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Witness (Delaunay) graphs

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ABSTRACT

Proximity graphs are used in several areas in which a neighborliness relationship for input data sets is a useful tool in their analysis, and have also received substantial attention from the graph drawing community, as they are a natural way of implicitly representing graphs. However, as a tool for graph representation, proximity graphs have some limitations that may be overcome with suitable generalizations.

We introduce a generalization, *witness graphs*, that encompasses both the goal of more power and flexibility for graph drawing issues and a wider spectrum for neighborhood analysis. We study in detail two concrete examples, both related to Delaunay graphs, and consider as well some problems on stabbing geometric objects and point set discrimination, that can be naturally described in terms of witness graphs.

1. Introduction and preliminary definitions

Proximity graphs are used in several areas in which a neighborliness relationship for input data sets is a useful tool in their analysis and use, see [33] for a survey. Examples of such areas are computer vision, geographic analysis, pattern classification, computational morphology, and spatial analysis. On the other hand, proximity graphs have also received substantial attention from the graph drawing community, as they are a natural way of implicitly representing graphs; a survey of such results appeared in [8] and has been extended and updated in [37].

As a tool for graph representation, proximity graphs have some limitations that may be overcome with suitable generalizations. An example of such an extension is the concept of *weak proximity graphs* [9]. Here we introduce a generalization that encompasses both the goal of more power and flexibility for graph drawing issues and a wider spectrum for neighborhood analysis.

In general, given a point set P and a set of geometric shapes S, a proximity graph is a graph $G = (P, E)$ with P as the vertex set and two points a and b being adjacent if and only if there is a suitable shape Γ, defined by a and b, from S—their *region of influence*—that covers them but no other point from P; the presence of another point is referred to as an *interference*. Whether the two points have to be on the boundary of the shape Γ, whether Γ is uniquely determined by them, and whether the interference is considered only if interior to Γ, depends on the specific problem studied as also does the family of shapes under consideration; see [33,37] for an extensive list of examples of proximity graphs.

A *witness graph* $G = (V, E)$ is defined by a quadruple $(P, S, W, \pm)$ in which $P = V$ is the set of *vertex points* (or just *vertices*), S provides the geometric shapes, and W is a second point set, consisting of the *witness points* (or just *witnesses*).

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In the *positive witness* version (+), the tentative adjacency between a and b is accepted if and only if a witness point is covered by at least one of the regions of influence defined by a and b. In the *negative witness* version (−), a witness inside the interaction region would destroy the tentative adjacency, hence there is an adjacency between a and b if at least one of their regions of influence is free of any witness. Notice that in both cases we only pay attention to the presence of witnesses in the regions of influence, not of points from P. In a third variation one may admit the presence of both negative and positive witnesses and use a combined decision rule; we do not pursue this possibility here.

To the best of our knowledge this family of graphs has not been introduced before in its full generality, yet, not surprisingly, the situation has been considered in more or less explicit form for some specific graphs. Ichino and Slansky [31] defined the *rectangular influence graph*, RIG(P), in which two points $p, q \in P$ are adjacent when the rectangle having them as opposite corners (the *box* they define) contains no point from P. In the same paper, they defined the *mutual neighborhood graph* MNG($P|Q$), in which $p, q \in P$ are adjacent when the associated box contains no point from Q, and they studied some properties that can be derived by considering simultaneously MNG($P|Q$) and MNG($Q|P$). In [10], De Berg, Cheong and Overmars defined the *dominance in a set P with respect to a set Q* and gave an efficient algorithm for its computation: $a \in P$ dominates $b \in P$ when $x(a) \geqslant x(b)$, $y(a) \geqslant y(b)$, and the box defined by a and b contains no point from Q. Finally, McMorris and Wang [41] defined the *sphere-of-attraction graphs* in which from every point of $p \in P$ taken as center a ball is grown until a first point from Q is encountered; the graph is then defined on P as a ball intersection graph. They obtained a characterization in dimension one and initiated the study in higher dimensions.

In the present paper, we consider two concrete examples, both related to Delaunay graphs, one for positive witnesses and one for negative ones. Other witness graphs such as the *witness Gabriel graph* and the *witness rectangle graph* are studied in the companion papers [6,5]. A systematic study is developed in [25].

We define the *witness Delaunay graph* of a point set P of *vertices* in the plane, with respect to a point set W of *witnesses*, denoted $\mathrm{DG}^-(P, W)$, as the graph with vertex set P in which two points $x, y \in P$ are adjacent if and only if there is an open disk that does not contain any witness $w \in W$ whose bounding circle passes through x and y. It is a negative-witness graph in which the shapes are all the disks in the plane whose boundary contains two points from P. When $W = \emptyset$ the graph $\mathrm{DG}^-(P, \emptyset)$ is simply the complete graph $K_{|P|}$. When $W = P$ the graph $\mathrm{DG}^-(P, P)$ is precisely the Delaunay graph DG(P), which under standard non-degeneracy assumptions is a triangulation and is denoted DT(P) (see, e.g., [7,30]). The latter example illustrates the fact that the use of a witness set gives a generalization of the basic Delaunay structure. The properties of $\mathrm{DG}^-(P, W)$ are studied in Section 2.

The *square graph* of a point set P in the plane, with respect to a point set W of witnesses, denoted $\mathrm{SG}^+(P, W)$, is the graph with vertex set P, in which two points $x, y \in P$ are adjacent when there is an axis-aligned square with x and y on its boundary whose interior contains some witness point $q \in W$. It is a positive-witness graph in which the shapes are all the axis-aligned squares in the plane whose boundary contains two points from P. Observe that a negative-witness version $\mathrm{SG}^-(P, W)$ of this graph, with $W = P$, would be the standard Delaunay graph for the L_∞ metric, and hence we are studying here the positive-witness-variant of this Delaunay structure. The graph $\mathrm{SG}^+(P, W)$ is discussed in Section 3.

In this work we describe algorithms for the computation of these graphs and prove several of their fundamental properties. We also give a complete characterization of the combinatorial graphs that admit a realization as $\mathrm{SG}^+(P, W)$ for suitable sets P and W, a kind of result that, however, remains elusive for $\mathrm{DG}^-(P, W)$. In Section 4, we also present some related results on stabbing geometric objects, which can be essentially described as follows: given a point set P, find a second point set W, as small as possible, such that no pair of points $p, q \in P$ have adjacent regions in the Voronoi diagram of $P \cup W$.

We use standard graph terminology as in [15]; in particular, for a graph $G = (V, E)$ we write $xy \in E$ or $x \sim y$ to indicate that $x, y \in V$ are adjacent vertices of G. The terms *closed* and *open* are used in the sense of closed and open sets (sets with or without their boundary).

2. Witness Delaunay graphs

Consider a witness Delaunay graph $\mathrm{DG}^-(P, W)$ of a point set P with respect to a witness set W. We assume that the set $P \cup W$ is *in general position*, i.e., that no three distinct points in $P \cup W$ are collinear and that no four distinct points in $P \cup W$ are concyclic. We denote by E the edge set of the graph, that will be drawn as segments as usual for Delaunay graphs. Let $n := \max\{|P|, |W|\}$. We say that a disk *covers* a witness if the witness lies in its interior.

Note that, by definition of the witness Delaunay graph, the presence of an edge between vertices $p, q \in P$ is independent of the fact that p and/or q might be witnesses, since any open disk whose boundary passes through p and q does not cover either point.

First, a simple geometric observation:

Observation 1. *If D is a closed disk containing points p and q then there exists a disk $D_{pq} \subset D$ whose boundary passes through p and q.*

Proof. Let c be the center of D. Shrink D while keeping its center at c until it is about to lose p or q. Let the resulting disk be D'. Without loss of generality, let $p \in \partial D'$. Shrink D' by a homothety with center p until it is about to lose q. The result is the desired disk D_{pq}. □

We start with the computation of the witness Delaunay graph, which requires some lemmas; the first one is immediate from the definition of $\mathrm{DG}^-(P,W)$:

Lemma 1. *Two points $p,q \in P$ are adjacent in $\mathrm{DG}^-(P,W)$ if and only if they are neighbors in $\mathrm{DT}(W \cup \{p,q\})$.*

Lemma 2. *Let $w_1,\ldots,w_t$ be the Delaunay neighbors of $p \in P$ in $\mathrm{DT}(W \cup \{p\})$ given in counterclockwise radial order, and let $q \in P$ be a point whose radial position around p is between w_i and w_{i+1}. If $\measuredangle w_i p w_{i+1} \geqslant \pi$, then p and q are adjacent in $\mathrm{DG}^-(P,W)$; if $\measuredangle w_i p w_{i+1} < \pi$, then p and q are adjacent in $\mathrm{DG}^-(P,W)$ if and only if q lies in the interior of the circle through p, w_i, and w_{i+1}.*

