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Interpolation in Grothendieck Institutions

Răzvan Diaconescu*

Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, Bucharest 014700, Romania

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Abstract

It is well known that interpolation properties of logics underlying specification formalisms play an important role in the study of structured specifications, they have also many other useful logical consequences.

In this paper, we solve the interpolation problem for Grothendieck institutions which have recently emerged as an important mathematical structure underlying heterogenous multi-logic specification. Our main result can be used in the applications in several different ways. It can be used to establish interpolation properties for multi-logic Grothendieck institutions, but also to lift interpolation properties from unsorted logics to their many sorted variants. The importance of the latter resides in the fact that, unlike other structural properties of logics, many sorted interpolation is a non-trivial generalisation of unsorted interpolation.

The concepts, results, and the applications discussed in this paper are illustrated with several examples from conventional logic and algebraic specification theory.

Keywords: Craig interpolation; Grothendieck institutions; Algebraic specification

1. Introduction

The theory of institutions [16] is a categorical abstract model theory which formalizes the intuitive notion of logical system, including syntax, semantics, and the satisfaction between them. Institutions become a common tool in the study of algebraic specification theory and can be considered its most fundamental mathematical structure. It is already an algebraic specification tradition to have an institution underlying each

* Tel.: +40-744-153703; fax: +40-21-2125126.
E-mail address: razvan.diaconescu@imar.ro (R. Diaconescu).

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language or system, in which all language/system constructs and features can be rigorously explained as mathematical entities. This has been first spelt out as a programme with a sample definition of specification language constructs in [26]. Most modern algebraic specification languages follow this tradition, including CASL [2], Maude [21], or CafeOBJ [13]. There is an increasing multitude of logics in use as institutions in algebraic specification and computing science. Some of them, such as first-order predicate (in many variants), second order, higher order, Horn, type theoretic, equational, modal (in many variants), infinitary logics, etc., are well known or at least familiar to the ordinary logicians, while others such as behavioural or rewriting logics are known and used mostly in computing science.

Grothendieck institutions have been introduced by Diaconescu [10] and were originally used for providing a simple homogeneous semantics for heterogeneous multi-logic specification with CafeOBJ [13] by replacing the theory of the so-called 'extra theory morphisms' [9]. Later they have been adopted (in a dual form) for providing semantics for heterogenous specification with CASL extensions [23].

The Craig Interpolation Property (abbreviated *CI*) is one of the basic properties of conventional first-order logic [8,7] but also an important desirable property for any logic. Interpolation properties have received much attention in specification theory especially due to its importance for module algebra based on first-order logic [3] or institution-independent [14,15], for structured specification [6,5], or for heterogeneous specification [30]. An institution-independent proof of Craig Interpolation Theorem having a multitude of instances for actual logics has been developed in [11] based on a very general concept of axiomatizability of the actual logic formalized as institution.

In this paper, we solve the interpolation problem for Grothendieck institutions. We also present several ways in which our main result can be used in the applications. We show how it can be used to establish interpolation properties for multi-logic Grothendieck institutions, but also how to lift interpolation properties from unsorted logics to their many sorted variants. While the former is important for the study of structured specifications in multi-logic institutions, the importance of the latter resides in the fact that, unlike other structural properties of logics, many sorted interpolation is a non-trivial generalisation of unsorted interpolation (see [4]).

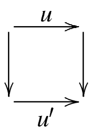
1.1. Categorical preliminaries

This work assumes some familiarity with category theory, and generally uses the same notations and terminology as MacLane [18], except that composition is denoted by ';' and written in the diagrammatic order. The application of functions (functors) to arguments may be written either normally using parentheses, or else in diagrammatic order without parentheses, or, more rarely, by using sub-scripts or super-scripts. We use $\Rightarrow$ rather than $\rightarrow$ in denoting natural transformations. The category of sets is denoted as $\mathbb{S}et$, and the category of categories [1] as $\mathbb{C}at$. The opposite of a category $\mathbb{C}$ is denoted by $\mathbb{C}^{\mathrm{op}}$. The class of objects of a category $\mathbb{C}$ is denoted by $|\mathbb{C}|$; also the

[1] We steer clear of any foundational problem related to the 'category of all categories'; several solutions can be found in the literature, see, for example [18].

set of arrows in $\mathbb{C}$ having the object a as source and the object b as target is denoted as $\mathbb{C}(a,b)$.

We say that a class of arrows $\mathscr{S}$ in a category $\mathbb{C}$ is *stable under pushouts* if and if for each pushout square in $\mathbb{C}$



$u' \in \mathscr{S}$ whenever $u \in \mathscr{S}$. By reversing the arrows in the definition above we can define that $\mathscr{S}$ is *stable under pullbacks*.

Let us now recall the concept of *indexed category* [24]. A good reference for indexed categories also discussing applications to algebraic specification theory is [31]. An *indexed category* [31] is a functor $B : I^{\mathrm{op}} \to \mathbb{C}at$; sometimes we denote $B(i)$ as B_i (or B^i) for an index $i \in |I|$ and $B(u)$ as B^u for an index morphism $u \in I$. The following 'flattening' construction providing the canonical fibration associated to an indexed category is known under the name of the *Grothendieck construction*, and plays an important role in mathematics. Given an indexed category $B : I^{\mathrm{op}} \to \mathbb{C}at$, let $B^\sharp$ be the *Grothendieck category*[2] having $\langle i, \Sigma \rangle$, with $i \in |I|$ and $\Sigma \in |B_i|$, as objects and $\langle u, \varphi \rangle : \langle i, \Sigma \rangle \to \langle i', \Sigma' \rangle$, with $u \in I(i, i')$ and $\varphi : \Sigma \to \Sigma' B^u$, as arrows. The composition of arrows in $B^\sharp$ is defined by $\langle u, \varphi \rangle ; \langle u', \varphi' \rangle = \langle u;u', \varphi ; (\varphi' B^u) \rangle$.

2. Institutions

Institutions [16] represent a mathematical meta-theory on logics, technically based on category theory, which abstracts the Tarskian concept of truth, and which builds on the idea of the invariance of truth with respect to translation of notation. This invariance of truth can also be interpreted that the meaning of a sentence does not depend on the context in which it is interpreted, which is surely a very basic intuition for classical logic.

Definition 1. An *institution* $I = (\mathbb{S}ign^I, Sen^I, \mathrm{M{\scriptsize OD}}^I, \models^I)$ consists of

(1) a category $\mathbb{S}ign^I$, whose objects are called *signatures*,
(2) a functor $Sen^I : \mathbb{S}ign^I \to \mathbb{S}et$, giving for each signature a set whose elements are called *sentences* over that signature,
(3) a functor $\mathrm{M{\scriptsize OD}}^I : (\mathbb{S}ign^I)^{\mathrm{op}} \to \mathbb{C}at$ giving for each signature Σ a category whose objects are called Σ-*models*, and whose arrows are called Σ-(*model*) *homomorphisms*, and
(4) a relation $\models^I_\Sigma \subseteq |\mathrm{M{\scriptsize OD}}^I(\Sigma)| \times Sen^I(\Sigma)$ for each $\Sigma \in |\mathbb{S}ign^I|$, called Σ-*satisfaction*, such that for each morphism $\varphi : \Sigma \to \Sigma'$ in $\mathbb{S}ign^I$, the satisfaction condition

$$M' \models^I_{\Sigma'} Sen^I(\varphi)(e) \quad \text{iff} \quad \mathrm{M{\scriptsize OD}}^I(\varphi)(M') \models^I_\Sigma e$$

[2] Notice that the terminology 'Grothendieck categories' is used in a rather different way in the context of Abelian categories [18].

holds for each $M' \in |\mathrm{Mod}^I(\Sigma')|$ and $e \in Sen^I(\Sigma)$. We may denote the reduct functor $\mathrm{Mod}^I(\varphi)$ by $\_\upharpoonright_\varphi$ and the sentence translation $Sen^I(\varphi)$ simply by $\varphi(\_)$. When $M = M'\upharpoonright_\varphi$ we will say that M' is an *expansion of* M *along* φ.

Example 2. Let **FOL** be the institution of *many sorted first-order logic with equality*. Its signatures (S, F, P) consist of a set of sort symbols S, a set F of function symbols, and a set P of relation symbols. Each function or relation symbol comes with a string of argument sorts, called *arity*, and for functions symbols, a result sort. $F_{w\to s}$ denotes the set of function symbols with arity w and sort s, and P_w the set of relation symbols with arity w.

