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Equivariant maps and bimodule projections

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Abstract

We construct a counterexample to Solel's [B. Solel, Contractive projections onto bimodules of von Neumann algebras, J. London Math. Soc. 45 (2) (1992) 169–179] conjecture that the range of any contractive, idempotent, MASA bimodule map on $B(\mathcal{H})$ is necessarily a ternary subalgebra. Our construction reduces this problem to an analogous problem about the ranges of idempotent maps that are equivariant with respect to a group action. Such maps are important to understand Hamana's theory [M. Hamana, Injective envelopes of C^*-dynamical systems, Tohoku Math. J. 37 (1985) 463–487] of G-injective operator spaces and G-injective envelopes.

Keywords: Injective; Multipliers; Operator space; Banach–Stone

1. Introduction

Solel [14] proved that if $\mathcal{H}$ is a Hilbert space, $\mathcal{M} \subseteq B(\mathcal{H})$ is a MASA, and $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ is a weak*-continuous, (completely) contractive, idempotent $\mathcal{M}$-bimodule map, then the range $\mathcal{R}(\Phi)$ of Φ is a *ternary subalgebra* of $B(\mathcal{H})$, i.e., $A, B, C \in \mathcal{R}(\Phi)$ implies that $AB^*C \in \mathcal{R}(\Phi)$. For another proof of this fact see [9]. Solel also conjectured that the same result would hold even when Φ was not weak*-continuous. In this paper, we give a counterexample to this conjecture.

Let $\mathbb{T}$ denote the unit circle with arc-length measure, and let $L^2(\mathbb{T})$ and $L^\infty(\mathbb{T})$ denote the square-integrable functions and essentially bounded functions, respectively. If we identify

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$L^\infty(\mathbb{T}) \subseteq B(L^2(\mathbb{T}))$ as the multiplication operators, then it is a maximal Abelian subalgebra (MASA). We construct a unital, completely positive, idempotent $L^\infty(\mathbb{T})$-bimodule map on $B(L^2(\mathbb{T}))$ whose range is not a subalgebra of $B(L^2(\mathbb{T}))$, and hence not a ternary subalgebra, since any ternary algebra that contains the identity is an algebra. Thus, this map will provide a counterexample to Solel's conjecture.

As a first step in our construction, we construct a unital, completely positive, idempotent map, $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$, that is equivariant with respect to the natural action of $\mathbb{Z}$ on $\ell^\infty(\mathbb{Z})$ and whose range is not a C^*-subalgebra. The construction of this map uses some results from Hamana's theory of G-injective envelopes [5], where G is a discrete group acting on all of the spaces.

In Section 2, we study $\mathbb{Z}$-equivariant projections on $\ell^\infty(\mathbb{Z})$, then in Section 3 we use these results together with a cross-product construction to build the counterexample.

2. $\mathbb{Z}$-Equivariant projections

In this section we take a careful look at $\ell^\infty(\mathbb{Z}) = C(\beta\mathbb{Z})$, where $\beta\mathbb{Z}$ denotes the Stone–Cech compactification of the integers [15], and study the $\mathbb{Z}$-equivariant maps on this space that are also idempotent. The identification of these two spaces comes by identifying a function $f \in C(\beta\mathbb{Z})$ with the vector $v = (f(n))_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$. The action of $\mathbb{Z}$ given by $\alpha(m)f(k) = f(k+m)$ corresponds to multiplication of the vector v by B^m where B denotes the backwards shift. This action also corresponds to the unique extension of the map $k \to k+m$ to a homeomorphism of $\beta\mathbb{Z}$ and so we shall denote this homeomorphism by $\omega \to m \cdot \omega$.

A linear map $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ is $\mathbb{Z}$-equivariant if and only if it commutes with the backwards shift. Given such a map Φ we let $\phi_n : \ell^\infty(\mathbb{Z}) \to \mathbb{C}$ denote the linear functional corresponding to the nth component, so that $\Phi(v) = (\phi_n(v))$.

Note that Φ commutes with B if and only if $\phi_n(v) = \phi_0(B^n v)$. Thus there is a one-to-one correspondence between linear functionals $\phi : \ell^\infty(\mathbb{Z}) \to \mathbb{C}$ and $\mathbb{Z}$-equivariant linear maps $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$. We shall denote the corresponding linear map by $\Phi = \Phi_\phi$.

Also, it is worth noting that Φ_ϕ is a completely positive map if and only if ϕ is a positive linear functional, Φ_ϕ is unital if and only if ϕ is unital and Φ_ϕ is completely contractive if and only if ϕ is contractive.

In this section we give some characterizations of idempotent $\mathbb{Z}$-equivariant maps. Note that this reduces to a question of which linear functionals will give rise to idempotents. We shall construct a positive, unital, $\mathbb{Z}$-equivariant idempotent, such that the range of Φ not a C^*-subalgebra.

