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On the relationships between G-preinvex functions and semistrictly G-preinvex functions[☆]

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Abstract

A new class of functions, termed semistrictly G-preinvex functions, is introduced in this paper. The relationships between semistrictly G-preinvex functions and G-preinvex functions are investigated under mild assumptions. Our results improve and extend the existing ones in the literature.

MSC: 90C26

Keywords: G-preinvex functions; Semistrictly G-preinvex functions; Invex set; Continuity

1. Introduction

Convexity and some generalizations of convexity play a crucial role in mathematical economics, engineering, management science, and optimization theory. Therefore, to consider wider and wider classes of generalized convex functions is important, but it is also important to seek practical criteria for convexity or generalized convexity (see [1–10] and the references therein). Two significant generalizations of convex functions are the so-called preinvex functions introduced in [7] and prequasi-invex functions given in [6]. Yang et al. [10] established characterizations of prequasi-invex functions under a semicontinuity condition; Luo et al. in [4,5] improved their results in [10] under much weaker assumptions. Yang and Li [8] presented some properties of preinvex functions; they in [9] introduced the semistrictly preinvex function and established relationships between preinvex functions and semistrictly preinvex functions under a certain set of conditions. Recently, Antczak [2,3] introduced the concept of the G-preinvex function, which includes the preinvex function [7] and r-preinvex function [1] as special cases. Relations of this G-preinvex function to preinvex functions and some properties of this class of functions were studied in [2].

In this paper, we introduce a new class of functions called semistrictly G-preinvex functions, which include semistrictly preinvex functions [9] as a special case. We investigate the relationship between G-preinvex functions

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and semistrictly G-preinvex functions under mild conditions. It is worth pointing out that the results obtained here improve and generalize the corresponding ones given in [9].

The rest of the paper is organized as follows. In Section 2, we give some preliminaries. The main results of the paper are presented in Section 3, and their proofs are given in Appendix. Section 4 gives some conclusions.

2. Preliminaries

In this setion, we will describe some definitions of generalized convexity.

Definition 1 (*[7]*). For a given set $K \subseteq \mathbb{R}^n$ and a given function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, K is said to be an invex set with respect to η iff

$$\forall x, y \in K,\ \forall \lambda \in [0, 1] \Rightarrow y + \lambda\eta(x, y) \in K.$$

Definition 2 (*[7]*). Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. The function $f : K \to \mathbb{R}$ is said to be preinvex on K iff, $\forall x, y \in K,\ \forall \lambda \in [0, 1]$,

$$f(y + \lambda\eta(x, y)) \le \lambda f(x) + (1 - \lambda) f(y).$$

Definition 3 (*[9]*). Let $K \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. The function $f : K \to \mathbb{R}$ is said to be semistrictly preinvex on K iff, $\forall x, y \in K,\ f(x) \neq f(y),\ \forall \lambda \in (0, 1)$,

$$f(y + \lambda\eta(x, y)) < \lambda f(x) + (1 - \lambda) f(y).$$

In [9], the relationship between preinvex functions and semistrictly preinvex functions was discussed under the following condition.

Condition C (*[7]*). Let $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. We say that the function η satisfies Condition C iff, $\forall x, y \in K, \lambda \in [0, 1]$,

$$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y),$$
$$\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y).$$

Let $I_f(K)$ be the range of f, i.e., the image of K under f, and G^{-1} be the inverse of G.

Definition 4 (*[2,3]*). Let $K \subseteq \mathbb{R}^n$ be an invex set (with respect to η). The function $f : K \to \mathbb{R}$ is said to be G-preinvex on K iff there exist a continuous real-valued increasing function $G : I_f(K) \to \mathbb{R}$ and a vector-valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that, $\forall x, y \in K,\ \forall \lambda \in [0, 1]$,

$$f(y + \lambda\eta(x, y)) \le G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))).$$

We note that the G-preinvex function in Definition 4 reduces to the preinvex function in Definition 2 and the r-preinvex function in [1] when setting $G(t) = t$ and $G(t) = \mathrm{e}^{rt}$, respectively.