Proof. If $\measuredangle w_i p w_{i+1} \geqslant \pi$, then p must be a vertex of the convex hull $\mathrm{CH}(W \cup \{p\})$ and the segment pq is external to this hull. Therefore there is a disk (in fact, a half-plane) containing p and q but covering no point from W, so they are adjacent in $\mathrm{DG}^-(P,W)$ by Observation 1. Assume now that $\measuredangle w_i p w_{i+1} < \pi$; then $p w_i w_{i+1}$ is a triangle in $\mathrm{DT}(W \cup \{p\})$ whose circumscribing disk D covers no points from W. If q is exterior to D then w_i and w_{i+1} are neighbors in $\mathrm{DT}(W \cup \{p,q\})$ and p and q cannot be adjacent in $\mathrm{DG}^-(P,W)$ because the segments pq and $w_i w_{i+1}$ cross. If q is interior to D then p and q are adjacent in $\mathrm{DG}^-(P,W)$, by Observation 1. □

Proposition 1. *Let P and W be two point sets in the plane, and $n := \max\{|P|,|W|\}$. The witness Delaunay graph $\mathrm{DG}^-(P,W)$ can be computed in $O(n^2)$ time, which is worst-case optimal.*

Proof. The radial order of the points in $P \setminus \{p\}$ around each point $p \in P$ can be obtained in overall time $O(n^2)$ [28], and the Delaunay triangulation $\mathrm{DT}(W)$ can be constructed in $O(n \log n)$ time [30]. Then for each point $p \in P$ we can obtain $\mathrm{DT}(W \cup \{p\})$ in additional $O(n)$ time and traverse the points of $P \setminus \{p\}$ in radial order within the same time bound, deciding for each one whether it is a neighbor of p in $\mathrm{DG}^-(P,W)$ in constant time, thanks to Lemma 2. □

Although the preceding algorithm is worst-case optimal because the output may have quadratic size (recall that $\mathrm{DG}^-(P,\emptyset)$ is the complete graph $K_{|P|}$), it is interesting to have an algorithm sensitive to the output size, even if it is more involved. We show next how to accomplish this.

We first observe that the problem is not interesting if $|P| \leqslant 1$, as there are no edges in the graph. Similarly, if there are no witnesses, the graph is complete. In fact, if there is only one witness, for any two vertices one of the two half-planes defined by them does not cover a witness; so the graph is again complete. Thus, for the remainder of this discussion we assume that $|W| > 1$ and $|P| > 1$.

Given the point sets P and W, with $|P| > 1$, $|W| > 1$, denote by $V(p)$ the (possibly unbounded polygonal) region of $p \in P$ in the Voronoi diagram $\mathrm{Vor}(W \cup \{p\})$; note that we allow the possibility that $p \in W$. Then, we have the following lemma:

Lemma 3. *The convex polygons $V(p)$, $p \in P$ behave as pseudodisks. More precisely, for distinct points $p,q \in P$, $V(p)$ and $V(q)$ are either disjoint or their boundaries $\partial V(p)$, $\partial V(q)$ either cross at most twice or overlap along a line segment.*

If $W \neq \{p,q\}$, $p \sim q$ in $\mathrm{DG}^-(P,W)$ if, and only if, $\partial V(p)$ and $\partial V(q)$ meet. If $W = \{p,q\}$, $p \sim q$ by definition of $\mathrm{DG}^-(P,W)$.

Proof. We assume that $W \neq \{p,q\}$, since the lemma is vacuously true otherwise. By definition, $p \sim q$ in $\mathrm{DG}^-(P,W)$ if and only if there exists a disk D_{pq} not covering any witnesses, with $p,q \in \partial D_{pq}$. Consider the set of all disks D whose bounding circle contains p and q; the union of the interiors of these disks cover the whole plane, except for a portion of the line pq. Since there are other witnesses besides p and q and they are not allowed to lie on this line, due to our general position assumptions, there is a disk D in this family whose boundary passes through p, q and another witness $w \in W \setminus \{p,q\}$ and such that D does not cover any witnesses. The center of the resulting disk D is equidistant from p, q and the witness $w \neq p,q$, and is no closer to any other witnesses. Hence it is a point of $\partial V(p) \cap \partial V(q)$, as claimed.

Conversely, suppose c is a point of $\partial V(p) \cap \partial V(q)$; let $r := d(c,p) (= d(c,q))$. By definition of the Voronoi regions, the distance from c to the closest witness is r. Hence the disk D_{pq} centered at c of radius r covers no witnesses and its boundary passes through p and q, certifying that $p \sim q$ in $\mathrm{DG}^-(P,W)$.

The first part of the proof implies that, if $V(p)$ and $V(q)$ meet, then their boundaries meet. To complete the proof of this lemma, it is enough to argue that the boundaries meet at most twice or overlap in a single segment. But this is clear, since the intersection of $\partial V(p)$ and $\partial V(q)$ lies on the perpendicular bisector of pq, which is a straight line meeting the boundary of the convex polygon $V(p)$, if at all, either in at most two points (where the two boundaries properly cross) or overlapping the boundaries of both cells in a single connected segment. □

We now use hierarchical representation techniques introduced by Dobkin and Kirkpatrick [21–23]; the properties we need were summarized by [24], who refer to [21,23,42] for the proofs:

Lemma 4. *(See Lemmas 5.2 and 5.3 in [24].) A three-dimensional polyhedron R with a total of n vertices, edges, and faces can be preprocessed in linear time into a data structure of linear size that supports the following operations in logarithmic time:*

(a) *given a directed line ℓ, find its first point of intersection with R, and*
(b) *given a line ℓ translating (within in a plane) from infinity, find the first point of contact of R and ℓ.*

Theorem 1. *Let P and W be two point sets in the plane, and $n := \max\{|P|, |W|\}$. The witness Delaunay graph $\mathrm{DG}^{-}(P, W)$ can be computed in time $O(k \log n + n \log^2 n)$, where k is the number of edges in the graph.*

Proof. As already mentioned, we will assume that there are at least two vertices and at least two witnesses. First suppose that no vertex is a witness point; we explain how to remove this assumption below.

By Lemma 3, the graph $\mathrm{DG}^{-}(P, W)$ is isomorphic to the intersection graph of the set of curves $\{\partial V(p) \mid p \in P\}$. Since any pair of curves cross at most twice (or overlap along a segment), it is sufficient to compute their arrangement and identify all vertices; a *vertex* is a point of crossing of two curves or an endpoint of a segment of overlap. We compute the arrangement by implementing a plane sweep from left to right [11], *without* representing the curves explicitly, since their worst-case combined complexity is easily seen to be $\Theta(n^2)$. We need the following operations:

(i) For a given $p \in P$, determine the leftmost and rightmost point of $\partial V(p)$. [Needed n times, once per p.]
(ii) For a given pair $p, q \in P$, determine the intersection points of $\partial V(p)$ and $\partial V(q)$, or confirm that the curves do not meet. [Needed $O(n+k)$ times, once for every pair of curves adjacent along the sweepline.]
(iii) For a given point $t(x, y)$ on a vertical line ℓ, and a given $p \in P$, such that $\partial V(p)$ meets ℓ, determine whether t lies in $V(p) \cap \ell$ and, if not, on what side of this intersection along the line. [Needed $O(n \log n)$ times, $O(\log n)$ times for each insertion of a new object into the data structure maintained by the sweepline; the point t is always the leftmost point of a newly discovered region.]

With the above three operations in hand, one can carry out a standard line sweep, sweeping a plane by a vertical line, say left-to-right, detecting appearances, intersections, and disappearances of curves and maintaining the order of their intersections with the line *without* explicitly computing the curves. It remains to describe how to implement each of the above operations to run in logarithmic time. The claimed running time bounds follow.

Recall the following standard *lifting transformation*: We transform a point $p(a, b) \in \mathbb{R}^2$ to the plane p^*: $z = 2ax + 2by - a^2 - b^2$ tangent to the *standard paraboloid* $z = x^2 + y^2$ in $\mathbb{R}^3$. The transformation has the following property: Given a set Q of points in the plane, consider the set $C(Q)$ of all points in space lying on or above the planes of $Q^* = \{q^* \mid q \in Q\}$. This set is an unbounded convex polyhedral region whose boundary is a convex monotone surface $\pi = \partial C(Q)$. The surface consists of convex portions (*faces*) of the planes of Q^*. The vertical projection of π to the plane coincides with the Voronoi diagram $\mathrm{Vor}(Q)$ and the faces project precisely to Voronoi regions [27].

We compute and store the polyhedron $C = C(W)$ in a data structure supporting operations (a) and (b) from Lemma 4. We translate the operations (i)–(iii) to operations on C. Operation (iii) involves determining, given t, ℓ, and p, the location of t along ℓ, in relation to $\ell \cap V(p)$. This can be accomplished in constant time, once we compute $\ell \cap V(p)$. "Lifting" the picture to three dimensions, consider the line $\ell' \subset p^*$ that projects vertically to ℓ. The desired intersection corresponds to $\ell' \cap C$ in $\mathbb{R}^3$. This set, in turn, can be computed in $O(\log n)$ time by shooting along ℓ' in both direction, using Lemma 4(a).

