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Bases for Modules

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Abstract: Let R be a ring and A an R-module. We examine different notions of bases or generating sets for A. Of particular interest is the notion of an irredundant basis for A, that is, a subset X of A that generates A but for which no proper subset of X generates A. We investigate the existence and cardinality of irredundant bases.

Keywords: Generating set, basis, irredundant basis, perfect ring

1. Introduction

The word "basis" is one of those overused words in mathematics whose meaning must often be deduced from context. On one hand the word "basis" is used for a linear independent spanning set in a vector space or free module while at the other extreme it is used to just mean any generating set for a module or ideal as in the Hilbert Basis Theorem. Equally ambiguous is the phrase "minimal basis" or "minimal generating set". The purpose of this article is to examine different types of "bases" or distinguished generating sets for modules. To compare and contrast the different types of bases we have included a number of well known results. Hopefully the less expert reader will enjoy our somewhat expository style and references to well known results while the expert reader will forgive us.

Let R be a ring (always with identity) and M an R-module (always a unitary left R-module unless otherwise noted). A subset $X \subseteq M$ is a *basis* (resp. *weak basis*, *i-basis*) for M if X generates M and for $x_1, \cdots, x_n \in X$ and $r_1, \cdots, r_n \in R$, $r_1x_1 + \cdots + r_nx_n = 0$ implies each $r_i = 0$ (resp. each $r_ix_i = 0$ but $x_i \neq 0$, each r_i is a nonunit). So M has a basis precisely when M is a free R-module and M has a weak basis if and only if M is a direct sum of cyclic modules. It is easy to see that X is an i-basis if and only if X is irredundant in the sense that no proper subset of X generates M.

In Section 2 these three types of bases are examined in some detail. A number of examples are given to illustrate the similarities and differences between the three types of bases.

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However, we will concentrate on i-bases. Of particular interest is what modules have an i-basis and how i-bases behave with respect to standard ring and module constructions.

In Section 3 we consider the possible different cardinalities for the different types of bases for a given module. It is of course well known that while a finitely generated free module may have bases of different (finite) cardinalities, this cannot happen for a free module with an infinite basis. We show that if a module M has an infinite i-basis X then any generating set Y for M has $|X| \leq |Y|$. In particular, if Y is another i-basis for M, then $|X| = |Y|$. We show that quasilocal rings are characterized by the property that any two i-bases for a module have the same cardinality. We also examine the possible different lengths of i-bases for a finitely generated module.

In Section 4 we consider the question of what rings R have the property that every R-module has an i-basis. We show that a left perfect ring has this property and give a partial converse.

As previously noted, "ring" will mean an associative ring with identity and "module" a unitary left R-module. We follow standard notation and terminology from [1] or [6].

2. Types of Bases and Examples

Throughout R is a ring with identity and module means left R-module. For a subset X of an R-module M, $\langle X\rangle$ denotes the submodule generated by X. The following definition gives three different notions of independence and basis.

Definition 2.1. *Let R be a ring and I an ideal of R. Let M be an R-module and let $X \subseteq M$. Then X is* I-independent (*resp.* weakly I-independent, irredundant) *if for $x_{\alpha_1}, \cdots, x_{\alpha_n} \in X$, $r_1 x_{\alpha_1} + \cdots + r_n x_{\alpha_n} = 0$ $(r_1, \cdots, r_n \in R)$ implies each $r_i \in I$ (resp. each $r_i x_{\alpha_i} \in IM$ but each $x_{\alpha_i} \notin IM$, each r_i is a nonunit). If X generates M and X is I-independent (resp., weakly I-independent, irredundant), then X is an* I-basis (*resp.,* weak I-basis, i-basis) for M. *If M has an i-basis, we say that M is* i-generated. *In the case $I = 0$, we just drop the I and say* independent, weakly independent, basis, *or* weak basis.

While in Definition 2.1 we have written $X \subseteq M$ as just a subset of M, we will usually be thinking of X as a "list" or as an indexed set. Thus X may have repeated elements. Of course, an irredundant set (and hence a basis or weak basis) can not have repeated elements.

Our first proposition gives an alternate formulation of each type of independence.

Proposition 2.2. *Let M be an R-module and let $X \subseteq M$.*
(1) X is independent if and only if $\langle X\rangle$ is a free R-module on X. So M has a basis if and only if M is a free R-module.
(2) X is weakly independent if and only if $\langle X\rangle = \oplus_{x\in X} Rx$ and each $Rx \neq 0$. So M has a weak basis if and only if M is a direct sum of cyclic R-modules.
(3) X is irredundant if and only if for each $x_\alpha \in X$, $x_\alpha \notin \langle X - \{x_\alpha\}\rangle$. So M has an i-basis if and only if M has a minimal generating set $\{g_\alpha\}$; i.e., $\{g_\alpha\}$ generates M but no proper subset of $\{g_\alpha\}$ generates M.

Proof. Clear. ∎

Let $X \subseteq M$ where M is an R-module. Then X independent $\Longrightarrow$ X is weakly independent $\Longrightarrow$ X is irredundant. However, neither of these implications can be reversed. For example, $\{\bar{1}\}$ is a weak basis for $\mathbb{Z}_2$ considered as a $\mathbb{Z}$-module but is not a basis for $\mathbb{Z}_2$ and $\{2, 3\}$ is an i-basis for $\mathbb{Z}$ but is not a weak basis for $\mathbb{Z}$. Note that for each ideal I, $\varnothing$ is I-independent, I-weakly independent, and irredundant. We next observe that division rings are precisely the rings for which the three types of independence coincide.

Theorem 2.3. *For a ring R the following conditions are equivalent.*
(1) R is a division ring.
(2) Every R-module has a basis.
(3) Every irredundant subset of an R-module is independent.
(4) Every irredundant subset of an R-module is weakly independent.
(5) Every weakly independent subset of an R-module is independent.

Proof. (1) $\Longrightarrow$ (2) This implication is well known; see, for example [6, Theorem IV.2.4]. (2) $\Longrightarrow$ (1) While this implication is also well known, we provide a proof. Let $\mathcal{M}$ be a maximal left ideal of R. Now $R/\mathcal{M}$ has a basis. Since $R/\mathcal{M}$ is simple, the basis has one element. So $R/\mathcal{M}$ is isomorphic to R. So 0 is a maximal left ideal of R. Hence R is a division ring. (1) $\Longrightarrow$ (3) This implication is clear as 0 is the only nonunit of a division ring R. (3) $\Longrightarrow$ (4) and (3) $\Longrightarrow$ (5) Clear. (5) $\Longrightarrow$ (1) Let $\mathcal{M}$ be a maximal left ideal of R. Then $\bar{1}$ in $R/\mathcal{M}$ is weakly independent and hence independent. But $\mathcal{M}\bar{1} = \bar{0}$, so $\mathcal{M} = 0$. Thus R is a division ring. (4) $\Longrightarrow$ (1) Suppose that R is not a division ring. So R has a nonzero proper left ideal Rr. Let $A = R \oplus (R/Rr)$ and $r_1 = (1, \bar{1})$ and $r_2 = (1, \bar{0})$. Then $\{r_1, r_2\}$ is an i-basis for A. However, $\{r_1, r_2\}$ is not weakly independent since $(0, \bar{0}) \neq (r, \bar{0}) = rr_1 = rr_2 \in Rr_1 \cap Rr_2$. ∎

We next give some examples of i-generated modules and non-i-generated modules.

