J. Math. Anal. Appl. 349 (2009) 10–20

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Existence and blow-up of solution of Cauchy problem for the generalized damped multidimensional Boussinesq equation

Necat Polat *, Abdulkadir Ertaş

Dicle University, Department of Mathematics, 21280 Diyarbakir, Turkey

ARTICLE INFO

Article history:
Received 21 April 2008
Available online 22 August 2008
Submitted by W. Layton

Keywords:
Boussinesq equation
Double dispersion equation
Cauchy problem
Local solution
Global solution
Blow-up of solution
Damping term

ABSTRACT

We consider the existence, both locally and globally in time, and the blow-up of solutions for the Cauchy problem of the generalized damped multidimensional Boussinesq equation.

1. Introduction

In this paper, we study the Cauchy problem of the generalized damped multidimensional Boussinesq equation with double dispersive term

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u - k\Delta u_t = \Delta f(u), \qquad (x,t) \in R^n \times (0, +\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in R^n, \tag{1.2}$$

where $u(x,t)$ denotes the unknown function, $f(s)$ is the given nonlinear function, $u_0(x)$ and $u_1(x)$ are the given initial value functions, k is a constant, the subscript t indicates the partial derivative with respect to t, n is the dimension of space variable x, and Δ denotes the Laplace operator in R^n.

Scott Russell's study [18] of solitary water waves motivated the development of nonlinear partial differential equations for the modeling wave phenomena in fluids, plasmas, elastic bodies, etc. It is well known that Boussinesq equation can be written in two basic forms

$$u_{tt} - u_{xx} + \delta u_{xxxx} = \left(u^2\right)_{xx}, \tag{1.3}$$

$$u_{tt} - u_{xx} - u_{xxtt} = \left(u^2\right)_{xx}. \tag{1.4}$$

Eq. (1.4) is an important model that approximately describes the propagation of long waves on shallow water like the other Boussinesq equations (with u_{xxxx}, instead of u_{xxtt}). In the case $\delta > 0$ Eq. (1.3) is linearly stable and governs small nonlinear

* Corresponding author.
E-mail address: npolat@dicle.edu.tr (N. Polat).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.08.025

transverse oscillations of an elastic beam (see [23] and references therein). It is called the “good” Boussinesq equation, while this equation with $\delta < 0$ received the name of the “bad” Boussinesq equation since it possesses the linear instability. Eq. (1.3) was first deduced by Boussinesq [1]. Eq. (1.4) is called improved Boussinesq equation (IBq equation).

There is a considerable mathematical interest in the Boussinesq equations which have been studied from various aspects (see [7,8,21,22] and references therein). A great deal of efforts has been made to establish sufficient conditions for the nonexistence of global solutions to various associated boundary value problems [10,21].

Levine and Sleeman [10] studied the global nonexistence of solutions for the equation

$$u_{tt} - u_{xx} - 3u_{xxxx} + 12\big(u^2\big)_{xx} = 0,$$

with periodic boundary conditions. Turitsyn [21] proved the blow-up in the Boussinesq equations

$$u_{tt} - u_{xx} + u_{xxxx} + \big(u^2\big)_{xx} = 0$$

and

$$u_{tt} - u_{xx} - u_{xxtt} + \big(u^2\big)_{xx} = 0$$

for the case of periodic boundary conditions and obtained exact sufficient criteria of the collapse dynamics.

The generalization of Boussinesq equation was studied in numerous papers [3,4,6,9,12,13,15,16,27,28]. Liu [12,13] studied the instability of solitary waves for a generalized Boussinesq type equation

$$u_{tt} - u_{xx} + \big(f(u) + u_{xx}\big)_{xx} = 0,$$

and studied existence, both locally and globally in time, and established some blow-up result for a nonlinear Pochhammer–Chree equation

$$u_{tt} - u_{xxtt} - f(u)_{xx} = 0. \tag{1.5}$$

Godefroy [3] showed the blow-up of the solutions of Cauchy problem for Eq. (1.5) and he focused on various perturbation of the equation. Guowang and Shubin [4] proved the existence and nonexistence of global solution for the generalized IMBq equation

$$u_{tt} - u_{xx} - u_{xxtt} = f(u)_{xx}.$$

Zhijian [27], Yang and Wang [28] studied respectively the existence and blow-up of solutions to the initial–boundary value problems for the generalized Boussinesq equations

$$u_{tt} - u_{xx} - bu_{xxxx} = \sigma(u)_{xx}$$

and

$$u_{tt} - u_{xx} - u_{xxtt} = \sigma(u)_{xx}.$$

Makhankov [14] pointed out that the IBq equation

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta\big(u^2\big)$$

can be obtained by starting with the exact hydro-dynamical set of equations in plasma, and a modification of the IBq equation analogous to the modified Korteweg–de Vries equation yields

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta\big(u^3\big). \tag{1.6}$$

Eq. (1.6) is the so-called IMBq (modified IBq) equation.

Wang and Chen [24,25] studied the existence, both locally and globally in time, and nonexistence of solution, and the global existence of small amplitude solution for the Cauchy problem of the multidimensional generalized IMBq equation

$$u_{tt} - \Delta u - \Delta u_{tt} = \Delta f(u). \tag{1.7}$$

In the Boussinesq equations, the effects of small nonlinearity and dispersion are taken into consideration, but in many real situations, damping effects are compared in strength to the nonlinear and dispersive one. Therefore the damped Boussinesq equations is considered as well

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta\big(u^2\big)_{xx}, \tag{1.8}$$

where u_{txx} is the damping term and $\alpha, b = \text{const} > 0$, $\beta = \text{const} \in \mathrm{R}$ (see [7,8,11,20,21] and references therein).

Varlamov [22,23] investigated the long-time behavior of solutions to initial value, spatially periodic, and initial–boundary value problems for Eq. (1.8) in two space dimensions. Polat et al. [15] established the blow-up of the solutions for the initial–boundary value problem of the damped Boussinesq equation

$$u_{tt} - bu_{xx} + \delta u_{xxxx} - ru_{xxt} = f(u)_{xx}.$$

Lai and Wu [7], Lai et al. [8] investigated respectively the global solution of the following generalized damped Boussinesq equations

$$u_{tt} - au_{ttxx} - 2bu_{txx} = -cu_{xxxx} + u_{xx} - p^2 u + \beta\left(u^2\right)_{xx} \tag{1.9}$$

and

$$u_{tt} - au_{ttxx} - 2bu_{txx} = -cu_{xxxx} + u_{xx} + \beta\left(u^2\right)_{xx}.$$

Polat and Kaya [16] established the blow-up of the solutions for the initial–boundary value problem of Eq. (1.9).

Polat [17] extended the result of [24] to the damped version of the problem (1.7).