We now introduce a new kind of generalized convex function termed semistrictly G-preinvex function as follows.

Definition 5. Let $K \subseteq \mathbb{R}^n$ be an invex set (with respect to η). The function $f : K \to \mathbb{R}$ is said to be semistrictly G-preinvex on K iff there exist a continuous real-valued increasing function $G : I_f(K) \to \mathbb{R}$ and a vector-valued function $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that, $\forall x, y \in K,\ f(x) \neq f(y),\ \forall \lambda \in (0, 1)$,

$$f(y + \lambda\eta(x, y)) < G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))).$$

We also observe that the semistrictly G-preinvex function in Definition 5 is a generalization of the semistrictly preinvex function in Definition 3 when taking $G(t) = t$.

Example 1. This example illustrates that a G-preinvex function is not necessarily a semistrictly G-preinvex function. Let

$$f(x) = \ln(|x| + 1),$$
$$\eta(x, y) = \begin{cases} x - y, & \text{if } xy < 0, \\ -x - y, & \text{if } xy \geq 0. \end{cases}$$

Then, f is a G-preinvex function on $\mathbb{R}$ with respect to η, where $G(t) = e^t$. But if we let $x = 0$, $y = 1$, $\lambda = 1/2$, we have

$$f(y) = f(1) = \ln 2 > 0 = f(0) = f(x),$$

and

$$f(y + \lambda\eta(x, y)) = f(1/2) = \ln(3/2) = G^{-1}(3/2) = G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))).$$

So, f is not a semistrictly G-preinvex function with respect to η on $\mathbb{R}$. Moreover, f is not a preinvex function on $\mathbb{R}$ with respect to η, since for such x, y, λ, we have

$$f(y + \lambda\eta(x, y)) > \lambda f(x) + (1 - \lambda) f(y).$$

Example 2. This example illustrates that a semistrictly G-preinvex function is not necessarily a G-preinvex function. Let

$$f(x) = \begin{cases} -\ln(|x| + 1), & \text{if } |x| \leq 1, \\ -\ln 2, & \text{if } |x| \geq 1, \end{cases}$$
$$\eta(x, y) = \begin{cases} x - y, & \text{if } x \geq 0,\ y \geq 0, \\ x - y, & \text{if } x \leq 0,\ y \leq 0, \\ x - y, & \text{if } x < -1,\ y > 1, \\ x - y, & \text{if } y < -1,\ x > 1, \\ y - x, & \text{if } -1 \leq x \leq 0,\ y \geq 0, \\ y - x, & \text{if } -1 \leq y \leq 0,\ x \geq 0, \\ y - x, & \text{if } 0 \leq x \leq 1,\ y \leq 0, \\ y - x, & \text{if } 0 \leq y \leq 1,\ x \leq 0, \end{cases}$$

Then, it is not difficult to show that f is a semistrictly G-preinvex function on $\mathbb{R}$ with respect to η, where $G(t) = \mathrm{e}^t$. However, let $x = 2$, $y = -2$, $\lambda = 1/2$, and we have

$$f(y + \lambda\eta(x, y)) = f(0) = 0 > \ln(1/2) = G^{-1}(1/2) = G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))).$$

Thus, f is not a G-preinvex function with respect to the same η on $\mathbb{R}$.

3. Main results

This section gives the main results of this paper; their proofs are given in Appendix. We assume always that:
(i) $K \subseteq \mathbb{R}^n$ is an invex set with respect to $\eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$;
(ii) η satisfies Condition C; f is a real valued function on K.

Theorem 1. *Let f be a G-preinvex function on K. If there exists an $\alpha \in (0, 1)$ such that, for each $x, y \in K$, $f(x) \neq f(y)$ implies*

$$f(y + \alpha\eta(x, y)) < G^{-1}(\alpha G(f(x)) + (1 - \alpha)G(f(y))), \tag{1}$$

then f is a semistrictly G-preinvex function on K.

