Topology and its Applications 28 (1988) 277-287
North-Holland

UNORIENTED BORDISM FOR ODD-ORDER GROUPS

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Received 3 September 1986
Revised 15 December 1986

We consider the action of the Burnside ring on equivariant unoriented bordism for an odd-order group. Known splittings of the bordism rings are shown to correspond to idempotents of the Burnside ring.

AMS (MOS) Subj. Class.: Primary 57R85;
secondary 55N22, 55N91, 55P91, 55Q91, 57S17

Burnside ring, equivariant bordism, fixed points, idempotents, splitting, equivariant homology, odd-order groups

0. Introduction

This paper is mainly an exposition of the use of the Burnside ring in equivariant bordism. The main results are the splittings of the unoriented geometric and homotopical bordism rings for odd-order groups, and a fairly explicit calculation of the multiplicative structures of these rings (Theorems 4.7 and 7.4). These splittings were already known (see [6], [10] and [12]), but by other techniques. In particular, [12, § 13] gives a similar method, using the 0-dimensional bordism ring, and at the end of that section a question is asked: is this splitting given by projections corresponding to idempotents in the bordism ring? Theorem 4.7 answers this affirmatively, at least for odd-order groups.

The study of the Burnside ring, and its use as in this paper to produce splittings of equivariant homology and cohomology theories, is due mainly to T. tom Dieck (see, for example, [4]). Another source for the general study and use of this ring is [9]. The arguments given here are, for the most part, just the general arguments of these references, written out in a concrete way for the specific example of bordism theory.

An important caveat is needed. The results in this paper depend very much on the fact that we are using odd-order groups. Since bordism is 2-local, we are really working away from the order of the group, and it is well-known that the phenomena we see here will be very much like those we see in the nonequivariant world. This

0166-8641/88/$3.50

is made precise by the complete splitting of the Burnside ring, and the corresponding splittings of equivariant homology theories, like bordism theory. The real interest and real work comes in studying 2-groups, where new things will happen (see, for example, [1] and [6]).

1. The Burnside ring

Let G be a finite odd-order group. Recall that the Burnside ring $A(G)$ is defined to be the Grothendieck group of finite G-sets, with multiplication given by Cartesian product. It is the free abelian group on the distinct orbits G/H. Since we are concerned with the action of $A(G)$ on the unoriented bordism groups, we really are interested in $A(G)\otimes\mathbb{Z}_2$. This ring has a particularly simple form when G has odd order.

For any subgroup H of G, let $\varphi_H : A(G)\otimes\mathbb{Z}_2 \to \mathbb{Z}_2$ be the ring homomorphism which assigns to a G-set S the number of points of S fixed by H, reduced mod 2. Writing (H) for the conjugacy class of H in G, let $\prod_{(H)} \mathbb{Z}_2$ denote the direct product of one copy of $\mathbb{Z}_2$ for each conjugacy class in G. Let $\varphi : A(G)\otimes\mathbb{Z}_2 \to \prod_{(H)} \mathbb{Z}_2$ be the ring map defined by the maps φ_H.

Theorem 1.1. *$\varphi : A(G)\otimes\mathbb{Z}_2 \to \prod_{(H)} \mathbb{Z}_2$ is a ring isomorphism when G is an odd-order group.*

Proof. φ is a ring map, so it suffices to show that it is an isomorphism of $\mathbb{Z}_2$-vector spaces. $A(G)\otimes\mathbb{Z}_2$ has a basis given by the classes of the orbits G/H. Since $\varphi_H(G/H)=1$ and $\varphi_K(G/H)=0$ if K is not subconjugate to H, it is easy to see that φ maps this basis to a basis, hence is an isomorphism. □

This theorem implies that every element of $A(G)\otimes\mathbb{Z}_2$ is an idempotent. We are interested in a particular set of idempotents.

Corollary 1.2. *There are idempotents $e_H \in A(G)\otimes\mathbb{Z}_2$, characterized by*

$$\varphi_K(e_H) = \begin{cases} 1 & \text{if } (K)=(H); \\ 0 & \text{if } (K)\neq(H). \end{cases}$$

Moreover, $e_H e_K = 0$ if $(H)\neq(K)$ and $\sum_{(H)} e_H = 1$.