Operation (ii) again reduces to shooting along a line. The points of $\partial V(p) \cap \partial V(q)$ lie on the bisector $b = b(p, q)$ and their corresponding three-dimensional points (i.e., the points of intersection of $p^* \cap \pi$ and $q^* \cap \pi$) lie on the line $b' = p^* \cap q^*$ that projects vertically to b. Indeed the lifted points in question are just the set $b' \cap \pi$ and can be computed by two directed-line-shooting queries along b', via Lemma 4(a).

Finally, operation (i) calls for finding the leftmost (i.e., x-minimum) point of $V(p)$; the rightmost point is handled similarly. This point is the projection of the x-minimum point of $p^* \cap C$ to xy-plane. The latter point is the first point of contact of the line $p^* \cap \{x = c\}$ with C, as c varies from $-\infty$ to $+\infty$, and so can be identified in logarithmic time, by Lemma 4(b). (It is also possible that $V(p)$ does not have a leftmost point—in this case we want to compute the infinite ray (or two) bounding $V(p)$ (or $p^* \cap C$ in three dimensions) and extending to infinity to the left; this is needed for properly initializing the state of the sweepline "at infinity." This can be done by preprocessing the intersection of C with "the plane" $x = +\infty$ for line intersection queries and intersecting it with p^*; being a two-dimensional problem, it is easier.) This concludes our description of the implementation of operations (i)–(iii).

What modifications are needed if some vertices are also witnesses? For such a vertex p, $V(p)$ coincides with Voronoi region of p in $\mathrm{Vor}(W)$; its lifted version is a facet of C, which in turn is precisely $p^* \cap C$. Hence the algorithm works as advertised, with the additional proviso that the data structure needs to handle the possibility that query lines and/or planes might be supporting lines/planes of C. $\square$

The idea of computing or detecting intersections among a set of objects by a sweepline algorithm, without explicitly computing the objects is not new; see, for example, [1,2].

The characterization of combinatorial graphs that are drawable as standard Delaunay graphs is a long-standing open problem (see Section 7.3.2 in [37]). Recall that every graph that is realizable as a Delaunay graph $\mathrm{DG}(P)$ is also a witness Delaunay graph, because $\mathrm{DG}^{-}(P, P) = \mathrm{DG}(P)$. In particular, all maximal outerplanar graphs are realizable, as proved by Dillencourt in [19,20] (better algorithms were later described in [45,36]).

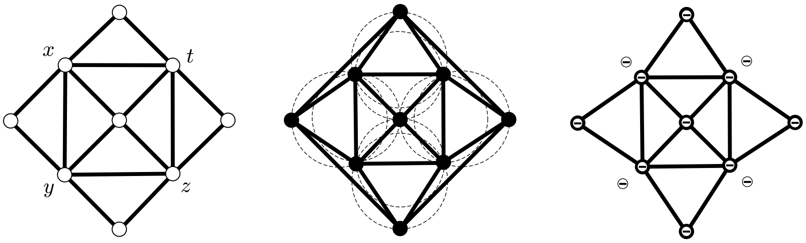

Fig. 1. The graph G on the left is not 1-tough, because the removal of vertices x, y, z and t yields five components. In the center a supergraph of G is realized as a Delaunay graph; from this a witness Delaunay graph realization of G is obtained by placing witnesses at the vertices plus four extra witnesses that force the removal of the convex hull edges.

Dillencourt also proved that every Delaunay graph must satisfy some necessary conditions [18], in particular that they are always 1-tough (the deletion of any k vertices cannot produce a graph with more than k components) which he used to construct some graphs that are not drawable as Delaunay graphs. For example the graph in Fig. 1, left, does not admit such a realization because it is not 1-tough. However it can be realized as a witness Delaunay graph, as shown in the figure.

Substantial effort has been devoted to drawing trees as proximity graphs [37,13,12,32,39,38,43]. We prove next that drawing a tree as witness Delaunay graph is always possible.

Theorem 2. *Every tree can be realized as witness Delaunay graph* $\mathrm{DG}^-(P, W)$ *for suitable point sets* P *and* W. *The realization can be carried out in time linear in the size of the tree, in infinite-precision-arithmetic model of computation.*

Proof. We show that every tree $T = (V, E)$, rooted at a vertex r, can be drawn as $\mathrm{DG}^-(P, W)$, rooted at a given point s, in such a way that:

(a) a witness is placed at each vertex, i.e., $P \subseteq W$;
(b) all the vertices P except for the root s are in the interior of an axis-parallel square box B, and s lies at the midpoint of the top side of B;
(c) there are disks D_{ab} incident to the endpoints $a, b \in P$ of each edge ab corresponding to edges of T, empty of witnesses and vertices, that certify the Delaunay edges; these disks D_{ab} lie inside B, except for some disks D_{sa}, incident to s; the disks D_{sa} have their centers inside B, and can only cross the top side of B;
(d) two witnesses are placed at the top-left and the top-right corners of B.

The proof is by induction on the height h of the tree. For $h = 0$, it is obvious as there is only one vertex and no edges; let B be an arbitrary square with s at the midpoint of its top side. Assume this is true for trees with heights up to k, $k \geqslant 0$, and let $T = (V, E)$ be a rooted tree of height $k + 1$. Subtrees $T_1 = (V_1, E_1), \ldots, T_m = (V_m, E_m)$ of the root have height at most k, and can be drawn as claimed, in boxes $B_1, \ldots, B_m$, by inductive assumption. By rescaling the boxes, if necessary, assume that each has side length 1. Place the boxes on a horizontal line, in order, $\frac{1}{2}$ apart; refer to Fig. 2. Draw an axis-parallel rectangle R with width $\frac{2m-1}{2}$, height 10 times its width, lower left corner at the upper left corner of B_1, and lower right corner at the upper right corner of B_m. Place s in the middle of the top side of R. Put three witnesses midway between consecutive boxes B_i and B_{i+1}, $1 \leqslant i \leqslant m - 1$, one aligned with the top of B_i, one with the bottom of B_i, and one midway between them. For $i = 1, \ldots, m$, consider a disk D_{sc_i} such that its boundary contains s and c_i, the root of T_i, and such that it is tangent to B_i. We construct an axis-parallel square box B with its upper midpoint at s, containing the smaller boxes B_1, $\ldots$, B_m and as narrow as possible but yet such that the disks D_{sc_i} intersect only its top side. We add a witness at s and two witnesses at the two upper corners of the new box B (see Fig. 2). Notice that the construction creates some collinearities. They can be easily removed by slightly perturbing the positions of the vertices and witnesses without changing the tree.

To confirm that the construction indeed realizes the tree T, we first prove a technical assertion: We claim that the sub-boxes $B_1, \ldots, B_m$ lie in the lower half of the box B. We prove this by induction, for which the base case is vacuously true. We calculate first the side length of B. Let r be the width of R. Let xc_m be the diameter of D_{sc_m} incident to c_m; refer to Fig. 2. Let y be the intersection of xc_m and the horizontal line through s. We obtain two congruent triangles $\triangle xys$ and $\triangle syc_m$. The distance xy is given by $\frac{(r/2-1/2)^2}{10r}$. Hence the radius of D_{sc_m} is $\frac{1}{2} \times (\frac{(r/2-1/2)^2}{10r} + 10r)$. Therefore the width of B is $\frac{(r/2-1/2)^2}{10r} + 10r + r - 1$ as the centers of D_{sc_1} and $D_{c_m s}$ are $r - 1$ apart. Now it is sufficient to prove that $\frac{(r/2-1/2)^2}{10r} + r - 1 < 10r$ to show that the smaller boxes $B_1, \ldots, B_m$ are in the lower half of B. As $r \geqslant 1$ and $\frac{(r/2-1/2)^2}{10r} < \frac{r}{40}$, we obtain that $\frac{(r/2-1/2)^2}{10r} + r - 1 < 2r$ and the claim follows.

