c_i<c}\alpha_ic_i$, $s_2^{(0)} = d + \sum_{d_j<d}\beta_jd_j$. Thus,

$$\begin{aligned} &s_1a_1a_2\cdots a_n - b_0b_1b_2\cdots b_m\cdot s_2 \\ &\quad = s_1db_1b_2\cdots b_m - cb_1b_2\cdots b_m\cdot s_2 \\ &\quad = \big(s_1db_1 - (cb_1)\cdot s_2\big)b_2\cdots b_m. \\ &\quad = \Bigg(s_1db_1 - (s_1b_1)s_2 + \sum_{c_i<c}\alpha_i(c_ib_1)s_2\Bigg)b_2\cdots b_m \\ &\quad = \Bigg(s_1b_1d + (b_1d)s_1 - (s_1b_1)s_2 + \sum_{c_i<c}\alpha_i(c_ib_1)s_2\Bigg)b_2\cdots b_m \end{aligned}$$

$$= \Big(s_1b_1(d-s_2) + (b_1d)s_1 + \sum_{c_i<c} \alpha_i(c_ib_1)s_2\Big)b_2\cdots b_m$$

$$= \Big(-\sum_{d_j<d} \beta_j s_1 b_1 d_j + (b_1d)s_1 + \Big(\sum_{c_i<c} \alpha_i c_i b_1\Big)\cdot s_2\Big)b_2\cdots b_m$$

$$= \Big(-\sum_{d_j<d} \beta_j s_1 d_j b_1 + \sum_{d_j<d} \beta_j(b_1d_j)s_1 + (b_1d)s_1 + \Big(\sum_{c_i<c} \alpha_i c_i b_1\Big)\cdot s_2\Big)b_2\cdots b_m$$

$$\equiv 0 \quad mod(S, w).$$

(2.22) If $c < a_1$, then $w = a_1ca_2\cdots a_n = b_0\langle b_1\cdots b_m d\rangle$ and $a_1 = b_0$. In this case, $d \geqslant b_1$, and then $b_1 = c$, $d = a_i$ for some $i \geqslant 2$. Otherwise, if $d < b_1$, then $d = c$. This implies $a_i = b_{i-1}$ for any $i \geqslant 1$ and $b_0 = a_1 \leqslant a_2 = b_1$, which is a contradiction. Therefore,

$$\begin{aligned} s_1a_1a_2\cdots a_n + b_0b_1b_2\cdots b_m\cdot s_2 &= -(a_1s_1)a_2\cdots a_n + a_1b_1b_2\cdots b_m\cdot s_2 \\ &= -\big((a_1s_1)d\big)a_2\cdots \hat{a}_i\cdots a_n + (a_1c)a_2\cdots \hat{a}_i\cdots a_n\cdot s_2 \\ &= \big((s_1a_1)\overline{s_2^{(0)}} + (a_1\bar{s}_1)\cdot s_2\big)a_2\cdots \hat{a}_i\cdots a_n \\ &= \big((s_1a_1)\big(\overline{s_2^{(0)}} - s_2^{(0)}\big) + \big(a_1(\bar{s}_1 - s_1)\big)\cdot s_2\big)a_2\cdots \hat{a}_i\cdots a_n \\ &\equiv 0 \quad mod(S, w). \end{aligned}$$

Case 3. $u_{s_1} = a_0a_1a_2\cdots a_n\cdot s_1$, $u_{s_2} = b_0b_1b_2\cdots b_n\cdot s_2$. We may assume that both $s_1$ and $s_2$ are (0)-monic. Suppose that $\overline{s_1^{(0)}} = c$ and $\overline{s_2^{(0)}} = d$. Then $w = a_0\langle a_1a_2\cdots a_nc\rangle = b_0\langle b_1b_2\cdots b_nd\rangle$ and $a_0 = b_0$.

(3.1) If $c = d$, then $a_i = b_i$ for all $i$ and

$$a_0a_1a_2\cdots a_n\cdot s_1 - b_0b_1b_2\cdots b_n\cdot s_2 = a_0a_1a_2\cdots a_n\cdot(s_1 - s_2).$$

If $s_1^{(1)} = s_2^{(1)} = 0$, i.e., $\bar{s}_1 = \overline{s_1^{(0)}} = \overline{s_2^{(0)}} = \bar{s}_2 = c$, then

$$a_0a_1a_2\cdots a_n\cdot(s_1 - s_2) = a_0a_1a_2\cdots a_n\cdot C_I(s_1, s_2) \equiv 0 \quad mod(S, w).$$

If $s_1^{(1)} \neq 0$, then

$$\begin{aligned} a_0a_1a_2\cdots a_n\cdot(s_1 - s_2) &= \big((a_0a_1)(s_1 - s_2)\big)a_2\cdots a_n \\ &= C_{VI}(s_1, s_2)_{w'}a_2\cdots a_n \\ &\equiv 0 \quad mod(S, w), \end{aligned}$$

where $w' = a_0\langle a_1c\rangle$.

(3.2) If $c \neq d$, say, $c > d$, then $w = a_0\langle cda_1\cdots \hat{a}_i\cdots a_n\rangle = a_0\langle cdb_1\cdots \hat{b}_j\cdots b_n\rangle$ for some $a_i$ and $b_j$.