Given any decomposition of the identity into orthogonal idempotents like this, we get a corresponding splitting of every $A(G)\otimes\mathbb{Z}_2$-module. Thus we have the following:

Corollary 1.3. *If U is an $A(G)\otimes\mathbb{Z}_2$-module, then*

$$U \cong \bigoplus_{(H)} e_H U$$

as $A(G)\otimes\mathbb{Z}_2$-modules. Moreover, if U is an $A(G)\otimes\mathbb{Z}_2$-algebra, then this is a splitting of algebras.

2. Unoriented bordism and families

We will now quickly review some basic facts about unoriented bordism. Let X be a G-space. We denote by $\mathcal{N}_n^G(X)$ the group of n-dimensional closed G-manifolds over X under the equivalence relation of G-bordism over X. This is the familiar generalization of the nonequivariant definition of unoriented bordism.

Recall the notion of restricted bordism introduced by Conner and Floyd [2]. Let $\mathscr{F}$ be a family of subgroups of G, i.e., a collection of subgroups closed under conjugation and passage to subgroups. An *$\mathscr{F}$-manifold* is a G-manifold, all of whose isotropy groups are members of $\mathscr{F}$. If we restrict our manifolds to be $\mathscr{F}$-manifolds, then we get a new theory, restricted bordism, denoted $\mathcal{N}_*^G[\mathscr{F}](X)$. If $\mathscr{F}'\subset\mathscr{F}$ are two families in G, then an $(\mathscr{F},\mathscr{F}')$-manifold is an $\mathscr{F}$-manifold whose boundary is an $\mathscr{F}'$-manifold. Considering such manifolds, we can define the restricted bordism groups $\mathcal{N}_*^G[\mathscr{F},\mathscr{F}'](X)$ (see [2] for details). If we agree that the only $\emptyset$-manifold is the empty set, then an $(\mathscr{F},\emptyset)$-manifold is the same thing as an $\mathscr{F}$-manifold, and $\mathcal{N}_*^G[\mathscr{F}](G)=\mathcal{N}_*^G[\mathscr{F},\emptyset](X)$. We will need the following fact.

Proposition 2.1. *Suppose H is a subgroup of G, $\mathscr{F}'\subset\mathscr{F}$ are families in G, and X is an H-space. Then*

$$\mathcal{N}_*^H[\mathscr{F}\,|\,H,\mathscr{F}'\,|\,H](X)\cong\mathcal{N}_*^G[\mathscr{F},\mathscr{F}'](G\times_H X).$$

Here $\mathscr{F}\,|\,H=\{K\in\mathscr{F}\,|\,K\subset H\}$, and $G\times_H X$ is $G\times X$ modulo the relation $(g,x)\sim(gh,h^{-1}x)$ for $h\in H$.

Proof. This follows from general stable homotopy theory, or we can establish the correspondence as follows. Given an H-manifold M over X, the corresponding G-manifold is $G\times_H M\to G\times_H X$. Conversely, if $N\to G\times_H X$ is a G-manifold, let $M\to X$ be the part of N lying over $X=H\times_H X\subset G\times_H X$. $\square$

We also need to recall that for each family $\mathscr{F}$ there is a G-CW-complex $E\mathscr{F}$ which is terminal in the homotopy category of $\mathscr{F}$-CW-spaces. That is, $E\mathscr{F}$ is an $\mathscr{F}$-space, and given any other $\mathscr{F}$-space Y there is a map $Y\to E\mathscr{F}$, which is unique up to G-homotopy. For the construction of $E\mathscr{F}$, see [4], [5] or [9]. For our purposes we need to know that

$$\mathcal{N}_*^G[\mathscr{F},\mathscr{F}'](X)\cong\mathcal{N}_*^G[\mathscr{F},\mathscr{F}'](E\mathscr{F}\times X),$$

which is obvious from the universal property of $E\mathscr{F}$. We also note that if X is a G-CW-complex, then $E\mathscr{F}\times X$ is a complex with all of its cells of the form $G/H\times D^n$ where $H\in\mathscr{F}$. Again, this is obvious.