Now we will check that the witness Delaunay graph of the set of vertices and witnesses described above is precisely T. Conditions (a) to (d) are clearly fulfilled by construction. By the inductive hypothesis, we obtain $T_1, T_2, \ldots, T_m$ as the witness Delaunay graphs of the constructions inside boxes $B_1, \ldots, B_m$, respectively. By construction, the witnesses we placed outside B_i do not interfere with the disks of the edges connecting two vertices within B_i. There are no edges $v_i \in B_i$, and $v_j \in B_j$,

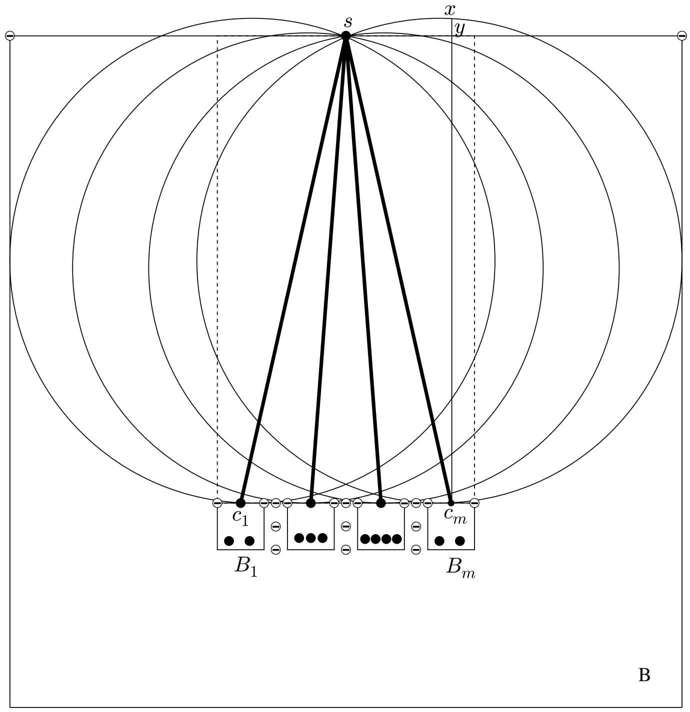

Fig. 2. The black points are the vertices and the white points, the witnesses. Rectangle R is dashed.

$i \neq j$, because the triples of witnesses between the smaller boxes prevent that. More precisely, we know that the edge $v_i v_j$ will cross the segment $w_1 w_2$ defined by two witnesses w_1 and w_2, vertically aligned at a vertical distance of $\frac{1}{4}$, lying between B_i and B_j. If we draw the disk $D_{w_1 w_2}$ with diameter $w_1 w_2$, it is empty of vertices by construction. As the edge $v_i v_j$ crosses it and v_i and v_j are outside $D_{w_1 w_2}$, any disk containing $v_i v_j$ contains a witness, for example, by Lemma 1.

The roots of $T_1, \ldots, T_m$ are adjacent to s by construction. It remains to check that s is not adjacent to any vertex interior to any of the boxes B_i. This is prevented by the three witnesses on the top edge of B_i. More precisely, let w_1, w_2, w_3, be the three witnesses on the top side of B_i. We consider the two vertices s and v, with v being a vertex inside the box B_i. A putative edge sv must cross either the segment $w_1 w_2$ or the segment $w_2 w_3$. Suppose that it crosses the segment $w_1 w_2$. Recall that all the interior vertices of B_i are in its lower half, hence the disk $D_{w_1 w_2}$ with diameter $w_1 w_2$ is empty of vertices. Therefore any Delaunay disk D_{sv} must contain either w_1 or w_2, or both, and sv is not an edge of the witness Delaunay graph. $\square$

We note that it might be interesting to investigate how large a grid one needs to draw a tree as a witness Delaunay graph if the vertices and witnesses are to be placed at points with integer coordinates. The above construction made no effort to optimize this quantity.

We conclude this section with a result on the negative side:

Theorem 3. *A non-planar bipartite graph cannot be realized as witness Delaunay graph* $\mathrm{DG}^-(P, W)$, *for any point sets P and W.*

Proof. The proof is by contradiction. If a realization of a non-planar bipartite graph G as a witness Delaunay graph $\mathrm{DG}^-(P, W)$ exists, it must contain two crossing edges $p_1 q_1$ and $p_2 q_2$, with p_1 and p_2 belonging to the same part of the bipartite graph, and q_1 and q_2 belonging to the other part. The vertices p_1, p_2, q_1, q_2 form a convex quadrilateral Q, and we may assume without loss of generality that they occur in this order along the boundary of Q (see Fig. 3).

As G is a bipartite graph, it does not contain $p_1 p_2$ or $q_1 q_2$. As the sum of the interior angles of a quadrilateral equals 360°, and the vertices are in general position, either $\measuredangle p_1 p_2 q_1 + \measuredangle q_1 q_2 p_1 < 180^\circ$ or $\measuredangle p_2 q_1 q_2 + \measuredangle q_1 q_2 p_1 < 180^\circ$. Without loss of generality, suppose $\measuredangle p_1 p_2 q_1 + \measuredangle q_1 q_2 p_1 < 180^\circ$. Then any disk D with $p_2 q_2$ as a chord will contain p_1, q_1, or both. Let $D_{p_2 q_2}$ be the witness-free disk certifying the edge $p_2 q_2 \in G$. Since $p_2 q_2$ is a chord of this disk, it must contain one of p_1, q_1. Suppose without loss of generality that $D_{p_2 q_2}$ contains q_1. By shrinking $D_{p_2 q_2}$ by a positive homothety with center at q_2 until its boundary passes through q_1, we obtain a disk $D_{q_1 q_2} \subseteq D_{p_2 q_2}$ not covering any witnesses, whose boundary contains q_1, q_2, contradicting $q_1 q_2 \notin G$. $\square$

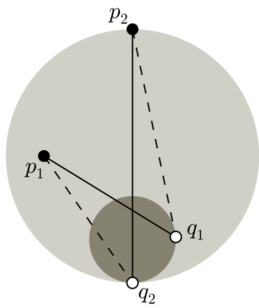

Fig. 3. Solid edges are present in the graph, while dashed ones may or may not be.

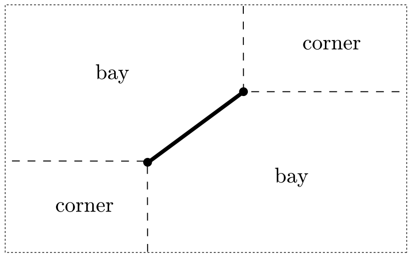

Fig. 4. Corners and bays.

3. Square graphs

In this section we use the term *square graph* as short for the square graph of a point set P (the *vertices*) with respect to a second point set W (the *witnesses*); we recall that two points $x, y \in P$ are adjacent in the graph $\mathrm{SG}^+(P, W)$ if and only if there is an axis-aligned square with x and y on its boundary whose interior contains some witness point $q \in W$. As mentioned in the introduction, this is the positive witness version on the Delaunay graph for the L_∞ metric. We assume that no two distinct points in $P \cup W$ have equal x- or y-coordinates and let $n := \max\{|P|, |W|\}$. (In this section, we do not require that no three points be collinear.) We denote by E the edge set of the graph; we partition E into E^+ and E^- according to the slope sign of the edges when drawn as segments.

First, a simple geometric observation:

Observation 2. *If R is a closed square containing points p and q then there exists a square $R_{pq} \subset R$ whose boundary passes through p and q.*

Proof. Let c be the center of R. Shrink R while keeping its center at c until it is about to lose p or q. Let the resulting square be R'. Without loss of generality, let $p \in \partial R'$. Shrink R' by a homothety with center p until it is about to lose q. The result is the desired square R_{pq}. $\square$

The isothetic rectangle (*box*) defined by two points p, q in the plane is denoted $B(p, q)$. For an edge $e = pq$ we also write $B(e)$ instead of $B(p, q)$. Every edge e, say in E^+, defines four regions in the plane as in Fig. 4, that we call *corners* and *bays*. A corner is a closed set while a bay is an open set.

If $p, q \in P$, then $pq \in E$ if and only if the union of the bays of pq contains some witness or, equivalently, if and only if W is not contained in the union of the corners. In particular, the placement of just two witnesses, for example, just outside and very close to the top corners of an axis-aligned rectangle enclosing P suffices to yield a complete graph, because every upper bay would contain a witness. Also, as the bays associated with a pair of points p, q cover the vertical open strip delimited by the lines $x = x(p)$ and $x = x(q)$ and the horizontal open strip delimited by the lines $y = y(p)$ and $y = y(q)$, we deduce the following useful fact.

Observation 3. *If there is a witness point $w \in W$ such that $x(w)$ is between $x(p)$ and $x(q)$ or $y(w)$ is between $y(p)$ and $y(q)$, then pq is an edge of* $\mathrm{SG}^+(P, W)$.

Computing how many witnesses are contained in quadrant I and quadrant IV for every $p \in P$ can be carried out in overall $O(n \log n)$ time with a line sweep from right to left, and by keeping the set of witnesses already encountered stored in a balanced search tree, sorted by the y-coordinate; a sweep in the opposite direction handles the remaining two quadrants.

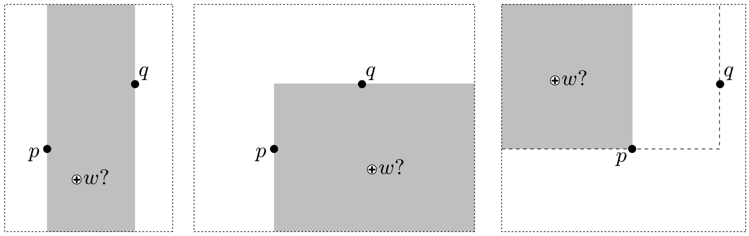

Fig. 5. Illustrating the proof of Theorem 4.