(3.21) If $d \geqslant b_1$, then $w = a_0b_1\langle cdb_2\cdots \hat{b}_j\cdots b_n\rangle = a_0a_1\langle cda_2\cdots \hat{a}_i\cdots a_n\rangle$, which implies $a_1 = b_1$, $a_2\cdots \hat{a}_i\cdots a_n = b_2\cdots \hat{b}_j\cdots b_n$. Thus,

$$\begin{aligned} a_0a_1a_2\cdots a_n\cdot s_1 - b_0b_1b_2\cdots b_n\cdot s_2 &= \big((a_0b_1d)\cdot s_1\big)a_2\cdots \hat{a}_i\cdots a_n - \big((a_0b_1c)\cdot s_2\big)b_2\cdots \hat{b}_j\cdots b_n \\ &= (a_0b_1d\cdot s_1 - a_0b_1c\cdot s_2)b_2\cdots \hat{b}_j\cdots b_n \end{aligned}$$

$$= \big(a_0b_1(d-s_2)\cdot s_1 - a_0b_1(c-s_1)\cdot s_2\big)b_2\cdots\hat{b}_j\cdots b_n$$
$$\equiv 0 \quad mod(S,w).$$

(3.22) If $d<b_1$, then $w=a_0db_1\cdots b_n=a_0a_1\langle a_2\cdots a_nc\rangle$, which implies $a_1=d$ and $c=b_i$ for some $i$.
(3.221) If $c=b_1<a_0$, then $a_i=b_i$ $(i\geqslant 2)$ and $w=a_0dcb_2\cdots b_n$. We have

$$a_0a_1a_2\cdots a_n\cdot s_1 - b_0b_1b_2\cdots b_n\cdot s_2 = \big((a_0d)\cdot s_1\big)a_2\cdots a_n - \big((a_0c)\cdot s_2\big)a_2\cdots a_n$$
$$= (a_0d\cdot s_1 - a_0c\cdot s_2)a_2\cdots a_n.$$

If $s_1^{(1)}=0$, then we may suppose that $s_1=c+\sum_{c_i<c}\alpha_ic_i$ and $s_2^{(0)}=d+\sum_{d_j<d}\beta_jd_j$. We have

$$(a_0ds_1-a_0cs_2)a_2\cdots a_n = \big((a_0s_1)d+s_1da_0-a_0c\cdot s_2\big)a_2\cdots a_n$$
$$= \Big((a_0s_1)s_2 - a_0c\cdot s_2 + s_1da_0 + \sum_{d_j<d}\beta_js_1a_0d_j\Big)a_2\cdots a_n$$
$$= \Big(a_0(s_1-c)s_2 + s_1da_0 + \sum_{d_j<d}\beta_js_1d_ja_0d_j - \sum_{d_j<d}\beta_j(a_0d_j)s_1\Big)a_2\cdots a_n$$
$$= \Big(\sum_{c_i<c}\alpha_i(a_0c_i)s_2 + s_1da_0 + \sum_{d_j<d}\beta_js_1d_ja_0d_j - \sum_{d_j<d}\beta_j(a_0d_j)s_1\Big)a_2\cdots a_n$$
$$\equiv 0 \quad mod(S,w).$$

If $s_2^{(1)}=0$, then we have

$$(a_0ds_1-a_0c\cdot s_2)a_2\cdots a_n = (a_0ds_1-a_0s_2c-s_2ca_0)a_2\cdots a_n$$
$$= \big(a_0(d-s_2)s_1 - a_0s_2(c-s_1) - s_2ca_0\big)a_2\cdots a_n$$
$$\equiv 0 \quad mod(S,w).$$

If $s_i^{(1)}\neq 0$ $(i=1,2)$, then let $w'=a_0dc$. We have $w=w'a_2\cdots a_n$ and

$$(a_0ds_1-a_0c\cdot s_2)a_2\cdots a_n = C_{VII}(s_2,s_1)_{w'}a_2\cdots a_n$$
$$\equiv 0 \quad mod(S,w).$$

(3.222) If $c=b_i>b_1$ for some $i\geqslant 2$, then

$$a_0a_1a_2\cdots a_n\cdot s_1 - b_0b_1b_2\cdots b_n\cdot s_2 = \big((a_0db_1)\cdot s_1 - (a_0b_1c)\cdot s_2\big)b_2\cdots\hat{b}_i\cdots b_n$$
$$= (a_0b_1d\cdot s_1 + b_1da_0\cdot s_1 - a_0b_1c\cdot s_2)b_2\cdots\hat{b}_i\cdots b_n$$
$$\equiv \big(a_0b_1\overline{s_2^{(0)}}\cdot s_1 - a_0b_1\overline{s_1^{(0)}}\cdot s_2\big)b_2\cdots\hat{b}_i\cdots b_n$$
$$\equiv \big(\big((a_0b_1)\cdot s_2\big)\cdot s_1 - \big((a_0b_1)\cdot s_1\big)\cdot s_2\big)b_2\cdots\hat{b}_i\cdots b_n$$
$$\equiv 0 \quad mod(S,w).$$