Wang and Chen [26] studied the existence and blow-up of the solution for the Cauchy problem of the generalized double dispersion equation

$$u_{tt} - u_{xx} - u_{xxtt} + u_{xxxx} - \alpha u_{xxt} = g(u)_{xx}. \tag{1.10}$$

As seen from above explanations, the works on the multidimensional Boussinesq type equation relatively scarce in contrast to the vast literature on similar results for one dimensional case.

The results in this paper are a development of results obtained in [8,17,26]. First, by using the contraction mapping principle, we establish the locally well posedness of the Cauchy problem. Then we derive the necessary a priori bounds that guarantee that every local solution is indeed global in time. Finally, we discuss that the local solution of the Cauchy problem with negative and nonnegative initial energy blows up in finite time by using the concavity method.

Throughout this paper, we use the following notations and lemmas.

L^p denotes the usual space of all L^p functions on R^n with norm $\|f\|_{L^p} = \|f\|_p$, H^s denotes the usual Sobolev space on R^n with norm $\|f\|_{H^s} = \|(I-\Delta)^{s/2} f\|_2$, where $1 \leqslant p \leqslant \infty$, $s \in \mathrm{R}$.

Lemma 1.1 *(Sobolev's Lemma). (See [19].) If $s > k + \frac{n}{2}$, where k is a nonnegative integer, then*

$$H^s\left(\mathrm{R}^n\right) \subset C^k\left(\mathrm{R}^n\right) \cap L^\infty\left(\mathrm{R}^n\right),$$

where the inclusion is continuous. In fact,

$$\sum_{|\alpha| \leqslant k} \left\|\partial^\alpha u\right\|_{L^\infty} \leqslant C_s \|u\|_{H^s},$$

where C_s is independent of u.

Lemma 1.2. *(See [5].) Let $q \in [1, n]$ and $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$, then for any $u \in H_1^q(\mathrm{R}^n)$,*

$$\|u\|_p \leqslant C(n,q)\|\nabla u\|_q.$$

where $C(n,q)$ is a constant dependent on n and q.

Lemma 1.3. *(See [25].) Assume that $f(u) \in C^k(\mathrm{R})$, $f(0) = 0$, $u \in W^{s,2} \cap L^\infty$ and $k = [s] + 1$, where $s \geqslant 0$. Then we have*

$$\left\|f(u)\right\|_{H^s} \leqslant K_1(M)\|u\|_{H^s}$$

if $\|u\|_\infty \leqslant M$, where $K_1(M)$ is a constant dependent on M.

Lemma 1.4. *(See [25].) Assume that $f(u) \in C^k(\mathrm{R})$, $u, v \in W^{s,2} \cap L^\infty$ and $k = [s] + 1$, where $s \geqslant 0$. Then*

$$\left\|f(u) - f(v)\right\|_{H^s} \leqslant K_2(M)\|u - v\|_{H^s}$$

if $\|u\|_\infty \leqslant M$, $\|v\|_\infty \leqslant M$, where $K_2(M)$ is a constant dependent on M.

Lemma 1.5 *(Minkowski's inequality for integrals). (See [2].) If $1 \leqslant p \leqslant \infty$, $u(x,t) \in L^p(\mathrm{R}^n)$ for a.e. t, and function $t \to \|u(\cdot,t)\|_p$ is in $L^1(I)$, where $I \subset [0,\infty)$ is an interval, then*

$$\left\|\int_I u(\cdot,t)\,dt\right\|_p \leqslant \int_I \left\|u(\cdot,t)\right\|_p dt.$$

The plan of this paper is as follows. In Section 2, we study the existence and uniqueness of the local solutions for problem (1.1), (1.2). The global well posedness of the problem is given in Section 3. In Section 4, we discuss the blow-up of solution to the problem.

2. Existence and uniqueness of local solution

In this section, we prove the existence and the uniqueness of the local solution for problem (1.1), (1.2) by the contraction mapping principle. For this, we construct the solution of the problem as a fixed point of the solution operator associated with related family of Cauchy problems for linear wave equation.

For this purpose, we can rewrite Eq. (1.1) as follows:

$$u_{tt} - \Delta u = L\big[f(u) + ku_t\big], \tag{2.1}$$

where $L = (I - \Delta)^{-1}\Delta$. By use of the Fourier transform, it is not difficult to check that

$$Lf = \Delta(G * f) = G * f - f,$$

where $G(x) = \frac{1}{2}e^{-|x|}$, and $u * v$ denotes the convolution of u and v, it is defined by

$$u * v = \int_{\mathrm{R}^n} u(y)v(x-y)\,dy.$$

Now, we proceed with the following linear wave equation:

$$u_{tt} - \Delta u = h(x,t), \quad x \in \mathrm{R}^n,\ t > 0 \tag{2.2}$$

with the initial value condition (1.2). Let us give some results which will be used in the following.

Lemma 2.1. *(See [19].) Let $s \in \mathrm{R}$. Let $u_0 \in H^s$, $u_1 \in H^{s-1}$ and $h(x,t) \in L^1([0,T], H^{s-1})$. Then for every $T > 0$, there is a unique solution $u \in C([0,T], H^s) \cap C^1([0,T], H^{s-1})$ of Cauchy problem of (2.2) and (1.2). Moreover, u satisfies*

$$\big\|u(t)\big\|_{H^s} + \big\|u_t(t)\big\|_{H^{s-1}} \leqslant C(1+T)\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \int_0^t \big\|h(\tau)\big\|_{H^{s-1}}\,d\tau\right)$$

for all $0 \leqslant t \leqslant T$, where C only depends on s.

Lemma 2.2. *The operator L is bounded on H^s for all $s \geqslant 0$ and*

$$\|Lu\|_{H^s} \leqslant \|u\|_{H^s}, \quad \forall u \in H^s.$$

Proof. For $u \in H^s$, $s \geqslant 0$, we have

$$\|Lu\|^2_{H^s} = \int_{\mathrm{R}^n} \big(1+\xi^2\big)^s \frac{\xi^4}{(1+\xi^2)^2} \big|\hat{u}(\xi)\big|^2\,d\xi \leqslant \|u\|^2_{H^s}.$$

Let us define the function space

$$X(T) = C\big([0,T], H^s\big) \cap C^1\big([0,T], H^{s-1}\big)$$

which equipped with the norm defined by

$$\|u\|_{X(T)} = \max_{0 \leqslant t \leqslant T}\big(\big\|u(t)\big\|_{H^s} + \big\|u_t(t)\big\|_{H^{s-1}}\big), \quad \forall u \in X(T).$$

It is easy to see that $X(T)$ is a Banach space. For $s > \frac{n}{2}$, and any initial values $u_0 \in H^s$, $u_1 \in H^{s-1}$, let $A = \|u_0\|_{H^s} + \|u_1\|_{H^s}$. Take the set

$$Y(A,T) = \big\{u \,\big|\, u \in X(T),\ \|u\|_{X(T)} \leqslant 2CA\big\}.$$

Obviously, $Y(A,T)$ is a nonempty bounded closed convex subset of $X(T)$ for any fixed $M > 0$ and $T > 0$.