After that every pair of points $p, q \in P$ can be checked for adjacency in constant time and therefore the square graph $\mathrm{SG}^+(P, W)$ can be computed in $O(n^2)$ time, which is worst-case optimal. We describe next an output-sensitive algorithm.

Theorem 4. *Let P and W two point sets in the plane, and $n := \max\{|P|, |W|\}$. The square graph $\mathrm{SG}^+(P, W)$ can be computed in optimal $O(k + n\log n)$ time, where k is the number of edges in $\mathrm{SG}^+(P, W)$.*

Proof. We first detect all pairs of points $p, q \in P$ such that the open strip bounded by the vertical lines through these points covers some witness, making them adjacent in the graph (Fig. 5, left). For this, it suffices to consider the projection z^* of all the points $z \in P \cup W$ onto the x-axis. Once the projections are sorted, it is clear that for every $p \in P$, if w is the first witness such that $x(p) < x(w)$, we can simply list all the $q \in P$ such that $x(w) < x(q)$. After the $O(n\log n)$ sorting step, a simple scan gets every adjacency listed once, and the global cost is proportional to their number. On the other hand, in $O(n)$ time after sorting, we can also store for each point $p \in P$ the number of witness projections to the right of p^*. This will later allow us to detect in constant time whether there is a witness in the vertical strip defined by $p, q \in P$.

Next we explain how to find the pairs of points $p, q \in P$, with $x(p) < x(q)$, such that the slope of the segment pq is positive, $p \sim q$ in $\mathrm{SG}^+(P, W)$, and the adjacency has not been reported in the previous step. The remaining case is handled in a symmetric manner.

Sweep from right to left with a vertical line ℓ, and maintain the lowest witness $w_R \in W$ to the right of ℓ and the highest witness $w_L \in W$ to the left of ℓ. In addition, maintain a y-sorted list L of the points of P to the right of ℓ. When the sweep line finds a point $p \in P$, we report all the points $q \in L$ that are above w_R and p, by a simple linear scan of the list from p, omitting those that have reported as adjacent to p in the preceding step that checked the vertical strip between them (Fig. 5, center).

If both w_L and w_R are above p, we additionally report all the points $q \in L$ that are above p and below w_R, performing a second linear scan of the list from p, again omitting those that have reported as adjacent to p in the first step (Fig. 5, right).

The involved costs are $\Theta(1)$ per edge found, $\Theta(\log n)$ to insert p into L, and $\Theta(1)$ to update w_L and w_R when a witness from W is encountered by the sweep line.

This process must be repeated from left to right for edges with negative slope. Overall, all edges will be found and each one reported exactly once, which proves that the graph can be computed in $O(k + n\log n)$ time, as claimed.

Let us now show that this is optimal. The lower bound $\Omega(k)$ is obvious. To see the $\Omega(n\log n)$ part, we use a reduction from the UNIQUENESS problem: "Given n positive integers, decide whether all of them are distinct" which is known to have an $\Omega(n\log n)$ lower bound in the algebraic computation tree model [47].

Now, given positive integers $S = \{x_1, \dots, x_n\}$, consider the point set $P = \{p_1, \dots, p_n\}$, with $p_i = (x_i - \frac{i}{10n}, x_i + \frac{i}{10n})$. It is easily checked that P is a set of points near the line $x = y$, such that, as long as $x_i \neq x_j$, for $i \neq j$, the slope of the segment $p_i p_j$ is positive. However, if $x_i = x_j$, for some $i \neq j$, the slope of $p_i p_j$ is -1. In particular, $SG^+(P, \{(0,0)\})$ has no edges if and only if all numbers in S are distinct. This completes the description of a linear-time reduction from UNIQUENESS to the computation of the square graph, hence proving the claimed complexity lower bound. □

Before describing the combinatorial structure of square graphs, we recall some well-known definitions.

Given points $a, b \in \mathbb{R}^d$, with $a = (a_1, \dots, a_d)$ and $b = (b_1, \dots, b_d)$ we say that *a dominates b* (denoted as $a \geqslant b$ or $b \leqslant a$) when $a_i \geqslant b_i$ for $i = 1, \dots, d$. Given a partially ordered set $\mathcal{P} = (X, \leqslant_{\mathcal{P}})$, a *$d$-dominance realization* of $\mathcal{P}$ is a function $f : X \to \mathbb{R}^d$ such that $x \leqslant_{\mathcal{P}} y$ if and only if $f(x) \leqslant f(y)$, for all $x, y \in X$.

The smallest d such that $\mathcal{P} = (X, \leqslant)$ admits a d-dominance realization is called the *dimension* of the partial order $\mathcal{P}$. Equivalently, d is the smallest integer such that $\mathcal{P}$ is the intersection of d total orders that are extensions of $\mathcal{P}$. The concept of dimension was introduced, and the equivalence of the definitions proved, in [26].

The undirected graphs underlying partial orders (i.e., for distinct x, y, $x \sim y$ when $x \leqslant y$ or $y \leqslant x$) are called *comparability graphs*. It has been proved (see Section 6.2 in [14]) that any two partial orders whose underlying comparability graphs are the same must have the same dimension, and therefore we can call this number the *dimension of the comparability graph*. The comparability graphs corresponding to two-dimensional partial orders are also called *permutation graphs* (this name arose in a different context yet equivalence was established).

We are now ready for our main result in this section, a complete characterization of square graphs:

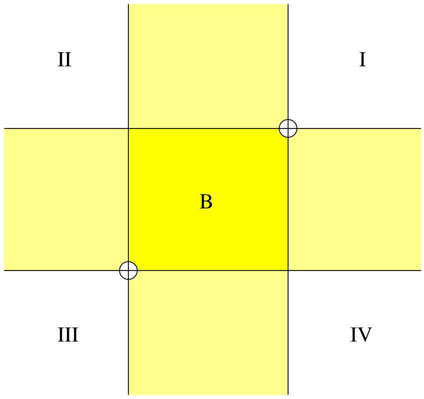

Fig. 6. Illustrating the proof of Theorem 5. The two open strips are lightly shaded. Their complement consists of the four quadrants numbered I through IV.

Theorem 5. *A combinatorial graph $G = (V, E)$ can be realized as a square graph $\mathrm{SG}^+(P, W)$ for suitable point sets P and W in the plane if and only it is a permutation graph. Moreover, any square graph can be realized using at most one witness.*

Proof. Let $G = \mathrm{SG}^+(P, W)$ be a square graph and G' its complement. Recall that we assume that no two distinct points in $P \cup W$ have equal x- or y-coordinates. Suppose first that $P \cap W = \emptyset$, i.e., no vertex is also a witness. We will remove this assumption below.

Draw a vertical line and a horizontal line through each witness, partitioning the plane into open *boxes*. From Observation 3 we know that no edge of G' crosses any of these lines.

Consider such a box B. By construction, the vertical open strip containing B covers no witnesses and the same is true of the horizontal open strip containing B. We partition the complement of the union of these strips into four closed *quadrants of* B, numbered I through IV (if B is unbounded in one or more directions, we simply treat two or more of the quadrants as empty sets), refer to Fig. 6.

Putting $P_B := B \cap P$, let $G_B = (P_B, E_B)$ be the subgraph of $\mathrm{SG}^+(P, W)$ induced on P_B, and let $G'_B = (P_B, E'_B)$ be its complement. If there is a pair of adjacent quadrants of B each containing a witness, the graph G_B is complete and its complement G'_B is the empty graph on P_B. The empty graph is certainly a two-dimensional comparability graph, as it suffices to take a sequence of points with increasing abscissae and decreasing ordinates.

There is only one case remaining: there are witnesses only in one pair of opposite quadrants of B (one of these opposite quadrants could be empty of witnesses); without loss of generality, these quadrants are I and III. Then $p, q \in P_B$ define an edge of G_B if, and only if, the slope of the segment pq is negative. Hence G'_B is the comparability graph underlying the dominance relation for P_B with the current system of coordinates.

Therefore, we have proved that the complement of $\mathrm{SG}^+(P, W)$ is the disjoint union of permutation graphs which itself is a permutation graph, if no vertex is also a witness. As the complement of a permutation graph is also a permutation graph (see [35]), we have proven that $\mathrm{SG}^+(P, W)$ is a permutation graph as well.

Now suppose $P \cap W \neq \emptyset$. The above argument applies verbatim to the subgraph of G' induced on $P \setminus W$, i.e., to the non-adjacencies between non-witness vertices. Let $q \in P \cap W$. Let the superbox H of q be the smallest open box enclosing the four open boxes (which we call $B_I, B_{II}, B_{III}, B_{IV}$ according to their position around q) adjacent to q; refer to Fig. 7. Let $\bar{H}$ be the closure of H. Using Observation 3, we conclude that q is adjacent in G to every vertex outside of $\bar{H}$. In particular, in G', all neighbors of q lie in $\bar{H}$.