The proof is complete. $\square$

**Theorem 2.9** *(Composition-Diamond lemma for metabelian Lie algebras). Let $S \subset \mathcal{L}_{(2)}(X)$ be a nonempty set of monic polynomials and $Id(S)$ be the ideal of $\mathcal{L}_{(2)}(X)$ generated by $S$. Then the following statements are equivalent.*

(i) *$S$ is a Gröbner–Shirshov basis.*
(ii) *$f \in Id(S) \Rightarrow \bar{f} = \overline{u_s}$ for some normal $S$-word $u_s$.*
(iii) *$Irr(S) = \{u \mid u \in N,\ u \neq \overline{v_s}$ for any normal $S$-word $v_s\}$ is a $\mathbf{k}$-basis for $\mathcal{L}_{(2)}(X|S) = \mathcal{L}_{(2)}(X)/Id(S)$.*

**Proof.** (i) ⇒ (ii). Let $S$ be a Gröbner–Shirshov basis and $0 \neq f \in Id(S)$. Then by Lemma 2.3 $f$ has an expression $f = \sum \alpha_i u_{s_i}$, where $0 \neq \alpha_i \in \mathbf{k}$, $u_{s_i}$ are normal $S$-words. Denote $w_i = \overline{u_{s_i}}$, $i = 1, 2, \ldots$. We may assume without loss of generality that

$$w_1 = w_2 = \cdots = w_l > w_{l+1} \geqslant w_{l+2} \geqslant \cdots$$

for some $l \geqslant 1$.

The claim of the theorem is obvious if $l = 1$.

Now suppose that $l > 1$. Then $\overline{u_{s_1}} = w_1 = w_2 = \overline{u_{s_2}}$. By Lemma 2.8, for some $\alpha \in \mathbf{k}$,

$$u_{s_2} \equiv \alpha u_{s_1} \quad mod(S, w_1).$$

Thus,

$$\begin{aligned}\alpha_1 u_{s_1} + \alpha_2 u_{s_2} &= (\alpha_1 + \alpha\alpha_2) u_{s_1} + \alpha_2 (u_{s_2} - \alpha u_{s_1}) \\ &\equiv (\alpha_1 + \alpha\alpha_2) u_{s_1} \quad mod(S, w_1).\end{aligned}$$

Therefore, if $\alpha_1 + \alpha\alpha_2 \neq 0$ or $l > 2$, then the result follows from the induction on $l$. For the case $\alpha_1 + \alpha\alpha_2 = 0$ and $l = 2$, we use the induction on $w_1$. Now the result follows.

(ii) ⇒ (iii). For any $f \in \mathcal{L}_{(2)}(X)$, we have

$$f = \sum_{\overline{u_{s_i}} \leqslant \bar{f}} \alpha_i u_{s_i} + \sum_{\overline{v_j} \leqslant \bar{f}} \beta_j v_j,$$

where $\alpha_i, \beta_j \in \mathbf{k}$, $v_j \in Irr(S)$ and $u_{s_i}$ are normal $S$-words. Therefore, the set $Irr(S)$ generates the algebra $\mathcal{L}_{(2)}(X)/Id(S)$.

On the other hand, suppose that $h = \sum \alpha_i v_i = 0$ in $\mathcal{L}_{(2)}(X)/Id(S)$, where $\alpha_i \in \mathbf{k}$, $v_i \in Irr(S)$. This means that $h \in Id(S)$. Then all $\alpha_i$ must be equal to zero. Otherwise, $\bar{h} = v_j$ for some $j$ which contradicts (ii).

(iii) ⇒ (i). For any $f, g \in S$, we have

$$C_\lambda(f, g)_w = \sum_{\overline{u_{s_i}} < w} \alpha_i u_{s_i} + \sum_{\overline{v_j} < w} \beta_j v_j.$$

Since $C_\lambda(f, g)_w \in Id(S)$ and by (iii), we have

$$C_\lambda(f, g)_w = \sum_{\overline{u_{s_i}} < w} \alpha_i u_{s_i}.$$

Therefore, $S$ is a Gröbner–Shirshov basis. □

**Lemma 2.10.** *(See [10].) Suppose that $f \in \mathcal{L}_{(2)}(X)$. Then there exists an element $f' \in \mathcal{L}_{(2)}(X)$ such that $Id(f) = Id(f')$, $\bar{f}' \leqslant \bar{f}$, $f'^{(0)} = f^{(0)}$ and no word occurring in $f'^{(1)}$ contains $\overline{f^{(0)}}$ as a strict subword.*