Signature morphisms map the three components in a compatible way. Models M are first-order structures interpreting each sort symbol s as a set M_s, each function symbol σ as a function M_σ from the product of the interpretations of the argument sorts to the interpretation of the result sort, and each relation symbol π as a subset M_π of the product of the interpretations of the argument sorts. Sentences are the usual first-order sentences built from equational and relational atoms by iterative application of logical connectives and quantifiers. Sentence translations rename the sorts, function, and relation symbols. For each signature morphism φ, the reduct $M'\upharpoonright_\varphi$ of a model M' is defined by $(M'\upharpoonright_\varphi)_x = M'_{\varphi(x)}$ for each x sort, function, or relation symbol from the domain signature of φ. The satisfaction of sentences by models is the usual Tarskian satisfaction defined inductively on the structure of the sentences.

The more conventional unsorted version of **FOL**, denoted **UFOL**, restricts many sorted first-order logic to $\{*\}$-sorted signatures for some fixed sort symbol $*$.

The institution **FOEQL** of *first-order equational logic* is obtained from the institution **FOL** of first-order logic by discarding the relation symbols and their interpretations. The signatures of **FOEQL** are called *algebraic signatures* and the **FOEQL** models are called *algebras*.

The institution **EQL** of *equational logic* can be obtained by restricting the sentences of **FOEQL** to universally quantified equations (either in conditional or unconditional form).

The institution **REL** of *relational logic* is obtained by eliminating from the institution **FOL** of first-order logic the function symbols and their interpretations. The signatures of **REL** are called *relational signatures*.

The very simple institution **MS** of *many sorted sets* can be regarded as a sub-institution of **FOL** determined by the signatures which have only sort symbols and no function or relation symbols. Notice that this institution has no sentences.

Example 3. In the institution **RWL** of *rewriting logic*, the signatures are just ordinary algebraic signatures. The models of rewriting logic are *preorder models* which are interpretations of the signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each function symbol as a preorder functor. A *preorder model homomorphism* is just a preorder functor which is an algebra homomorphism.

The sentences are either universal ordinary equations or *transitions*, both in their conditional and unconditional form. An unconditional transition $t => t'$ is satisfied by a preorder model M when the interpretations of the terms are in the preorder relation of the carrier, i.e. $M_t \leqslant M_{t'}$.

For reasons of simplicity of presentation our definition of rewriting logic restricts the full definition of rewriting logic [20] to the unlabelled case. This unlabelled version of rewriting logic has been also adopted by CafeOBJ [13].

The institution **RWL** can also be extended to *first-order rewriting logic* **FORWL** by allowing any first-order logic sentences formed by equational and transitional atoms.

Definition 4. In any institution, (Σ, E) is a *presentation* when Σ is a signature and E is a set of Σ-sentences. A *presentation morphism* $\phi : (\Sigma, E) \to (\Sigma', E')$ is a signature morphism such that $E' \models \phi(E)$.[3]

The relationships between various institutions are captured mathematically by 'institution morphisms'. However, there are several concepts of such structure preserving mappings between institutions. The original one, introduced by Goguen and Burstall [16], is adequate for encoding a 'forgetful' operation from a 'richer' institution to a 'poorer' one.

Definition 5. An *institution morphism* $(\Phi, \alpha, \beta) : (\mathbb{S}ign', Sen', \mathrm{Mod}', \models') \to (\mathbb{S}ign, Sen, \mathrm{Mod}, \models)$ consists of
(1) a functor $\Phi : \mathbb{S}ign' \to \mathbb{S}ign$,
(2) a natural transformation $\alpha : \Phi; Sen \Rightarrow Sen'$, and
(3) a natural transformation $\beta : \mathrm{Mod}' \Rightarrow \Phi^{\mathrm{op}}; \mathrm{Mod}$,
such that the following *satisfaction condition* holds:

$$M' \models'_{\Sigma'} \alpha_{\Sigma'}(e) \quad \text{iff} \quad \beta_{\Sigma'}(M') \models_{\Sigma'\Phi} e$$

for each signature $\Sigma' \in |\mathbb{S}ign'|$ for each Σ'-model M', and each $\Sigma'\Phi$-sentence e.

Under obvious composition, the institution morphisms form a category denoted as $\mathbb{I}ns$.

Institution comorphisms [17], previously know as 'plain map' in [19] or 'representation' in [29,30], capture the idea of embedding of a 'poorer' institution into a 'richer' one.

Definition 6. An *institution comorphism* $(\Phi, \alpha, \beta) : (\mathbb{S}ign, Sen, \mathrm{Mod}, \models) \to (\mathbb{S}ign', Sen', \mathrm{Mod}', \models')$ consists of
(1) a functor $\Phi : \mathbb{S}ign \to \mathbb{S}ign'$,
(2) a natural transformation $\alpha : Sen \Rightarrow \Phi; Sen'$, and
(3) a natural transformation $\beta : \Phi^{\mathrm{op}}; \mathrm{Mod}' \Rightarrow \mathrm{Mod}$,

[3] Each model M' satisfying E' also satisfies $\phi(E)$.

such that the following *satisfaction condition* holds:

$$M' \models'_{\Sigma\Phi} \alpha_\Sigma(e) \quad \text{iff} \quad \beta_\Sigma(M') \models_\Sigma e$$

for each signature $\Sigma \in |\mathbb{S}ign|$ for each $\Sigma\Phi$-model M', and each Σ-sentence e.

Under obvious composition, the institution comorphisms form a category denoted as $co\mathbb{I}ns$.

The following duality relationship between institution morphisms and comorphisms was first observed in [32] and established in [1].

Theorem 7. *An adjunction* $(\Phi, \bar{\Phi}, \zeta, \bar{\zeta})$ *between the categories of signatures*[4] *of institutions* $(\mathbb{S}ign, Sen, \text{Mod}, \models)$ *and* $(\mathbb{S}ign', Sen', \text{Mod}', \models')$ *determines a canonical bijection between institution morphisms* $(\Phi, \alpha, \beta) : (\mathbb{S}ign', Sen', \text{Mod}', \models') \to (\mathbb{S}ign, Sen, \text{Mod}, \models)$ *and institution comorphisms* $(\bar{\Phi}, \bar{\alpha}, \bar{\beta}) : (\mathbb{S}ign, Sen, \text{Mod}, \models) \to (\mathbb{S}ign', Sen', \text{Mod}', \models')$ *given by the following equalities*:

$$\bar{\alpha} = \zeta Sen; \bar{\Phi}\alpha \quad \text{and} \quad \bar{\beta} = \bar{\Phi}^{\text{op}}\beta; \zeta^{\text{op}}\text{Mod}$$

and

$$\alpha = \Phi\bar{\alpha}; \bar{\zeta}Sen' \quad \text{and} \quad \beta = \bar{\zeta}^{\text{op}}\text{Mod}; \Phi^{\text{op}}\bar{\beta}.$$

(Φ, α, β) *and* $(\bar{\Phi}, \bar{\alpha}, \bar{\beta})$ *are called* adjoint *institution morphism*, *respectively*, *comorphism*.

Example 8. The 'forgetful' institution morphism **FOL** → **FOEQL** forgets the relation symbols, i.e. maps any **FOL**-signature (S, F, P) to the algebraic signature (S, F), maps any (S, F, P)-model to its underlying (S, F)-algebra, and regards each (S, F)-sentence as an (S, F, P)-sentence. It is an adjoint institution morphism, the left adjoint to the 'forgetful' functor $\mathbb{S}ign^{\mathbf{FOL}} \to \mathbb{S}ign^{\mathbf{FOEQL}}$ mapping any algebraic signature (S, F) to the first-order logic signature $(S, F, \emptyset)$.

The 'forgetful' institution morphism **FOEQL** → **EQL** is an identity on signatures and models, and regards each equation as a first-order sentence.

The 'forgetful' institution morphism **FOL** → **REL** forgets the function symbols, i.e. maps any **FOL**-signature (S, F, P) to the relational signature (S, P), the model mapping forgets the interpretation of the function symbols F, and the sentence translation regards each (S, P)-sentence as an (S, F, P)-sentence. Notice that this is an adjoint institution morphism too, the left adjoint functor $\mathbb{S}ign^{\mathbf{REL}} \to \mathbb{S}ign^{\mathbf{FOL}}$ maps any relational signature (S, P) to the first-order logic signature $(S, \emptyset, P)$.

Notice that the comorphism **UFOL** → **FOL** embedding unsorted first-order logic into many sorted first-order logic is *not* an adjoint comorphism.

[4] $\Phi : \mathbb{S}ign' \to \mathbb{S}ign$ is the right adjoint, $\bar{\Phi}$ is the left adjoint, ζ is the unit, and $\bar{\zeta}$ is the counit of the adjunction.

Definition 9. A signature morphism $\phi : \Sigma \to \Sigma'$ is *conservative* when each Σ-model has an expansion along ϕ.

Example 10. It is rather easy to check that in first-order logic **FOL**, a signature morphism $\varphi : (S,F,P) \to (S',F',P')$ is conservative when φ is injective on the sort, function, and relation symbols and does not add new function symbols of sorts in S that are 'empty' (i.e., without F-terms). Consequently, if (S,F,P) has only 'non-empty' sorts, then each injective signature morphism $\varphi : (S,F,P) \to (S',F',P')$ is conservative.