We first consider the special case $W = \{q\}$. As argued above, disregarding q, the complement of G is the disjoint union of at most four permutation graphs, one for each of the boxes surrounding q. By definition of a square graph, q is not adjacent to anything in G and hence G' is formed by taking the disjoint union of four or fewer permutation graphs and adding a vertex adjacent to all other vertices. We argue that then G' is a single permutation graph and hence G is a permutation graph as well. Indeed, form a 4-by-4 grid in the plane and draw each of the permutation graphs of G' in the diagonal boxes of the grid, top-left to bottom-right, so that each coincides with the comparability graph of their xy-dominance relation. There are no dominance relations between the diagonal boxes, so we have a realization of their disjoint union. Now place the vertex corresponding to q below and to the left of the grid, completing the realization of G' as the comparability graph of a 2-dimensional dominance relation.

For the remainder of this proof, we assume that q is not the only witness.

We first prove the following claim: the subgraph $G'_{P \cap W}$ induced by vertex-witnesses in G' is a collection of paths, and each path is a chain or an antichain in the dominance relation on P, i.e., the vertices along the path either have increasing x- and y-coordinates, or increasing x- and decreasing y-coordinates.

Indeed, by construction and by our assumption that no two vertex-witnesses share x- or y-coordinates, $\bar{H}$ can contain at most two vertex-witnesses besides q. Any such vertex-witness must lie at a corner of $\bar{H}$, and if there are two of them, they must occupy diagonally opposite corners of $\bar{H}$. As already observed, q is not adjacent in G' to any vertex outside $\bar{H}$,

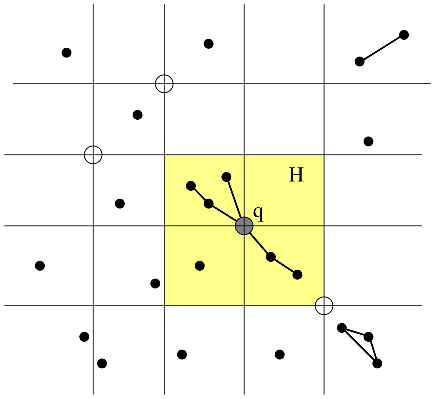

Fig. 7. The superbox H is lightly shaded, $q \in P \cap W$ is grey, points of $P \setminus W$ are black, and witnesses of $W \setminus P$ are white with a cross.

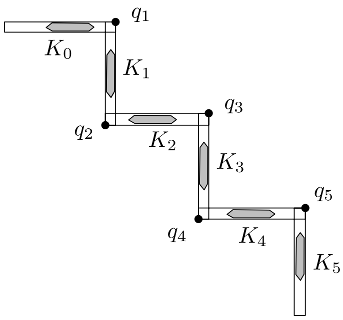

Fig. 8. In most general case, a connected component of G' is a comparability graph of a two-dimensional dominance relation. The shaded regions represent two-dimensional dominance realizations of graphs K_i.

hence it has degree at most two in $G'_{P\cap W}$ and if it does has two neighbors, they form a chain or an antichain with q in the dominance relation on P. The claim easily follows.

Let G_H (G'_H) be the subgraph of G (respectively, G') induced by the vertices in the open box H, i.e., by q and the vertices in $B_I, \ldots, B_{IV}$. Consider the union of the open vertical and horizontal strips defined by H; its complement is a union of (at most) four quadrants, which we refer to as *the quadrants of H* and number in the usual manner, I through IV. The quadrants contain all the witnesses besides q, and we have assumed there exists at least one such witness. As above, if there is a pair of adjacent quadrants containing witnesses, G_H is complete and therefore G'_H is the empty graph; in fact it is easy to check that $G_{\bar{H}}$ is complete, $G'_{\bar{H}}$ is empty, and therefore q is an isolated vertex in G' in this case.

There remains the case that there is a pair of opposite quadrants, say I and III, one or both of which contain a witness. Then $G_{B_{II}}$ and $G_{B_{IV}}$ are complete graphs and therefore $G'_{B_{II}}$ and $G'_{B_{IV}}$ are empty graphs; q is not adjacent in G' to any vertex in these two boxes. On the other hand, G'_{B_I} and $G'_{B_{III}}$ represent the dominance relation in B_I and B_{III}, respectively, and q is adjacent to every vertex in those two boxes. If either of the quadrant I or quadrant III corners of H is a vertex-witness, they are the neighbors of q along its path in $G'_{P\cap W}$.

To summarize, G' decomposes into disjoint subgraphs of the form

$$\langle K_0, q_1, K_1, q_2, \ldots, q_\ell, K_\ell \rangle,$$

with $\ell \geqslant 0$, where each K_i is a permutation graph (possibly with no vertices), each q_i is a vertex-witness, $q_1, q_2, \ldots, q_\ell$ is a simple path in G' (in fact, it is a chain or an antichain in the 2-dimensional dominance relation on P) and all vertices of K_i are adjacent to q_i and q_{i+1}, for $0 < i < \ell$. Vertices of K_0 are adjacent to q_1, vertices of K_ℓ are adjacent only to q_ℓ. There are no adjacencies between vertices of different K_i's.

The first part of the proof of the theorem is complete once we argue that each $\langle K_0, q_1, \ldots \rangle$ is isomorphic to the comparability graph of the dominance relation of some two-dimensional set of points; such a realization in depicted in Fig. 8 (notice that this realization uses a set of points unrelated to P); it follows that the complement of all $\langle K_0, q_1, \ldots \rangle$ is a two-dimensional comparability graph.

Conversely, let $G = (V, E)$ be a comparability graph of dimension two. Consider a set P of points in the plane with no repeated coordinate values, whose dominance graph (for (x, y)-coordinates) is isomorphic to G. Points $p, q \in P$ are adjacent in G if and only if pq has positive slope, hence $\mathrm{SG}^+(P, \{w\}) \cong G$ if we place the sole witness w to right and below all points of P. $\square$

Recognizing whether a combinatorial graph $G = (V, E)$ is a permutation graph can be done in time $O(|V| + |E|)$ [40]. Combining this result with the preceding theorem, we immediately obtain:

Corollary 6. *One can decide in $O(|V| + |E|)$ time whether a given combinatorial graph $G = (V, E)$ can be realized as a square graph $\mathrm{SG}^+(P, W)$ for some point sets P, W in the plane. If a realization exists, it can be constructed within the same time bounds.*

4. Size of stabbing sets

Consider a point set P in the plane. Let $\mathcal{S}$ be a family of geometric objects with nonempty interiors, each one associated to a finite subset of P. We say that a point w *stabs* an object $Q \in \mathcal{S}$ if w lies in the interior of Q. In this section we consider the problem of how many points are required to stab all the elements of $\mathcal{S}$, which we denote by $st_{\mathcal{S}}(P)$, and how large this number can be when all the point sets with $|P| = n$ are considered. We denote this extremal value by $st_{\mathcal{S}}(n) = \max_{|P|=n} st_{\mathcal{S}}(P)$. We will see that these problems can be rephrased in terms of witness graphs, and therefore the results from the previous sections be used for their study. On the other hand, the more natural and interesting formulation is in terms of Voronoi discrimination, a description that we present in Section 4.3.

Similar problems have already been considered for the family of all convex polygons with vertices from among the points of P. In particular, if $\mathcal{T}$ and $\mathcal{Q}$ are the family of triangles and convex quadrilaterals, respectively, with vertices in the given point set, it has been proved that $st_{\mathcal{T}}(n) = 2n - 5$ and that $st_{\mathcal{Q}}(n) = 2n - o(n)$ [34,16,44]. Those families of shapes are finite, while the ones we consider here are infinite and continuous: the disks and the isothetic squares whose boundary contains two points from P. For these problems the stabbing set can be viewed as a witness set that yields a specific type of corresponding witness graph, a connection that allows us to use the preceding results and that we make precise below.