**Proof.** If no word occurring in $f^{(1)}$ contains $\overline{f^{(0)}}$ as a strict subword, then we are done. If $\overline{f^{(1)}}$ contains $\overline{f^{(0)}}$ as a strict subword, say $\bar{f} = \overline{f^{(1)}} = a_0a_1 \cdots a_n$, $\overline{f^{(0)}} = a_i$ for some $i \geqslant 2$ or $\overline{f^{(0)}} = a_1$ and $a_0 > a_2$, then let $f_1$ be the composition of type II of $f$ and itself:

$$f_1 = C_{II}(f, f)_{\bar{f}} = f - \beta^{-1}a_0a_1 \cdots \hat{a}_i \cdots a_n \cdot f,$$

where $a_i = \overline{f^{(0)}}$. It is obvious that $Id(f) = Id(f_1)$, and $\bar{f}_1 < \bar{f}$, $f_1^{(0)} = f^{(0)}$. If $\overline{f_1^{(1)}}$ contains $\overline{f^{(0)}}$ as a strict subword, we again consider the composition $f_2 = C_{II}(f_1, f_1)_{\bar{f}_1}$, and so on. By induction on the leading word, we obtain an element $f'$ such that $Id(f) = Id(f')$, $\bar{f}' \leqslant \bar{f}$, $f'^{(0)} = f^{(0)}$, and either $f' = f'^{(0)}$ or $\overline{f'^{(1)}}$ does not contain $\overline{f^{(0)}}$ as a strict subword.

Arguments analogous to the one given above for the leading word also apply to other regular $R$-words occurring in the expansion of $f$ and containing $\overline{f^{(0)}}$ as a strict subword. Finally, we have the one we want. □

**Lemma 2.11.** *Suppose that $\bar{f} = \overline{f^{(1)}} = a_0a_1 \cdots a_n$, $g^{(1)} \neq 0$, $\overline{g^{(0)}} = a_1$ and $a_0 \leqslant a_2$ or $n = 1$. If $f^{(0)} = 0$, then for $a = a_1 < a_0$ and $w = a_0\langle a_1 \cdots a_n a\rangle$, the composition of type IV of $f$ and $g$ is trivial.*

**Proof.** We may suppose that $g$ is (0)-monic. Then

$$\begin{aligned} C_{IV}(f, g)_w &= fa_1 - \bar{f} \cdot g \\ &= r_f^{(1)} \cdot \overline{g^{(0)}} - \bar{f} \cdot r_g^{(0)} \\ &= r_f^{(1)} \cdot \overline{g^{(0)}} - \bar{f} \cdot r_g^{(0)} + r_f^{(1)} \cdot r_g^{(0)} - r_f^{(1)} \cdot r_g^{(0)} \\ &= r_f^{(1)}\big(\overline{g^{(0)}} + r_g^{(0)}\big) - \big(\bar{f} + r_f^{(1)}\big) \cdot r_g^{(0)} \\ &= r_f^{(1)} \cdot g - f \cdot r_g^{(0)} \\ &\equiv 0 \quad mod\big(\{f, g\}, w\big), \end{aligned}$$

where $f = f^{(1)} = \bar{f} + r_f^{(1)}$ and $g^{(0)} = \overline{g^{(0)}} + r_g^{(0)}$. □

**Lemma 2.12.** *The compositions of type I, V and VI formed by $f$ itself are always trivial.*

**Proof.** For type I and VI, the result is obvious. We only check type V. Suppose that $\bar{f} = \overline{f^{(1)}} = a_0a_1 \cdots a_n$, $\overline{f^{(0)}} = b \notin \{a_i\}_{i=1}^n$, and $w = a_0\langle a_1 \cdots a_n b\rangle$. We have

$$\begin{aligned} C_V(f, f)_w &= fb - \beta^{-1}a_0a_1 \cdots a_n \cdot f \\ &= f \cdot \overline{f^{(0)}} - \beta^{-1}\bar{f} \cdot f \\ &= f \cdot \overline{f^{(0)}} - f \cdot \beta^{-1}\big(r^{(1)} + \beta\overline{f^{(0)}} + r^{(0)}\big) \\ &= -\beta^{-1}f \cdot \big(r^{(1)} + r^{(0)}\big) \\ &= \beta^{-1}r^{(1)} \cdot f - \beta^{-1}f \cdot r^{(0)} \\ &\equiv 0 \quad mod(f, w), \end{aligned}$$

where $f^{(1)} = \bar{f} + r^{(1)}$ and $f^{(0)} = \beta\overline{f^{(0)}} + r^{(0)}$, $\beta \in \mathbf{k}$. □

**Remark.** If a subset $S$ of $\mathcal{L}_{(2)}(X)$ is not a Gröbner–Shirshov basis, then one can add all nontrivial compositions of polynomials of $S$ to $S$. Continuing this process repeatedly, we finally obtain a Gröbner–Shirshov basis $S^C$ that generates the same ideal as $S$. Such a process is called Shirshov's algorithm and $S^C$ is called a Gröbner–Shirshov complement of $S$. By Lemma 2.10, we may assume that any element of the original relation set $S$ has no composition of type II formed by itself and Shirshov's algorithm does not involve compositions discussed in Lemmas 2.11 and 2.12.