From Lemma 1.1, $u \in C([0,T], L^\infty)$ and $\|u\|_{L^\infty} \leqslant C_s\|u\|_{H^s}$ if $u \in X(T)$.

For $w \in Y(A,T)$, we consider the linear wave equation

$$u_{tt} - \Delta u = L\big[f(w) + kw_t\big], \tag{2.3}$$

and we let H denote the map which carries w into the unique solution (2.3), (1.2). Our object is to show that H has a unique fixed point in $Y(A,T)$ for appropriately chosen T. For this purpose we shall employ the contraction mapping principle and Lemma 2.1. Firstly, we prove the following lemma. □

Lemma 2.3. *Assume that $s > \frac{n}{2}$, $u_0 \in H^s$, $u_1 \in H^{s-1}$ and $f(s) \in C^{[s]+1}(\mathrm{R})$. Then H is contractive mapping from $Y(A,T)$ into itself for T sufficiently small relative to M.*

Proof. We first prove that H maps $Y(M,T)$ into itself for T small enough. Let $w \in Y(A,T)$ be given. Let us define $h(x,t)$ by

$$h(x,t) = L\big[f(w) + kw_t\big].$$

By use of Lemma 1.3 and 2.2, it is easily obtained that

$$\big\|h(t)\big\|_{H^{s-1}} \leqslant C_s\big\|f(w)\big\|_{H^s} + |k|\|w_t\|_{H^{s-1}} \leqslant K_1(A,s)\|w\|_{H^s} + |k|\|w_t\|_{H^{s-1}},$$

where $K_1(A,s)$ is a constant dependent on A and s. From the above inequality we conclude that $h(x,t) \in L^1([0,T],H^{s-1})$. From Lemma 2.1 the solution $u = Hw$ of problem (2.3), (1.2) belongs to $C([0,T],H^s) \cap C^1([0,T],H^{s-1})$ and

$$\big\|u(t)\big\|_{H^s} + \big\|u_t(t)\big\|_{H^{s-1}} \leqslant C(1+T)\left(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}} + \int_0^t \big\|h(\tau)\big\|_{H^{s-1}}\,d\tau\right)$$
$$\leqslant CA + C\big[1 + 2C\big(K_1(A,s) + |k|\big)(1+T)\big]AT.$$

By choosing T small enough, in order to have

$$\big[1 + 2C\big(K_1(A,s) + |k|\big)(1+T)\big]T \leqslant 1, \tag{2.4}$$

then we get

$$\|Hw\|_{X(T)} \leqslant 2CA. \tag{2.5}$$

Therefore, if condition (2.5) holds, then H maps $Y(A,T)$ into $Y(A,T)$.

Now we are going to prove that the map H is strictly contractive. Let $T > 0$ and $w, \bar{w} \in Y(A,T)$ be given. For w and $\bar{w}$ there are the corresponding solutions $u = Hw$ and $\bar{u} = H\bar{w}$ for problem (2.3), (1.2). Set $U = u - \bar{u}$, $W = w - \bar{w}$, and note that

$$U_{tt} - \Delta U = H(x,t), \quad (x,t) \in \mathrm{R}^n \times (0,+\infty), \tag{2.6}$$

$$U(x,0) = U_t(x,0) = 0, \tag{2.7}$$

where $H(x,t)$ is defined by

$$H(x,t) = L\big[f(w) - f(\bar{w})\big] + kL[W_t]. \tag{2.8}$$

It is observed that H has the smoothness required to apply Lemma 2.1 to (2.6), (2.7). Making use of Lemmas 2.1, 2.2 and 1.4, we get from (2.8) that

$$\big\|U(t)\big\|_{H^s} + \big\|U_t(t)\big\|_{H^{s-1}} \leqslant C(1+T)\int_0^t \big\|f\big(w(\tau)\big) - f\big(\bar{w}(\tau)\big)\big\|_{H^{s-1}} + |k|\|W_t\|_{H^{s-1}}\,d\tau$$
$$\leqslant C(1+T)\Big[K_2(A,s)\max_{0\leqslant t\leqslant T}\big\|W(t)\big\|_{H^s} + |k|\max_{0\leqslant t\leqslant T}\big\|W_t(t)\big\|_{H^{s-1}}\Big]T.$$

Thus, we have

$$\big\|U(t)\big\|_{X(T)} \leqslant C(1+T)\big[K_2(A,s) + |k|\big]T\big\|W(t)\big\|_{X(T)}.$$

By choosing T so small enough that (2.4) holds and

$$(1+T)\big[K_2(A,s) + |k|\big]T < \frac{1}{C}, \tag{2.9}$$

then

$$\|Hw - H\bar{w}\|_{X(T)} < \|w - \bar{w}\|_{X(T)}.$$

This shows that $H : Y(A,T) \to Y(A,T)$ is strictly contractive. The lemma is proved. □

Theorem 2.1. *Assume that the conditions of Lemma 2.3 hold, then problem (1.1), (1.2) admits a unique local solution $u(x,t)$ defined on a maximal time interval $[0,T_0)$ with $u(x,t) \in C([0,T_0),H^s) \cap C^1([0,T_0),H^{s-1})$. Moreover, if*

$$\sup_{t\in[0,T_0)}\big(\big\|u(t)\big\|_{H^s} + \big\|u_t(t)\big\|_{H^{s-1}}\big) < \infty, \tag{2.10}$$

then $T_0 = \infty$.

Proof. From Lemma 2.3 and the contraction mapping principle, it follows that for appropriately chosen $T > 0$, H has a unique fixed point $u(x,t) \in Y(A,T)$, which is a strong solution of problem (1.1), (1.2). It is not difficult to prove the uniqueness of the solution which belongs to $X(T')$ for each $T' > 0$.