Theorem 2. *Let f be a semistrictly G-preinvex function on K. If there exists an $\alpha \in (0, 1)$ such that, for every $x, y \in K$,*

$$f(y + \alpha\eta(x, y)) \leq G^{-1}(\alpha G(f(x)) + (1 - \alpha)G(f(y))), \tag{2}$$

then f is a G-preinvex function on K.

Theorem 3. *Let f be continuous on K and satisfy $f(y+\eta(x,y)) \leq f(x)$ for all $x, y \in K$. Assume that $G : I_f(K) \to \mathbb{R}$ is an increasing continuous function. If there exists an $\alpha \in (0,1)$ such that, for every $x, y \in K$, $f(x) \neq f(y)$ implies (1), then f is both a G-preinvex function on K and a semistrictly G-preinvex function on K.*

We note from the proof of Theorem 3 that when $G(t) = t$, the continuity of f in conditions of Theorem 3 can be weakened to lower semi-continuity, based on Theorem 3.2 in [8].

Theorems 1–3 generalize and improve Theorems 4.2, 4.1 and 5.3 in [9] from the preinvex case to the G-preinvex case. Indeed, Theorem 4.1 of [9] requires that there exists $\alpha \in (0,1)$ such that, for each $x, y \in K$,

$$f(y+\alpha\eta(x,y)) \leq \alpha f(x) + (1-\alpha) f(y), \tag{3}$$

$$f(y+(1-\alpha)\eta(x,y)) \leq \alpha f(y) + (1-\alpha) f(x), \tag{4}$$

while in conditions of Theorem 2, (3) is only needed when $G(t) = t$. On the other hand, Theorems 4.2 and 5.3 in [9] also require that there exists $\alpha \in (0,1)$ such that, for each $x, y \in K$, $f(x) \neq f(y)$ implies that (3) and (4) strictly hold, but Theorems 1 and 3 only require that (3) strictly holds when $G(t) = t$.

4. Conclusions

In this paper, we have introduced a new kind of generalized convex function called a semistrictly G-preinvex function. The relationship between semistrictly G-preinvex functions and G-preinvex functions are mainly investigated under weaker conditions. Our results improve and extend the existing ones in the literature.

An interesting topic for our future research is to investigate G-preinvex functions and semicontinuity. Another interesting research topic is to explore some properties of these semistrictly G-preinvex functions and their applications in optimization problems.

Appendix. Proofs of theorems

In this section, the proofs of theorems are given, in which a useful lemma is used:

Lemma 1 ([2]). *G^{-1} is increasing if and only if G is increasing.*

Proof of Theorem 1. By contradiction: suppose that there exist $x, y \in K$, $\lambda \in (0,1)$ such that $f(x) \neq f(y)$ and

$$f(y+\lambda\eta(x,y)) \geq G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))). \tag{5}$$

Let $z = y + \lambda\eta(x,y)$. There are two cases to be considered as follows.

Case (i): $f(x) > f(y)$. By the assumption and Definition 4, G is an increasing function and hence, by Lemma 1, G^{-1} is also increasing. Thus, the inequality (5) implies

$$f(z) \geq G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))) > f(y). \tag{6}$$

Since f is a G-preinvex function, we have

$$f(z) \leq G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))),$$

which, together with (6), leads to

$$f(y) < f(z) = G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))). \tag{7}$$

Define a sequence $\{z_k\}$ by induction as follows:

$$\begin{cases} z_1 = y + \alpha\eta(z,y), \\ z_2 = z_1 + \alpha\eta(z,z_1), \\ \quad \cdots \\ z_k = z_{k-1} + \alpha\eta(z,z_{k-1}), \quad \forall k \geq 2. \end{cases} \tag{8}$$