## 3. Applications

Suppose that $\mathcal{A}$ is a metabelian Lie algebra and $Y=\{a_i, i\in I\}\cup\{b_j, j\in J\}$ is a $\mathbf{k}$-basis of $\mathcal{A}$, where $\{a_i\}$ is a basis of $\mathcal{A}^{(1)}$ and $\{b_j, j\in J\}$ is linear independent modulo $\mathcal{A}^{(1)}$. Suppose that $I$ and $J$ are well-ordered sets. The set of multiplications of $Y$, say $M$, consists of the following:

$$m_{1ij}\colon\ a_ib_j-\sum\gamma_{ij}^k a_k,$$

$$m_{2ij}\colon\ b_ib_j-\sum\delta_{ij}^k a_k\quad (i>j),$$

$$m_{3ij}\colon\ a_ia_j\quad (i>j),$$

where $\gamma_{ij}^k,\delta_{ij}^k\in\mathbf{k}$. Then we have $\mathcal{A}=\mathcal{L}_{(2)}(Y|M)$ and since $Irr(M)=Y$, by Theorem 2.9, $M$ is a Gröbner–Shirshov basis for $\mathcal{A}$ with respect to $a_i>b_j$.

Let $\mathcal{S}$ denote the free metabelian Lie product of $\mathcal{A}$ and a free metabelian Lie algebra generated by a well-ordered set $X=\{x_h\mid h\in H\}$, i.e.,

$$\mathcal{S}=\mathcal{A}*\mathcal{L}_{(2)}(X)=\mathcal{L}_{(2)}(X\cup Y|M).$$

**Theorem 3.1.** *Let the notion be as above. Then with respect to $x_h>a_i>b_j$, a Gröbner–Shirshov complement $M^C$ of $M$ in $\mathcal{L}_{(2)}(X\cup Y)$ consists of $M$ and some $X$-homogeneous polynomials without $(0)$-part, whose leading words are of the form $xy\cdots$ with an $a_i$ as a strict subword, $x\in X$, $a_i, y\in Y$.*

**Proof.** For convenience, we call the $X$-homogeneous polynomials described in the theorem to satisfy property $P_X$.

Since $M$ is a Gröbner–Shirshov basis in $\mathcal{L}_{(2)}(Y)$, we need to check the compositions which are formed by $M$ itself and involve some elements in $X$. The possible types are VI and VII.

First, we check type VI. Suppose that $\overline{m_{1ij}^{(0)}}=\overline{m_{1st}^{(0)}}=a_l$ and the corresponding $w$ is of the forms $xx'a_l$, $xba_l$ and $x\langle aa_l\rangle$ for some $x,x'\in X$, $b\in\{b_j\}$ and $a\in\{a_i\}$.

If $w=xx'a_l$, then

$$\begin{aligned}C_{VI}(m_{1ij},m_{1st})_w&=(xx')\big((\gamma_{ij}^l)^{-1}m_{1ij}-(\gamma_{st}^l)^{-1}m_{1st}\big)\\&=-\sum_{k<l}(\gamma_{ij}^l)^{-1}\gamma_{ij}^k xx'a_k+\sum_{k<l}(\gamma_{st}^l)^{-1}\gamma_{st}^k xx'a_k\\&=-\sum_{k<l}(\gamma_{ij}^l)^{-1}\gamma_{ij}^k xa_kx'+\sum_{k<l}(\gamma_{st}^l)^{-1}\gamma_{st}^k xa_kx'\\&\quad+\sum_{k<l}(\gamma_{ij}^l)^{-1}\gamma_{ij}^k x'a_kx-\sum_{k<l}(\gamma_{st}^l)^{-1}\gamma_{st}^k x'a_kx\end{aligned}$$

and obviously it satisfies $P_X$.

If $w = xba_l$, then

$$C_{VI}(m_{1ij}, m_{1st})_w = (xb)\big(\big(\gamma_{ij}^l\big)^{-1} m_{1ij} - \big(\gamma_{st}^l\big)^{-1} m_{1st}\big)$$
$$= -\sum_{k<l}\big(\gamma_{ij}^l\big)^{-1}\gamma_{ij}^k xba_k + \sum_{k<l}\big(\gamma_{st}^l\big)^{-1}\gamma_{st}^k xba_k$$

and it still satisfies $P_X$.

If $w = xaa_l$, then

$$\begin{aligned} C_{VI}(m_{1ij}, m_{1st})_w &= (xa)\big(\big(\gamma_{ij}^l\big)^{-1} m_{1ij} - \big(\gamma_{st}^l\big)^{-1} m_{1st}\big) \\ &= -\sum_{k<l}\big(\gamma_{ij}^l\big)^{-1}\gamma_{ij}^k xaa_k + \sum_{k<l}\big(\gamma_{st}^l\big)^{-1}\gamma_{st}^k xaa_k \\ &\equiv -\sum_{a_k<a}\big(\gamma_{ij}^l\big)^{-1}\gamma_{ij}^k xa_k a + \sum_{a_k<a}\big(\gamma_{st}^l\big)^{-1}\gamma_{st}^k xa_k a \\ &\quad - \sum_{a_k\geqslant a}\big(\gamma_{ij}^l\big)^{-1}\gamma_{ij}^k xaa_k + \sum_{a_k\geqslant a}\big(\gamma_{st}^l\big)^{-1}\gamma_{st}^k xaa_k \quad mod(M, w), \end{aligned}$$

and again the remainder satisfies $P_X$.