Proposition 2.4.
(1) A direct sum of cyclic modules is i-generated. More generally, if $\{M_\alpha\}$ is a family of i-generated R-modules, then $M = \oplus M_\alpha$ is i-generated.
(2) A uniserial R-module (i.e., the set of submodules of M is totally ordered by inclusion) is i-generated if and only if it is cyclic. Thus, the abelian group $\mathbb{Z}_{p^\infty}$ is not i-generated.
(3) Suppose that M is a finitely generated R-module. Then any generating set X for M has a finite subset $Y \subseteq X$ that is an i-generating set for M. Thus a finitely generated module is i-generated.
(4) A nonzero divisible abelian group G is not i-generated. More generally, if D is an integral domain that is not a field, then a nonzero divisible D-module cannot be i-generated.
(5) A nonzero i-generated R-module A must have a maximal submodule. More generally, if A has an i-basis X, then any submodule B of A that can be generated by fewer than $|X|$ elements is contained in a maximal submodule.

Proof.
(1) Let $\{m_{\alpha\beta}\}$ be an i-basis for M_α. Identify $m_{\alpha\beta}$ with its image in $M = \oplus M_\alpha$. Then $\bigcup_\alpha \{m_{\alpha\beta}\}$ is an i-basis for M. For $\bigcup_\alpha \{m_{\alpha\beta}\}$ certainly generates M and $\sum_\alpha \sum_\beta r_{\alpha\beta} m_{\alpha\beta} = 0$

implies $\sum r_{\alpha\beta}m_{\alpha\beta} = 0$ for each α. But then $\{m_{\alpha\beta}\}$ an i-basis for M_α gives that each $r_{\alpha\beta}$ is a nonunit. Hence $\bigcup_\alpha \{m_{\alpha\beta}\}$ is an i-basis for $M = \oplus M_\alpha$.
(2) Let M be a uniserial R-module. We may assume that M is nonzero. If $M = Rm$ is cyclic, then $\{m\}$ is an i-basis for M. Conversely, suppose that M is i-generated; say $\{m_\alpha\}_{\alpha\in\Lambda}$ is an i-basis. Suppose $|\Lambda| > 1$; so $m_{\alpha_1}, m_{\alpha_2}$ are distinct elements of $\{m_\alpha\}$. But then since M is uniserial, Rm_{α_1} and Rm_{α_2} are comparable; say $Rm_{\alpha_1} \subseteq Rm_{\alpha_2}$. So $\langle\{m_\alpha\} - \{m_{\alpha_1}\}\rangle = M$, a contradiction. Thus M must be cyclic.
(3) Let M be finitely generated and suppose that X generates M. Then some finite subset $X' = \{x_1, \cdots, x_n\}$ of X generates M. If X' is an i-basis for M, we are done. So suppose that some $x_{i_0} \in \langle\{x_i\} - \{x_{i_0}\}\rangle$, so $X' - \{x_{i_0}\}$ generates M. Continuing we get that some subset of $X' - \{x_{i_0}\} \subseteq X' \subseteq X$ is an i-basis for M.
(4) Let $\{d_\alpha\}$ be an i-basis for G. Let $d_0 \in \{d_\alpha\}$. Then $\langle\{d_\alpha\} - \{d_0\}\rangle = G_0 \subsetneq G$ and G/G_0 is a nonzero cyclic abelian group, necessarily divisible. But this is a contradiction. For the generalizaton, it suffices to observe that if D/I is a divisible D-module for some proper ideal I, then $I = 0$ and hence D is a field (for more detail see the proof of Lemma 4.4).
(5) First, suppose that A is finitely generated with i-basis $x_1, \cdots, x_n$. Then the module $A/\langle x_1, \cdots, x_{n-1}\rangle$ is cyclic, say $A/\langle x_1, \cdots, x_{n-1}\rangle$ is isomorphic to R/I for some left ideal I of R. But then for a maximal left ideal $\mathcal{M} \supseteq I$, $\mathcal{M}/I$ is a maximal submodule of R/I. So A has a maximal submodule. Thus for the second statement we can assume that X is infinite. Let $B = \langle Y\rangle$ where $|Y| < |X|$. Since each element of Y is a finite linear combination of elements of X, $B \subseteq \langle X'\rangle$ for some $X' \subsetneq X$. Let $x_0 \in X - X'$; so $B \subseteq \langle X - \{x_0\}\rangle$. Now $A/\langle X - \{x_0\}\rangle$ is a nonzero cyclic R-module and hence has a maximal submodule. Thus $\langle X - \{x_0\}\rangle$, and hence B, is contained in a maximal submodule. ■

Now observe that a module can have a maximal submodule without being i-generated. For example, $\mathbb{Z}_2 \oplus \mathbb{Z}_{2^\infty}$ has a maximal submodule $0 \oplus \mathbb{Z}_{2^\infty}$, but $\mathbb{Z}_2 \oplus \mathbb{Z}_{2^\infty}$ is easily checked (or see Corollary 2.10) to not be i-generated. Also, the condition in Proposition 2.4(5) that B be generated by fewer than $|X|$ elements is necessary. For $G = (\oplus_{n=1}^\infty \mathbb{Z}_2) \oplus \mathbb{Z}_{2^\infty}$ is i-generated (Example 2.11), but $\oplus_{n=1}^\infty \mathbb{Z}_2$ is not contained in a maximal submodule since $G/\oplus_{n=1}^\infty \mathbb{Z}_2 \approx \mathbb{Z}_{2^\infty}$. While Proposition 2.4(1) gives that a direct sum of i-generated modules is i-generated, the next example shows that a direct product of i-generated modules need not be i-generated.