In fact, let $u_1, u_2 \in X(T')$ be two solutions of problem (1.1), (1.2). Let $u = u_1 - u_2$; then

$$u_{tt} - \Delta u - \Delta u_{tt} + \Delta^2 u - k\Delta u_t = \Delta\big[f(u_1) - f(u_2)\big].$$

Multiplying the above equation by $(-\Delta)^{-1}u_t$ and integrating the product with respect to x, we obtain that

$$\frac{1}{2}\frac{d}{dt}\big[\big\|(-\Delta)^{-1/2}u_t\big\|_2^2 + \|u\|_2^2 + \|u_t\|_2^2 + \|\nabla u\|_2^2\big] + k\|u_t\|_2^2 = \int_{\mathrm{R}^n}\big[f(u_1) - f(u_2)\big]u_t\,dx. \tag{2.11}$$

From the definition of the space $X(T')$, $s > \frac{n}{2}$ and Sobolev imbedding theorem we have $\|u_i(t)\|_\infty \leqslant C_1(T')$ for $i = 1, 2$ and $0 \leqslant t \leqslant T' < T$, where $C_1(T')$ is a constant dependent on T'. Thus, we get from Cauchy inequality that

$$\left|\int_{\mathrm{R}^n}\big[f(u_1) - f(u_2)\big]u_t\,dx\right| \leqslant \big\|f(u_1) - f(u_2)\big\|_2\|u_t\|_2 \leqslant C_2(T')\|u\|_2\|u_t\|_2,$$

where $C_2(T')$ is a constant dependent on $C_1(T')$. From Young inequality it follows that

$$\big[\big\|(-\Delta)^{-1/2}u_t\big\|_2^2 + \|u\|_2^2 + \|u_t\|_2^2 + \|\nabla u\|_2^2\big] + k\int_0^t \|u_\tau\|_2^2\,d\tau \leqslant C_2(T')\int_0^t \big[\|u\|_2^2 + \|u_t\|_2^2\big]\,d\tau.$$

From the above inequality we have

$$\|u\|_2^2 + \|u_t\|_2^2 \leqslant \big[C_2(T') + 2|k|\big]\int_0^t \big[\|u\|_2^2 + \|u_t\|_2^2\big]\,d\tau. \tag{2.12}$$

By Gronwall's inequality, we get from (2.12) that $\|u\|_2^2 + \|u_t\|_2^2 \equiv 0$ for $0 \leqslant t \leqslant T'$. Hence $u \equiv 0$ for $0 \leqslant t \leqslant T'$, i.e., problem (1.1), (1.2) has at most one solution which belongs to $X(T')$.

Now, let $[0, T_0)$ be the maximal time interval of existence for $u \in X(T_0)$. We want to show that if (2.10) is satisfied, then $T_0 = \infty$.

Suppose that (2.10) holds and $T_0 < \infty$. For each $T' \in [0, T_0)$, we consider the Cauchy problem

$$v_{tt} - \Delta v = L\big[f(v) + kv_t\big], \tag{2.13}$$

$$v(x,0) = u(x,T'), \qquad v_t(x,0) = u_t(x,T'). \tag{2.14}$$

By (2.10),

$$\big\|u(\cdot,t)\big\|_{2,p} + \big\|u_t(\cdot,t)\big\|_{2,p} + \big\|u(\cdot,t)\big\|_\infty + \big\|u_t(\cdot,t)\big\|_\infty \leqslant K,$$

where K is a positive constant independent of $T' \in [0, T_0)$. From Lemma 2.3 and the contraction mapping principle we see that there exists a constant $T_1 \in (0, T_0)$ such that for each $T' \in [0, T_0)$, problem (2.13), (2.14) has a unique solution $v(x,t) \in X(T_1)$. In particular, (2.4) and (2.9) reveal that T_1 can be selected independently of $T' \in [0, T_0)$. Take $T' = T_0 - T_1/2$ and define

$$\tilde{u}(x,t) = \begin{cases} u(x,t), & t \in [0, T'], \\ v(x, t - T'), & t \in [T', T_0 + T_1/2], \end{cases}$$

then $\tilde{u}(x,t)$ is a solution of Eqs. (1.1), (1.2) on interval $[0, T_0 + T_1/2]$, and by the uniqueness, $\tilde{u}$ extends u, which violates the maximality of $[0, T_0)$. Therefore, if (2.10) holds, then $T_0 = \infty$. Theorem 2.1 is proved. □

3. Existence and uniqueness of global solution

In this section, we prove the existence and the uniqueness of the global solutions for problem (1.1). (1.2). For this purpose we are going to make a priori estimates of the local solutions for problem (1.1), (1.2).

Lemma 3.1. *Suppose that $f(u) \in C(\mathrm{R})$, $F(u) = \int_0^u f(s)\,ds$, $u_0 \in H^1$, $(-\Delta)^{-1/2}u_1 \in L^2$, $u_1 \in L^2$, and $F(u_0) \in L^1$. Then for the solution $u(x,t)$ of problem* (1.1), (1.2), *we have the energy identity*

$$E(t) = \big\|(-\Delta)^{-1/2}u_t\big\|_2^2 + \|u_t\|_2^2 + \|u\|_2^2 + \|\nabla u\|_2^2 + 2k\int_0^t \|u_\tau\|_2^2\,d\tau + 2\int_{\mathrm{R}^n} F(u)\,dx = E(0), \tag{3.1}$$

here and in the sequel $(-\Delta)^{-\alpha}u(x)=\mathcal{F}^{-1}[|x|^{-2\alpha}\mathcal{F}u(x)]$, $\mathcal{F}$ *and* $\mathcal{F}^{-1}$ *denote respectively Fourier transformation and inverse Fourier transformation in* R^n (*see* [20]).

Proof. Multiplying Eq. (1.1) by $(-\Delta)^{-1}u_t$ and integrating the product with respect to x, we obtain that

$$\big(u_{tt}-\Delta u-\Delta u_{tt}+\Delta^2 u-k\Delta u_t-\Delta f(u),(-\Delta)^{-1}u_t\big)=0,$$
$$\big((-\Delta)^{-1}u_{tt}+u+u_{tt}-\Delta u+ku_t+f(u),u_t\big)=0,$$
$$\big((-\Delta)^{-1/2}u_{tt},(-\Delta)^{-1/2}u_t\big)+(u,u_t)+(u_{tt},u_t)-(\Delta u,u_t)+k(u_t,u_t)+\big(f(u),u_t\big)=0,$$
$$\frac{1}{2}\frac{d}{dt}\bigg[\big\|(-\Delta)^{-1/2}u_t\big\|_2^2+\|u_t\|_2^2+\|u\|_2^2+\|\nabla u\|_2^2+2\int_{\mathrm{R}^n}F(u)\,dx\bigg]+k\|u_t\|_2^2=0,$$

where $(\cdot,\cdot)$ denotes the inner product of L^2 space. Integrating the above equality with respect to t over $[0,t]$, we get (3.1). The lemma is proved. □

Lemma 3.2. *Suppose that the assumptions of Lemma* 3.1 *hold and* $F(u)\geqslant 0$ *or* $f'(u)$ *is bounded below, i.e. there is a constant* A_0 *such that* $f'(u)\geqslant A_0$ *for any* $u\in\mathrm{R}$, *then the solution* $u(x,t)$ *of problem* (1.1), (1.2) *has the estimation*

$$E_1(t)=\big\|(-\Delta)^{-1/2}u_t\big\|_2^2+\|u_t\|_2^2+\|u\|_2^2+\|\nabla u\|_2^2\leqslant M_1(t),\quad \forall t\in[0,T], \tag{3.2}$$

here and in the sequel $M_i(T)$ $(i=1,2,\ldots)$ *are constants dependent on* T.