Definition 11. An institution is *compact* if for each set of sentences E and each sentence e, if $E \models e$ then there exists a finite sub-set $E' \subseteq E$ such that $E' \models e$.

Definition 12. An institution *has conjunctions* when for each family of sentences $\{e_i \mid i \in I\}$ (for the same signature), there exists a sentence e' such that the models satisfying e' are exactly the models satisfying $\{e_i \mid i \in I\}$.

When we consider only finite families of sentences we say that the institution has *finite* conjunctions.

Similarly, we can define that an institution has *implications*, *negations*, etc.

Exactness properties for institutions formalize the possibility of amalgamating models of different signatures when they are consistent on some kind of 'intersection' of the signatures (formalized as a pushout square).

Definition 13. An institution $(\mathbb{S}ign, Sen, \mathrm{M{\small OD}}, \models)$ is *exact* if and only if the model functor $\mathrm{M{\small OD}} : \mathbb{S}ign^{\mathrm{op}} \to \mathbb{C}at$ preserves finite limits. The institution is *semi-exact* if and only if MOD preserves pullbacks.

Semi-exactness is everywhere. Virtually all institutions formalising conventional or non-conventional logics are at least semi-exact. In general, the institutions of many sorted logics are exact, while those of unsorted (or one sorted) logics are only semi-exact [14]. However, in applications the important amalgamation property is the semi-exactness rather than the full exactness. Moreover, in practice often the weak[5] version of exactness suffices [9,30].

The following amalgamation property is a direct consequence of semi-exactness.

Definition 14. The commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\phi_1} & \Sigma_1 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \phi_1'} \\ \Sigma_2 & \xrightarrow[\phi_2']{} & \Sigma' \end{array}$$

[5] In the sense of 'weak' universal properties [18] not requiring uniqueness.

is an *amalgamation square* if and only if for each Σ_1-model M_1 and a Σ_2-model M_2 such that $M_1\restriction_{\phi_1} = M_2\restriction_{\phi_2}$, there exists an unique Σ'-model M' such that $M'\restriction_{\phi_1'} = M_1$ and $M'\restriction_{\phi_2'} = M_2$.

Corollary 15. *In a semi-exact institution each pushout square of signature morphisms is an amalgamation square.*

3. Institution-independent Craig interpolation

In the algebraic specification literature there are many institution-independent formulations of CI, for example [27] being one of the first and most representative ones. All these formulations generalise the conventional intersection–union (of signatures) framework to squares of signature morphisms which almost always are required to be a pushout (see for example [28,5,4,15]) and when this is not the case the signature morphisms are required to be (abstract) inclusions [14].

It has been noticed in [11] that the mere formulation of CI does not require any extra technical assumptions besides a commuting square of signature morphisms, the role of such assumptions having to do with the proof of CI rather than with its formulation.

Definition 16. A commuting square of signature morphisms

$$\begin{array}{ccc} \Sigma & \xrightarrow{\phi_1} & \Sigma_1 \\ {\scriptstyle \phi_2}\downarrow & & \downarrow{\scriptstyle \phi_1'} \\ \Sigma_2 & \xrightarrow[\phi_2']{} & \Sigma' \end{array}$$

is a *Craig Interpolation square* if and only if for each set E_1 of Σ_1-sentences and set E_2 of Σ_2-sentences such that $\phi_1'(E_1) \models \phi_2'(E_2)$ there exists a set E of Σ-sentences such that $E_1 \models \phi_1(E)$ and $\phi_2(E) \models E_2$. The set E is called the *interpolant* of E_1 and E_2.

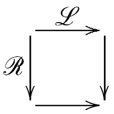
We agree with [25,14] that this is more natural then the formulations of CI considering single sentence rather than sets of sentences; in particular, note that (cf. [25]) equational logic satisfies Definition 16 but not the single-sentence versions (given in [27] for example).

Fact 17. *In a compact institution, if E_2 of Definition* 16 *is finite, then the interpolant E can be chosen to be finite too.*

The immediate consequence of this fact is that in compact institutions which has finite conjunctions, the sets of sentence formulation of CI implies the single-sentence formulation.

In principle, in the actual examples, CI is expected for pushout squares of signature morphisms, however, in many situations only some pushout squares satisfy it. This intuition has been formulated first time in [5]. The interpolation concept below is slightly simpler and more general than the so-called $(\mathscr{D},\mathscr{T})$-interpolation of [5].

Definition 18. For any classes of signature morphisms $\mathscr{L},\mathscr{R}\subseteq\mathbb{S}ign$ in any institution $(\mathbb{S}ign, Sen, \mathrm{Mod}, \models)$, we say that the institution has the *Craig* $(\mathscr{L},\mathscr{R})$*-Interpolation* if each pushout square of signature morphisms of the form



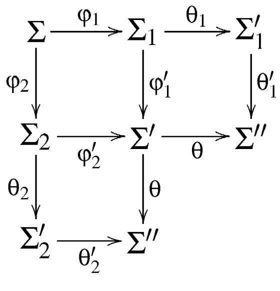
is a Craig Interpolation square.

Example 19. The institution **FOL** of first-order logic has Craig $(\mathscr{S},\mathscr{S})$-interpolation where $\mathscr{S}$ is the class of all sort injective signature morphisms [4]. This generalises its unsorted version which states that unsorted first-order logic **UFOL** has Craig $(\mathbb{S}ign^{\mathbf{UFOL}},\mathbb{S}ign^{\mathbf{UFOL}})$-interpolation [15].

The institution **EQL** of equational logic has Craig $(\mathbb{S}ign^{\mathbf{EQL}},\mathscr{C})$-interpolation where $\mathscr{C}$ is the class of the conservative algebraic signature morphisms [11]. The paper [11] gives a very general proof of CI based on the abstract Birkhoff-style axiomatizability properties of the actual institution and gives a large list of sub-institutions I of **FOL** having $(\mathbb{S}ign^{I},\mathscr{C})$-interpolation.

The following 'horizontal' and 'vertical' composability properties of CI squares are used crucially in the study of CI in Grothendieck institutions.

Proposition 20. *In any institution, consider the commuting squares of signature morphisms*



Then

(1a) $[\Sigma,\Sigma'_1,\Sigma_2,\Sigma'']$ *is a CI square if* $[\Sigma,\Sigma_1,\Sigma_2,\Sigma']$ *and* $[\Sigma_1,\Sigma'_1,\Sigma',\Sigma'']$ *are CI squares, and*

(1b) $[\Sigma,\Sigma_1,\Sigma_2,\Sigma']$ *is a CI square if* $[\Sigma,\Sigma'_1,\Sigma_2,\Sigma'']$ *is CI square and* θ_1 *is conservative,*

(2a) $[\Sigma,\Sigma_1,\Sigma'_2,\Sigma'']$ *is a CI square if* $[\Sigma,\Sigma_1,\Sigma_2,\Sigma']$ *and* $[\Sigma_2,\Sigma',\Sigma'_2,\Sigma'']$ *are CI squares,*

(2b) $[\Sigma,\Sigma_1,\Sigma_2,\Sigma']$ *is a CI square if* $[\Sigma,\Sigma_1,\Sigma'_2,\Sigma'']$ *is a CI square and* θ_2 *is conservative.*

Proof.

Lemma 21. *For any signature morphism* $\varphi : \Sigma \to \Sigma'$, *for any sets of sentences* $E_1, E_2 \subseteq Sen(\Sigma)$, *if* $E_1 \models E_2$ *then* $\varphi(E_1) \models \varphi(E_2)$.

(1a) Let $E_1' \subseteq Sen(\Sigma_1')$ and $E_2 \subseteq Sen(\Sigma_2)$ such that $\theta_1'(E_1') \models (\varphi_2'; \theta)(E_2)$.

Because $[\Sigma_1, \Sigma_1', \Sigma', \Sigma'']$ is a CI square there exists $E_1 \subseteq Sen(\Sigma_1)$ such that $E_1' \models \theta_1(E_1)$ and $\varphi_1'(E_1) \models \varphi_2'(E_2)$.

From $\varphi_1'(E_1) \models \varphi_2'(E_2)$ and because $[\Sigma, \Sigma_1, \Sigma_2, \Sigma']$ is a CI square, there exists $E \subseteq Sen(\Sigma)$ such that $E_1 \models \varphi_1(E)$ and $\varphi_2(E) \models E_2$.

We therefore have that $E_1' \models \theta_1(E_1) \models \theta_1(\varphi_1(E))$ (by Lemma 21) $= (\varphi_1; \theta_1)(E)$ and $\varphi_2(E) \models E_2$ holds directly from the argument above. This shows that $[\Sigma, \Sigma_1', \Sigma_2, \Sigma'']$ is a CI square.