Example 2.5. *(A direct product of i-generated (even cyclic) modules need not be i-generated.) Let $G = \prod_p \mathbb{Z}_p$ where the product runs over all primes p of $\mathbb{N}$. Then the torsion subgroup $tG = \oplus_p \mathbb{Z}_p$ and G/tG is divisible. Suppose that G is i-generated with i-basis X. Then X is not countable. Since tG is countably generated, by Proposition 2.4(5) tG is contained in a maximal subgroup. But then the divisible group G/tG has a maximal subgroup, a contradiction. Thus G is not i-generated.*

In a vector space, any set of vectors can be cut down to an independent set of vectors with the same span. In particular, any spanning set can be cut down to a basis. This need not be true for a generating set of a module. For example, $\{1/2^n + \mathbb{Z}\}_{n=1}^\infty$ is a generating set for $\mathbb{Z}_{2^\infty}$, but no subset is an i-basis for $\mathbb{Z}_{2^\infty}$ since $\mathbb{Z}_{2^\infty}$ is not i-generated. However, if X is a finite subset of a module, then there is a subset $Y \subseteq X$ with Y irredundant and $\langle X\rangle = \langle Y\rangle$. Also, in a vector space any independent set can be enlarged to a basis. However, even a finite irredundant subset of a finitely generated module cannot necessarily be enlarged to an i-basis. For example, for $\bar{2} \in \mathbb{Z}_4$, $\{\bar{2}\}$ is an irredundant subset of $\mathbb{Z}_4$ that can not be enlarged to an i-basis for $\mathbb{Z}_4$. For a second example, let k be a field and $R = k[\{X_\alpha\}]$ where $\{X_\alpha\}$ is a set of indeterminates over k. Then $\{X_\alpha\}$ is an irredundant subset of R as is $\{1\}$. However $\{X_\alpha\}$ cannot be extended to an i-basis for R. If we take $\{X_\alpha\}$ to be infinite, we see that $(\{X_\alpha\}) \subsetneq R$ where $(\{X_\alpha\})$ has an infinite i-basis while R has a finite i-basis.

We next relate the notions of basis and I-basis.

Theorem 2.6. *Let R be a ring, I an ideal of R, A an R-module and $X \subseteq A$. Put $\bar{R} = R/I$ and $\bar{A} = A/IA$. Suppose that $\langle X \rangle = A$. Then the following conditions are equivalent.*
(1) X is an I-basis for A.
(2) For $r_1, \cdots, r_n \in R$ and $x_1, \cdots, x_n \in X$, $r_1x_1 + \cdots + r_nx_n \in IA \Longrightarrow$ each $r_i \in I$.
(3) $\bar{X}$ is an $\bar{R}$-basis for $\bar{A}$.

Proof.
(1) $\Longrightarrow$ (2) Suppose that $r_1x_1 + \cdots + r_nx_n \in IA$, say $r_1x_1 + \cdots + r_nx_n = i_1x_1 + \cdots + i_nx_n + i_{n+1}x_{n+1} + \cdots + i_mx_m$ where $i_1, \cdots, i_m \in I$ and $x_{n+1}, \cdots, x_m \in X - \{x_1, \cdots, x_n\}$. Then $0 = (i_1 - r_1)x_1 + \cdots + (i_n - r_n)x_n + i_{n+1}x_{n+1} + \cdots + i_mx_m$ implies each $i_j - r_j \in I$ and hence each $r_j \in I$.
(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) Clear. ■

Note that in (3) $\Longrightarrow$ (1) of Theorem 2.6 it is crucial that we think of X as an indexed set rather than just a set. Indeed, for $X = \{1, 3\}$, X is not a (2)-basis for $\mathbb{Z}$, while as sets $\{\bar{1}, \bar{3}\} = \{\bar{1}\}$ is a $\bar{\mathbb{Z}} = \mathbb{Z}_2$-basis for $\mathbb{Z}_2$.

Recall that a ring R is quasilocal if it has a unique maximal left ideal $\mathcal{M}$. In this case $\mathcal{M}$ is also the unique maximal right ideal and $R/\mathcal{M}$ is a division ring (for example, see [13, Lemma 4.42]). We will write $(R, \mathcal{M})$ to indicate that R is a quasilocal ring with maximal left ideal $\mathcal{M}$. We have the following two well known corollaries to Theorem 2.6.

Corollary 2.7. *Suppose that $(R, \mathcal{M})$ is quasilocal, A is an R-module and $X \subseteq A$ with $\langle X \rangle = A$. Then the following are equivalent.*
(1) X is an $\mathcal{M}$-basis for A.
(2) For $r_1, \cdots, r_n \in R$ and $x_1, \cdots, x_n \in X$, $r_1x_1 + \cdots + r_nx_n \in \mathcal{M}A \Longrightarrow$ each $r_i \in \mathcal{M}$.
(3) $\bar{X}$ is an $R/\mathcal{M}$-vector space basis for $\bar{A} = A/\mathcal{M}A$.
(4) X is an i-basis for A.

Corollary 2.8. *Suppose that $(R, \mathcal{M})$ is quasilocal and A is an R-module with the property that for a submodule B of A, $A = B + \mathcal{M}A$ implies $A = B$ (e.g., A is finitely generated or $\mathcal{M}$ is nilpotent). Let $X \subseteq A$. Then the following are equivalent.*
(1) X is an $\mathcal{M}$-basis for A.
(2) $\langle X \rangle = A$ and for $r_1, \cdots, r_n \in R$ and $x_1, \cdots, x_n \in X$, $r_1x_1 + \cdots + r_nx_n \in \mathcal{M}A \Longrightarrow$ each $r_i \in \mathcal{M}$.
(3) $\bar{X}$ is an $R/\mathcal{M}$-vector space basis for $\bar{A} = A/\mathcal{M}A$.
(4) X is an i-basis for A.

Proof. It suffices to show that for (1)-(4) we have $\langle X \rangle = A$. This is assumed for (1), (2), and (4). For (3), $\bar{X}$ is an $R/\mathcal{M}$-vector space basis gives that $A = \langle X \rangle + \mathcal{M}A$. Hence by hypothesis, $\langle X \rangle = A$. ■

We have remarked (Proposition 2.4) that if $\{M_\alpha\}$ is a family of R-modules each having an i-basis, then $\oplus M_\alpha$ has an i-basis. Similar statements hold for bases and weak bases. Conversely, we can ask whether $A \oplus B$ i-generated implies A and B are i-generated. First observe that if $A \oplus B$ has a basis (resp., weak basis), A and B need not have a basis (resp., weak basis). Indeed, a direct summand of a free module is a projective module and a projective module need not be free. For example, let Q be a nonprincipal ideal of a Dedekind domain. Then Q is a direct summand of a free module but is not free and hence doesn't have a basis or even a weak basis. Of course, Q being finitely generated has an i-basis. We give an example of a direct summand of an i-generated module that is not i-generated.