Note that G and G^{-1} are increasing. Then, under the condition of (1), $f(y) < f(z)$ implies

$$\begin{aligned}
&f(z_1) = f(y + \alpha\eta(z, y)) < G^{-1}(\alpha G(f(z)) + (1-\alpha)G(f(y))) < f(z),\\
&f(z_2) = f(z_1 + \alpha\eta(z, z_1)) < G^{-1}(\alpha G(f(z)) + (1-\alpha)G(f(z_1))) < f(z),\\
&\dots\\
&f(z_k) = f(z_{k-1} + \alpha\eta(z, z_{k-1})) < G^{-1}(\alpha G(f(z)) + (1-\alpha)G(f(z_{k-1}))) < f(z), \forall k \geq 2.
\end{aligned} \tag{9}$$

Using Condition C, we from (8) have

$$\begin{aligned}
\eta(z, z_i) &= \eta(z, z_{i-1} + \alpha\eta(z, z_{i-1}))\\
&= (1-\alpha)\eta(z, z_{i-1})\\
&= \cdots\\
&= (1-\alpha)^{i-1}\eta(z, z_1)\\
&= (1-\alpha)^{i-1}\eta(z, y + \alpha\eta(z, y))\\
&= (1-\alpha)^{i}\eta(z, y),
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\eta(z, y) &= \eta(y + \lambda\eta(x, y), y)\\
&= \eta(y + \lambda\eta(x, y), y + \lambda\eta(x, y) - \lambda\eta(x, y))\\
&= \eta(y + \lambda\eta(x, y), y + \lambda\eta(x, y) + \eta(y, y + \lambda\eta(x, y)))\\
&= -\eta(y, y + \lambda\eta(x, y))\\
&= \lambda\eta(x, y).
\end{aligned} \tag{11}$$

It then follows from (8), (10) and (11) that

$$\begin{aligned}
z_k &= z_1 + \alpha\sum_{i=1}^{k-1}\eta(z, z_i)\\
&= y + \left[\alpha\sum_{i=0}^{k-1}(1-\alpha)^i\right]\eta(z, y)\\
&= y + \lambda[1 - (1-\alpha)^k]\eta(x, y).
\end{aligned} \tag{12}$$

Since $0 < \alpha < 1$, then $z_k \to y + \lambda\eta(x, y) = z$ as $k \to \infty$. Choose $k_1 \in N$ such that

$$\alpha(1-\alpha)^{k_1-1} < (1-\lambda)/\lambda.$$

Denote

$$\begin{aligned}
&\beta_1 = \lambda[1 - (1-\alpha)^{k_1}], \qquad \beta_2 = \lambda[1 + \alpha(1-\alpha)^{k_1-1}],\\
&\hat{x} = y + \beta_1\eta(x, y), \qquad \hat{y} = y + \beta_2\eta(x, y).
\end{aligned}$$

It is easy to verify that $0 < \beta_1 < \lambda < \beta_2 < 1$ and,

$$\alpha\beta_1 + (1-\alpha)\beta_2 = \lambda. \tag{13}$$

Also, from (12),

$$z_{k_1} = y + \lambda[1 - (1-\alpha)^{k_1}]\eta(x, y) = y + \beta_1\eta(x, y) = \hat{x}.$$

It then follows from (9) that

$$f(\hat{x}) = f(z_{k_1}) < f(z). \tag{14}$$

(a) If $f(\hat{x}) \geq f(\hat{y})$. From Condition C and (13), we deduce

$$\begin{aligned}\hat{y}+\alpha\eta(\hat{x},\hat{y}) &= y+\beta_2\eta(x,y)+\alpha\eta(y+\beta_1\eta(x,y),y+\beta_2\eta(x,y))\\ &= y+\beta_2\eta(x,y)+\alpha\eta(y+\beta_1\eta(x,y),y+\beta_1\eta(x,y)+(\beta_2-\beta_1)\eta(x,y))\\ &= y+\beta_2\eta(x,y)+\alpha\eta(y+\beta_1\eta(x,y),y+\beta_1\eta(x,y)\\ &\quad +[(\beta_2-\beta_1)/(1-\beta_1)]\eta(x,y+\beta_1\eta(x,y)))\\ &= y+\beta_2\eta(x,y)-[\alpha(\beta_2-\beta_1)/(1-\beta_1)]\eta(x,y+\beta_1\eta(x,y))\\ &= y+[\beta_2-\alpha(\beta_2-\beta_1)]\eta(x,y)\\ &= y+\lambda\eta(x,y)\\ &= z. \end{aligned} \tag{15}$$

Since G and G^{-1} are increasing, the G-preinvexity of f and equality (15) give rise to

$$f(z) \leq G^{-1}(\alpha G(f(\hat{x}))+(1-\alpha)G(f(\hat{y}))) \leq f(\hat{x})$$

which contradicts (14).