(1b) Let $E_1 \subseteq Sen(\Sigma_1)$ and $E_2 \subseteq Sen(\Sigma_2)$ such that $\varphi_1'(E_1) \models \varphi_2'(E_2)$.

Then $\theta_1'(\theta_1(E_1)) = \theta(\varphi_1'(E_1)) \models \theta(\varphi_2'(E_2))$. Now we can apply the fact that $[\Sigma, \Sigma_1', \Sigma_2, \Sigma'']$ is a CI square for $\theta_1(E_1)$ and E_2 and deduce that there exists $E \subseteq Sen(\Sigma)$ such that $\theta_1(E_1) \models \theta_1(\varphi_1(E))$ and $\varphi_2(E) \models E_2$. For resuming the proof that E is the interpolant we are looking for, we still have to prove is that $E_1 \models \varphi_1(E)$.

For showing that $E_1 \models \varphi_1(E)$, consider a model M_1 for Σ_1 such that $M_1 \models E_1$. Because θ_1 is conservative, consider M_1' an expansion of M_1 along θ_1. Then by the satisfaction condition of the institution $M_1' \models \theta_1(E_1) \models \theta_1(\varphi_1(E))$. Again by the satisfaction condition but applied in the opposite direction, we have that $M_1 \models \varphi_1(E)$.

(2a+2b) By similar reasoning to (1a) and (1b). □

4. Grothendieck institutions

Grothendieck institutions [10] generalise the Grothendieck construction from indexed categories to indexed institutions. The idea behind the Grothendieck construction for institutions is to put together a system of institutions into a single institution such that their individual identities and the relationships between them are fully retained. This can be interpreted as regarding a heterogenous (multi-logic) environment in a homogenous way without any loss of information.

Definition 22. Given a category I of indices, an *indexed institution* $\mathcal{J}$ is a functor $\mathcal{J} : I^{\mathrm{op}} \to \mathbb{I}ns$. For each index $i \in |I|$ let us denote the institution $\mathcal{J}^i$ by $(\mathbb{S}ign^i, \mathrm{Mod}^i, Sen^i, \models^i)$ and for each index morphism $u \in I$ let us denote the institution morphism $\mathcal{J}^u$ by $(\Phi^u, \alpha^u, \beta^u)$.

The *Grothendieck institution* $\mathcal{J}^\sharp$ of an indexed institution $\mathcal{J} : I^{\mathrm{op}} \to \mathbb{I}ns$ is defined as follows:

(1) its category of signatures $\mathbb{S}ign^\sharp$ is the Grothendieck category of the *indexed* category of signatures $\mathbb{S}ign : I^{\mathrm{op}} \to \mathbb{C}at$ of the indexed institution $\mathcal{J}$;
(2) its model functor $\mathrm{Mod}^\sharp : (\mathbb{S}ign^\sharp)^{\mathrm{op}} \to \mathbb{C}at$ is given by
   - $\mathrm{Mod}^\sharp(\langle i, \Sigma \rangle) = \mathrm{Mod}^i(\Sigma)$ for each index $i \in |I|$ and signature $\Sigma \in |\mathbb{S}ign^i|$, and
   - $\mathrm{Mod}^\sharp(\langle u, \varphi \rangle) = \beta^u_{\Sigma'}; \mathrm{Mod}^i(\varphi)$ for each $\langle u, \varphi \rangle : \langle i, \Sigma \rangle \to \langle i', \Sigma' \rangle$;

(3) its sentence functor $Sen^\sharp : \mathbb{S}ign^\sharp \to \mathbb{S}et$ is given by
- $Sen^\sharp(\langle i, \Sigma\rangle) = Sen^i(\Sigma)$ for each index $i \in |I|$ and signature $\Sigma \in |\mathbb{S}ign^i|$, and
- $Sen^\sharp(\langle u, \varphi\rangle) = Sen^i(\varphi);\ \alpha^u_{\Sigma'}$ for each $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$,

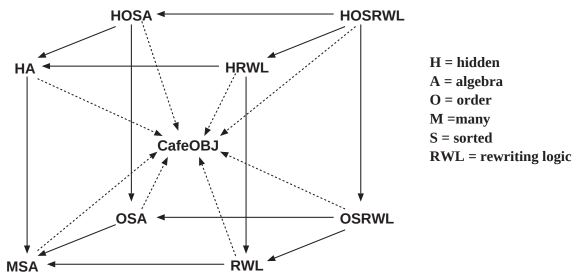
(4) $M \models^\sharp_{\langle i,\Sigma\rangle} e$ iff $M \models^i_\Sigma e$ for each index $i \in |I|$, signature $\Sigma \in |\mathbb{S}ign^i|$, model $M \in |\mathrm{Mod}^\sharp(\langle i, \Sigma\rangle)|$, and sentence $e \in Sen^\sharp(\langle i, \Sigma\rangle)$.

Both Grothendieck categories and Grothendieck institutions are shown in [10] to be special cases of the more abstract concept of ‘Grothendieck object’ in a 2-category, defined as a lax colimit of the indexing (1-)functor.

When the Grothendieck construction is regarded from the viewpoint of fibrations, Grothendieck institutions are the same as the so-called ‘split fibred institution’ of [10], which are institutions with a split fibration projection from their category of signatures. In the fibration language, we may call the institution $\mathscr{I}^i$ as the ‘fibre of $\mathscr{I}^\sharp$ at i’.

The following example characterises the use of Grothendieck institutions for multi-logic specification and historically constitute the origin of the development of Grothendieck institutions.

Example 23. The institution underlying the CafeOBJ language and system [12,13] is the Grothendieck institution of the indexed institution below, called the CafeOBJ cube. (The actual CafeOBJ cube consists of the full arrows, the dotted arrows denote the morphisms from components of the indexed institution to the Grothendieck institution.)



The definition of the institutions and of the institution morphisms of the CafeOBJ cube can be found in [13].

The example below has a rather different flavour from the previous one, and shows how ‘many-sorted’ institutions can be naturally presented as Grothendieck institutions. This example presents **FOL** as a Grothendieck institution of the indexed institution determined by it many sorted structure; however, such presentation can be applied to any other actual institution having a many sorted structure.

Example 24. For any set S, let the institution of *S-sorted first-order logic* $\mathbf{FOL}^S = (\mathbb{S}ign^S, Sen^S, \mathrm{Mod}^S, \models)$ be the sub-institution of the first-order logic institution **FOL** determined by fixing the set of sort symbols to S. The category of signatures $\mathbb{S}ign^S$

consists of all pairs (F,P) where F is an S-sorted set of function symbols and P is an S-sorted set of relation symbols, morphisms of signatures in $\mathbb{S}ign^S$ being just morphisms of signatures in first-order logic which are identities on the sets S of sort symbols. Then the (F,P)-sentences, respectively, models in $\mathbf{FOL}^S$ are the (S,F,P)-sentences, respectively, models in **FOL**. The satisfaction relation between models and sentences is also inherited from **FOL**.

Any function $u: S \to S'$ determines an institution morphism $(\Phi^u, \alpha^u, \beta^u): \mathbf{FOL}^{S'} \to \mathbf{FOL}^S$ such that for each S'-sorted signature (F', P')

- $\Phi^u(F',P') = (F,P)$ with $F_{w\to s} = F'_{u(w)\to u(s)}$ and $P_w = P'_{u(w)}$ for each string of sort symbols $w \in S^*$ and each sort symbol $s \in S$. The canonical first-order logic signature morphism $(S,F,P) \to (S',F',P')$ thus determined is denoted by $\varphi^u_{(F',P')}$,
- $\alpha^u_{(F',P')}: Sen^S(F,P) \to Sen^{S'}(F',P')$ is defined as $Sen(\varphi^u_{(F',P')})$ and, informally, maps each (F,P)-sentence to itself but regarded as an (F',P')-sentence, and
- $\beta^u_{(F',P')}: \mathrm{Mod}^{S'}(F',P') \to \mathrm{Mod}^S(F,P)$ is defined as $\mathrm{Mod}(\varphi^u_{(F',P')})$.

This situation is common to all 'many sorted' logics formalized as institutions and follows from the fact that the category $\mathbb{S}ign$ of the first-order logic signatures is fibred over $\mathbb{S}et$ by the projection of each signature to its set of sorts.

This determines a $\mathbb{S}et$-indexed institution $fol: \mathbb{S}et^{\mathrm{op}} \to \mathbb{I}ns$ such that $fol(S) = \mathbf{FOL}^S$. Then the institution **FOL** of first-order logic can be presented as the Grothendieck institution $fol^\sharp$.

The Grothendieck construction for institutions can be also done with comorphisms rather than morphisms. Comorphism-based Grothendieck institutions have been introduced in [22] by dualization of the morphism-based Grothendieck construction, and they seem to behave more friendly with respect to model amalgamation properties than their morphism-based counterpart.