Proposition 2.9. *Let R be a ring and let A be an R-module i-generated by $\{a_\alpha\}_{\alpha\in\Lambda}$. Let B be an R-module and $\{b_\alpha\}_{\alpha\in\Lambda} \subseteq B$. Suppose there exist $\{d_\alpha\}_{\alpha\in\Lambda} \subseteq R$ with $d_\alpha a_\alpha = 0$ and $\langle\{d_\alpha b_\alpha\}_{\alpha\in\Lambda}\rangle = B$. Then $G = A \oplus B$ is i-generated by $\{\gamma_\alpha\}$ where $\gamma_\alpha = (a_\alpha, b_\alpha)$.*

Proof. We first show that $\{\gamma_\alpha\}$ generates G. Since $d_\alpha\gamma_\alpha = (0, d_\alpha b_\alpha)$, $0 \oplus B \subseteq \langle\{\gamma_\alpha\}\rangle$. But then $(a_\alpha, 0) = \gamma_\alpha - (0, b_\alpha) \in \langle\{\gamma_\alpha\}\rangle$; so $A \oplus B = \langle\{\gamma_\alpha\}\rangle$. Suppose that $0 = \sum c_\alpha\gamma_\alpha$, so $0 = \sum c_\alpha a_\alpha$. Since $\{a_\alpha\}$ is an i-basis for A, each c_α is a nonunit. Hence $\{\gamma_\alpha\}$ is an i-basis for $G = A \oplus B$. ■

Corollary 2.10. *Let A be an i-generated torsion abelian group and let B be a divisible abelian group with $|A| \geq |B|$. Then $G = A \oplus B$ is i-generated. Conversely, let A and B be abelian groups with B divisible. If $A \oplus B$ is i-generated, then $|A| \geq |B|$.*

Proof. Let $X = \{x_\alpha\}_{\alpha\in\Lambda}$ be an i-basis for A. If $X = \varnothing$, $A = 0$; so $G = 0$ is i-generated. So assume $X \neq \varnothing$. Now X finite gives A finite and hence $B = 0$. So we can assume that X is infinite. Then $|X| = |A| \geq |B|$. So B can be generated by $|X|$ elements, say $\{d_\alpha\}_{\alpha\in\Lambda}$ generates B. Choose $0 \neq r_\alpha \in R$ with $r_\alpha x_\alpha = 0$ and $d'_\alpha \in B$ with $d_\alpha = r_\alpha d'_\alpha$. Then by Proposition 2.9 $\{(x_\alpha, d'_\alpha)\}_{\alpha\in\Lambda}$ is an i-basis for G.

For the partial converse, suppose that A and B are abelian groups with B divisible and $A \oplus B$ are i-generated. Suppose that $|A| < |B|$. By Proposition 2.4(5), $A \oplus 0 \subseteq C \subsetneq A \oplus B$ where C is a maximal subgroup of $A \oplus B$. But then $C/A \oplus 0$ is a maximal subgroup of the divisible group $A \oplus B/A \oplus 0 \approx B$, a contradiction. ■

Using Corollary 2.10 we get a number of interesting examples of i-generated abelian groups.

Example 2.11.
(1) Let $\{d_n\}_{n=1}^{\infty}$ be a sequence of integers where each $d_n \geq 2$ and let p be a prime. Then by Corollary 2.10 $(\oplus_{n=1}^{\infty}\mathbb{Z}_{d_n}) \oplus \mathbb{Z}_{p^\infty}$ and $(\oplus_{n=1}^{\infty}\mathbb{Z}_{d_n}) \oplus \mathbb{Q}$ are i-generated abelian groups but $\mathbb{Z}_{p^\infty}$ and $\mathbb{Q}$ being divisible are not i-generated (Proposition 2.4). Hence $A \oplus B$ i-generated need not imply that A and B are i-generated.
(2) Let p be a prime. Then $(\oplus_{n=1}^{\infty}\mathbb{Z}_p) \oplus \mathbb{Z}_{p^\infty}$ is an i-generated p-primary abelian group.

(3) Let $p \neq q$ be primes. Let $G = (\oplus_{n=1}^{\infty}\mathbb{Z}_p) \oplus \mathbb{Z}_{q^\infty}$. Then G is i-generated, but its q-primary component $\mathbb{Z}_{q^\infty}$ is not i-generated.
(4) Let $\{p_n\}_{n=1}^{\infty}$ be the set of primes. Put $G_n = \mathbb{Z}_{p_n} \oplus \mathbb{Z}_{p_n^\infty}$, so G_n is the p_n-primary component of $G = \oplus_{n=1}^{\infty} G_n = (\oplus_{n=1}^{\infty}\mathbb{Z}_{p_n}) \oplus (\oplus_{n=1}^{\infty}\mathbb{Z}_{p_n^\infty})$. By Corollary 2.10 G is an i-generated torsion abelian group (since $\oplus_{n=1}^{\infty}\mathbb{Z}_{p_n^\infty}$ is countably generated). However, since each G_n is not i-generated (see the paragraph after Proposition 2.4), no primary component of G is i-generated.
(5) Let p be a prime and let $G = (\oplus_{n=1}^{\infty}\mathbb{Z}_p) \oplus \mathbb{Q}$. Then G is i-generated but $\mathbb{Q}$ is not. Here $G = tG \oplus G/tG$ where $tG = \oplus_{n=1}^{\infty}\mathbb{Z}_p$ is the torsion subgroup of G, but $G/tG = \mathbb{Q}$ is not i-generated.

Example 2.11 shows some of the pitfalls in attempting to characterize the i-generated abelian groups. For example, a torsion abelian group can be i-generated without its p-primary components being i-generated and while a divisible abelian group cannot be i-generated, an i-generated abelian group need not be reduced. Also, note that Example 2.5 gives an example of a reduced abelian group that is not i-generated.

Problem 2.12. *Characterize the i-generated abelian groups.*

We have noted that if R-modules A and C have a basis (resp. weak basis, i-basis), then so does $A \oplus C$. For a short exact sequence of R-modules $0 \to A \to B \to C \to 0$ this raises the general question of what is the relationship of A, B, or C having a basis, weak basis, or i-basis. Of course, B can have a basis (and hence a weak basis) without A or C having a basis or even a weak basis. But if A and C have a basis, then the short exact sequence splits so $B \approx A \oplus C$ has a basis. However, unlike the case for bases, A and C can have weak bases without B having a weak basis (e.g., over $\mathbb{Q}[X,Y]$, $0 \to (X) \to (X,Y) \to (X,Y)/(X) \to 0$). We next consider the i-generated case. Now B i-generated does not imply that A or C is i-generated. In fact, A and B (resp., B and C) i-generated does not imply that C (resp., A) is i-generated as seen by $0 \to \oplus_{n=1}^{\infty}\mathbb{Z}_2 \to (\oplus_{n=1}^{\infty}\mathbb{Z}_2) \oplus \mathbb{Z}_{2^\infty} \to \mathbb{Z}_{2^\infty} \to 0$ (resp., $0 \to \mathbb{Z}_{2^\infty} \to (\oplus_{n=1}^{\infty}\mathbb{Z}_2) \oplus \mathbb{Z}_{2^\infty} \to \oplus_{n=1}^{\infty}\mathbb{Z}_2 \to 0$). However, if $X \subseteq B$ and $\bar{X}$ is irredundant in C, then X is irredundant in B, see Proposition 2.13. It remains open whether A and C i-generated implies B is i-generated. We suspect not. But our next result gives a special case where this is true.