It is well known that the range of such a map is completely isometrically isomorphic to a C^*-algebra, but what we are interested in is whether or not it is actually a C^*-subalgebra. These questions are the analogues of Solel's results that the ranges of *weak**-continuous MASA bimodule idempotents on $B(H)$ are necessarily ternary subalgebras, since any ternary subalgebra containing the unit is a C^*-subalgebra.

We let $c_0(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$ denote the functions that vanish at infinity.

Proposition 2.1. *Let $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be a $\mathbb{Z}$-equivariant linear map and decompose $\phi_0 = \phi^{ac} + \phi^{s}$ into its weak*-continuous and singular parts. If we define $\Phi^{ac}(v) = (\phi^{ac}(B^n v))$ and $\Phi^{s}(v) = (\phi^{s}(B^n v))$, then both these maps are $\mathbb{Z}$-equivariant, Φ^{ac} is weak*-continuous, $\Phi^{ac}(c_0(\mathbb{Z})) \subseteq c_0(\mathbb{Z})$, $\Phi^{s}(c_0(\mathbb{Z})) = 0$, $\Phi = \Phi^{ac} + \Phi^{s}$, and this decomposition is the unique decomposition of Φ into a weak*-continuous part and singular part.*

Proof. We only prove that $\Phi^{\mathrm{ac}}(c_0(\mathbb{Z})) \subseteq c_0(\mathbb{Z})$. There is a vector $a = (a_n) \in \ell^1(\mathbb{Z})$ such that $\phi^{\mathrm{ac}}(v) = a \cdot v = \sum_{n\in\mathbb{Z}} a_n v_n$. Hence, $\phi^{\mathrm{ac}}(B^k v) = \sum_{n\in\mathbb{Z}} a_n v_{n+k} \to 0$ as $k \to \pm\infty$. □

Note that if we set $\hat{a}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$, then since $a \in \ell^1(\mathbb{Z})$ this series converges to give a continuous function on the circle. Identifying, $v \in \ell^\infty(\mathbb{Z})$ with the formal series, $\hat{v}(e^{i\theta}) = \sum_{n\in\mathbb{Z}} v_n e^{in\theta}$, we have that $\widehat{\Phi^{\mathrm{ac}}(v)}(e^{i\theta}) = \hat{a}(e^{i\theta})\hat{v}(e^{i\theta})$.

Proposition 2.2. *Let $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be a $\mathbb{Z}$-equivariant linear map. If Φ is weak*-continuous and idempotent, then Φ is either the identity map or* 0.

Proof. We have that $\Phi = \Phi^{\mathrm{ac}}$ and so Φ is given as "multiplication" by the continuous function $\hat{a}$. It is easily checked that $\Phi \circ \Phi$ is also weak*-continuous and is given as multiplication by $\hat{a}^2$. Since Φ is idempotent, $\hat{a}^2 = \hat{a}$ and since this function is continuous it must be either constantly 0 or constantly 1, from which the result follows. □

Theorem 2.3. *Let $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be a $\mathbb{Z}$-equivariant linear map and let $\mathcal{I}$ denote the identity map. If Φ is idempotent, then either $\Phi = \Phi^{\mathrm{s}}$ or $\Phi = \mathcal{I} - \Psi$ where Ψ is singular and idempotent.*

Proof. Write $\Phi = \Phi^{\mathrm{ac}} + \Phi^{\mathrm{s}}$. Then $\Phi = \Phi \circ \Phi = \Phi^{\mathrm{ac}} \circ \Phi^{\mathrm{ac}} + \Phi^{\mathrm{ac}} \circ \Phi^{\mathrm{s}} + \Phi^{\mathrm{s}} \circ \Phi^{\mathrm{ac}} + \Phi^{\mathrm{s}} \circ \Phi^{\mathrm{s}}$ and the first term in this sum is easily seen to be weak*-continuous and each of the last three terms annihilate $c_0(\mathbb{Z})$ and hence are singular. Thus, by uniqueness of the decomposition, we have that $\Phi^{\mathrm{ac}} = \Phi^{\mathrm{ac}} \circ \Phi^{\mathrm{ac}}$. Hence, either Φ^{ac} is 0 or the identity. If $\Phi^{\mathrm{ac}} = 0$, then $\Phi = \Phi^{\mathrm{s}}$. If Φ^{ac} is the identity, then equating the singular parts of the above equation yields, $\Phi^{\mathrm{s}} = 2\Phi^{\mathrm{s}} + \Phi^{\mathrm{s}} \circ \Phi^{\mathrm{s}}$. Thus, $\Phi^{\mathrm{s}} \circ \Phi^{\mathrm{s}} = -\Phi^{\mathrm{s}}$ and so $\Psi = -\Phi^{\mathrm{s}}$ is singular and idempotent. □

Thus, we see that to construct all the idempotent maps, it is sufficient to construct all of the singular idempotents and the singular part of an idempotent is either idempotent or the negative of an idempotent.