4.1. Stabbing disks

Let P be a set of n points, and let $\mathcal{D}$ be the set of disks whose boundary contains at least two points from P. If $p, q \in P$, a set of points W stabs every disk with p and q on its boundary if and only if pq is not an edge of $\mathrm{DG}^-(P, W)$. In other words,

$$st_{\mathcal{D}}(n) = \max_{|P|=n} st_{\mathcal{D}}(P) = \max_{|P|=n} \min\bigl\{|W|:\ \mathrm{DG}^-(P, W) = \emptyset\bigr\}.$$

Lemma 5. *Let P be a set of n points and let $\mathcal{D}'$ be a set of disks with pairwise disjoint interiors, such that the boundary of any of them contains two points from P. Then, $|\mathcal{D}'| \leqslant n$, which is tight.*

Proof. A point $p \in P$ can lie on the boundary of at most two interiorly disjoint disks from $\mathcal{D}'$, and which would necessarily be tangent at p. As every bounding circle contains two points from P, $|\mathcal{D}'| \leqslant n$. This bound is achievable, for example, in a necklace of disks in which each one touches its two neighbors, with P being the set of contact points. $\square$

Lemma 6. *Let P be a set of n points. If none of the edges of $\mathrm{DT}(P)$ are in $\mathrm{DG}^-(P, W)$, then $\mathrm{DG}^-(P, W) = \emptyset$.*

Proof. Let p and q be two points from P and let C be any circle through them. If p and q are neighbors in $\mathrm{DT}(P)$, we know by hypothesis that $p \nsim q$ in $\mathrm{DG}^-(P, W)$. If p and q are not adjacent in $\mathrm{DT}(P)$, then there is at least one point from P in the interior of the disk D bounded by C, and we can find a disk D' contained in D, tangent to C at p, that has a second point, say p', from P on its boundary but none interior. (D' can be obtained, for example, by shrinking D with center p until the moment it contains no point of P in its interior.) Therefore $pp' \in \mathrm{DT}(P)$. Since pp' is not an edge of $\mathrm{DG}^-(P, W)$, D' must contain a witness point. This witness stabs D as well, therefore $p \nsim q$ in $\mathrm{DG}^-(P, W)$. $\square$

The preceding lemma implies that to stab all disks whose boundaries contain pairs of points from P, it is enough to stab only the disks corresponding to pairs of Delaunay neighbors. This can easily be done by placing a witness point very close to the midpoint of each Delaunay edge, the witness point being external for the convex hull edges, yielding roughly a total of $3n$ witnesses. We show next a better upper bound.

Theorem 7. *For $n \geqslant 2$, $n \leqslant st_{\mathcal{D}}(n) \leqslant 2n - 2$.*

Proof. The lower bound comes from the existence of sets, as shown in Lemma 5, that admit n disks with pairwise disjoint interiors, each containing two of the points on its boundary, because each disk requires a distinct stabbing witness. For the upper bound we place a witness p_T inside each Delaunay triangle T in $\mathrm{DT}(P)$, in such a way that p_T sees every side of

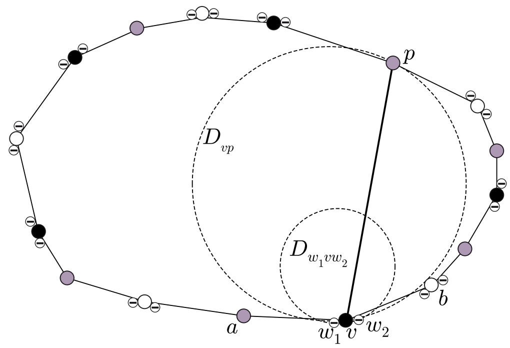

Fig. 9. Illustration for the proof of Proposition 2.

T with an angle greater than $\pi/2$ (for example, one may place p_T on an internal height, very close to its foot). We also place a witness for every edge of the convex hull, external and very close to its midpoint. In this way every disk having a Delaunay edge from DT(P) as a chord will be stabbed at least on one side of the edge. If the size of the convex hull is h, the number of triangles in DT(P) is $2n-h-2$, therefore the total number of witnesses we have placed is $2n-2$, as claimed. □

We have obtained a better bound for points in convex position, described in the following proposition. However, we believe both in the convex case and the generic case that the tight values should be closer to the n lower bound rather than to our upper bounds.[3]

Proposition 2. *Let P be a set of n points in convex position, then $st_{\mathcal{D}}(P) \leqslant \frac{4}{3}n$. In other words, a suitable set W of at most $\frac{4}{3}n$ witnesses is always sufficient to have* $\mathrm{DG}^-(P, W) = \emptyset$.

Proof. Recall from Lemma 6 that to eliminate all edges in a witness Delaunay graph of a set of points P, it is sufficient to eliminate the edges of the Delaunay triangulation of P. Color the vertices of the Delaunay triangulation with three colors [29], white, gray and black. Pick the color that covers the largest number of vertices, suppose this is color is gray. For each vertex v of color black or white, and its two incident edges on the convex hull va and vb, put a witness w_1 outside of CH(P) very close to v and va, and another witness w_2, outside of CH(P), and very close to v and vb (see Fig. 9). The witnesses w_1 and w_2 are close enough to v so that the disk $D_{w_1vw_2}$ defined by w_1, v, and w_2 is empty of vertices. Consider any Delaunay edge vp incident to v. By construction, p is outside of $D_{w_1vw_2}$ and vp intersects the interior of $D_{w_1vw_2}$. Therefore $\measuredangle vw_1p + \measuredangle vw_2p > 180°$ and there is no disk with v and p on its boundary that is empty of witnesses.

As at most $\frac{2}{3}n$ vertices are surrounded by two witnesses, $\frac{4}{3}n$ witnesses are sufficient to remove all the edges in the Delaunay triangulation of P, and the claim follows. □

4.2. Stabbing squares

Let P be a set of n points, such that no two of them have equal abscissa or ordinate, and let $\mathcal{S}$ be the set of isothetic squares whose boundary contains two points of P. Recall that $\mathrm{SG}^-(P, W)$ is the negative witness square graph of P with respect to W, in which two points p and q from P are adjacent if and only if there is a square that has p and q on its boundary but covers no point from W. Equivalently, $\mathrm{SG}^-(P, W)$ is the Delaunay graph of P with respect to W for the L_∞ metric.

If $p, q \in P$, a set of points W stabs all the squares whose boundary contains p and q if and only if pq is not an edge of $\mathrm{SG}^-(P, W)$. Hence we see that

$$st_{\mathcal{S}}(n) = \max_{|P|=n} st_{\mathcal{S}}(P) = \max_{|P|=n} \min\bigl\{|W|:\ \mathrm{SG}^-(P, W) = \emptyset\bigr\}.$$

The extrema are taken over pairs of sets (P, W) so that $P \cup W$ is in general position, i.e., with no two distinct points on the same vertical or the same horizontal line.

Lemma 7. *There is a set P of n points, no two of them with equal abscissa or ordinate, that admits a set of $\frac{5}{4}n - \Theta(\sqrt{n})$ squares with pairwise disjoint interiors, each with two points of P on its boundary.*

[3] After this paper was submitted, improvements on our upper bounds were claimed in a contribution to a recent conference [4]: the upper bound in Theorem 7 is improved from $2n-2$ to $3n/2$, and the upper bound in Proposition 2 from $4n/3$ to $5n/4$.

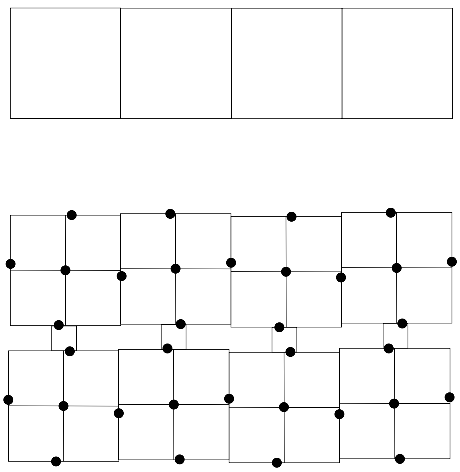

Fig. 10. Top: Initial row of basic squares. Bottom: Two rows of basic squares, after perturbation, subdivision, and point insertion, are connected by smaller squares.

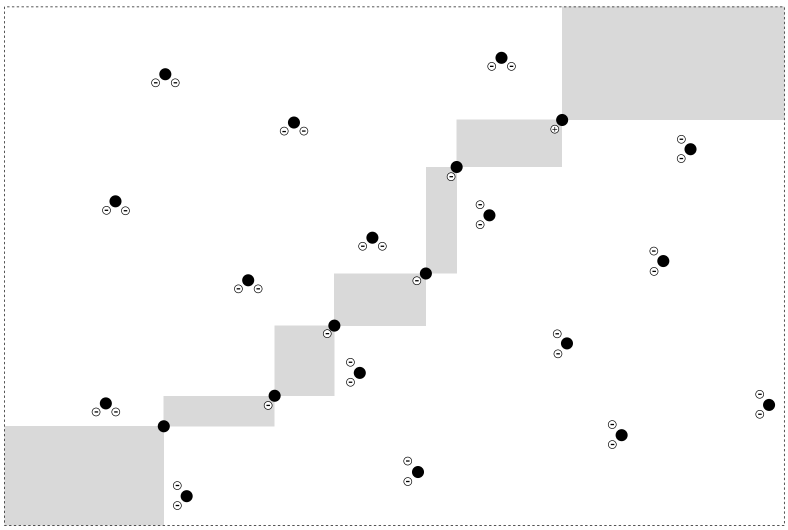

Fig. 11. Illustration for the proof of Theorem 8.

Proof. Consider a horizontal row of t equal size basic squares each sharing vertical sides with its neighbors (Fig. 10, top). We apply a different infinitesimal vertical translation to each square, and then subdivide it into four equal squares; one point is placed at the center and four other points very close to the midpoints of the initial square edges, as shown in Fig. 10. The inserted points are shown in solid, and the union of all of them will form the desired set P.