Definition 25. Given a category I of indices, an indexed coinstitution $\mathcal{J}$ is a functor $\mathcal{J}: I^{\mathrm{op}} \to co\mathbb{I}ns$. Its Grothendieck institution $\mathcal{J}^\sharp$ is defined as follows:

(1) its category of signatures is $((\mathbb{S}ign;(\_)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$ where $\mathbb{S}ign: I^{\mathrm{op}} \to \mathbb{C}at$ is the *indexed* category of signatures of the indexed coinstitution $\mathcal{J}$, $(\_)^{\mathrm{op}}: \mathbb{C}at \to \mathbb{C}at$ is the 'opposite' functor, and $(\mathbb{S}ign;(\_)^{\mathrm{op}})^\sharp$ is its Grothendieck category; this means that
  - ○ signatures are pairs $\langle i, \Sigma\rangle$ for $i \in |I|$ index and $\Sigma \in |\mathbb{S}ign^i|$, and
  - ○ signature morphisms are pairs $\langle u, \varphi\rangle : \langle i, \Sigma\rangle \to \langle i', \Sigma'\rangle$ where $u \in I(i', i)$ and $\varphi \in \mathbb{S}ign^{i'}(\Sigma\Phi^u, \Sigma')$;

(2) its model functor $\mathrm{Mod}^\sharp: (\mathbb{S}ign;(\_)^{\mathrm{op}})^\sharp \to \mathbb{C}at$ is given by
  - ○ $\mathrm{Mod}^\sharp(\langle i, \Sigma\rangle) = \mathrm{Mod}^i(\Sigma)$ for each index $i \in |I|$ and signature $\Sigma \in |\mathbb{S}ign^i|$, and
  - ○ $\mathrm{Mod}^\sharp(\langle u, \varphi\rangle) = \mathrm{Mod}^{i'}(\varphi); \Sigma\beta^u$ for each $\langle u, \varphi\rangle : \langle i', \Sigma'\rangle \to \langle i, \Sigma\rangle$;

(3) its sentence functor $Sen^\sharp: ((\mathbb{S}ign;(\_)^{\mathrm{op}})^\sharp)^{\mathrm{op}} \to \mathbb{S}et$ is given by
  - ○ $Sen^\sharp(\langle i, \Sigma\rangle) = Sen^i(\Sigma)$ for each index $i \in |I|$ and signature $\Sigma \in |\mathbb{S}ign^i|$, and
  - ○ $Sen^\sharp(\langle u, \varphi\rangle) = \Sigma\alpha^u; Sen^{i'}(\varphi)$ for each $\langle u, \varphi\rangle : \langle i', \Sigma'\rangle \to \langle i, \Sigma\rangle$;

(4) $M \models^\sharp_{\langle i,\Sigma\rangle} e$ iff $M \models^i_\Sigma e$ for each index $i \in |I|$, signature $\Sigma \in |\mathbb{S}ign^i|$, model $M \in |\mathrm{Mod}^\sharp(\langle i, \Sigma\rangle)|$, and sentence $e \in Sen^\sharp(\langle i, \Sigma\rangle)$;

where $\mathscr{I}^i = (\mathbb{S}ign^i, \mathrm{MOD}^i, Sen^i, \models^i)$ for each index $i \in |I|$ and $\mathscr{I}^u = (\Phi^u, \alpha^u, \beta^u)$ for $u \in I$ index morphism.

Definition 26. An *adjoint-indexed institution* is an indexed institution $\mathscr{I} : I^{\mathrm{op}} \to \mathbb{I}ns$ for which all institution morphisms $\mathscr{I}^u$ are adjoint morphisms for all index morphisms $u \in I$. Adjoint-indexed coinstitutions are defined similarly.

Remark 27. For each adjoint-indexed institution $\mathscr{I} : I^{\mathrm{op}} \to \mathbb{I}ns$ there exists an adjoint-indexed coinstitution $\bar{\mathscr{I}} : (I^{\mathrm{op}})^{\mathrm{op}} \to co\mathbb{I}ns$ such that

- $\bar{\mathscr{I}}^i = \mathscr{I}^i$ for each index $i \in I$, and
- $\bar{\mathscr{I}}^u$ is the comorphism dual to the morphism $\mathscr{I}^u$ for each index morphism u.

Example 28. By following the details of [13] we can notice that the CafeOBJ cube is an adjoint-indexed institution.

Example 29. The $\mathbb{S}et$-indexed institution fol determined by the fibred institution **FOL** of first-order logic is adjoint-indexed. For each function $u : S \to S'$, let $\overline{\Phi^u} : \mathbb{S}ign^S \to \mathbb{S}ign^{S'}$ map each S-sorted signature (F, P) to the S'-sorted signature (F^u, P^u) defined by $F^u_{w' \to s'} = \bigcup_{u(ws) = w's'} F_{w \to s}$ and $P^u_{w'} = \bigcup_{u(w) = w'} P_w$ for each string of sort symbols $w \in S^*$ and sort symbol $s \in S$. [6] Notice that $\overline{\Phi^u}$ is a left adjoint to the 'forgetful' functor $\Phi^u : \mathbb{S}ign^{S'} \to \mathbb{S}ign^S$.

In the case of adjoint institution morphisms/comorphisms, the Grothendieck construction on institution is independent of the choice between using morphisms or comorphisms.

Proposition 30 (Mossakowski [23]). *For each dual pair of an adjoint-indexed institution $\mathscr{I}$ and an adjoint-indexed coinstitution $\bar{\mathscr{I}}$ their Grothendieck institutions $\mathscr{I}^\sharp$ and $\bar{\mathscr{I}}^\sharp$ are isomorphic.*

Example 31. The institution **FOL** of first-order logic can be also obtained as a comorphism-based Grothendieck institution by using the indexed coinstitution determined by the sorting structure.

5. Interpolation in Grothendieck institutions

In this section we show that interpolation at the 'global' level of the Grothendieck institutions is equivalent to the 'local' interpolation at the level of each index institution, plus an interpolation property at the level of the indexed institution, plus an interpolation property of the institution mappings corresponding to the index morphisms. This is similar to the situation of the semi-exactness property in Grothendieck institutions [10,22]. As noticed by Mossakowski [22], comorphisms interact in a simpler

[6] The unions defining F' and P' should be disjoint.

way with the semi-exactness of Grothendieck institutions, and this is also the case for interpolation. Therefore, for the study of interpolation we work with comorphism-based Grothendieck institutions.

Definition 32. A *Craig Interpolation square of institution comorphisms* is any commuting square of institution comorphisms

$$\begin{array}{ccc}
(\mathbb{S}ign, Sen, \text{MOD}, \models) & \xrightarrow{(\Phi_1,\alpha_1,\beta_1)} & (\mathbb{S}ign_1, Sen_1, \text{MOD}_1, \models_1) \\
{\scriptstyle (\Phi_2,\alpha_2,\beta_2)}\downarrow & & \downarrow{\scriptstyle (\Phi_1',\alpha_1',\beta_1')} \\
(\mathbb{S}ign_2, Sen_2, \text{MOD}_2, \models_2) & \xrightarrow[(\Phi_2',\alpha_2',\beta_2')]{} & (\mathbb{S}ign', Sen', \text{MOD}', \models')
\end{array}$$

such that for each $\Sigma \in |\mathbb{S}ign|$, for each set E_1 of $\Sigma\Phi_1$-sentences and for each set E_2 of $\Sigma\Phi_2$-sentences, if $(\alpha_1')_{\Sigma\Phi_1}(E_1) \models' (\alpha_2')_{\Sigma\Phi_2}(E_2)$, then there exists a set E of Σ-sentences such that $E_1 \models_1 (\alpha_1)_\Sigma(E)$ and $(\alpha_2)_\Sigma(E) \models_2 E_2$.

Example 33. The square of embedding institution comorphisms

$$\begin{array}{ccc}
\mathbf{MS} & \longrightarrow & \mathbf{EQL} \\
\downarrow & & \downarrow \\
\mathbf{REL} & \longrightarrow & \mathbf{FOL}
\end{array}$$

is trivially a CI square because for each set S there are no $(S, \emptyset)$-sentences in **EQL** and no $(S, \emptyset)$-sentences in **REL**.

In general, in the case of multi-logic Grothendieck institutions this property seem to be quite trivially satisfied.

Example 34. The example of the sorting fibres in the institution **FOL** of first-order logic is, however, more interesting.