$C_{VI}(m_{1ij}, m_{2st})_w$, $C_{VI}(m_{2ij}, m_{2st})_w$ are similar to $C_{VI}(m_{1ij}, m_{1st})_w$.

Second, we check type VII. Suppose that $\overline{m_{1ij}^{(0)}} = a_p > a_q = \overline{m_{1st}^{(0)}}$ and $w = xa_q a_p$. Then

$$\begin{aligned} C_{VII}(m_{1ij}, m_{1st})_w &= \big(\gamma_{ij}^p\big)^{-1}(xa_q)m_{1ij} - \big(\gamma_{st}^q\big)^{-1}(xa_p)m_{1st} \\ &= -\sum_{k<p}\big(\gamma_{ij}^p\big)^{-1}\gamma_{ij}^k xa_q a_k + \sum_{k<q}\big(\gamma_{st}^q\big)^{-1}\gamma_{st}^k xa_p a_k - x(a_p a_q) \\ &= -\sum_{q\leqslant k<l}\big(\gamma_{ij}^p\big)^{-1}\gamma_{ij}^k xa_q a_k - \sum_{k<q}\big(\gamma_{ij}^p\big)^{-1}\gamma_{ij}^k xa_k a_q - \sum_{q\leqslant k<l}\big(\gamma_{ij}^p\big)^{-1}\gamma_{ij}^k x(a_q a_k) \\ &\quad + \sum_{k<q}\big(\gamma_{st}^l\big)^{-1}\gamma_{st}^k xa_k a_p + \sum_{k<q}\big(\gamma_{st}^l\big)^{-1}\gamma_{st}^k x(a_p a_k) - x(a_p a_q) \\ &\equiv -\sum_{q\leqslant k<l}\big(\gamma_{ij}^p\big)^{-1}\gamma_{ij}^k xa_q a_k - \sum_{k<q}\big(\gamma_{ij}^p\big)^{-1}\gamma_{ij}^k \\ &\quad + xa_k a_q \sum_{k<q}\big(\gamma_{st}^l\big)^{-1}\gamma_{st}^k xa_k a_p \quad mod(M, w), \end{aligned}$$

and the remainder has property $P_X$. One may check that $C_{VII}(m_{1ij}, m_{2st})_w$ and $C_{VII}(m_{2ij}, m_{2st})_w$ are the same as $C_{VII}(m_{1ij}, m_{1st})_w$, which have property $P_X$.

Observing the above and the definition of compositions, we know that the nontrivial compositions of polynomials satisfying $P_X$ themselves are only of type I and the results again satisfy $P_X$. Also by the definition of compositions and property $P_X$, the compositions of $M$ and polynomials satisfying $P_X$ are only of type II and the results still satisfy $P_X$. The theorem is proved. □

Observing the proof of the above theorem, we have the following proposition.

**Proposition 3.2.** *Let $\mathcal{A}_i = \mathcal{L}_{(2)}(X_i|S_i)$, where $S_i \subset \mathcal{L}_{(2)}(X_i)^{(1)}$, $i = 1, 2$. Then $S_1^C \cup S_2^C$ is a Gröbner–Shirshov basis for the free metabelian Lie product $\mathcal{A}_1 * \mathcal{A}_2$, where $S_i^C$ is a Gröbner–Shirshov complement of $S_i$ in $\mathcal{L}_{(2)}(X_i)$, $i = 1, 2$.*

Now, we consider partial commutative metabelian Lie algebras related to some graphs.

Let $\Gamma = (V, E)$ be a graph, where $V$ is the set of vertices and $E$ the set of edges. For $e \in E$ we call $o(e)$ the origin of $e$ and $t(e)$ the terminus. We say a metabelian Lie algebra is partial commutative related to a graph $\Gamma = (V, E)$, denoted by $\mathcal{ML}_\Gamma$, if

$$\mathcal{ML}_\Gamma = \mathcal{L}_{(2)}\big(V \bigm| \big[o(e), t(e)\big] = 0,\ e \in E\big).$$

In this section, we find Gröbner–Shirshov bases for partial commutative metabelian Lie algebras related to circuits, trees and 3-cube.

The following algorithm gives a Gröbner–Shirshov basis for partial commutative metabelian Lie algebras with a finite relation set.

**Algorithm 3.3.** Input: relations $f_1, \ldots, f_s$ of $\mathcal{L}_{(2)}(X)$, $f_i = xx'$, $F = \{f_1, \ldots, f_s\}$.

Output: a Gröbner–Shirshov basis $H = \{h_1, \ldots, h_t\}$ for $\mathcal{L}_{(2)}(X|F)$.