Proof. If $F(u)\geqslant 0$, then from energy identity (3.1) we get

$$E_1(t)\leqslant E(0)+2|k|\int_0^t\|u_\tau\|_2^2\,d\tau.$$

It follows from Gronwall's inequality and the above inequality that

$$E_1(t)\leqslant E(0)e^{2|k|T}. \tag{3.3}$$

If $f'(u)$ is bounded below, let $f_0(u)=f(u)-k_0u$, where $k_0=\min\{A_0,0\}$ $(\leqslant 0)$, then $f_0(0)=0$, $f_0'(u)=f'(u)-k_0\geqslant 0$ and $f_0(u)$ is a monotonically increasing function. Then $F_0(u)=\int_0^u f_0(s)\,ds\geqslant 0$ and $F(u)=\int_0^u f(s)\,ds=\int_0^u(f_0(s)+k_0s)\,ds=F_0(u)+\frac{k_0}{2}u^2$. From (3.1)

$$\begin{aligned}E_1(t)+2\int_{\mathrm{R}^n}F_0(u)\,dx&=E(0)-2k\int_0^t\|u_\tau\|_2^2\,d\tau-k_0\|u\|_2^2\\&=E(0)-2k\int_0^t\|u_\tau\|_2^2\,d\tau-k_0\|u_0\|_2^2+\int_0^t\big(k_0^2\|u\|_2^2+\|u_\tau\|_2^2\big)\,d\tau\\&\leqslant E(0)-k_0\|u_0\|_2^2+\big(2|k|+1+k_0^2\big)\int_0^t\big(\|u\|_2^2+\|u_\tau\|_2^2\big)\,d\tau.\end{aligned}$$

It follows from Gronwall's inequality and the above inequality that

$$E_1(t)\leqslant\big(E(0)-k_0\|u_0\|_2^2\big)\exp\big[\big(2|k|+1+k_0^2\big)T\big]. \tag{3.4}$$

We get (3.2) from inequalities (3.3) and (3.4). The lemma is proved. □

Lemma 3.3. *Under the conditions of Lemma* 3.2, *assume that* $1\leqslant n\leqslant 4$, $f(u)\in C^2(\mathrm{R})$ *and* $|f'(u)|\leqslant A|u|^\rho+B$, $0<\rho\leqslant\infty$ *for* $2\leqslant n\leqslant 4$, $u_0\in H^2$ *and* $u_1\in H^1$, *then the solution* $u(x,t)$ *of problem* (1.1), (1.2) *has the estimation*

$$E_2(t)=\|u_t\|_2^2+\|\nabla u\|_2^2+\|\nabla u_t\|_2^2+\|\Delta u\|_2^2\leqslant M_2(T),\quad\forall t\in[0,T]. \tag{3.5}$$

Proof. Multiplying Eq. (1.1) by u_t and integrating the product over R^n, we obtain that

$$\frac{d}{dt}E_2(t)+2k\|\nabla u_t\|_2^2+2\big(\nabla f(u),\nabla u_t\big)=0. \tag{3.6}$$

When $n=1$, we conclude from Lemmas 1.1 and 3.2 that $u\in L^\infty$. Therefore, from (3.6), Hölder inequality, Cauchy inequality, Lemma 1.3 and (3.2), we get

$$\begin{aligned}\frac{d}{dt}E_2(t) &\leqslant 2|k|\|\nabla u_t\|_2^2 + 2\big|\big(\nabla f(u), \nabla u_t\big)\big| \\ &\leqslant 2|k|\|\nabla u_t\|_2^2 + 2\big\|\nabla f(u)\big\|_2\|\nabla u_t\|_2 \\ &\leqslant 2|k|\|\nabla u_t\|_2^2 + 2K_1\big(\|u\|_\infty\big)\big(\|u\|_2 + \|\nabla u\|_2\big)\|\nabla u_t\|_2 \\ &\leqslant C_1\big(M_1(t)\big)\big(\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2\big), \end{aligned} \tag{3.7}$$

where, and in the sequel $C_i(M_j(t))$ $(i = 1, 2, \ldots,\ j = 1, 2, \ldots)$ are constants dependent on $M_j(t)$. Integrating (3.7) with respect to t and using the Gronwall's inequality, we obtain (3.5).

In the case $2 \leqslant n \leqslant 4$, from Hölder inequality, Lemma 1.2, Cauchy inequality and (3.2) we have

$$\begin{aligned}\int_{R^n} \nabla f(u)\nabla u_t\,dx &\leqslant A\|u^\rho\|_\infty\|\nabla u\|_2\|\nabla u_t\|_2 + B\|\nabla u\|_2\|\nabla u_t\|_2 \\ &\leqslant \frac{A}{2}\big(C^2\|\Delta u\|_2^2\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2\big) + \frac{B}{2}\big(\|\nabla u\|_2^2 + \|\nabla u_t\|_2^2\big) \\ &\leqslant \frac{A}{2}\big(C_2\big(M_1(t)\big)\|\Delta u\|_2^2 + \|\nabla u_t\|_2^2\big) + \frac{B}{2}\big(M_1(t) + \|\nabla u_t\|_2^2\big). \end{aligned}$$

Substitute the above inequality into (3.6) to get

$$\frac{d}{dt}E_2(t) \leqslant 2|k|\|\nabla u_t\|_2^2 + 2\big|\big(\nabla f(u), \nabla u_t\big)\big| \leqslant BM_1(t) + C_3\big(M_1(t)\big)\big(\|\Delta u\|_2^2 + \|\nabla u_t\|_2^2\big). \tag{3.8}$$

Integrating (3.8) with respect to t and using the Gronwall's inequality, we obtain (3.5). The lemma is proved. □

Lemma 3.4. *Under the conditions of Lemma* 3.3, *assume that* $s \geqslant 2$, $f(u) \in C^{[s]}(R)$, $u_0 \in H^s$, $u_1 \in H^{s-1}$, *then the solution* $u(x,t)$ *of problem* (1.1), (1.2) *has the estimation*

$$E_3(t) = \big\|\nabla^{s-2}u_t\big\|_2^2 + \big\|\nabla^{s-1}u\big\|_2^2 + \big\|\nabla^{s-1}u_t\big\|_2^2 + \big\|\nabla^s u\big\|_2^2 \leqslant M_3(T), \quad \forall t \in [0, T]. \tag{3.9}$$

Proof. Multiplying Eq. (1.1) by $\Delta^{s-2}u_t$ and integrating the product over R^n, we obtain that

$$\frac{d}{dt}E_3(t) + 2k\big\|\nabla^{s-1}u_t\big\|_2^2 + 2\big(\nabla^{s-1}f(u), \nabla^{s-1}u_t\big) = 0. \tag{3.10}$$