3. The splitting of unoriented bordism

$A(G)$ acts on $\mathcal{N}_*^G(X)$ in an obvious way: if $M \to X$ is a G-manifold over X and S is a G-set, then $[S][M \to X] = [S \times M \to X]$. Said another way, there is a ring homomorphism $A(G) \to \mathcal{N}_0^G(*)$ which considers a finite G-set to be a 0-dimensional manifold. As we noted before, this action factors through $A(G) \otimes \mathbb{Z}_2$ because $\mathcal{N}_*^G(X)$ is a $\mathbb{Z}_2$-vector space. Thus we can apply Corollary 1.3 to the $A(G) \otimes \mathbb{Z}_2$-module $\mathcal{N}_*^G(X)$ to get a splitting. With some more work we shall identify the summands. From now on, X will denote a G-CW-complex.

Proposition 3.1. *Suppose that $e \in A(G) \otimes \mathbb{Z}_2$, $\mathcal{F}' \subset \mathcal{F}$ are families in G, and $\varphi_H(e) = 1$ for every $H \in \mathcal{F} - \mathcal{F}'$. Then e acts as the identity on $\mathcal{N}_*^G[\mathcal{F}, \mathcal{F}'](X)$.*

Proof. We know that $\mathcal{N}_*^G[\mathcal{F}, \mathcal{F}'](X) \cong \mathcal{N}_*^G[\mathcal{F}, \mathcal{F}'](E\mathcal{F} \times X)$, and that the cells of $E\mathcal{F} \times X$ are all of the form $G/H \times D^n$ for $H \in \mathcal{F}$. Using induction on the cells and a Mayer–Vietoris argument, it suffices to show that e acts as the identity on $\mathcal{N}_*^G[\mathcal{F}, \mathcal{F}'](G/H)$ when $H \in \mathcal{F}$. But this group is isomorphic to $\mathcal{N}_*^H[\mathcal{A}, \mathcal{F}' | H](*)$, where $\mathcal{A} = \mathcal{F} | H$ is the family of all subgroups of H, and e acts on this group via its restriction to $A(H)$.

Now in $A(H)$ we can write

$$e = [H/H] + \sum_{(K) \subset \mathcal{F}' | H} a_K [H/K]$$

for some coefficients $a_K \in \mathbb{Z}_2$. For, certainly we can write $e = \sum_{(K)} a_K [H/K]$ where (K) runs over all conjugacy classes in H. Since $\varphi_H(e) = 1$, we must have $a_H = 1$. If (K) is a maximal proper conjugacy class with $a_K = 1$, then $\varphi_K(e) = a_H + a_K = 0$, so we must have $(K) \subset \mathcal{F}' | H$, and hence this is true of all (K) with nontrivial coefficient.

Now we just notice that $[H/H]$ is the identity, while $[H/K]$ acts as 0 if $K \in \mathcal{F}' | H$. This last statement is true since $H/K \times M$ is an $\mathcal{F}'$-manifold for any H-manifold M, and an $\mathcal{F}'$-manifold represents the 0 class by the definition of $(\mathcal{F}, \mathcal{F}')$-bordism. □

Proposition 3.2. *Suppose $\mathcal{F}' \subset \mathcal{F}$ are families in G, and $e \in A(G) \otimes \mathbb{Z}_2$ is the element with $\{H \mid \varphi_H(e) = 1\} = \mathcal{F} - \mathcal{F}'$. Then*

$$e\mathcal{N}_*^G(X) \cong \mathcal{N}_*^G[\mathcal{F}, \mathcal{F}'](X).$$

Proof. Let $\mathcal{A}$ be the family of all subgroups of G. Then $1 - e$ acts as the identity on $\mathcal{N}_*^G[\mathcal{A}, \mathcal{F}](X)$, by the previous Proposition, so e annihilates this group. From the long exact sequence of the pair $(\mathcal{A}, \mathcal{F})$ we see that

$$e\mathcal{N}_*^G[\mathcal{F}](X) \cong e\mathcal{N}_*^G(X).$$

Similarly, $1 - e$ acts as the identity on $\mathcal{N}_*^G[\mathcal{F}'](X)$, so e annihilates it, and the long exact sequence of the pair $(\mathcal{F}, \mathcal{F}')$ shows that

$$e\mathcal{N}_*^G[\mathcal{F}](X) \cong e\mathcal{N}_*^G[\mathcal{F}, \mathcal{F}'](X).$$