(b) If $f(\hat{x}) < f(\hat{y})$. Equality (15) and the condition of (1) imply

$$f(z) < G^{-1}(\alpha G(f(\hat{x}))+(1-\alpha)G(f(\hat{y}))). \tag{16}$$

Again by $\hat{x}=y+\beta_1\eta(x,y)$, $\hat{y}=y+\beta_2\eta(x,y)$ and the G-preinvexity of f, we have

$$f(\hat{x}) \leq G^{-1}(\beta_1 G(f(x))+(1-\beta_1)G(f(y))),$$
$$f(\hat{y}) \leq G^{-1}(\beta_2 G(f(x))+(1-\beta_2)G(f(y))),$$

which in turn imply that

$$G(f(\hat{x})) \leq \beta_1 G(f(x))+(1-\beta_1)G(f(y)), \tag{17}$$
$$G(f(\hat{y})) \leq \beta_2 G(f(x))+(1-\beta_2)G(f(y)). \tag{18}$$

Adding (17) multiplied by α to (18) multiplied by $1-\alpha$, by (13), we obtain

$$\alpha G(f(\hat{x}))+(1-\alpha)G(f(\hat{y})) \leq \lambda G(f(x))+(1-\lambda)G(f(y)). \tag{19}$$

Since G^{-1} is increasing, inequality (16) together with (19) leads to

$$f(z) < G^{-1}(\lambda G(f(x))+(1-\lambda)G(f(y))), \tag{20}$$

which contradicts (7).

Case (ii): $f(x) < f(y)$. Similarly, we have

$$f(x) < f(z) = G^{-1}(\lambda G(f(x))+(1-\lambda)G(f(y))). \tag{21}$$

Also define an induction sequence $\{z_k\}$ as follows:

$$z_1 = z+(1-\alpha)\eta(x,z), \quad z_k = z_{k-1}+\alpha\eta(z,z_{k-1}), \quad \forall k \geq 2. \tag{22}$$

Since G and G^{-1} are increasing, the G-preinvexity of f and $f(x) < f(z)$ imply that

$$\begin{aligned} &f(z_1) = f(z+(1-\alpha)\eta(x,z)) \leq G^{-1}((1-\alpha)G(f(x))+\alpha G(f(z))) < f(z),\\ &f(z_k) = f(z_{k-1}+\alpha\eta(z,z_{k-1})) \leq G^{-1}(\alpha G(f(z))+(1-\alpha)G(f(z_{k-1}))) < f(z), \quad \forall k \geq 2. \end{aligned} \tag{23}$$

From Condition C and (22), we have

$$z_k = z+(1-\alpha)^k\eta(x,z) = y+[\lambda+(1-\alpha)^k(1-\lambda)]\eta(x,y). \tag{24}$$

Choose $k_1 \in N$ such that

$$(1-\alpha)^{k_1}/\alpha < \lambda/(1-\lambda). \tag{25}$$

Let

$$\bar{\beta}_1 = \lambda + (1-\alpha)^{k_1}(1-\lambda), \quad \bar{\beta}_2 = \lambda - (1-\lambda)(1-\alpha)^{k_1+1}/\alpha, \tag{26}$$

$$\bar{x} = y + \bar{\beta}_1\eta(x, y), \quad \bar{y} = y + \bar{\beta}_2\eta(x, y). \tag{27}$$