From Lemmas 1.2 and 3.3, we know that $u \in L^\infty$. we get from Hölder inequality, Cauchy inequality, Lemma 1.3 and (3.2) that

$$\begin{aligned}\frac{d}{dt}E_3(t) &\leqslant 2|k|\big\|\nabla^{s-1}u_t\big\|_2^2 + 2\big|\big(\nabla^{s-1}f(u), \nabla^{s-1}u_t\big)\big| \\ &\leqslant 2|k|\big\|\nabla^{s-1}u_t\big\|_2^2 + 2K_2\big(\|u\|_\infty\big)\big(\|u\|_2 + \|\nabla^{s-1}u\|_2\big)\big\|\nabla^{s-1}u_t\big\|_2 \\ &\leqslant C_4\big(M_1(t)\big)\big(\big\|\nabla^{s-1}u\big\|_2^2 + \big\|\nabla^{s-1}u_t\big\|_2^2\big). \end{aligned}$$

Integrating the above inequality with respect to t and using the Gronwall's inequality, we obtain (3.9). The lemma is proved. □

Theorem 3.1. *Assume that* $1 \leqslant n \leqslant 4$, $s \geqslant \frac{n+1}{2}$, $f(u) \in C^{[s]+1}(R)$, $F(u) = \int_0^u f(s)\,ds$, $F(u) \geqslant 0$ *or* $f'(u)$ *is bounded below, i.e. there is a constant* A_0 *such that* $f'(u) \geqslant A_0$ *for any* $u \in R$, $|f'(u)| \leqslant A|u|^\rho + B$, $0 < \rho \leqslant \infty$ *for* $2 \leqslant n \leqslant 4$, $(-\Delta)^{-1/2}u_1 \in L^2$, $u_0 \in H^{s+1}$ *and* $u_1 \in H^s$, $F(u_0) \in L^1$. *Then problem* (1.1), (1.2) *admits a unique global solution* $u(x,t) \in C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-1})$ *and* $(-\Delta)^{-1/2}u_t \in L^2$.

Proof. By virtue of Theorem 2.1, it is enough to show that $\sup_{t\in[0,T_0)}(\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-1}}) < \infty$. From Lemmas 3.2–3.4, we know that

$$\big\|u(t)\big\|_{H^s} + \big\|u_t(t)\big\|_{H^{s-1}} < M_4(T), \quad \forall t \in [0, T),$$

where $M_4(T)$ is a constant dependent on T. Therefore, we see from the above inequality that problem (1.1), (1.2) has a unique global solution $u(x,t) \in C([0,\infty), H^s) \cap C^1([0,\infty), H^{s-1})$ and $(-\Delta)^{-1/2}u_t \in L^2$. The theorem is proved. □

4. Blow-up of solution

In this section, we are going to consider the blow-up of the solution for problem (1.1), (1.2) by the concavity method. For this purpose, we give the following lemma [6] which is a generalization of Levine's result [9].

Lemma 4.1. *Suppose that a positive, twice differentiable function $\psi(t)$ satisfies on $t \geqslant 0$ the inequality*

$$\psi''(t)\psi(t) - (1+\upsilon)\big(\psi'(t)\big)^2 \geqslant -2M_1\psi(t)\psi'(t) - M_2\big(\psi(t)\big)^2$$

where $\upsilon > 0$ and $M_1, M_2 \geqslant 0$ are constants. If $\psi(0) > 0$, and $\psi'(0) > -\gamma_2\upsilon^{-1}\psi(0)$, and $M_1 + M_2 > 0$, then $\psi(t)$ tends to infinity as

$$t \to t_1 \leqslant t_2 = \frac{1}{2\sqrt{M_1^2 + \upsilon M_2}} \ln \frac{\gamma_1\psi(0) + \upsilon\psi'(0)}{\gamma_2\psi(0) + \upsilon\psi'(0)},$$

where $\gamma_{1,2} = -M_1 \mp \sqrt{M_1^2 + \upsilon M_2}$. If $\psi(0) > 0$ and $\psi'(0) > 0$, and $M_1 = M_2 = 0$, then $\psi(t) \to \infty$ as $t \to t_1 \leqslant t_2 = \psi(0)/\upsilon\psi'(0)$.

Theorem 4.1. *Assume that $k \geqslant 0$, $f(u) \in C(\mathrm{R})$, $u_0 \in H^1$, $u_1 \in L^2$, $(-\Delta)^{-1/2}u_0, (-\Delta)^{-1/2}u_1 \in L^2$, $F(u) = \int_0^u f(s)\,ds$, $F(u_0) \in L^1$, and there exists a constant $\alpha > 0$ such that*

$$f(u)u \leqslant (\alpha + k + 2)F(u) + \frac{\alpha}{2}u^2, \quad \forall u \in \mathrm{R}. \tag{4.1}$$

Then the solution $u(x,t)$ of problem (1.1), (1.2) *blows up in finite time if one of the following conditions is valid*:

(i) $E(0) = \|(-\Delta)^{-1/2}u_1\|_2^2 + \|u_1\|_2^2 + \|u_0\|_2^2 + \|\nabla u_0\|_2^2 + 2\int_{\mathrm{R}^n} F(u_0)\,dx < 0$,
(ii) $E(0) = 0$ *and* $((-\Delta)^{-1/2}u_0, (-\Delta)^{-1/2}u_1) + (u_0, u_1) > 0$,
(iii) $E(0) > 0$ *and* $((-\Delta)^{-1/2}u_0, (-\Delta)^{-1/2}u_1) + (u_0, u_1) > \sqrt{2\frac{4\alpha+k+2}{\alpha+2}E(0)(\|(-\Delta)^{-1/2}u_0\|_2^2 + \|u_0\|_2^2)}$.

Proof. Suppose that the maximal time of existence of the solution for problem (1.1), (1.2) is infinite. A contradiction will be obtained by Lemma 4.1. Let

$$\psi(t) = \big\|(-\Delta)^{-1/2}u\big\|_2^2 + \|u\|_2^2 + \beta(t+\tau)^2, \tag{4.2}$$

where β and τ are nonnegative constants to be specified later. Obviously we have

$$\psi'(t) = 2\big[\big((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}u_t\big) + (u, u_t) + \beta(t+\tau)\big]. \tag{4.3}$$

Using the Schwarz inequality and the inequality,

$$(a_1b_1 + \cdots + a_nb_n)^2 \leqslant \big(a_1^2 + \cdots + a_n^2\big)\big(b_1^2 + \cdots + b_n^2\big),$$

where $a_i, b_i \geqslant 0$, $i = 1, \ldots, n$ we have

$$\begin{aligned}\big(\psi'(t)\big)^2 &\leqslant 4\big[\big\|(-\Delta)^{-1/2}u\big\|_2^2 + \|u\|_2^2 + \beta(t+\tau)^2\big]\big[\big\|(-\Delta)^{-1/2}u_t\big\|_2^2 + \|u_t\|_2^2 + \beta\big] \\ &= 4\psi(t)\big[\big\|(-\Delta)^{-1/2}u_t\big\|_2^2 + \|u_t\|_2^2 + \beta\big].\end{aligned} \tag{4.4}$$