The result now follows when we notice that e acts as the identity on $\mathcal{N}_*^G[\mathcal{F}, \mathcal{F}'](X)$. □

It is interesting to see this isomorphism described explicitly in terms of manifolds. The map $e\mathcal{N}_*^G(X) \to \mathcal{N}_*^G[\mathcal{F}, \mathcal{F}'](X)$ is easy: it takes a manifold of the form $e[M]$, which is an $\mathcal{F}$-manifold, and considers it to be an $(\mathcal{F}, \mathcal{F}')$-manifold. The other direction is harder. If we start with an $(\mathcal{F}, \mathcal{F}')$-manifold, the proof of the last Proposition shows that it can be represented in its cobordism class by a closed $\mathcal{F}$-manifold (in other words, the boundary must bound an $\mathcal{F}'$-manifold). The inverse isomorphism then takes a cobordism class to the class of one of its closed representatives, considered as a G-manifold, multiplied by e (which has the effect of eliminating the ambiguity introduced by our choice of representative).

Combining Corollary 1.3 with Proposition 3.2 we get the following Corollary.

Corollary 3.3. *$\mathcal{N}_*^G(X) \cong \bigoplus_{(H)} \mathcal{N}_*^G[\mathcal{A}(H), \mathcal{P}(H)](X)$ where $\mathcal{A}(H) = \{K \mid (K) \leq (H)\}$ and $\mathcal{P}(H) = \{K \in \mathcal{A}(H) \mid (K) \neq (H)\}$.*

4. Calculation of the summands

To complete the calculation we analyze the groups $\mathcal{N}_*^G[\mathcal{A}(H), \mathcal{P}(H)](X)$. Let NH be the normalizer of H in G.

Proposition 4.1. $\mathcal{N}_*^G[\mathcal{A}(H), \mathcal{P}(H)](X) \cong \mathcal{N}_*^{NH}[\mathcal{A}(H), \mathcal{P}(H)](X)$.

Proof. An $(\mathcal{A}(H), \mathcal{P}(H))$-$G$-manifold M is determined by the submanifold $M^{(H)}$ of points with isotropy a conjugate of H, and the normal bundle to the inclusion $M^{(H)} \hookrightarrow M$. We can write $M^{(H)} = G \times_{NH} M^H$ where M^H is the submanifold of points with isotropy exactly H (in this case the same as the set of points fixed by H), and the normal bundle decomposes as well. The correspondence $M \leftrightarrow M^H$ and its normal bundle establishes the desired isomorphism. □

For the rest of this analysis write N for NH, and W for N/H. Let $B_NO(k)$ be the classifying space for k-dimensional N-vector bundles. The fixed-point set $B_NO(k)^H$, a W-space, is nonequivariantly the disjoint union of components corresponding to the various k-dimensional real representations of H [7]. We let $B_NO(k)^H_{\mathcal{P}}$ denote the union of those components corresponding to representations having no H-trivial summand. It is a W-subspace of $B_NO(k)^H$.

Proposition 4.2. *$\mathcal{N}_*^N[\mathcal{A}(H), \mathcal{P}(H)](X) \cong \tilde{\mathcal{N}}_*^W[\text{Free}](\mathcal{B}_H \wedge (X^H)^+)$ where $\mathcal{B}_H = \bigvee_{k=0}^{\infty} \Sigma^k(B_NO(k)^H_{\mathcal{P}})^+$, and Y^+ denotes Y with a disjoint W-fixed basepoint adjoined.* Free *denotes the family consisting of the identity subgroup only.*

Proof. As usual, an $(\mathscr{A}(H), \mathscr{P}(H))$-manifold is cobordant to a normal tube around its H-fixed submanifold. Thus, we may regard $\mathcal{N}_*^N[\mathscr{A}(H), \mathscr{P}(H)](X)$ as the bordism group formed from manifolds with isotropy exactly H, equipped with bundles containing no H-trivial summands in any fiber. Since $B_N O(k)_{\mathscr{P}}^H$ classifies bundles of this sort, we see that

$$\mathcal{N}_*^N[\mathscr{A}(H), \mathscr{P}(H)](X) \cong \bigoplus_{k=0}^{n} \mathcal{N}_{n-k}^W[\text{Free}](B_N O(k)_{\mathscr{P}}^H \times X^H)$$
$$= \tilde{\mathcal{N}}_n^W[\text{Free}](\mathscr{B}_H \wedge (X^H)^+). \qquad \square$$