It is clear that $0 < \bar{\beta}_2 < \lambda < \bar{\beta}_1 < 1$ and $(1-\alpha)\bar{\beta}_1 + \alpha\bar{\beta}_2 = \lambda$. Also from (24),

$$z_{k_1} = z + (1-\alpha)^{k_1}\eta(x, z) = y + \bar{\beta}_1\eta(x, y) = \bar{x},$$

which by (23) yields

$$f(\bar{x}) = f(z_{k_1}) < f(z). \tag{28}$$

Again using Condition C, we derive

$$\bar{x} + \alpha\eta(\bar{y}, \bar{x}) = y + [\bar{\beta}_1 - \alpha(\bar{\beta}_1 - \bar{\beta}_2)]\eta(x, y) = y + \lambda\eta(x, y) = z. \tag{29}$$

(a) $f(\bar{x}) \geq f(\bar{y})$. By (29) and the G-preinvexity of f, we get a contradiction with (28).

(b) $f(\bar{x}) < f(\bar{y})$. By (29) and condition (1), similarly to (16)–(20) of Case (i)(b), we also get a contradiction with (21). □

Proof of Theorem 2. By contradiction: we assume that there exist $x, y \in K$, $\lambda \in (0, 1)$ such that

$$f(y + \lambda\eta(x, y)) > G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))). \tag{30}$$

Since G is an increasing function, by Lemma 1, G^{-1} is also increasing. Let $z = y + \lambda\eta(x, y)$. If $f(x) \neq f(y)$, then the semistrict G-preinvexity of f implies

$$f(z) < G^{-1}(\lambda G(f(x)) + (1-\lambda)G(f(y))),$$

which contradicts (30). So, we have $f(x) = f(y)$. Then (30) gives

$$f(z) > f(x) = f(y). \tag{31}$$

Take a sequence $\{z_k\}$ defined as in (22). By the semistrict G-preinvexity of f and $f(x) < f(z)$, similar to (22)–(29) of Case (ii) in the proof of Theorem 1, we obtain

$$f(\bar{x}) = f(z_{k_1}) < f(z), \tag{32}$$

$$\bar{x} + \alpha\eta(\bar{y}, \bar{x}) = z, \tag{33}$$

where k_1 is defined in (25), the pair $\bar{x}$, $\bar{y}$ and the pair $\bar{\beta}_1$, $\bar{\beta}_2$ are given as in (27) and (26), respectively. We consider two cases as follows.

Case (i): $f(\bar{x}) = f(\bar{y})$. Condition (2) and the equality (33) yield

$$f(z) \leq G^{-1}(\alpha G(f(\bar{y})) + (1-\alpha)G(f(\bar{x}))) = f(\bar{x}),$$

contradicting (32).

Case (ii): $f(\bar{x}) \neq f(\bar{y})$. The semistrict G-preinvexity of f and equality (33) imply

$$f(z) < G^{-1}(\alpha G(f(\bar{y})) + (1-\alpha)G(f(\bar{x}))).$$

Thus,

$$G(f(z)) < \alpha G(f(\bar{y})) + (1-\alpha)G(f(\bar{x})),$$

which by (32) gives

$$f(z) < f(\bar{y}), \tag{34}$$

since G is increasing. From Condition C, we have

$$y + \frac{\bar{\beta}_2}{\lambda}\eta(z, y) = y + \bar{\beta}_2\eta(x, y) = \bar{y}.$$

Note from (26) that $0 < \frac{\bar{\beta}_2}{\lambda} < 1$. Also, G and G^{-1} are increasing. Since f is a semistrictly G-preinvex function and $f(z) > f(y)$ by (31), we obtain

$$f(\bar{y}) < G^{-1}\left(\frac{\bar{\beta}_2}{\lambda}G(f(z)) + \left(1 - \frac{\bar{\beta}_2}{\lambda}\right)G(f(y))\right) < f(z),$$

contradicting (34). □

Proof of Theorem 3. First, we shall prove that f is a G-preinvex function on K. By Theorem 10 in [2], we need to show that for each $x, y \in K$, there exists $\lambda \in (0, 1)$ such that

$$f(y + \lambda\eta(x, y)) \leq G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))).$$

