Isomorphisms between Infinite Matrix Rings

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ABSTRACT

A characterization of ring isomorphisms between endomorphism rings of infinitely generated free modules (over arbitrary rings) is given in terms of Morita equivalences. In addition, we indicate the interrelationship between our characterization and the classical description which uses semilinear maps.

Let R and S denote arbitrary associative rings with identity, and let $R^{(N)}$ and $S^{(N)}$ be free right modules with countably infinite basis over R and S respectively. We wish to describe the isomorphisms between $\mathrm{End}_R(R^{(N)})$ (the ring of column finite matrices over R) and $\mathrm{End}_S(S^{(N)})$ (the ring of column finite matrices over S). These isomorphisms are known when R and S are principal ideal domains [6], and when $R=S$ and R has the property that certain projective modules are free [4]. The descriptions obtained in both were in terms of semilinear maps between the underlying modules. A similar situation occurred in describing the isomorphisms for free modules with finite basis, and it was realized [1] that the semilinear description must, and can, be replaced by a categorical description if one wishes to include *all* rings. Thus we pursue a categorical description for the isomorphisms between rings of column finite matrices, and show how the semilinear descriptions in [4] and [6] can be recovered from the categorical.

For sake of brevity, we set $U=R^{(N)}$ and $V=S^{(N)}$. Let $\{u_i\}$ be a basis for U, $\{v_i\}$ be a basis for V, $\{e_{ij}\}$ be the standard matrix units for $\{u_i\}$ [i.e., $e_{ij}(u_j)=u_i$, and $e_{ij}(u_k)=0$ for $k\neq j$], and $\{f_{ij}\}$ be the standard matrix units for $\{v_i\}$. Let Φ be any ring isomorphism:

$$\Phi:\mathrm{End}_R(U)\xrightarrow{\sim}\mathrm{End}_S(V).$$

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It is shown by Camillo in [2] that $\text{End}_R(U)$ and $\text{End}_S(V)$ are isomorphic if and only if R and S are Morita equivalent. We recall now some facts from the proof of his result. The right S-module $\Phi(e_{11})V$ is a progenerator over S, with

$$\begin{aligned} R &\simeq e_{11}\,\text{End}_R(U)\,e_{11} \\ &= e_{11}\Phi^{-1}(\text{End}_S(V))e_{11} \\ &\simeq \Phi(e_{11})\,\text{End}_S(V)\,\Phi(e_{11}) \\ &\simeq \text{End}_S(\Phi(e_{11})V). \end{aligned} \tag{1}$$

In addition, one has

$$\Phi(e_{i1}) : \Phi(e_{11})V \xrightarrow{\sim} \Phi(e_{ii})V, \tag{2}$$

an S-module isomorphism for any i. Moreover, $\sum_i \Phi(e_{ii})V$ is direct.

LEMMA. *$\sum_i \Phi(e_{ii})V$ is all of V.*

Proof. We show that $v_i s$ is in $\sum_i \Phi(e_{ii})V$ for any i, $s \in S$. Consider f_{ii}; using an argument similar to one in Camillo's proof, one can find an integer k such that

$$[\Phi(e_{11}) + \Phi(e_{22}) + \cdots + \Phi(e_{kk})]\, f_{ii} = f_{ii}.$$

Now,

$$\begin{aligned} v_i s \in f_{ii}V &= [\Phi(e_{11}) + \cdots + \Phi(e_{kk})]\, f_{ii}V \\ &\subseteq [\Phi(e_{11}) + \cdots + \Phi(e_{kk})]\, V \\ &\subseteq \sum_i \Phi(e_{ii})V. \end{aligned}$$ ■

Let $\mathcal{M}_R$ denote the category of unital right R-modules; similarly $\mathcal{M}_S$. We are now ready to prove our result.

THEOREM. *Let R, S be arbitrary associative rings with identity. Let $U = R^{(N)}$, $V = S^{(N)}$, where N is a countably infinite set. Let*

$$\Phi : \text{End}_R(U) \xrightarrow{\sim} \text{End}_S(V)$$

be a ring isomorphism. Then there exists a unique (up to natural isomorphism) category equivalence

$$F_\Phi : \mathcal{M}_R \to \mathcal{M}_S$$

such that $F_\Phi(U) = V$, *and* $F_\Phi(f) = \Phi(f)$ *for all* $f \in \mathrm{End}_R(U)$.