Proposition 2.13. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules.*
(1) Let $X = \{x_\alpha\} \subseteq B$ and let $\bar{X} = \{\bar{x}_\alpha\}$ be its image in C. If $\bar{X}$ is irredundant, then X is irredundant. Hence if X generates B and $\bar{X}$ is an i-basis for C, then X is an i-basis for B.
(2) Suppose that A is finitely generated and C is i-generated. Then B is i-generated.

Proof.
(1) If $Y \subsetneq X$ with $\langle Y \rangle = \langle X \rangle$, then $\bar{Y} \subsetneq \bar{X}$ (for $\alpha \neq \beta$, $\bar{x}_\alpha \neq \bar{x}_\beta$ since $\bar{X}$ cannot have repeated elements) and $\langle \bar{Y} \rangle = \langle \bar{X} \rangle$, a contradiction.
(2) Let $a_1, \cdots, a_n$ generate A and choose $\{b_\alpha\} \subseteq B$ so that $\{\bar{b}_\alpha\}$ is an i-basis for C. Then $\{a_1, \cdots, a_n\} \cup \{b_\alpha\}$ certainly generates B. Note that no $b_{\alpha_0} \in \langle (\{a_1, \cdots, a_n\} \cup \{b_\alpha\}) - \{b_{\alpha_0}\} \rangle$, for then $\{\bar{b}_\alpha\} - \{\bar{b}_{\alpha_0}\}$ would generate C. Suppose that some $a_i \in \langle \{a_1, \cdots, \hat{a}_i, \cdots, a_n\} \cup \{b_\alpha\} \rangle$.

Then $\langle\{a_1, \cdots, \hat{a}_i, \cdots, a_n\} \cup \{b_\alpha\}\rangle = B$. Continuing, we get a subset $\{a_{i_1}, \cdots, a_{i_s}\} \subseteq \{a_1, \cdots, a_n\}$ (possibly empty) with $\{a_{i_1}, \cdots, a_{i_s}\} \cup \{b_\alpha\}$ an i-basis for B. ■

We end this section with a result which in principal gives all i-generated R-modules. This result was remarked to us by Victor Camillo.

Theorem 2.14. *An R-module M is i-generated if and only if it has a presentation $F \xrightarrow{\pi} M \longrightarrow 0$ where F is a free R-module on a set $Y = \{y_\alpha\}$ and $\ker \pi \subseteq \oplus \mathcal{M}_\alpha y_\alpha$ for some collection $\{\mathcal{M}_\alpha\}$ of maximal left ideals of R.*

Proof. Suppose that M has i-basis $X = \{x_\alpha\}_{\alpha \in \Lambda}$. Let $Y = \{y_\alpha\}_{\alpha \in \Lambda}$ be a set disjoint from X indexed by the same set Λ. Let F be a free R-module on Y and let $\pi\colon F \to M$ be the R-module homomorphism induced by $\pi(y_\alpha) = x_\alpha$. For each $\alpha \in \Lambda$, let $\mathcal{N}_\alpha = (\langle X - \{x_\alpha\}\rangle : x_\alpha)$. Since X is irredundant, $\mathcal{N}_\alpha$ is a proper left ideal of R. For each α choose a maximal left ideal $\mathcal{M}_\alpha \supseteq \mathcal{N}_\alpha$. If $r_1 y_{\alpha_1} + \cdots + r_n y_{\alpha_n} \in \ker \pi$, then $r_1 x_{\alpha_1} + \cdots + r_n x_{\alpha_n} = 0$ and hence $r_i \in \mathcal{N}_{\alpha_i} \subseteq \mathcal{M}_{\alpha_i}$. Thus $\ker \pi \subseteq \oplus \mathcal{N}_\alpha y_\alpha \subseteq \oplus \mathcal{M}_\alpha y_\alpha$. If conversely we have such a presentation, $X = \{\pi(y_\alpha)\}$ is easily seen to be an i-basis for M. ■

3. Number of Generators

In this short section we consider the different possible cardinalities for a basis, weak basis, or i-basis of a module having such a basis. As is the case (see below) for bases for (free) modules where if a free module has an infinite basis, then any other basis has the same cardinality, if a module has an infinite weak basis (resp. i-basis), then any other weak basis (resp. i-basis) has the same cardinality (Theorem 3.3). We show (Theorem 3.4) that quasilocal rings are characterized by the property that any two weak bases (resp. i-bases) for a module have the same cardinality. Finally, we consider (Theorem 3.5) the possible cardinalities of an i-basis for a finitely generated module.

It is well known that for a vector space over a division ring, any two bases have the same cardinality (for example, see [6, Theorem IV.2.7]). Now for a general ring R, if F is a free R-module with infinite basis X, then for any other basis Y of F we have $|X| = |Y|$ (see, for example [6, Theorem IV.2.6]). However, two bases for a finitely generated free R-module need not have the same cardinality as the following classic example shows (for example, see [6, Exercise IV.2.13]).

Example 3.1. *Let K be a ring and let F be a free K-module with basis $\{e_n\}_{n=1}^{\infty}$. Then for $R = \operatorname{Hom}_K(F, F)$, the free R-module ${}_RR$ has a basis with one element, namely 1_R, and also a basis with two elements $\{f_1, f_2\}$ where f_i is defined by $f_1(e_{2n}) = f_2(e_{2n-1}) = e_n$ and $f_1(e_{2n-1}) = f_2(e_{2n}) = 0$. Alternatively, observe that ${}_RR = \operatorname{Hom}_K(F, F) \approx \operatorname{Hom}_K(F \oplus F, F) \approx \operatorname{Hom}_K(F, F) \oplus \operatorname{Hom}_K(F, F) = {}_RR \oplus {}_RR$. In fact, it is easily shown that ${}_RR$ has a basis of length n for each $n \geq 1$.*

It is even easier to give examples of modules having weak bases or i-bases of different lengths. Indeed, taking $R = \mathbb{Z}$, we see that $\mathbb{Z}/6\mathbb{Z} \approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ has weak bases of length one and two and $\mathbb{Z}$ itself has i-bases of length one and two, namely $\{1\}$ and $\{2, 3\}$.

This leads us to the following definition.

Definition 3.2. *A ring R satisfies the* invariant basis number property (IBN) (*resp.* invariant weak basis number property (IwBN), invariant i-basis number property (IiBN)) *if for each R-module M with a basis (resp. weak basis, i-basis), any two bases (resp. weak bases, i-bases) for M have the same cardinality.*

Strictly speaking each of the previous definitions should have a left and a right version and as we are using "module" to mean a left R-module, we have given the left version. For the case of IBN, it is well known (and follows from an easy matrix argument [5, Proposition 2.2]) that the notions of left and right IBN coincide. It follows from Theorem 3.4 that the notions of left and right $IiBN$ and left and right $IwBN$ coincide, indeed, the rings satisfying $IiBN$ or $IwBN$ are just the quasilocal rings.