Proposition 35. *For each pushout of sets*

$$\begin{array}{ccc}
S & \xrightarrow{u1} & S1 \\
{\scriptstyle u2}\downarrow & & \downarrow{\scriptstyle v1} \\
S2 & \xrightarrow[v2]{} & S'
\end{array}$$

its corresponding square of institution comorphisms

$$\begin{array}{ccc} \mathbf{FOL}^{S} & \xrightarrow{(\overline{\Phi^{u1}},\overline{\alpha^{u1}},\overline{\beta^{u1}})} & \mathbf{FOL}^{S1} \\ {\scriptstyle(\overline{\Phi^{u2}},\overline{\alpha^{u2}},\overline{\beta^{u2}})}\downarrow & & \downarrow{\scriptstyle(\overline{\Phi^{v1}},\overline{\alpha^{v1}},\overline{\beta^{v1}})} \\ \mathbf{FOL}^{S2} & \xrightarrow[(\overline{\Phi^{v2}},\overline{\alpha^{v2}},\overline{\beta^{v2}})]{} & \mathbf{FOL}^{S'} \end{array}$$

is a CI square when either $u1$ *or* $u2$ *are injective.*

Proof. For each function u recall that $\overline{\Phi^u}$ denotes the left adjoint to the 'forgetful' functor Φ^u. Fix a signature (F,P) of S-sorted function and relation symbols. Let $\overline{\Phi^{u1}}(F,P)=(F_1,P_1)$, $\overline{\Phi^{u2}}(F,P)=(F_2,P_2)$, and $\overline{\Phi^{ui;vi}}(F,P)=(F',P')$. Consider E_1 a set of (F_1,P_1)-sentences and E_2 a set of (F_2,P_2)-sentences such that $\overline{\alpha^{v1}_{(F,P)}}(E_1)\models \overline{\alpha^{v2}_{(F,P)}}(E_2)$.

If $u1$ is injective, then $\overline{\alpha^{u1}_{(F,P)}}$ is bijection, hence let us define $E=(\overline{\alpha^{u1}_{(F,P)}})^{-1}(E_1)$. Obviously $E_1\models\overline{\alpha^{u1}}(E)$. We show that $\overline{\alpha^{u2}}(E)\models E_2$ too. Let M_2 be an (F_2,P_2)-model such that $M_2\models\overline{\alpha^{u2}}(E)$. By the satisfaction condition this means that $\overline{\beta^{u2}}(M_2)\models E$.

Because $u1$ is injective it is easy to notice that we can find an (F_1,P_1)-model M_1 such that $\overline{\beta^{u1}}(M_1)=\overline{\beta^{u2}}(M_2)$. By the semi-exactness of **FOL** we can notice that there exists an unique (F',P')-model M' such that $\overline{\beta^{v1}}(M')=M_1$ and $\overline{\beta^{v2}}(M')=M_2$. By the satisfaction condition $M'\models\overline{\alpha^{ui;vi}}(E)$ which means that $M'\models\overline{\alpha^{v1}}(E_1)$ which by hypothesis implies $M'\models\overline{\alpha^{v2}}(E_2)$, which by the satisfaction condition means $M_2\models E_2$.

The other case, when $u2$ is injective, gets a similar proof. □

This result is essentially based on the fact that $\overline{\alpha^{u}_{(F,P)}}$ is surjective, which follows by the injectivity of u, hence similarly one can show the same type of result for other many sorted logics presented as institutions, including for example rewriting logic **RWL**, etc.

Definition 36. For a fixed class $\mathscr{S}\subseteq\mathbb{S}ign$ of signature morphisms, we say that an institution comorphism $(\Phi,\alpha,\beta):(\mathbb{S}ign,Sen,\text{Mod},\models)\to(\mathbb{S}ign',Sen',\text{Mod}',\models')$

- has the *Craig $\mathscr{S}$-left Interpolation property* when for each $\varphi:\Sigma\to\Sigma_1$ signature morphism in $\mathscr{S}$, for each set E_1 of Σ_1-sentences and each set E_2 of $\Sigma\Phi$-sentences such that $\alpha_{\Sigma_1}(E_1)\models'(\varphi\Phi)(E_2)$, there exists a set of Σ-sentences E such that $E_1\models\varphi(E)$ and $\alpha_\Sigma(E)\models' E_2$, and
- has the *Craig $\mathscr{S}$-right Interpolation property* when for each $\varphi:\Sigma\to\Sigma_2$ signature morphism in $\mathscr{S}$, for each set E_1 of $\Sigma\Phi$-sentences and each set E_2 of Σ_2-sentences such that $(\varphi\Phi)(E_1)\models'\alpha_{\Sigma_2}(E_2)$, there exists a set of Σ-sentences E such that $E_1\models\alpha_\Sigma(E)$ and $\varphi(E)\models' E_2$.

Example 37. It is rather easy to notice that the comorphism **REL** → **FOL** embedding the institution of relational logic into the institution of first-order logic has both the Craig $\mathbb{S}ign^{\mathbf{REL}}$ left and right interpolation properties.

Indeed, for any relational signature morphism $\varphi:(S,P)\to(S_1,P_1)$, if $E_1 \models^{\mathbf{FOL}}_{(S_1,\emptyset,P_1)} \varphi'(E_2)$ (where $\varphi':(S,\emptyset,P)\to(S_1,\emptyset,P_1)$ is the trivial expansion of φ to a **FOL** signature morphism), then because any $(S,\emptyset,P)$-sentence in **FOL** is a (S,P)-sentence in **REL**, we can take the interpolant to be just E_2. The interpolant property of E_2 can be checked very easily. This shows the left interpolation property, the right one can be shown similarly. A similar example is given by the comorphism **FOEQL** → **FOL** embedding first-order equational logic into first-order logic.

In principle, in the actual examples, the interpolation property for an institution comorphism holds easily when the sentences of the source and of the target institution have the same expressive power.

Example 38. A rather subtle example is provided by the comorphism **EQL** → **RWL** embedding equational logic into rewriting logic. We may recall from [10,9] that this embedding comorphism is problematic for multi-logic systems because it destroys the semi-exactness property for Grothendieck institutions.

Let $\mathscr{S}$ be the class of algebraic signature morphisms $\varphi:(S,F)\to(S',F')$ which are injective on the sort symbols and such that for each $\sigma\in F'_{w\to s}$ if $s\in S'\backslash\varphi(S)$ then $w\in(S'\backslash\varphi(S))^*$, and let $\mathscr{S}_c$ be the class of signature morphisms of $\mathscr{S}$ which are in addition conservative.

Proposition 39.

- *The comorphism* **EQL** → **RWL** *embedding the institution of equational logic into the institution of rewriting logic has Craig $\mathscr{S}_c$-right interpolation, and*
- *the comorphism* **FOEQL** → **FORWL** *embedding the institution of first-order equational logic into the institution of first-order rewriting logic has both Craig $\mathscr{S}$-left and right interpolation.*

Proof. In order to establish these properties some non-trivial work is needed.

Recall that a *universal Horn sentence* for a first-order signature (S,F,P) is a sentence of the form $(\forall X)H\to C$, where H is a finite conjunction of (relational or equational) atoms and C is a (relational of equational) atom, and $H\to C$ is the implication of C by H. The sub-institution **HCL** of **FOL** has the same signatures and models as **FOL** but only universal Horn sentences as sentences. Each algebraic signature (S,F) can be mapped to the **HCL** presentation $((S,F,\{\leqslant_s\}_{s\in S}), pre_{(S,F)})$ such that for each sort symbol $s\in S$ the arity of $\leqslant_s$ is ss, and $pre_{(S,F)}$ contains the preorder axioms for each $\leqslant_s$ and all axioms stating the preorder functoriality of the interpretations of the function symbols of F. Moreover, each (S,F)-sentence e in **RWL** can be canonically mapped to a universal Horn $(S,F,\{\leqslant_s\}_{s\in S})$-sentence $\bar{e}$.[7] In the case of **FORWL** a similar mapping can be done to **FOL** rather than **HCL**.

[7] Below we also extend this notation to sets of sentences.

Now let $\varphi:(S,F)\to(S_2,F_2)$ be any algebraic signature morphism in $\mathscr{S}_c$, E_1 be a set of (S,F)-sentences in **RWL**, and E_2 be a set of (S_2,F_2)-equations such that $\varphi'(E_1)\models^{\mathbf{RWL}}E_2$, where $\varphi':(S,F,\{\leqslant_s\}_{s\in S})\to(S_2,F_2,\{\leqslant_s\}_{s\in\varphi(S)})$ is the canonical extension of φ to a **HCL**-signature morphism.

We first remark that $\varphi'(\overline{E_1})\cup\varphi'(pre_{(S,F)})\models^{\mathbf{HCL}}E_2$. For this, it is crucial to observe that each $(S,F,\{\leqslant_s\}_{s\in S})$-model M satisfying $\varphi'(pre_{(S,F)})$, since $\varphi\in\mathscr{S}$, can be trivially regarded as an (S_2,F_2)-model in **RWL** by defining $M_{\leqslant_{s'}}$ as the diagonal relation for each $s'\in S_2\backslash\varphi(S)$.