Proof. Existence: Retain all notation from the above. For any $r \in R$, let

$$A_r = \begin{bmatrix} r & 0 & 0 & \cdots \\ 0 & r & 0 & \\ 0 & 0 & r & \\ \vdots & & & \ddots \end{bmatrix} \quad \text{and} \quad B_r = \begin{bmatrix} r & 0 & 0 & \cdots \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ \vdots & & & \ddots \end{bmatrix}.$$

From (1) there exists a natural left R-structure on $P = \Phi(e_{11})V$ given by

$$\begin{aligned} r \cdot p &= r \cdot (\Phi(e_{11})v) \\ &= \Phi(e_{11})\Phi(A_r)\Phi(e_{11})v \\ &= \Phi(e_{11}A_r e_{11})v \\ &= \Phi(B_r)v. \end{aligned}$$

We have a natural isomorphism $\gamma : U \otimes_R P \to P^{(N)}$, and using the isomorphisms of (2) we have an obvious isomorphism $\delta : P^{(N)} \to \sum_i \Phi(e_{ii})V = V$. Consider [for any $f \in \mathrm{End}_R(U)$] the diagram

$$\begin{array}{ccccc} & U \otimes_R P & \xrightarrow{f \otimes 1} & U \otimes_R P & \\ \gamma & \downarrow & & \downarrow & \gamma \\ & P^{(N)} & & P^{(N)} & \\ \delta & \downarrow & & \downarrow & \delta \\ & V & \xrightarrow{\Phi(f)} & V & \end{array}$$

We will show this diagram commutes. Since all maps are additive, we need only check on a generator $u_i r\otimes p$ of $U\otimes P$. We want

$$\delta\circ\gamma\circ(f\otimes 1)(u_i r\otimes p)=\Phi(f)\circ\delta\circ\gamma(u_i r\otimes p). \tag{3}$$

In the arguments below, $u_i=(0,\dots,1,0,\dots)$, and f will be considered to be a column finite matrix.

For the left hand side of (3),

$$\delta\circ\gamma\circ(f\otimes 1)((0,\dots,r,0,\dots)\otimes p)$$

$$=\delta\circ\gamma(f(0,\dots,r,0,\dots)\otimes p)$$

$$=\delta\circ\gamma((\alpha_1 r,\alpha_2 r,\dots,\alpha_n r,0,\dots)\otimes p)$$

[where $(\alpha_1,\alpha_2,\dots,\alpha_n,0,\dots)$ forms the ith column of f]

$$=\delta((\alpha_1 r)p,(\alpha_2 r)p,\dots,(\alpha_n r)p,0,\dots)$$

$$=\delta(b_1\cdot p,b_2\cdot p,\dots,b_n\cdot p,0,\dots)\qquad(\text{where}\quad b_i=\alpha_i r)$$

$$=\delta(b_1\cdot\Phi(e_{11})v,\dots,b_n\cdot\Phi(e_{11})v,0,\dots)$$

[since $P=\Phi(e_{11})v$]

$$=\delta\big(\Phi(B_{b_1})v,\dots,\Phi(B_{b_n})v,0,\dots\big)$$

(this comes from the R-structure on P)

$$=\sum_i\Phi(e_{i1})\Phi(B_{b_i})v$$

$$=\left[\sum_i\Phi(e_{i1}B_{b_1})\right]v$$

$$=\Phi\left(\sum_i e_{i1}B_{b_i}\right)v$$

$$=\Phi(C)v,$$

where C is the matrix with $(b_1,\dots,b_n,0,\dots)$ in the first column, and zero everywhere else.

For the right hand side of (3),

$$\begin{aligned}
&\Phi(f)\circ\delta\circ\gamma((0,\ldots,r,0,\ldots)\otimes p)\\
&\quad=\Phi(f)\circ\delta(0,\ldots,rp,0,\ldots)\\
&\quad=\Phi(f)\circ\gamma(0,\ldots,\Phi(B_r)v,0,\ldots)\\
&\quad=\Phi(f)(\Phi(e_{i1})\Phi(B_r)v)\\
&\quad=\Phi(fe_{i1}B_r)v\\
&\quad=\Phi(C)v,
\end{aligned}$$

where C is the same matrix as above. Therefore, the diagram commutes.