A "useful" characterization of the rings satisfying IBN is not known. However, any commutative ring satisfies IBN. For if $f\colon R \to S$ is a ring homomorphism and S satisfies IBN, then so does R. Since a division ring satisfies IBN, a ring having a division ring as a homomorphic image (e.g., a commutative ring) satisfies IBN (for example, see [6, Proposition IV.2.11, Corollary IV.2.12]). For further results on IBN, the reader is referred to [5].

However, it is easy to characterize the rings satisfying $IwBN$ or $IiBN$. Recall that a ring R is quasilocal if R has a unique maximal left ideal $\mathcal{M}$. In this case $\mathcal{M}$ is also the unique maximal right ideal of R and $R/\mathcal{M}$ is actually a division ring. Dually, a ring is quasilocal if it has a unique maximal right ideal $\mathcal{N}$, for then $\mathcal{N}$ is also the unique maximal left ideal of R. Now by Theorem 3.4, R satisfies (left) $IwBN$ or (left) $IiBN$ if and only if R is (left) quasilocal. Thus the right version of Theorem 3.4 gives that R satisfies right $IwBN$ or right $IiBN$ if and only if R is (right) quasilocal. Since the notions of "left quasilocal" and "right quasilocal" coincide, the notions of left and right $IwBN$ and left and right $IiBN$ coincide.

But we first show that if an R-module M has an infinite i-basis, then any other i-basis for M has the same cardinality. Since a weak basis is an i-basis, the same result also holds for weak bases. This of course also gives a proof of the previously mentioned result that if F is free with an infinite basis X, then any other basis Y for F has $|X| = |Y|$.

Theorem 3.3. *Let R be a ring and M an R-module. Suppose that M has an infinite i-basis (resp. weak basis, basis) X. Then for any generating set Y of M, $|X| \leq |Y|$. Hence if Y is another i-basis (resp. weak basis, basis) for M, then $|X| = |Y|$.*

Proof. Since a basis is a weak basis and a weak basis is an i-basis, it suffices to do the case where X is an i-basis. So let X be an i-basis for M and let Y be a generating set for M. Suppose that $|Y| < |X|$. For each $y \in Y$, choose a representation $y = r_1 x_{\alpha_1} + \cdots + r_n x_{\alpha_n}$ where $\{x_{\alpha_1}, \cdots, x_{\alpha_n}\}$ is some finite subset of X. Put $y(X) = \{x_{\alpha_1}, \cdots, x_{\alpha_n}\}$. Then $Y' = \bigcup_{y\in Y} y(X) \subseteq X$. If Y is finite, then Y' is also finite and hence $|Y'| < |X|$. If Y is infinite, then $|Y'| \leq |Y|$ since Y' is a union of $|Y|$ finite subsets of X and hence $|Y'| < |X|$. In either case, $Y' \subsetneq X$ and $\langle Y' \rangle = M$, a contradiction. Hence $|X| \leq |Y|$. If Y is actually an i-basis, then reversing the roles of X and Y gives $|Y| \leq |X|$ and hence $|X| = |Y|$. ■

Note that Theorem 3.3 gives that if an R-module M is isomorphic to a direct sum of α nonzero cyclic R-modules where α is infinite, then all i-bases for M also have cardinality α.

Theorem 3.4. *For a ring R, the following conditions are equivalent.*
(1) R is quasilocal.
(2) R satisfies $IiBN$.
(3) R satisfies $IwBN$.

Proof. (1) $\Longrightarrow$ (2) Suppose that R is quasilocal with maximal ideal $\mathcal{M}$. By Theorem 3.3 it suffices to show that for a finitely generated R-module M, any two i-bases for M have the same cardinality. By Nakayama's Lemma $m_1, \cdots, m_n$ generate M as an R-module if and only if $\bar{m}_1, \cdots, \bar{m}_n$ generate $\bar{M} = M/\mathcal{M}M$ as an $\bar{R} = R/\mathcal{M}$-module. Hence $m_1, \cdots, m_n$ is an i-basis for M if and only if $\bar{m}_1, \cdots, \bar{m}_n$ is a $\bar{R}$-basis for $\bar{M}$. So every i-basis for M has cardinality $\dim_{\bar{R}} \bar{M}$. (This is really just Corollary 2.7) (2) $\Longrightarrow$ (3) This implication is clear since any weak basis for M is also an i-basis for M. (3) $\Longrightarrow$ (1) Suppose that R has more than one maximal left ideal; say $\mathcal{M}$ and $\mathcal{N}$ are two distinct maximal left ideals of R. Since $\mathcal{M} + \mathcal{N} = R$, the map $R \to R/\mathcal{M} \oplus R/\mathcal{N}$ given by $r \to (r + \mathcal{M}, r + \mathcal{N})$ is surjective and hence $R/\mathcal{M} \cap \mathcal{N} \approx R/\mathcal{M} \oplus R/\mathcal{N}$. Thus $R/\mathcal{M} \cap \mathcal{N}$ has weak bases of cardinalities one and two, a contradiction. ■

We next determine the possible different lengths of i-bases (all are finite by Theorem 3.3) for a finitely generated R-module. While this has already been done by Ratliff and Robson [12], our treatment is different. Our proof uses the Tarski Irredundant Basis Theorem, or rather its corollary, given below. The Irredundant Basis Theorem was proved by A. Tarski [14]; also, see S. Burris and H. P. Sankappanavar [2, Theorem 4.4] for a very readable account of this result.

Theorem 3.5. *(Tarski [14]) Let R be a ring and M a finitely generated R-module. Let $\mathrm{Irr}(M) = \{n \in \mathbb{N}_0 | M$ has an i-basis with n elements$\}$. Then $\mathrm{Irr}(M)$ is a convex subset of $\mathbb{N}_0$.*

Let M be a nonzero finitely generated R-module. Now each proper submodule of M is contained in a maximal submodule of M. Recall that the Jacobson radical of M is $J(M) = \cap\{N \mid N$ is a maximal submodule of $M\}$. (So $J(R)$ is the Jacobson radical of R.) Observe that $m_1, \cdots, m_n \in M$ generate M if and only if $\bar{m}_1, \cdots, \bar{m}_n$ generate $\bar{M} = M/J(M)$.