Proposition 4.3. *If W is an odd-order group, then $\mathcal{N}_*^W[\text{Free}](Y) \cong \mathcal{N}_*(Y)^W$ for any W-space Y. Here $\mathcal{N}_*$ denotes the nonequivariant unoriented bordism theory, and W acts on $\mathcal{N}_*(Y)$ via its action on Y.*

Proof. Let $\alpha : \mathcal{N}_*^W[\text{Free}](Y) \to \mathcal{N}_*(Y)$ be the forgetful map. Notice that its image is contained in the W-invariant part. Let $\beta : \mathcal{N}_*(Y) \to \mathcal{N}_*^W[\text{Free}](Y)$ be the map defined by $\beta[M] = [W \times M]$. Now $\beta\alpha$ is multiplication by $[W]$, which, by Proposition 3.1, acts as the identity of $\mathcal{N}_*^W[\text{Free}](Y)$. $\alpha\beta$ is given by

$$\alpha\beta(m) = \sum_{w \in W} w \cdot m,$$

so on the W-invariant part it is just multiplication by $|W|$, an odd number, hence the identity. $\square$

Let $\mathscr{I}_H$ be the set of nontrivial irreducible real representations of H. If $V \in \mathscr{I}_H$, let $d_V = \dim V$. Notice that $\text{Hom}_H(V, V) = \mathbb{C}$ since H has odd order.

Proposition 4.4. $\mathscr{B}_H \simeq \bigwedge_{V \in \mathscr{I}_H} (\bigvee_{k=0}^{\infty} \Sigma^{d_V k} BU(k)^+)$ *nonequivariantly.*

Proof. As mentioned earlier, $B_N O(k)^H$ is the disjoint union of components corresponding to the various k-dimensional representations of H. If Z is a k-dimensional representation of H with no trivial summands, then we can write $Z = \bigoplus_{V \in \mathscr{I}_H} V^{i_V}$ for some integers i_V, with $\sum d_V i_V = k$. The component corresponding to Z then has the form $\prod_{V \in \mathscr{I}_H} BU(i_V)$, so

$$B_N O(k)_{\mathscr{P}}^H \simeq \coprod_{\sum d_V i_V = k} \left(\prod_{V \in \mathscr{I}_H} BU(i_V) \right).$$

Thus,

$$\mathscr{B}_H \simeq \bigvee_{k=0}^{\infty} \Sigma^k \left(\bigvee_{\sum d_V i_V = k} \left(\bigwedge_{V \in \mathscr{I}_H} BU(i_V)^+ \right) \right)$$
$$= \bigvee_{k=0}^{\infty} \left(\bigvee_{\sum d_V i_V = k} \left(\bigwedge_{V \in \mathscr{I}_H} \Sigma^{d_V i_V} BU(i_V)^+ \right) \right) = \bigwedge_{V \in \mathscr{I}_H} \left(\bigvee_{i_V=0}^{\infty} \Sigma^{d_V i_V} BU(i_V)^+ \right).$$

Writing k for i_V, we get the desired result. $\square$

$\mathscr{B}_H$ has a product, given by the Whitney sum maps $BU(i)\times BU(j)\to BU(i+j)$. We compute $\tilde{\mathcal{N}}_*(\mathscr{B}_H)$ with the induced product. It suffices to consider each smash factor separately.

Proposition 4.5. *Let $d\geq 0$. Then*

$$\tilde{\mathcal{N}}_*\Big(\bigvee_{k=0}^{\infty}\Sigma^{dk}BU(k)^+\Big)\cong\mathcal{N}_*[\gamma_1,\gamma_2,\ldots],\quad |\gamma_i|=2(i-1)+d.$$

γ_i is represented by $\mathbb{C}P^{i-1}\hookrightarrow BU(1)$. $|\gamma_i|$ is the dimension in which γ_i lives.

This was shown for the real case when $d=1$ in [1], and with suitable changes the same proof will work here. The argument is essentially a dimension count in homology, using the known structure of the homology of BU.