By Craig interpolation in **HCL** (see [11]), the pushout square below

$$\begin{array}{ccc}(S,F,\emptyset) & \longrightarrow & (S,F,\{\leq_s\}_{s\in S})\\ \varphi\big\downarrow & & \big\downarrow\varphi'\\ (S_2,F_2,\emptyset) & \longrightarrow & (S_2,F_2,\{\leq_s\}_{s\in\varphi(S)})\end{array}$$

is a CI square in **HCL** (notice that the conservativeness of φ is essentially used at this step). Therefore, there exists a set E of (S,F)-equations such that $\overline{E_1}\cup pre_{(S,F)}\models^{\mathbf{HCL}}E$ and $\varphi(E)\models^{\mathbf{HCL}}E_2$.

The first relation tells us that $E_1\models^{\mathbf{RWL}}E$ and the second one that $\varphi(E)\models^{\mathbf{EQL}}E_2$.

A similar proof can be invoked for establishing that the embedding comorphism **FOEQL** $\to$ **FORWL** has the Craig $\mathscr{S}$-interpolation property. For this we have to map **FORWL** into **FOL** rather than **HCL** and to involve CI in many sorted first-order logic [4]. Notice that in this case, because of the difference of CI in **HCL** and **FOL**, we can relax the requirement on φ to belong to $\mathscr{S}$ rather than $\mathscr{S}_c$.

Finally, for the Craig $\mathscr{S}$-left interpolation property of the comorphism **FOEQL** $\to$ **FORWL**, we let $\varphi:(S,F)\to(S_1,F_1)$ be any algebraic signature morphism in $\mathscr{S}$, E_1 be a set of (S_1,F_1)-sentences in **FOEQL**, and E_2 be a set of (S,F)-sentences in **FORWL** such that $E_1\models^{\mathbf{FORWL}}\varphi(E_2)$.

Similar to the above, we remark that $E_1\cup\varphi'(pre_{(S,F)})\models^{\mathbf{FOL}}\varphi'(\overline{E_2})$ where $\varphi':(S,F,\{\leqslant_s\}_{s\in S})\to(S_1,F_1,\{\leqslant_s\}_{s\in\varphi(S)})$ is the canonical extension of φ to a **FOL**-signature morphism.

Then, similarly to the above, we notice that the pushout square below

$$\begin{array}{ccc}(S,F,\emptyset) & \xrightarrow{\ \varphi\ } & (S_1,F_1,\emptyset)\\ \big\downarrow & & \big\downarrow\\ (S,F,\{\leq_s\}_{s\in S}) & \xrightarrow[\ \varphi'\ ]{} & (S_1,F_1,\{\leq_s\}_{s\in\varphi(S)})\end{array}$$

is a CI square. Because **FOL** has finite conjunctions and implications it is also a Craig–Robinson interpolation square [15]. Therefore, there exists a set E of $(S,F,\emptyset)$-sentences such that $E_1\models^{\mathbf{FOL}}\varphi(E)$ and $E\cup pre_{(S,F)}\models^{\mathbf{FOL}}\overline{E_2}$.

The first relation implies $E_1 \models^{\mathbf{FOEQL}} \varphi(E)$, while the second one implies $E \models^{\mathbf{FORWL}} E_2$.

Let us also notice that this proof for the left interpolation cannot be applied to the comorphism **EQL** → **RWL** because **HCL** lacks Craig–Robinson interpolation. □

Example 40. Let us now turn our look to the sorting fibres of the institution **FOL** of first-order logic.

Proposition 41. *For each injective function* $u : S \to S'$, *the institution comorphism* $(\overline{\Phi^u}, \overline{\alpha^u}, \overline{\beta^u}) : \mathbf{FOL}^S \to \mathbf{FOL}^{S'}$ *has both the* $\mathbb{S}ign^S$*-left and right interpolation properties.*

Proof. Let us consider a signature morphism $\varphi : (F,P) \to (F_1,P_1)$ in $\mathbb{S}ign^S$. Let $\varphi^u : (F^u,P^u) \to (F_1^u,P_1^u)$ denote $\overline{\Phi^u}(\varphi)$. Let E_1 be a set of (F_1,P_1)-sentences and E_2 be a set of (F^u,P^u)-sentences such that $\overline{\alpha^u_{(F_1,P_1)}}(E_1) \models \varphi^u(E_2)$.

Because u is injective, $\overline{\alpha^u_{(F,P)}}$ is surjective, hence we can find a set E of (F,P)-sentences such that $E_2 = \overline{\alpha^u_{(F,P)}}(E)$. Since the other part of the interpolant property of E is trivial, we have to prove only that $E_1 \models \varphi(E)$.

Consider a model M_1 of E_1. Because u is injective we can find an (F^u,P^u)-model M' such that $\overline{\beta^u_{(F,P)}}(M') = M_1\upharpoonright_\varphi$. By the semi-exactness of **FOL**, we can find a model M_1' of (F_1^u,P_1^u) such that $\overline{\beta^u_{(F_1,P_1)}}(M_1') = M_1$ and $M_1'\upharpoonright_{\varphi^u} = M'$. By the satisfaction condition we have that $M_1' \models \overline{\alpha^u_{(F_1,P_1)}}(E_1)$, hence by the hypothesis we have $M_1' \models \varphi^u(E_2)$. By the satisfaction condition we have that $\overline{\beta^u_{(F,P)}}(M_1'\upharpoonright_{\varphi^u}) \models E$ which by the naturality of $\overline{\beta^u}$ means that $M_1\upharpoonright_\varphi \models E$. By the satisfaction condition we finally deduce that $M_1 \models \varphi(E)$.

We have thus proved the left interpolation property, the right interpolation property gets a similar proof. □

This result is essentially based on the fact that $\overline{\alpha^u_{(F,P)}}$ is surjective, which follows by the injectivity of u, hence similarly one can show the same type of result for other many sorted logics presented as institutions, including got example rewriting logic **RWL**, etc.

Now we are ready to formulate and prove the main result of this paper.

Theorem 42. *Let* $\mathscr{J} : I^{\mathrm{op}} \to co\mathbb{I}ns$ *be an indexed coinstitution such that*

- *there are fixed classes of index morphisms* $\mathscr{L}, \mathscr{R} \subseteq I$ *containing all identities,*
- *for each index* i *there are fixed classes of signature morphisms* $\mathscr{L}^i, \mathscr{R}^i \subseteq \mathbb{S}ign^i$ *containing all identities,*

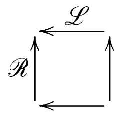
such that

- $\mathscr{L}$ *and* $\mathscr{R}$ *are stable under pullbacks,*
- $\Phi^u(\mathscr{R}^i) \subseteq \mathscr{R}^j$ *for each index morphism* $u : j \to i$ *in* $\mathscr{L}$,
- $\Phi^u(\mathscr{L}^i) \subseteq \mathscr{L}^j$ *for each index morphism* $u : j \to i$ *in* $\mathscr{R}$,

Let $\mathscr{L}^\sharp$, *and* $\mathscr{R}^\sharp$, *be the classes of signature morphisms* $\langle u : j \to i, \varphi \rangle$ *of the Grothendieck institution such that* $u \in \mathscr{L}$, *respectively,* $u \in \mathscr{R}$, *and* $\varphi \in \mathscr{L}^j$, *respectively,* $\varphi \in \mathscr{R}^j$.

Then the Grothendieck institution $\mathcal{I}^\sharp$ has the Craig $(\mathcal{L}^\sharp, \mathcal{R}^\sharp)$-interpolation property if and only if

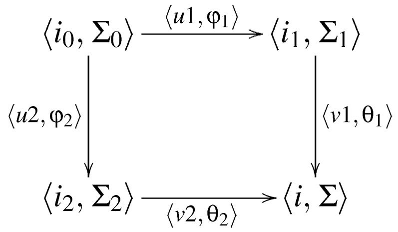
- *for each index i the institution $\mathcal{I}^i$ has the $(\mathcal{L}^i, \mathcal{R}^i)$-interpolation property,*
- *each pullback square of index morphisms*



determines a Craig interpolation square of institution comorphisms

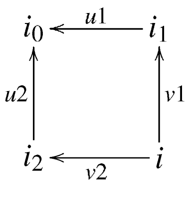
- *for each $u : j \to i$ in $\mathcal{L}$ the corresponding institution comorphism has the Craig $\mathcal{R}^i$-right interpolation property,*
- *for each $u : j \to i$ in $\mathcal{R}$ the corresponding institution comorphism has the Craig $\mathcal{L}^i$-left interpolation property.*

Proof. For the 'sufficient' part, we consider an arbitrary pushout of signatures in the Grothendieck institution



such that $u1 \in \mathcal{L}$, $\varphi_1 \in \mathcal{L}^{i_1}$, and $u2 \in \mathcal{R}$, $\varphi_2 \in \mathcal{R}^{i_2}$.