Theorem 3.6. *(Ratliff and Robson [12]) Let R be a ring and M a finitely generated R-module. Let $\mathrm{Irr}(M) = \{n \in \mathbb{N}_0 \mid M$ has an i-basis of cardinality $n\}$ and let $\mu(M) = \min\{n \mid n \in \mathrm{Irr}(M)\}$. If $M/J(M)$ has finite length λ, then $\mathrm{Irr}(M) = \{\mu(M), \mu(M)+1, \cdots, \lambda\}$ while if $M/J(M)$ has infinite length, then $\mathrm{Irr}(M) = \{\mu(M), \mu(M)+1, \cdots\}$.*

Proof. By Theorem 3.5, $\mathrm{Irr}(M)$ is a convex subset of $\mathbb{N}_0$. Of course, $\mu(M)$ is its least element. First, suppose that $M/J(M)$ has a finite length λ. Let $m_1, \cdots, m_s$ be an i-basis for M. Then in $\bar{M} = M/J(M)$, $\bar{0} \subsetneq R\bar{m}_1 \subsetneq R\bar{m}_1 + R\bar{m}_2 \subsetneq \cdots \subsetneq R\bar{m}_1 + \cdots + R\bar{m}_s = \bar{M}$. Hence $s \leq \lambda$. But since $\bar{M}$ has length λ, $\bar{M}$ being semisimple, is a direct sum of λ simple R-modules and hence has a weak basis and thus an i-basis of length λ.

Next suppose that $\bar{M} = M/J(M)$ does not have finite length. Then $J(M)$ cannot be a finite intersection of maximal submodules of M. (For if $J(M) = M_1 \cap \cdots \cap M_n$ where M_i is a maximal submodule of M, then $\bar{M} = M/M_1\cap\cdots\cap M_n$ embeds into the finite length R-module $M/M_1\oplus\cdots\oplus M/M_n$ and hence has finite length, a contradiction.) Thus there is a countably infinite collection $\{M_n\}_{n=1}^{\infty}$ of maximal submodules of M with $M \supsetneq M_1 \supsetneq M_1 \cap M_2 \supsetneq \cdots \supsetneq M_1 \cap \cdots \cap M_n \supsetneq \cdots$. Now $M/M_1 \cap \cdots \cap M_n$ has finite length n, so from the previous paragraph $\bar{M} = M/M_1\cap\cdots\cap M_n$ has an i-basis $\{\bar{\beta}_1, \cdots, \bar{\beta}_n\}$ where $\beta_1, \cdots, \beta_n \in M$. Choose a generating set $\{a_\alpha\}$ for $M_1\cap\cdots\cap M_n$. Then $\langle\{\beta_1, \cdots, \beta_n\} \cup \{a_\alpha\}\rangle = M$. Since M is finitely generated, $M = \langle\beta_1, \cdots, \beta_n, a_{\alpha_1}, \cdots, a_{\alpha_m}\rangle$ for some finite subset $\{a_{\alpha_1}, \cdots, a_{\alpha_m}\} \subseteq \{a_\alpha\}$. Note that $\langle\beta_1, \cdots, \hat{\beta}_i, \cdots, \beta_n, a_{\alpha_1}, \cdots, a_{\alpha_m}\rangle \subsetneq M$. For if not, then $\left\langle\bar{\beta}_1, \cdots, \hat{\bar{\beta}}_i, \cdots, \bar{\beta}_n\right\rangle = \bar{M}$. But this is a contradiction since $\{\bar{\beta}_1, \cdots, \bar{\beta}_n\}$ is an i-basis for $\bar{M}$. Thus for some subset $\{a_{i_1}, \cdots, a_{i_n}\}$, $\{\beta_1, \cdots, \beta_n, a_{i_1}, \cdots, a_{i_s}\}$ is an i-basis for M. But then $n + s \in \operatorname{Irr}(M)$. Since $\operatorname{Irr}(M)$ is convex and contains arbitrarily large integers, $\operatorname{Irr}(M) = \{\mu(M), \mu(M) + 1, \cdots\}$. ■

The following example is an applicaton of Theorem 3.6.

Example 3.7.
(1) Let R be a ring such that $R/J(R)$ is not left Artinian (e.g., $\mathbb{Z}$). Then for each $n \geq 1$, ${}_RR$ has an i-basis of length n.
(2) Let I be a nonzero finitely generated ideal of $K[\{X_\alpha\}]$, K a field. Then I has i-bases of arbitrarily long finite length.

For a finitely generated R-module M we can also ask what is the minimum cardinality of an i-basis for M. The following result from [12] gives the answer when M has finite length.

Theorem 3.8. *(Ratliff and Robson [12]) Let R be a ring and M a finite length nonzero R-module. For each isomorphism class of simple R-module S appearing in composition series for M, let $e(S)$ denote the number of copies of S in the composition series and let $f(S)$ be the length of $R/\operatorname{ann}(S)$. Then $\mu(M)$ is the least integer $\geq \sup\{1, e(S)/f(S)\}$. (If $R/\operatorname{ann}(S)$ has infinite length, then $e(S)/f(S) = 0$.)*

For a finitely generated R-module M, we can also consider the sets $\operatorname{Bas}(M) = \{n \in \mathbb{N}_0 \mid M$ has a basis of length $n\}$ and $w\operatorname{Bas}(M) = \{n \in \mathbb{N}_0 \mid M$ has a weak basis of length $n\}$. So $\operatorname{Bas}(M) \subseteq w\operatorname{Bas}(M) \subseteq \operatorname{Irr}(M)$. Of course, $\operatorname{Bas}(M) = \varnothing$ (resp. $w\operatorname{Bas}(M) = \varnothing$) unless M is free (resp. M is a direct sum of cyclic modules). Now $\operatorname{Bas}(M)$ need not be a convex subset of $\mathbb{N}_0$. In fact, the subsets of $\mathbb{N}_0$ that can be a $\operatorname{Bas}(M)$ for some finitely generated free module M have been characterized by W. G. Leavitt [9, 10]; also see [5].

For a finite abelian group M, it is easy to see that $w\operatorname{Bas}(M) = \operatorname{Irr}(M) = \{m, m+1, \cdots, n\}$ where M has an invariant factor decomposition with m summands and M has an elementary divisor decomposition with n summands. For a finitely generated abelian group with free part of rank $r > 0$, $w\operatorname{Bas}(M) = \{r+m, \cdots, r+n\}$ and $\operatorname{Irr}(M) = \{r+m, \cdots\}$ where m and n are as above.

4. Strongly i-generated Rings

After discussing modules that have a basis, weak basis, or i-basis, it is natural to ask which rings have the property that every module has a basis, has a weak basis, or has an i-basis.

The rings for which every module has a basis (or equivalently, is free) are of course just the division rings. We have already noted this in Theorem 2.3. It is perhaps worth noting that every (left) R-module has a basis if and only if every right R-module has a basis.

The rings for which every module has a weak basis are just the rings with the property that every module is a direct sum of cyclic modules. Köthe [8] proved that a left Artinian principal ideal ring has this property and that a commutative Artinian ring for which all modules are a direct sum of cyclics is a principal ideal ring. Cohen and Kaplansky [4] showed that a commutative ring R has every R-module a direct sum of cyclics if and only if R is an Artinian principal ideal ring. The question of what rings have the property that every module is a direct sum of cyclics appears to be open. Nakayama [11] showed that such a ring need not be a principal ideal ring, but Chase [3] showed that such a ring is left Artinian. More precisely, Chase showed that a ring with the property that every module is a direct sum of finitely generated modules must be left Artinian.