Assume, by contradiction, that there exist $x, y \in K$ such that

$$f(y + \lambda\eta(x, y)) > G^{-1}(\lambda G(f(x)) + (1 - \lambda)G(f(y))), \quad \forall\lambda \in (0, 1). \tag{35}$$

If $f(x) \neq f(y)$, condition (1) implies that

$$f(y + \alpha\eta(x, y)) < G^{-1}(\alpha G(f(x)) + (1 - \alpha)G(f(y))),$$

which contradicts (35). Thus, we have $f(x) = f(y)$, and then (35) implies

$$f(y + \lambda\eta(x, y)) > f(x) = f(y), \quad \forall\lambda \in (0, 1). \tag{36}$$

Note that $0 < \alpha < 1$. Take $\lambda = \frac{\alpha-1}{\alpha-2} \in (0, 1)$, and let $z = y + \lambda\eta(x, y)$. Then,

$$f(z) > f(x) = f(y). \tag{37}$$

Define

$$z_1 = z + \alpha\eta(x, z), \quad z_2 = z_1 + \alpha\eta(y, z_1).$$

From Condition C, we obtain

$$\begin{aligned} z_1 &= y + [(1 - \alpha)\lambda + \alpha]\eta(x, y) = y + 1/(2 - \alpha)\eta(x, y), \\ z_2 &= y + (1 - \alpha)/(2 - \alpha)\eta(x, y) = y + \lambda\eta(x, y) = z. \end{aligned}$$

Since $1/(2 - \alpha) \in (0, 1)$, it follows from (36) that

$$f(z_1) = f(y + 1/(2 - \alpha)\eta(x, y)) > f(y). \tag{38}$$

By the assumption, G is an increasing function. Hence, by Lemma 1, G^{-1} is also increasing. So, $f(z) > f(x)$ by (37) and condition of (1) give

$$f(z_1) = f(z + \alpha\eta(x, z)) < G^{-1}(\alpha G(f(x)) + (1 - \alpha)G(f(z))) < f(z). \tag{39}$$

Also, from (38) and condition of (1), we get

$$f(z_2) = f(z_1 + \alpha\eta(y, z_1)) < G^{-1}(\alpha G(f(y)) + (1 - \alpha)G(f(z_1))) < f(z_1). \tag{40}$$

The combination of (39) and (40), together with $z_2 = z$, implies $f(z) = f(z_2) < f(z)$, which is a contradiction.

From Theorem 1 above, f is also a semistrictly G-preinvex function on K. □

References

[1] T. Antczak, r-pre-invexity and r-invexity in mathematical programming, Comput. Math. Appl. 50 (2005) 551–566.
[2] T. Antczak, G-pre-invex functions in mathematical programming, J. Comput. Appl. Math. (2007), doi:10.1016/j.cam.2007.06.026.
[3] T. Antczak, New optimality conditions and duality results of G-type in differentiable mathematical programming, Nonlinear Anal. 66 (2007) 1617–1632.
[4] H.Z. Luo, Z.K. Xu, On characterizations of prequasi-invex functions, J. Optim. Theory Appl. 120 (2004) 429–439.
[5] H.Z. Luo, H.X. Wu, Y.H. Zhu, Remarks on Criteria of Prequasi-invex Functions, Appl. Math. J. Chinese Univ. 19 (2004) 335–341.
[6] R. Pini, Invexity and generalized convexity, Optimization 22 (1991) 513–525.
[7] T. Weir, B. Mond, Pre-invex functions in multiple objective optimization, J. Math. Anal. Appl. 136 (1988) 29–38.

[8] X.M. Yang, D. Li, On properties of preinvex functions, J. Math. Anal. Appl. 256 (2001) 229–241.
[9] X.M. Yang, D. Li, Semistrictly preinvex functions, J. Math. Anal. Appl. 258 (2001) 287–308.
[10] X.M. Yang, X.Q. Yang, K.L. Teo, Characterizations and Applications of Prequasi-Invex Functions, J. Optim. Theory Appl. 110 (2001) 645–668.