Initialization: $H := F$

While: $f_i = x_{i_0}x_{i_1}\cdots x_{i_n}$, $f_i = x_{j_0}x_{j_1}\cdots x_{j_m}$, and $x_{i_0} = x_{j_0}$, $x_{i_1} \neq x_{j_1}$

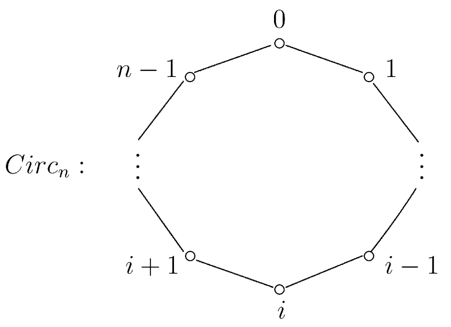
Then Do: $h := \max\{x_{i_1}, x_{j_1}\}\min\{x_{i_1}, x_{j_1}\}\langle x_{t_1}x_{t_2}\cdots x_{t_l}\rangle$
where $\{x_{t_1}, x_{t_2}, \ldots, x_{t_l}\} = \{x_{i_0}, x_{i_2}, \ldots, x_{i_n}\} \cup \{x_{j_2}, \ldots, x_{j_m}\}$

If: there is no $f_j \in H$ such that $f_j$ is a subword of $h$

Do: $H := H \cup \{h\}$

End

**Definition 3.4.** Let $n$ be a positive integer. A *circuit* (*of length n*), denoted by *Circ$_n$*, is a graph for which the set of vertices is $\mathbf{Z}/n\mathbf{Z}$ and the orientation is given by $n$ edges $e_{i,i+1}$, $i \in \mathbf{Z}/n\mathbf{Z}$, with $o(e_{i,i+1}) = i$ and $t(e_{i,i+1}) = i + 1$.




**Theorem 3.5.** *For the partial commutative metabelian Lie algebra related to Circ$_n$*

$$\mathcal{ML}_{Circ_n} = \mathcal{L}_{(2)}\big(\mathbf{Z}/n\mathbf{Z} \bigm| [i+1, i] = 0,\ i \in \mathbf{Z}/n\mathbf{Z}\big),$$

*with the usual ordering on natural numbers, a Gröbner–Shirshov basis for $\mathcal{ML}_{Circ_n}$ consists of the following relations*:

$$f_0\colon\ [n-1, 0] = 0,$$

$$f_i\colon\ [i, i-1] = 0, \quad 1 \leqslant i \leqslant n-1,$$

$$g_j\colon\ [j,0,j+1,j+2,\ldots,n-1]=0,\quad 2\leqslant j\leqslant n-2,$$

*where the brackets* $[\cdots]$ *are the left-normed brackets.*

**Proof.** The only possible compositions are of type $I$ by $f_{n-1}, f_0$ and $g_j, f_j$, where the corresponding $w$'s are $[n-1,0,n-2]$ and $[j,0,j-1,j+1,j+2,\ldots,n-1]$ respectively.

For the first one, $w=[n-1,0,n-2]$ and

$$\begin{aligned} C_I(f_{n-1},f_0)_w &= [n-1,n-2]\cdot 0-[n-1,0,n-2] \\ &= [n-2,0,n-1] \\ &\equiv 0 \quad mod(g_{n-2},w). \end{aligned}$$

For the second one, $w=[j,0,j-1,j+1,j+2,\ldots,n-1]$ and

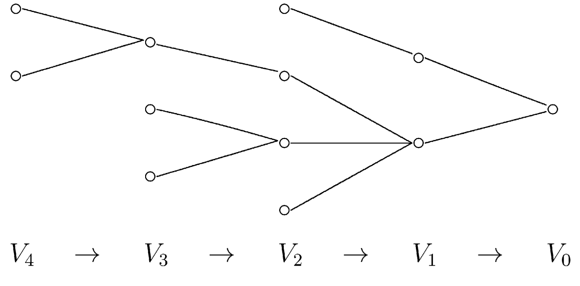
$$\begin{aligned} C_I(g_j,f_j)_w &= [j,0,j+1,j+2,\ldots,n-1]\cdot(j-1)-[j,j-1,0,j+2,\ldots,n-1] \\ &= [j-1,0,j,j+1,j+2,\ldots,n-1]. \end{aligned}$$

Then it is trivial modulo $f_2$ if $j=2$ and modulo $g_{j-1}$ if $j\geqslant 3$. $\square$

**Definition 3.6.** A *tree* is a connected nonempty graph without circuits.

A geodesic in a tree is a path without backtracking. The length of the geodesic from $v$ to $v'$ is called the distance from $v$ to $v'$, and is denoted by $l(v,v')$.

Fix a vertex $v_0$ of a tree $\Gamma$. For each integer $n\geqslant 0$, let $V_n$ be the set of vertices $v$ of $\Gamma$ such that $l(v_0,v)=n$. Then the set of vertices of $\Gamma$ is the union of $V_n$ and $V_i\cap V_j=\emptyset,\ i\neq j$. If $v\in V_n$ with $n\geqslant 1$, there is a single vertex $v'\in V_{n-1}$ from $v_0$ to which $v$ is adjacent.




We linearly order the set of vertices $V=\bigcup_{n\geqslant 0}V_n$ such that $v_0$ is the smallest element and for any $v\in V_i,\ v'\in V_j,\ v<v'$ if $i<j$. Then the partial commutative metabelian Lie algebra related to the tree $\Gamma$ is defined by:

$$\mathcal{ML}_\Gamma=\mathcal{L}_{(2)}(V|R),$$

where

$$R=\bigl\{\bigl[v',v\bigr]=0 \bigm| v'\in V_{n+1},\ v\in V_n,\ v' \text{ and } v \text{ are adjacent},\ n\geqslant 0\bigr\}.$$

**Theorem 3.7.** *The relation set R forms a Gröbner–Shirshov basis for the partial commutative metabelian Lie algebra* $\mathcal{ML}_\Gamma$ *related to the tree* $\Gamma$.