Corollary 4.6. *$\tilde{\mathcal{N}}_*(\mathscr{B}_H)\cong\mathcal{N}_*[\gamma_{V,i}\mid V\in\mathscr{I}_H,\, i=1,2,\ldots]$ where $|\gamma_{V,i}|=2(i-1)+d_V$ with $d_V=\dim_{\mathbb{R}} V$.*

These results together imply our main result:

Theorem 4.7. *If G is an odd-order group, and X is a G-CW-complex, then*

$$\begin{aligned}\mathcal{N}_*^G(X)&\cong\bigoplus_{(H)}(\mathcal{N}_*[\gamma_{H,V,i}]\otimes_{\mathcal{N}_*}\mathcal{N}_*(X^H))^{WH}\\&\cong\bigoplus_{(H)}(\mathcal{N}_*[\gamma_{H,V,i}]\otimes_{\mathbb{Z}_2}H_*(X^H))^{WH},\end{aligned}$$

where the sum is taken over all conjugacy classes of subgroups of G, V ranges over the nontrivial irreducible representations of H, and $i=1,2,\ldots$, $|\gamma_{H,V,i}|=2(i-1)+d_V$ as in Corollary 4.6. $WH=NH/H$ acts on the polynomial generators by permuting the representations. In particular,

$$\mathcal{N}_*^G\cong\prod_{(H)}(\mathcal{N}_*[\gamma_{H,V,i}])^{WH}$$

as rings. Moreover, the action of $A(G)$ is given by the fact that

$$A(G)\otimes\mathbb{Z}_2\cong\mathcal{N}_0^G\cong\prod_{(H)}\mathbb{Z}_2,$$

so an $e\in A(G)\otimes\mathbb{Z}_2$ acts by projecting onto the factors corresponding to those (H) for which $\varphi_H(e)=1$.

5. Abelian groups

Theorem 4.7 takes on a particularly simple form when G is an abelian group of odd order. In this case, $WH=G/H$ acts trivially on H, for any subgroup H of G, hence it acts trivially on the representations of H. Thus we get the following corollary.

Corollary 5.1. *If G is an abelian group of odd order, and X is a G-CW-complex, then*

$$\mathcal{N}_*^G(X) \cong \bigoplus_H \mathcal{N}_*[\gamma_{H,V,i}] \otimes_{\mathcal{N}_*} (\mathcal{N}_*(X^H))^{G/H} \cong \bigoplus_H \mathcal{N}_*[\gamma_{H,V,i}] \otimes_{\mathbb{Z}_2} (H_*(X^H))^{G/H}.$$

In particular,

$$\mathcal{N}_*^G \cong \prod_H \mathcal{N}_*[\gamma_{H,V,i}]$$

as rings.

We can find manifolds representing the polynomial generators $\gamma_{H,V,i}$ by unwinding the isomorphisms used in finding the splitting. When we do this we find that

$$\gamma_{H,V,i} = e_H[G \times_H \mathbb{R}\mathrm{P}(\mathbb{R} \oplus (\eta_{i-1} \otimes_{\mathbb{C}} V))]$$

where e_H is the idempotent mentioned in Proposition 1.2, and η_{i-1} is the canonical line bundle over $\mathbb{C}\mathrm{P}^{i-1}$ given trivial H-action. V is given either of its complex structures. This should be compared to § 4.1 of [6], which gives a similar set of multiplicative generators. To make the similarity clearer, we may notice that the H-representation V may be extended to a G-representation, which we also call V, and then we can write

$$\gamma_{H,V,i} = e_H[\mathbb{R}\mathrm{P}(\mathbb{R} \oplus (\eta_{i-1} \otimes_{\mathbb{C}} V))].$$

6. Bordism and homology

Nonequivariantly, we know that unoriented bordism is given by singular homology with coefficients in the bordism ring (see, e.g., [11]). The computation of Theorem 4.7 allows us to deduce a similar result for odd-order groups.

The analog of singular homology theory that we will use is Bredon homology, as discussed in [8], for example. This theory depends on the G-space X, and also on a Mackey functor M, which is an algebraic functor on the category of orbits of G, having both restriction maps and transfers. For any such M, the value $M(G/H)$ will be an $A(H)$-module. The example that we are interested in is $M = \underline{\mathcal{N}}_k^G$, where $\underline{\mathcal{N}}_k^G(G/H) = \mathcal{N}_k^H$.