Now define $F_\Phi:\mathscr{M}_R\to\mathscr{M}_S$ by $F_\Phi(M)=M\otimes_R P$, except $F_\Phi(U)=V$, and for $f:M\to N$, then $F_\Phi(f)=T_N(f\otimes 1)T_M^{-1}$, with $T_M=\mathrm{Id}_M$ except $T_U=(\gamma\circ\delta)$. It is straightforward to show that F_Φ is an equivalence with the desired properties.

Uniqueness: Suppose $G:\mathscr{M}_R\to\mathscr{M}_S$ is a category equivalence such that $G(U)=V$, $G(f)=\Phi(f)$. We want to show G is naturally isomorphic to F_Φ. Choose an inverse for G, and set

$$H=G^{-1}\circ F_\Phi.$$

It will suffice to show that H is naturally isomorphic to the identity autoequivalence on $\mathscr{M}_R$.

By the properties of G and the definition of G^{-1}, we have an isomorphism $\alpha:H(U)\xrightarrow{\sim}U$ such that

$$\begin{array}{ccc}
U & \xrightarrow{\ f\ } & U\\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\alpha}\\
H(U) & \xrightarrow{\ H(f)\ } & H(U)
\end{array}\qquad (4)$$

commutes for all $f\in\mathrm{End}(U)$.

From Morita theory, there exists an R-R bimodule such that H is naturally isomorphic to $\_\otimes_R P$. This fact together with (4) yields a diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & U \\ {\scriptstyle \tau}\uparrow & & \uparrow{\scriptstyle \tau} \\ U\otimes P & \xrightarrow{\quad f\otimes 1 \quad} & U\otimes P \end{array} \tag{5}$$

which commutes for all $f \in \operatorname{End}_R(U)$, and τ is an R-module isomorphism. In particular, inserting $e_{11} \in \operatorname{End}_R(U)$, we find that $P \simeq \operatorname{image}(e_{11}\otimes 1)$ is right R-isomorphic to $R \simeq \operatorname{image}(e_{11})$. It follows now that P is actually equal to the bimodule ${}_\sigma R_1$ where $\sigma: R \to R$ is a ring automorphism, $H \simeq \_\otimes_\sigma R_1$, and H is naturally isomorphic to the identity functor if σ is an inner automorphism (see [5] for details).

Extend σ to $\bar{\sigma}: \operatorname{End}_R(U) \xrightarrow{\sim} \operatorname{End}_R(U)$ by acting on each element of the matrix. Let $\psi: U\otimes_\sigma R_1 \to U$ be the R-isomorphism given by

$$(r_1, r_2, \dots, r_n, 0, \dots)\otimes r \to (r_1^\sigma r, r_2^\sigma r, \dots, r_n^\sigma r, 0, \dots).$$

Using (5), we can put together a commuting diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad f \quad} & U \\ {\scriptstyle \tau}\uparrow & & \uparrow{\scriptstyle \tau} \\ U\otimes_\sigma R_1 & \xrightarrow{\quad f\otimes 1 \quad} & U\otimes_\sigma R_1 \\ {\scriptstyle \psi^{-1}}\uparrow & & \uparrow{\scriptstyle \psi^{-1}} \\ U & \xrightarrow{\quad \bar{\sigma}(f) \quad} & U \end{array}$$

for any $f \in \operatorname{End}_R(U)$. (It is a straightforward check to see that the bottom square commutes.)

In particular let $r \in R$, and consider A_r, the infinite matrix with r on the diagonal and 0 elsewhere. Let $(t_1, \dots, t_n, 0, \dots)$ be the image of u_1 under

$\tau \circ \psi^{-1}$. Then by commutativity

$$\begin{aligned} A_r(\tau \circ \psi^{-1})u_1 &= A_r(t_1, \ldots, t_n, 0, \ldots) \\ &= (rt_1, \ldots, rt_n, 0, \ldots) \\ &= (\tau \circ \psi^{-1})\bar{\sigma}(A_r)u_1 \\ &= (\tau \circ \psi^{-1})(r^\sigma, 0, \ldots) \\ &= (\tau \circ \psi^{-1})u_1 \cdot r^\sigma \\ &= (t_1 r^\sigma, \ldots, t_n r^\sigma, 0, \ldots), \end{aligned}$$

and thus

$$rt_i = t_i r^\sigma \tag{6}$$

for all $1 \leqslant i \leqslant n$, and all $r \in R$.