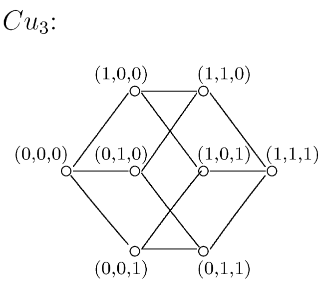
**Proof.** It is obvious that for any $v' \in V_{n+1}$, there is only one element $v \in V_n$ such that the relation $[v', v] = 0$ lies in $R$, which means there is no composition in $R$ at all. Thus, $R$ is a Gröbner–Shirshov basis automatically. $\square$

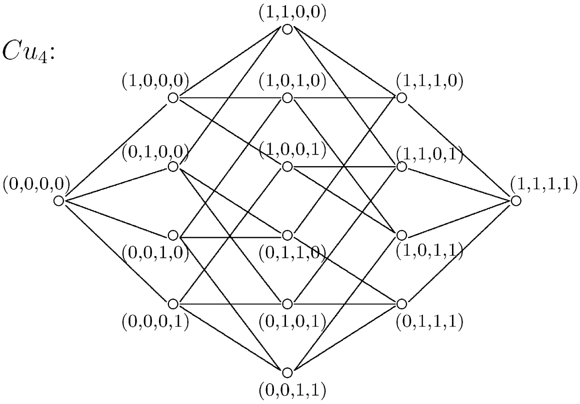
By Theorems 2.9 and 3.7, we have the following corollary.

**Corollary 3.8.** *A linear basis of $\mathcal{ML}_\Gamma$ consists of regular words $v_0v_1\cdots v_n$ $(n \geqslant 0)$ on $V$ satisfying the following condition: if $v_0 > v_i$ $(i \geqslant 1)$, then $l(v_0, v_i) \neq 1$.*

**Definition 3.9.** Let $n$ be a positive integer. An *n-cube*, denoted by $Cu_n$, is a graph for which the set of vertices $V_n = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{R}^n \mid \varepsilon_i = 0 \text{ or } 1\}$ and two vertices $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$ are adjacent if $\exists i$, such that $\varepsilon_i = \delta_i + 1$ *mod* 2 and $\varepsilon_j = \delta_j$ for any $j \neq i$.

For example, 3-cube and 4-cube are the followings:




We order all vertices lexicographically. The distance of $\varepsilon$ and $\delta$ is $d(\varepsilon, \delta) = \sum_{i=1}^{n} |\varepsilon_i - \delta_i|$. Then the partial commutative metabelian Lie algebra related to the $n$-cube $Cu_n$ is defined by:

$$\mathcal{ML}_\Gamma = \mathcal{L}_{(2)}\big(V_n \,\big|\, \varepsilon\delta = 0,\ d(\varepsilon, \delta) = 1\big).$$

**Theorem 3.10.** *A Gröbner–Shirshov basis $S$ for the partial commutative metabelian Lie algebra related to 3-cube*

$$\mathcal{ML}_{Cu_3} = \mathcal{L}_{(2)}\big(V_3 \,\big|\, \varepsilon\delta,\ d(\varepsilon, \delta) = 1, \varepsilon > \delta\big)$$

*is the union of the following*:

$$R_2 = \big\{\lfloor\varepsilon\delta\rfloor \,\big|\, d(\varepsilon, \delta) = 1\big\},$$
$$R_3 = \big\{\lfloor\varepsilon\delta\rfloor\mu \,\big|\, d(\varepsilon, \delta) = 2,\ \mu\varepsilon, \mu\delta \in R_1\big\},$$
$$R_4 = \big\{\lfloor\varepsilon\delta\rfloor\mu\gamma \,\big|\, d(\varepsilon, \delta) = 3,\ \mu\varepsilon \in R_2, \mu\delta\gamma \in R_3\big\},$$
$$R_5 = \big\{\lfloor\delta_1\delta_2\rfloor\gamma\langle\mu_1\mu_2\rangle \,\big|\, d(\delta_1, \delta_2) = 2,\ \gamma\delta_i\mu_i \in R_3,\ i = 1, 2\big\},$$
$$R_5' = \big\{\lfloor\delta_1\delta_2\rfloor\gamma\mu\mu' \,\big|\, d(\delta_1, \delta_2) = 2,\ \gamma\delta_1 \in R_2,\ \gamma_2\mu\mu' \in R_4,\ d(\mu, \delta_1) \neq 1\big\},$$

*where* $\lfloor\varepsilon\delta\rfloor = \max\{\varepsilon, \delta\}\min\{\varepsilon, \delta\}$.

By Algorithm 3.3, we have that a reduced Gröbner–Shirshov basis (it means there is no composition of type I, II, III) for the partial commutative metabelian Lie algebra related to 4-cube $\mathcal{ML}_{Cu_4}$ consists of 268 relations.

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