We get from Eq. (1.1)

$$\begin{aligned}\psi''(t) &= 2\big\|(-\Delta)^{-1/2}u_t\big\|_2^2 + 2\|u_t\|_2^2 + 2\big((-\Delta)^{-1/2}u, (-\Delta)^{-1/2}u_{tt}\big) + 2(u, u_{tt}) + 2\beta \\ &= 2\big\|(-\Delta)^{-1/2}u_t\big\|_2^2 + 2\|u_t\|_2^2 + 2\beta + 2\big(u, (-\Delta)^{-1}u_{tt} + u_{tt}\big) \\ &= 2\big\|(-\Delta)^{-1/2}u_t\big\|_2^2 + 2\|u_t\|_2^2 + 2\beta - 2\big(u, u - \Delta u + ku_t + f(u)\big) \\ &= 2\big\|(-\Delta)^{-1/2}u_t\big\|_2^2 + 2\|u_t\|_2^2 + 2\beta - 2\|u\|_2^2 - 2\|\nabla u\|_2^2 - 2k(u, u_t) - 2\int_{\mathrm{R}^n} uf(u)\,dx.\end{aligned} \tag{4.5}$$

By the aid of the Cauchy inequality and equality (3.1) we have

$$2k(u, u_t) \leqslant k\big(\|u\|_2^2 + \|u_t\|_2^2\big) = k\left[E(0) - \big\|(-\Delta)^{-1/2}u_t\big\|_2^2 - \|\nabla u\|_2^2 - 2k\int_0^t \|u_\tau\|_2^2\,d\tau - 2\int_{\mathrm{R}^n} F(u)\,dx\right]. \tag{4.6}$$

From (4.2)–(4.6) we obtain that

$$\begin{aligned}\psi(t)\psi''(t)-\left(1+\frac{\alpha}{4}\right)\left(\psi'(t)\right)^2 &\geqslant \psi(t)\psi''(t)-(4+\alpha)\psi(t)\left[\left\|(-\Delta)^{-1/2}u_t\right\|_2^2+\|u_t\|_2^2+\beta\right]\\ &\geqslant \psi(t)\left\{(k-\alpha-2)\left\|(-\Delta)^{-1/2}u_t\right\|_2^2+(-2-\alpha)\|u_t\|_2^2+(-4-\alpha)\beta+(k-2)\|\nabla u\|_2^2\right.\\ &\quad\left.+\int_{R^n}\left[2kF(u)-2uf(u)-2u^2\right]dx+2k^2\int_0^t\|u_\tau\|_2^2\,d\tau-kE(0)\right\}. \qquad (4.7)\end{aligned}$$

From equality (3.1) we have

$$\begin{aligned}(k-\alpha-2)\left\|(-\Delta)^{-1/2}u_t\right\|_2^2+(-2-\alpha)\|u_t\|_2^2+(k-2)\|\nabla u\|_2^2 &\geqslant (-\alpha-2)\left(\left\|(-\Delta)^{-1/2}u_t\right\|_2^2+\|u_t\|_2^2+\|\nabla u\|_2^2\right)\\ &=(\alpha+2)\left(\|u\|_2^2+2k\int_0^t\|u_\tau\|_2^2\,d\tau+2\int_{R^n}F(u)\,dx-E(0)\right).\end{aligned}$$

Thus, from the above inequality, inequality (4.7) and (4.1), we get

$$\begin{aligned}\psi(t)\psi''(t)-\left(1+\frac{\alpha}{4}\right)\left(\psi'(t)\right)^2 &\geqslant \psi(t)\left\{-(4+\alpha)\beta-(2+\alpha+k)E(0)\right.\\ &\quad\left.+\int_{R^n}\left[2(2+\alpha+k)F(u)+\alpha u^2-2uf(u)\right]dx+\left(2k(2+\alpha)+2k^2\right)\int_0^t\|u_\tau\|_2^2\,d\tau\right\}.\\ &\geqslant -\left[(4+\alpha)\beta+(2+\alpha+k)E(0)\right]\psi(t). \qquad (4.8)\end{aligned}$$

If $E(0)<0$, taking $\beta=-\frac{2+\alpha+k}{4+\alpha}E(0)>0$, then

$$\psi(t)\psi''(t)-\left(1+\frac{\alpha}{4}\right)\left(\psi'(t)\right)^2\geqslant 0.$$

We may now choose τ so large that $\psi'(0)>0$. From Lemma 4.1 we know that $\psi(t)$ becomes infinite at a time T_1 at most equal to

$$T_2=\frac{4\psi(0)}{\alpha\psi'(0)}<\infty.$$

If $E(0)=0$, taking $\beta=0$, then we get from (4.8)

$$\psi(t)\psi''(t)-\left(1+\frac{\alpha}{4}\right)\left(\psi'(t)\right)^2\geqslant 0.$$

Also $\psi'(0)>0$ by assumption (ii). Thus, we obtain from Lemma 4.1 that $\psi(t)$ becomes infinite at a time T_1 at most equal to

$$T_2=\frac{4\psi(0)}{\alpha\psi'(0)}<\infty.$$

If $E(0)>0$, then taking $\beta=0$, inequality (4.8) becomes

$$\psi(t)\psi''(t)-\left(1+\frac{\alpha}{4}\right)\left(\psi'(t)\right)^2\geqslant -(2+\alpha+k)E(0)\psi(t). \qquad (4.9)$$

Define $J(t)=(\psi(t))^{-\upsilon}$, where $\upsilon=\alpha/4$. Then

$$\begin{aligned}J'(t)&=-\upsilon\left(\psi(t)\right)^{-\upsilon-1}\psi'(t),\\ J''(t)&=-\upsilon\left(\psi(t)\right)^{-\upsilon-2}\left[\psi(t)\psi''(t)-(1+\upsilon)\left(\psi'(t)\right)^2\right]\leqslant \upsilon(2+k+4\upsilon)E(0)\left(\psi(t)\right)^{-\upsilon-1}, \qquad (4.10)\end{aligned}$$

where inequality (4.9) is used. Assumption (iii) implies $J'(0)<0$. Let

$$t^*=\sup\left\{t \mid J'(\tau)<0,\ \tau\in(0,t)\right\}. \qquad (4.11)$$

By the continuity of $J'(t), t^*$ is positive. Multiplying (4.10) by $2J'(t)$ yields

$$\left[\left(J'(t)\right)^2\right]' \geqslant -2\upsilon^2(2+k+4\upsilon)E(0)\left(\psi(t)\right)^{-2\upsilon-2}\psi'(t) = 2\upsilon^2\frac{(2+k+4\upsilon)}{2\upsilon+1}E(0)\left[\left(\psi(t)\right)^{-2\upsilon-1}\right]', \quad \forall t\in[0,t^*). \tag{4.12}$$