Now use commutativity of (5) applied to the matrix $A_r - A_r e_{11}$, and acting on u_1, to show that

$$(0, 0, \ldots) = (0, rt_2, rt_3, \ldots, rt_n, 0, \ldots),$$

and (for $r = 1$) deduce that $t_2 = t_3 = \cdots = t_n = 0$. Now $(t_1, 0, \ldots)$ is the image of u_1 under the isomorphism $\tau \circ \psi^{-1}$, and it follows that t_1 must be invertible.

From (6), we have $rt_1 = t_1 r^\sigma$ for all $r \in R$, and since t_1 is invertible, σ must be inner. ■

The semilinear descriptions can now be recovered under certain conditions. Recall that the isomorphism $\Phi: \mathrm{End}_R(U) \to \mathrm{End}_S(V)$ is said to be induced by a semilinear if there exists:

(1) a ring isomorphism $\sigma: R \xrightarrow{\sim} S$,
(2) a σ-semilinear isomorphism $\varphi: U \to V$ [i.e., φ is group isomorphism with $\varphi(ur) = \varphi(u)r^\sigma$ for all $u \in U$, $r \in R$]

such that $\Phi(f) = \varphi f \varphi^{-1}$ for all $f \in \mathrm{End}_R(U)$.

Corollary. *Let $R = S$, together with the hypothesis of the Theorem (in particular, $F_\Phi \in \mathrm{Pic}(R)$). Then Φ is induced by a semilinear if F_Φ is in the* *image of*

$$\psi: \quad \text{Aut}(R) \to \text{Pic}(R)$$

$(\psi(\sigma) = \_\otimes_\sigma R_1)$.

Proof. If F_Φ is in the image of ψ, then by our Theorem we have a commuting diagram

$$\begin{array}{ccc} U & \xrightarrow{F_\Phi(f)} & U \\ \text{Id}\downarrow & & \downarrow\text{Id} \\ U & \xrightarrow{\Phi(f)} & U \\ \alpha\downarrow & & \downarrow\alpha \\ U\otimes_\sigma R_1 & \xrightarrow{f\otimes \text{Id}} & U\otimes_\sigma R_1 \\ \text{nat}\downarrow & & \downarrow\text{nat} \\ U_{\sigma^{-1}} & \xrightarrow{f} & U_{\sigma^{-1}} \end{array}$$

for all $f \in \text{End}_R(U)$, and some fixed α. Let ϕ be the isomorphism of groups, $\phi: U \to U$ given by $\alpha^{-1} \circ \text{nat}^{-1}$. Clearly ϕ is σ-semilinear, and $\Phi(f) = \phi f \phi^{-1}$. ∎

The semilinear description in [6] now follows easily, since Morita equivalent principal ideal domains are isomorphic, and for such rings the map ψ above is onto (in a principal ideal domain projectives are free).

Similarly, in the particular case when $R = S$, McDonald in [5] has a semilinear description for automorphisms of $\text{End}_R(U)$, provided R has the property:

($*$) if P is projective over R and $\text{End}(P) \simeq R$, then P is free of rank 1.

This result is more general than an automorphism version of our Theorem in that he considers infinitely generated projectives (with at least one unimodular element) instead of free modules. On the other hand, for free modules our corollary provides a semilinear description under the less restrictive condition that $\phi: \text{Aut}(R) \to \text{Pic}(R)$ is onto. To see that our property is less restrictive than property ($*$), consider the ring $R = \Pi_{p_i} \mathbb{Z}_{p_i}$ where p_i ranges over all primes. This ring does not satisfy property ($*$), since $\text{End}_R(\Sigma_{p_i} \mathbb{Z}_{p_i}) \simeq R$ with

$\sum_{p_i} \mathbb{Z}_{p_i}$ projective and not free. However, ψ is onto for this ring. Since R is commutative, ψ induces a splitting: $\text{Picent}(R) \oplus \text{Aut}(R) \simeq \text{Pic}(R)$. But $\text{Picent}(\mathbb{Z}_{p_i})$ is trivial for all p_i, and by [3] we know that $\text{Picent}(R)$ must be trivial.

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