By following the construction of colimits in Grothendieck categories [31] for the special case of pushouts notice that



is a pullback in the index category I. By the stability hypothesis we have that $v1 \in \mathcal{R}$ and $v2 \in \mathcal{L}$.

By the same result, we can also notice that the pushout square of signatures in the Grothendieck institution can be expressed as the following composition of four pushout squares:

$$
\begin{array}{ccccc}
\langle i_0, \Sigma_0\rangle & \xrightarrow{\langle u1, 1_{\Sigma_0\Phi^{u1}}\rangle} & \langle i_1, \Sigma_0\Phi^{u1}\rangle & \xrightarrow{\langle 1_{i_1}, \varphi_1\rangle} & \langle i_1, \Sigma_1\rangle \\
\Big\downarrow{\scriptstyle \langle u2, 1_{\Sigma_0\Phi^{u2}}\rangle} & & \Big\downarrow{\scriptstyle \langle v1, 1_{\Sigma_0\Phi^{u1}\Phi^{v1}}\rangle} & & \Big\downarrow{\scriptstyle \langle v1, 1_{\Sigma_1\Phi^{v1}}\rangle} \\
\langle i_2, \Sigma_0\Phi^{u2}\rangle & \xrightarrow[\langle v2, 1_{\Sigma_0\Phi^{u2}\Phi^{v2}}\rangle]{} & \langle i, \Sigma_0\Phi^{ui}\Phi^{vi}\rangle & \xrightarrow[\langle 1_i, \varphi_1\Phi^{v1}\rangle]{} & \langle i, \Sigma_1\Phi^{v1}\rangle \\
\Big\downarrow{\scriptstyle \langle 1_{i_2}, \varphi_2\rangle} & & \Big\downarrow{\scriptstyle \langle 1_i, \varphi_2\Phi^{v2}\rangle} & & \Big\downarrow{\scriptstyle \langle 1_i, \theta_1\rangle} \\
\langle i_2, \Sigma_2\rangle & \xrightarrow[\langle v2, 1_{\Sigma_2\Phi^{v2}}\rangle]{} & \langle i, \Sigma_2\Phi^{v2}\rangle & \xrightarrow[\langle 1_i, \theta_2\rangle]{} & \langle i, \Sigma\rangle
\end{array}
$$

- The up-left pushout square is a CI square by applying the fact that the corresponding square of institution comorphisms is a CI square and by considering the signature Σ_0.
- The down-right pushout square is a CI square because it is a CI square in the institution $\mathcal{J}^i$ as a pushout square of a signature morphism in $\mathcal{L}^i$ with a signature morphism in $\mathcal{R}^i$. Here we have to notice that $\varphi_1\Phi^{v1} \in \mathcal{L}^i$ because $\varphi_1 \in \mathcal{L}^{i_1}$ and $v1 \in \mathcal{R}$, and that $\varphi_2\Phi^{v2} \in \mathcal{R}^i$ because $\varphi_2 \in \mathcal{R}^{i_2}$ and $v2 \in \mathcal{L}$.
- The up-right pushout square is a CI square because $\varphi_1 \in \mathcal{L}^{i_1}$ and $v1 \in \mathcal{R}$ and by the assumption that $(\Phi^{v1}, \alpha^{v1}, \beta^{v1})$ has the Craig $\mathcal{L}^{i_1}$-left interpolation property.
- The down-left pushout square is a CI square by an argument symmetrical to the argument of the item above.

Therefore, all four components of the big pushout square in the Grothendieck institution are CI squares. By Proposition 20, we can now conclude that the original pushout square of signature morphisms in the Grothendieck institution is a CI square. This completes the proof of the 'sufficient' part of the theorem.

For the 'necessary' part, we have only to notice the following:

- For each index i, by considering 1_i as index morphism, a Craig $(\mathcal{L}^i, \mathcal{R}^i)$-interpolation square in $\mathcal{J}^i$ is just a Craig $(\mathcal{L}^\sharp, \mathcal{R}^\sharp)$-interpolation square in the Grothendieck institution.
- For $\langle v1, v2\rangle$ a pullback of $\langle u1, u2\rangle$ in the index category I, by the colimit construction in Grothendieck categories, the following squares

$$
\begin{array}{ccc}
\langle i_0, \Sigma\rangle & \xrightarrow{\langle u1, 1_{\Sigma\Phi^{u1}}\rangle} & \langle i1, \Sigma\Phi^{u1}\rangle \\
\Big\downarrow{\scriptstyle \langle u2, 1_{\Sigma\Phi^{u2}}\rangle} & & \Big\downarrow{\scriptstyle \langle v1, 1_{\Sigma\Phi^{u1}\Phi^{v1}}\rangle} \\
\langle i2, \Sigma\Phi^{u2}\rangle & \xrightarrow[\langle v2, 1_{\Sigma\Phi^{u2}\Phi^{v2}}\rangle]{} & \langle i, \Sigma\Phi^{ui}\Phi^{vi}\rangle
\end{array}
$$

  are pushouts in the category of signatures $((\mathbb{S}ign; (\_)^{\mathrm{op}})^\sharp)^{\mathrm{op}}$ of the Grothendieck institution for each signature Σ in $|\mathbb{S}ign^{i_0}|$. Therefore, they are CI squares if and only if the square of index morphisms determines a CI square of institution comorphisms.

- For each $u : j \to i$ in $\mathscr{L}$ and each signature morphism $\varphi : \Sigma_1 \to \Sigma_2$ in $\mathscr{R}^i$, the square below

$$\begin{array}{ccc}
\langle i, \Sigma_1 \rangle & \xrightarrow{\langle u, 1_{\Sigma_1 \Phi^u} \rangle} & \langle j, \Sigma_1 \Phi^u \rangle \\
{\scriptstyle \langle 1_i, \varphi \rangle} \downarrow & & \downarrow {\scriptstyle \langle 1_j, \varphi \Phi^u \rangle} \\
\langle i, \Sigma_2 \rangle & \xrightarrow[\langle u, 1_{\Sigma_2 \Phi^u} \rangle]{} & \langle j, \Sigma_2 \Phi^u \rangle
\end{array}$$

is a pushout in the category of signatures of the Grothendieck institution. Moreover, these squares are CI squares if and only if $(\Phi^u, \alpha^u, \beta^u)$ has the Craig $\mathscr{R}^i$-right interpolation property.

- By replacing $\mathscr{L}$ by $\mathscr{R}$, $\mathscr{R}^i$ by $\mathscr{L}^i$, and 'right' by 'left' in the argument above, we can deduce the symmetrical conclusion. □

Besides establishing CI properties for multi-logic Grothendieck institutions, Theorem 42 can be used as a uniform method to lift CI properties from unsorted logics to their many sorted extensions. We illustrate this with the example of **FOL**, however other many sorted logics can be treated similarly.

Corollary 43 (Borzyszkowski [14]). *In the institution* **FOL** *of first-order logic, each pushout square of signature morphisms which are injective on the sort symbols is a CI square.*

Proof. We start from the fact that any pushout of unsorted first-order logic signatures is a CI square [15], and then we notice easily that in the many sorted context this generalizes trivially to any fixed set of sorts. The next step regards the institution **FOL** of (many sorted) first-order logic as the Grothendieck institution determined by the sorting and take $\mathscr{L}$ and $\mathscr{R}$ as the class of all injective functions, and for each set S, we take $\mathscr{L}^S$ and $\mathscr{R}^S$ as the class $\mathbb{S}ign^S$ of all S-sorted signature morphisms. Finally, Propositions 35 and 41 together with the stability of injective functions under pullbacks, show that all hypotheses of Theorem 42 hold. □

6. Conclusions and future research

We gave a necessary and sufficient condition for Craig interpolation in Grothendieck institutions based on 'local' interpolation at the level of indice institutions, interpolation at the level of the indexed institution, and an interpolation property for the institution mappings (i.e. comorphisms) involved.

We have provided an analysis of the latter two conditions by illustrating them with several significant examples. We have seen that in many cases they can be easily established, but the interpolation property of institution comorphisms can be rather subtle in some cases.

We also showed how our main result also provides a uniform method for lifting interpolation from the unsorted situations to many sorted situations, many sorted interpolation being a non-trivial generalisation of unsorted interpolation. We have illustrated this with the example of first-order logic. An interesting open question is whether by varying the parameters of the main theorem one could obtain interpolation results for many sorted first-order logic different from Corollary 43.

Future research might emphasise further investigations of applications for our main result Theorem 42. A somehow obligatory future research project is to use then main result of this paper for providing a complete analysis of CI properties of actual multi-logic institutions in use in computing, such as CafeOBJ or extended CASL. The size of such enterprise recommends it as a different project from this paper.

Finally, we also think our main result can be easily extended from CI to Craig–Robinson Interpolation which seems to be the right interpolation property needed for theorem proving [8] and semantics [15] of structured specifications.

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