As mentioned earlier, the technique of splitting homology theories along the splittings of $A(G)$ works quite generally. Details are given in [4] or [9]. If we apply this to homology, we get the following theorem.

Theorem 6.1. *Let G be an odd-order group, and let M be a Mackey functor having $\mathbb{Z}_2$-vector spaces as values. Then for any G-CW-complex X we have*

$$H_*^G(X; M) \cong \bigoplus_{(H)} H_*(X^H; e_H^H M(G/H))^{WH}$$

$$\cong \bigoplus_{(H)} (e_H^H M(G/H) \otimes_{\mathbb{Z}_2} H_*(X^H))^{WH}$$

where $e_H^H \in A(H) \otimes \mathbb{Z}_2$ *is the idempotent satisfying*

$$\varphi_K(e_H^H) = \begin{cases} 1 & \text{if } K = H, \\ 0 & \text{if } K \neq H. \end{cases}$$

If we notice, from Theorem 4.7 that

$$e_H^H \mathcal{N}_*^H \cong \mathcal{N}_*[\gamma_{H,V,i}],$$

and then compare Theorems 4.7 and 6.1, we get

Corollary 6.2. *If G is an odd-order group and X is a G-CW-complex, then*

$$\mathcal{N}_*^G(X) \cong H_*^G(X; \underline{\mathcal{N}}_*^G).$$

If we use the general techniques of [9] to express these results as stable splittings, we can conclude, as in the nonequivariant case, that the equivariant spectrum representing $\mathcal{N}_*^G(-)$ (which is found, and called mO_G, in [3]), splits as a wedge of equivariant Eilenberg–MacLane spectra.

7. Homotopical bordism

We can carry through much the same analysis for *homotopical bordism.* This is the theory defined by the equivariant Thom spectrum MO_G. Precisely, we let

$$MO_n^G(X) = \operatorname*{colim}_V\, [S^V, X^+ \wedge MO_G(|V|)]_G,$$

where $|V| = \dim_{\mathbb{R}} V$ and $MO_G(k)$ is the Thom space of the universal G-k-plane bundle. We refer to $\mathcal{N}_*^G(-)$ as *geometric bordism,* in contrast. As shown in [3], we can interpret the homotopical theory as the bordism groups of *stable manifolds* over X, where a stable manifold is a map $f:(M, \partial M) \to (D(V), S(V))$ with M a G-manifold and V a real representation of G. The relation of stable equivalence is the one generated by considering f to be equivalent to

$$f \times 1: \ (M \times D(W), \partial(M \times D(W))) \to (D(V \oplus W), S(V \oplus W)),$$

where W is any other representation of G.

Using either this interpretation, or the fact that our results through Proposition 4.1 hold for any G-homology theory, we can split $MO_*^G(X)$:

$$MO_*^G(X) \cong \bigoplus_{(H)} MO_*^{NH}[\mathcal{A}(H), \mathcal{P}(H)](X).$$

It remains only to compute the summands. We use the notations introduced in Section 4. Let $B_H = \prod_{V \in \mathcal{I}_H} BU$ where $BU = \operatorname{colim}_k BU(k)$. Let $\widetilde{RO}(H)$ be the subgroup of the representation ring $RO(H)$ generated by the nontrivial irreducible representations of H. Finally, let $\bar{\mathcal{B}}_H = \bigvee_{\alpha \in \widetilde{RO}(H)} \Sigma^{|\alpha|} B_H^+$ where $|\alpha|$ is the virtual dimension of α (so $\bar{\mathcal{B}}_H$ is a spectrum, not a space).

Proposition 7.1. *Suppose that H is a normal subgroup of N and W = N/H. Then*

$$\begin{aligned} MO_*^N[\mathscr{A}(H), \mathscr{P}(H)](X) &\cong \widetilde{MO}_*^W[\text{Free}](\bar{\mathscr{B}}_H \wedge (X^H)^+) \\ &\cong \tilde{\mathcal{N}}_*^W[\text{Free}](\bar{\mathscr{B}}_H \wedge (X^H)^+) \\ &\cong \tilde{\mathcal{N}}_*(\bar{\mathscr{B}}_H \wedge (X^H)^+)^W. \end{aligned}$$