Integrate (4.12) with respect to t over $[0,t)$ to get

$$\begin{aligned}\left(J'(t)\right)^2 &\geqslant 2\upsilon^2\frac{(2+k+4\upsilon)}{2\upsilon+1}E(0)\left(\psi(t)\right)^{-2\upsilon-1} + \left(J'(0)\right)^2 - 2\upsilon^2\frac{(2+k+4\upsilon)}{2\upsilon+1}E(0)\left(\psi(0)\right)^{-2\upsilon-1}\\ &\geqslant \left(J'(0)\right)^2 - 2\upsilon^2\frac{(2+k+4\upsilon)}{2\upsilon+1}E(0)\left(\psi(0)\right)^{-2\upsilon-1}.\end{aligned}$$

By assumption (iii)

$$\left(J'(0)\right)^2 - 2\upsilon^2\frac{(3+4\upsilon)}{2\upsilon+1}E(0)\left(\psi(0)\right)^{-2\upsilon-1} > 0.$$

Hence by continuity of $J'(t)$, we obtain

$$J'(t) \leqslant -\left[\left(J'(0)\right)^2 - 2\upsilon^2\frac{(2+k+4\upsilon)}{2\upsilon+1}E(0)\left(\psi(0)\right)^{-2\upsilon-1}\right]^{1/2} \tag{4.13}$$

for $0 \leqslant t < t^*$. By the continuity of t^*, it follows that inequality (4.13) holds for all $t \geqslant 0$. Therefore

$$J(t) \leqslant J(0) - \left[\left(J'(0)\right)^2 - 2\upsilon^2\frac{(2+k+4\upsilon)}{2\upsilon+1}E(0)\left(\psi(0)\right)^{-2\upsilon-1}\right]^{1/2} t, \quad \forall t > 0.$$

So $J(T_1) = 0$ for some T_1 and

$$0 < T_1 \leqslant T_2 = J(0)/\left[\left(J'(0)\right)^2 - \left[\alpha^2(2+\alpha+k)/(4\alpha+8)\right]E(0)\left(\psi(0)\right)^{-(\alpha+2)/2}\right]^{1/2}.$$

Thus, $\psi(t)$ becomes infinite at a time T_1.

Therefore, $\psi(t)$ becomes infinite at a time T_1 under either assumptions (i), (ii) or (iii). We have a contradiction with the fact that the maximal time of existence is infinite. Hence the maximal time of existence is finite. This completes the proof. □

References

[1] J.V. Boussinesq, Théorie générale des mouvements qui sont propagés dans un canal rectangulaire horizontal, C. R. Acad. Sci. Paris 73 (1871) 256–260.
[2] G.B. Folland, Real Analysis, Modern Techniques and Their Applications, Wiley, New York, 1984.
[3] A.D. Godefroy, Blow up of solutions for the damped Boussinesq equation, IMA J. Appl. Math. 60 (1998) 123–138.
[4] C. Guowang, W. Shubin, Existence and nonexistence of global solutions for the generalized IMBq equation, Nonlinear Anal. 36 (1999) 961–980.
[5] E. Hebey, Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities, Amer. Math. Soc., 2000.
[6] V.K. Kalantarov, O.A. Ladyzhenskaya, The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type, J. Soviet Math. 10 (1978) 53–70.
[7] S. Lai, Y. Wu, The asymptotic solution of the Cauchy problem for a generalized Boussinesq equation, Discrete Contin. Dyn. Syst. Ser. B 3 (3) (2003) 401–408.
[8] S. Lai, Y. Wu, X. Yang, The global solution of an initial boundary value problem for the damped Boussinesq equation, Commun. Pure Appl. Anal. 3 (2) (2004) 319–328.
[9] H.A. Levine, Instability and nonexistence of global solutions of nonlinear wave equations of the form $Pu_{tt} = Au + F(u)$, Trans. Amer. Math. Soc. 192 (1974) 1–21.
[10] H.A. Levine, B.D. Sleeman, A note on the non-existence of global solutions of initial boundary value problems for the Boussinesq equation $u_{tt} - u_{xx} - 3u_{xxxx} + 12(u^2)_{xx} = 0$, J. Math. Anal. Appl. 107 (1985) 206–210.
[11] Y.A. Li, P.J. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, J. Differential Equations 162 (2000) 27–63.
[12] Y. Liu, Instability of solitary waves for generalized Boussinesq equation, J. Dynam. Differential Equations 5 (1993) 537–558.
[13] Y. Liu, Existence and blow up of solutions of a nonlinear Pochhammer–Chree equation, Indiana Univ. Math. J. 45 (1996) 797–816.
[14] V.G. Makhankov, Dynamics of classical solitons (in non-integrable systems), Phys. Rep. 35 (1978) 1–128.
[15] N. Polat, D. Kaya, H.I. Tutalar, Blow up of solutions for the damped Boussinesq equation, Z. Naturforsch A 60a (2005) 473–476.
[16] N. Polat, D. Kaya, Blow up of solutions for the generalized Boussinesq equation with damping term, Z. Naturforsch A 61a (2006) 235–238.
[17] N. Polat, Existence and blow up of solution of Cauchy problem of the generalized damped multidimensional improved modified Boussinesq equation, Z. Naturforsch A 63a (2008) 543–552.
[18] J. Scott Russell, Report on water waves, British Assoc. Report, 1844.
[19] S. Selberg, Lecture Notes Mat., 632, PDE, http://www.math.ntnu.no/~sselberg, 2001.
[20] E.M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, NJ, 1970.
[21] S.K. Turitsyn, Blow-up in the Boussinesq equation, Phys. Rev. E 47 (1993) 796–799.
[22] V.V. Varlamov, Asymptotic behavior of solutions of the damped Boussinesq equation in two space dimensions, Int. J. Math. Math. Sci. 22 (1999) 131–145.
[23] V.V. Varlamov, Eigenfunction expansion method and the long-time asymptotics for the damped Boussinesq equation, Discrete Contin. Dyn. Syst. 7 (4) (2001) 675–702.
[24] S. Wang, G. Chen, The Cauchy problem for the generalized IMBq equation in $W^{s,p}(\mathbb{R}^n)$, J. Math. Anal. Appl. 266 (2002) 38–54.
[25] S. Wang, G. Chen, Small amplitude solutions of the generalized IMBq Equation, J. Math. Anal. Appl. 274 (2002) 846–866.
[26] S. Wang, G. Chen, Cauchy problem of the generalized double dispersion equation, J. Math. Anal. Appl. 64 (2006) 159–173.
[27] Y. Zhijian, On local existence of solutions of initial boundary value problems for the "bad" Boussinesq-type equation, Nonlinear Anal. 51 (2002) 1259–1271.
[28] Z. Yang, X. Wang, Blow up of solutions for improved Boussinesq type equation, Nonlinear Anal. 51 (2002) 1259–1271.