We place t copies of this construction nearly covertically, but applying different slight horizontal shifts to each row, ensuring that no two points of P get equal x or y coordinates. Any two consecutive rows are at distance slightly smaller than half the side of the original basic square, and we place t connecting squares between the two rows, each touching two points of P, as in the figure.

The point set P constructed in this way has a total of $n = 4t^2 + t$ points and admits a set of $5t^2 - t = \frac{5}{4}n - \Theta(\sqrt{n})$ squares with pairwise disjoint interiors, each one with two points from P on its boundary. □

Theorem 8. *The function* $st_{\mathcal{S}}(n)$ *satisfies* $\frac{5}{4}n - \Theta(\sqrt{n}) \leqslant st_{\mathcal{S}}(n) \leqslant 2n - \Theta(\sqrt{n})$.

Proof. The lower bound follows from the preceding lemma. We show that $2n - \Theta(\sqrt{n})$ witness stabbing points are always sufficient. Notice that a square containing two points always contains the rectangle they define as opposite corners: We prove a stronger claim, namely, that $2n - \Theta(\sqrt{n})$ points are always sufficient for stabbing the rectangles such that two opposite corners belong to a given set P of n points. Using Dillworth's theorem for partially ordered sets (or Erdős–Székeres theorem for sequences) we get a maximal subset P' of P of at least $\sqrt{n}$ points with increasing x, such that their ordinates

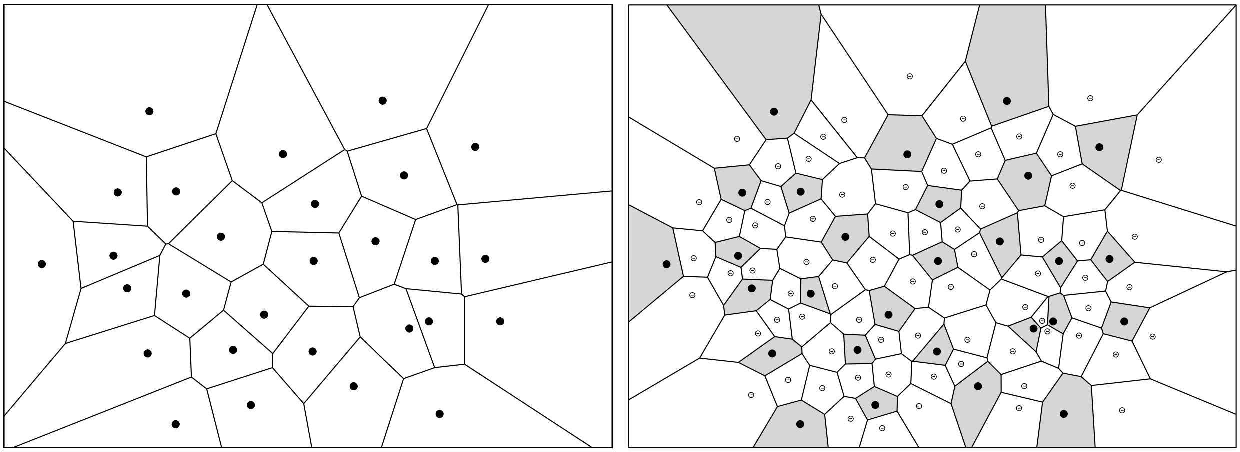

Fig. 12. The witnesses (right) prevent original points (left) from being Voronoi neighbors with the metric L_2.

strictly decrease or strictly increase; we assume the latter without loss of generality. Consider the boxes that have as opposite corners consecutive points in this sequence (adding points $(-\infty, -\infty)$ and $(+\infty, +\infty)$). The interiors of these boxes, shown shaded in Fig. 11, cannot contain any other point from P because of the maximality of P'.

Let ε_x and ε_y be the minimum gap between the x-coordinates and the y-coordinates of the points in P, respectively, and define $\varepsilon := \min\{\varepsilon_x, \varepsilon_y\}/3$, which is by assumption a positive number.

We put a witness inside every finite shaded box, namely at position $(x - \varepsilon, y - \varepsilon)$, if (x, y) is the upper right corner of the box. For every point $(x, y) \in P \setminus P'$ in the upper bay we put witnesses in its relative third and fourth quadrant, at positions $(x - \varepsilon, y - \varepsilon)$ and $(x + \varepsilon, y - \varepsilon)$. Finally, for every point $(x, y) \in P \setminus P'$ in the lower bay we put witnesses in its relative second and third quadrant, at positions $(x - \varepsilon, y + \varepsilon)$ and $(x - \varepsilon, y - \varepsilon)$ (see Fig. 11). In this way any rectangle with two opposite corners in P is stabbed, and the total number of used witnesses is at most $2n - \sqrt{n}$. $\square$

4.3. Voronoi discrimination

Given a set P of n "black" points in the plane, how many "white" points are needed in the worst case to completely separate the Voronoi regions of the black points from each other? Observe that this problem is precisely the one we have been considering throughout this section, as can be formulated in terms of finding a set W of witnesses (the white points) such that $\mathrm{DG}^-(P, W) = \emptyset$, with the Euclidean metric (Fig. 12), or that $\mathrm{SG}^-(P, W) = \emptyset$, when the L_∞ metric is considered (Fig. 13).

This interesting discrimination problem seems potentially useful in several applications. However, to the best of our knowledge, this problem had not been explored before, either from the combinatorial viewpoint, or from the viewpoint of computation. Various related problems without satisfactory solutions exist as well, for example, finding placements for points such that their Voronoi regions will cover maximal area [17], delineating boundaries [46], or competing for area as modeled by a two-players game [3].

5. Concluding remarks

We have introduced in this paper the generic concept of witness graphs and described several properties and computation algorithms for two specific examples, one with negative witnesses and Euclidean metric balls as interaction regions, another one using isothetic squares, the L_∞ balls, and positive witnesses.

Several open problems remain. In particular, we have characterized some graphs that can be realized as witness Delaunay graphs, and some others that cannot. A complete combinatorial characterization would certainly be desirable. Closing the gaps between the bounds in Theorem 7 and Theorem 8 on the maximum number of witnesses needed to eliminate all edges in a witness Delaunay graph, and a square graph, respectively, also seem to us interesting problems on the combinatorial side.

As for algorithms, it can be easily proved that designing an *output-sensitive* algorithm for constructing a witness Delaunay graph with k edges has a lower bound complexity $\Omega(k + n \log n)$, given its set of n vertices and witnesses, yet the most efficient algorithm we have found has running time, $O(k \log n + n \log^2 n)$, hence there is still a complexity gap to be resolved.

However, we consider that the most prominent issue in this regard is that we have not obtained any complexity results on computing an optimal discriminating set of witnesses for a given point set, i.e., given a set P, find a minimum set W such that no two Voronoi regions of points from P are adjacent in the Voronoi diagram $\mathrm{VD}(P \cup W)$, which we know is equivalent to having $\mathrm{DG}^-(P, W) = \emptyset$, for the Euclidean metric, and $\mathrm{SG}^-(P, W) = \emptyset$, for L_∞. From practical point of view,

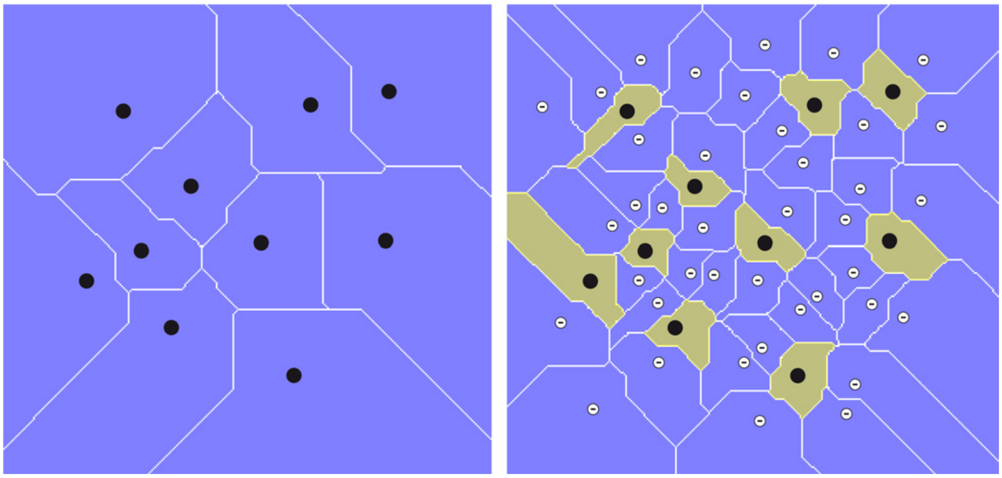

Fig. 13. The witnesses (right) prevent original points (left) from being Voronoi neighbors with the metric L_∞.

the question of computing efficiently a small (i.e., approximating the smallest-size one) discriminating set seems possibly the most relevant.

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