We next ask what rings have the property that every module has an i-basis. This leads us to the following definition.

Definition 4.1. *A ring R is* strongly i-generated *if every R-module has an i-basis.*

Of course we could also define a strongly right i-generated ring. Recall that a ring R is *left perfect* if every R-module has a projective cover. A number of conditions equivalent to being left perfect are known. For example [1, Theorem 28.4], the following conditions are equivalent: (1) R is left perfect, (2) $R/J(R)$ is left Artinian semisimple and every nonzero R-module has a maximal submodule, and (3) $R/J(R)$ is left Artinian semisimple and $J(R)$ is left t-nilpotent (i.e., given a sequence $a_1, a_2, \cdots$ in $J(R)$ there is an n with $a_1 a_2 \cdots a_n = 0$). We show that a left perfect ring is strongly i-generated and conversely that a commutative coherent strongly i-generated ring is perfect.

Theorem 4.2. *A left perfect ring is strongly i-generated.*

Proof. Suppose that R is left perfect. Let A be an R-module. Since $\bar{R} = R/J(R)$ is left Artinian semisimple, $\bar{A} = A/J(R)A$ is a semisimple $\bar{R}$-module. Hence $\bar{A}$ has an i-basis $\{\bar{a}_\alpha\}$ where $a_\alpha \in A$. Now $A = \langle\{a_\alpha\}\rangle + J(R)A$, so $J(R)(A/\langle\{a_\alpha\}\rangle) = A/\langle\{a_\alpha\}\rangle$. Since $J(R)$ is t-nilpotent, $A/\langle\{a_\alpha\}\rangle = 0$ and hence $A = \langle\{a_\alpha\}\rangle$ [1, Lemma 28.3]. But $\{a_\alpha\}$ is irredundant since $\{\bar{a}_\alpha\}$ is (Proposition 2.13). Thus $\{a_\alpha\}$ is an i-basis for A. ■

To establish the partial converse for the commutative case we need several lemmas. Note that Lemma 4.3 does not require R to be commutative.

Lemma 4.3. *If R is strongly i-generated, then so is each factor ring R/I.*

Proof. Each R/I-module A is an R-module and hence has an i-basis $\{a_\alpha\}$ when considered as an R-module. But $\{a_\alpha\}$ is then also an i-basis for A considered as an R/I-module. ■

Lemma 4.4. *Let R be a strongly i-generated commutative ring. Then R has Krull dimension zero.*

Proof. Let P be a prime ideal of R. By Lemma 4.3 the integral domain $\bar{R} = R/P$ is strongly i-generated. Let K be the quotient field of $\bar{R}$. Then K has an i-basis X. Suppose that $|X| > 1$. Let $A = \langle X - \{x_0\} \rangle$ for some fixed $x_0 \in X$. Then K/A is a cyclic divisible $\bar{R}$-module, say $K/A \approx \bar{R}/I$ for some ideal I of $\bar{R}$. For $\bar{0} \neq r \in I$, $r(K/A) = K/A$, so $r(\bar{R}/I) = \bar{R}/I$. Hence $I = \bar{0}$. So $\bar{R}$ is a divisible $\bar{R}$-module; i.e., $\bar{R} = K$ is a field. Hence P is a maximal ideal of R. ■

Recall that a ring R is *left coherent* if every finitely generated R-module is finitely related.

Theorem 4.5. *Let R be a coherent (e.g. Noetherian) commutative strongly i-generated ring. Then R is perfect.*

Proof. By Lemma 4.4, $\dim R = 0$. Hence $\bar{R} = R/J(R)$ is von Neumann regular. We show that $\bar{R}$ is Artinian. Now by Proposition 2.4 every nonzero R-module has a maximal submodule. Hence by the previously mentioned [1, Theorem 28.4], R is (left) perfect.

We claim that $\bar{R}$ is Artinian. Let $\bar{I} = I/J(R)$ be an ideal of $\bar{R}$. Now R/I is a finitely generated R-module and hence is a finitely related R-module since R is coherent. Thus $\bar{R}/\bar{I}$ is a finitely related $\bar{R}$-module. Since $\bar{R}$ is von Newmann regular, $\bar{R}/\bar{I}$ is a flat $\bar{R}$-module [13, Theorem 4.16] and hence is projective since $\bar{R}/\bar{I}$ is finitely related [13, Theorem 3.58]. Thus the short exact sequence $0 \to \bar{I} \to \bar{R} \to \bar{R}/\bar{I} \to 0$ splits; so $\bar{I}$ is finitely generated (even generated by an idempotent). Thus every ideal of $\bar{R}$ is finitely generated; so $\bar{R}$ is Noetherian and hence Artinian being zero-dimensional. ■

We end with the following result.

Theorem 4.6. *Let R be a ring.*
(1) Let M be an i-generated R-module. Suppose that $IM{=}M$ where $I \subseteq J(R)$. Then $M = 0$.
(2) R is strongly i-generated if and only if $J(R)$ is left t-nilpotent and $R/J(R)$ is strongly i-generated.

Proof. (1) This is just Nakayama's Lemma. Suppose $M \neq 0$. Let $\{m_\alpha\} \neq \varnothing$ be an i-basis for M. For $m_0 \in \{m_\alpha\}$, $m_0 = \sum i_\alpha m_\alpha$ where each $i_\alpha \in I$; so $(1 - i_0)m_0 \in \langle \{m_\alpha\} - \{m_0\} \rangle$ and hence $m_0 \in \langle \{m_\alpha\} - \{m_0\} \rangle$ since $i_0 \in J(R)$ gives that $1 - i_0$ is a unit.
(2) ($\Longrightarrow$) Suppose that R is strongly i-generated. By Lemma 4.3, $R/J(R)$ is strongly i-generated. Since R is strongly i-generated, for each R-module M, $J(R)M = M$ implies $M = 0$. By [1, Lemma 28.3], $J(R)$ is left t-nilpotent. ($\Longleftarrow$) Let A be an R-module. Now $R/J(R)$ is strongly i-generated, so $A/J(R)A$ is i-generated as an $R/J(R)$-module and hence as an R-module. Choose $\{a_\alpha\} \subseteq A$ so that $\{\bar{a}_\alpha\}$ is an i-basis for $A/J(R)A$. By Proposition 2.13, $\{a_\alpha\}$ is irredundant. Also $\langle \{a_\alpha\} \rangle + J(R)A = A$; so $\langle \{a_\alpha\} \rangle = A$ since $J(R)$ is left t-nilpotent. Thus $\{a_\alpha\}$ is an i-basis for A. ■

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