Proof. The first isomorphism follows when we use the interpretation of $MO_*^N(-)$ as bordism of stable manifolds. The normal bundle to the H-fixed submanifold is now a stable bundle with no H-fixed summands, and bundles of this sort are classified by B_H. Hence

$$\begin{aligned} MO_*^N[\mathscr{A}(H), \mathscr{P}(H)](X) &\cong \bigoplus_{\alpha \in \widetilde{RO}(H)} MO_{n-|\alpha|}^W[\text{Free}](B_H \times X^H) \\ &\cong MO_n^W[\text{Free}](\bar{\mathscr{B}}_F \wedge (X^H)^+). \end{aligned}$$

In general, $MO_*^W[\text{Free}](Y) \cong \mathcal{N}_*^W[\text{Free}](Y)$ for any W-space Y, because equivariant transversality works for maps out of free manifolds (see, e.g., [3]). The final isomorphism in our Proposition comes from Proposition 4.3. □

Just as in Proposition 4.4 we can prove the following:

Proposition 7.2. $\bar{\mathscr{B}}_H \simeq \bigwedge_{V \in \mathscr{I}_H} \bigvee_{k=-\infty}^{\infty} \Sigma^{d_V k} BU^+$ *nonequivariantly.*

Analogous to Proposition 4.5 we have the following:

Proposition 7.3. $\tilde{\mathcal{N}}_*(\bigvee_{k=-\infty}^{\infty} \Sigma^{dk} BU^+) \cong \mathcal{N}_*[\gamma_1^{-1}, \gamma_1, \gamma_2, \ldots]$, $|\gamma_i| = 2(i-1) + d$.

The proof is similar to that of Proposition 4.5. We may also notice that

$$\bigvee_{k=-\infty}^{\infty} \Sigma^{dk} BU^+ = BU^+ \wedge \bigvee_{k=-\infty}^{\infty} S^{dk}.$$

γ_i is represented just as before, in fact it is the image of the γ_i of Proposition 4.5 under the obvious map. If we think of γ_1 as a point with the virtual representation V, or as the bundle $V \to *$, then γ_1^{-1} may be thought of as a point with the virtual representation $-V$ or as the stable manifold $* \hookrightarrow D(V)$, inclusion of the origin. Our main result is now the following theorem.

Theorem 7.4. *If G is an odd order group, and X is a G-CW-complex, then*

$$\begin{aligned} MO_*^G(X) &\cong \bigoplus_{(H)} (\mathcal{N}_*[\gamma_{H,V,1}^{-1}, \gamma_{H,V,i}] \otimes_{\mathcal{N}_*} \mathcal{N}_*(X^H))^{WH} \\ &\cong \bigoplus_{(H)} (\mathcal{N}_*[\gamma_{H,V,i}^{-1}, \gamma_{H,V,i}] \otimes_{\mathbb{Z}_2} H_*(X^H))^{WH}. \end{aligned}$$

In particular,

$$MO_*^G \cong \prod_{(H)} (\mathcal{N}_*[\gamma_{H,V,1}^{-1}, \gamma_{H,V,i}])^{WH}$$

as rings. $e \in A(G) \otimes \mathbb{Z}_2$ *acts by projecting to the summands corresponding to those* (H) *for which* $\varphi_H(e) = 1$.

Again, if G is abelian, WH acts trivially on the $\gamma_{H,V,i}$, and the results simplify. In this case the generators $\gamma_{H,V,i}$ can be taken to be the images of the generators of geometric bordism of the same names, hence are represented in the same way. The generator $\gamma_{H,V,1}^{-1}$ is represented as follows. Extend V to a representation of G. Then

$$\gamma_{H,V,i}^{-1} = e_H[* \hookrightarrow D(V)],$$

where the object inside the brackets is a stable G-manifold.

Just as in the last section, we can show the following corollary.

Corollary 7.5. *If* G *is an odd-order group and* X *is a* G-*CW-complex, then*

$$MO_*^G(X) \cong H_*^G(X; \underline{MO_*^G}).$$

We may also conclude that the Thom spectrum MO_G splits as a wedge of equivariant Eilenberg-MacLane spectra.

Acknowledgements

I would like to thank J. Peter May for suggesting that the Burnside ring should be applied to equivariant bordism in this way, and Bob Stong for pointing out several mistakes in an earlier version of this paper.

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