Corollary 2.4. *Let $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be a $\mathbb{Z}$-equivariant linear map and let $\mathcal{I}$ denote the identity map. If Φ is idempotent and contractive, then either $\Phi = \Phi^{\mathrm{s}}$ or $\Phi = \mathcal{I}$.*

Proof. We must show that if $\Phi = \mathcal{I} - \Psi$ with Ψ idempotent and Φ is contractive, then $\Psi = 0$. Assume that Ψ is not 0, and choose $v = (v_n)$ with $\|v\| = 1$ and $\Psi(v) = v$. Pick a component k such that $|v_k| \geqslant 1/2$ and let e_k denote the canonical basis vector that is 1 in the kth entry and 0 elsewhere. Then $\|2v_k e_k - v\| \leqslant 1$, but $\Phi(2v_k e_k - v) = (2v_k e_k - v) - (-v) = 2v_k e_k$ which has norm greater than 1. □

We would now like to define a *spectrum* for idempotent maps. To this end, for each $\lambda \in \mathbb{T}$, where $\mathbb{T}$ denotes the unit circle in the complex plane, let $x_\lambda = (\lambda^n) \in \ell^\infty(\mathbb{Z})$. Note that these vectors satisfy, $B(x_\lambda) = \lambda x_\lambda$ and that the eigenspace of B corresponding to the eigenvalue λ is one-dimensional. Hence, if Φ is any $\mathbb{Z}$-equivariant linear map, then $\Phi(x_\lambda) = c_\lambda x_\lambda$ for some scalar c_λ. Moreover, if Φ is idempotent then $c_\lambda^2 = c_\lambda$ and hence, c_λ is 0 or 1.

Definition 2.5. Let $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be a $\mathbb{Z}$-equivariant linear map, then we set $\sigma(\Phi) = \{\lambda \in \mathbb{T} \colon c_\lambda \neq 0\}$ and we call this set the *spectrum* of Φ.

Remark 2.6. Recall that every character on $\mathbb{Z}$ is of the form, $\rho_\lambda(n) = \lambda^n$, for some $\lambda \in \mathbb{T}$. Thus, under the identification between bounded functions on $\mathbb{Z}$ and vectors in $\ell^\infty(\mathbb{Z})$, the vector x_λ is just the character, ρ_λ. Note that since Φ is $\mathbb{Z}$-equivariant, the range of Φ, $\mathcal{R}(\Phi)$ is a $\mathbb{Z}$-invariant subspace and $\lambda \in \sigma(\Phi)$ if and only if $x_\lambda \in \mathcal{R}(\Phi)$, i.e., if and only if $\rho_\lambda \in \mathcal{R}(\Phi)$. With these identifications, the set $\sigma(\Phi)$ is the same as the "spectrum" of the subspace, $sp(\mathcal{R}(\Phi))$ studied in the theory of spectral synthesis, although the latter definition is usually only made for weak*-closed subspaces. See, for example, [1, Definition 1.4.1].

We will show later that $\sigma(\Phi)$ is not always a closed subset of $\mathbb{T}$. The difficulty is that if $\lambda_n \to \lambda$ in $\mathbb{T}$, then $x_{\lambda_n} \to x_\lambda$ only in the wk*-topology of $\ell^\infty(\mathbb{Z})$, but the map Φ is generally singular.

Proposition 2.7. *Let $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be a $\mathbb{Z}$-equivariant linear map, then $\sigma(\Phi) = \{\lambda\colon \phi_0(x_\lambda) \neq 0\}$. If Φ is also idempotent, then $\phi_0(x_\lambda)$ is always either* 0 *or* 1 *and $\sigma(\Phi) = \{\lambda\colon \phi_0(x_\lambda) = 1\}$.*

The following result is fairly well known. In particular, it can be deduced from Kadison's results on isometries of C^*-algebras [8]. We present a different proof that uses Choi's theory of multiplicative domains [3] and our off-diagonalization method.

Lemma 2.8. *Let $\mathcal{A}$ be a C^*-algebra and let $\Phi : \mathcal{A} \to \mathcal{A}$ be completely contractive. If U_1, U_2, U_3 are unitaries and $\Phi(U_i) = U_i$, then $\Phi(U_1 U_2^* U_3) = U_1 U_2^* U_3$.*

Proof. We may regard $\mathcal{A}$ as a C^*-subalgebra of $B(\mathcal{H})$ for some Hilbert space, $\mathcal{H}$. By [12, Theorem 8.3], there exist unital completely positive maps, $\Phi_i : \mathcal{A} \to B(\mathcal{H})$, $i = 1, 2$, such that the map $\Psi : M_2(\mathcal{A}) \to M_2(B(\mathcal{H}))$ defined by

$$\Psi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \Phi_1(a) & \Phi(b) \\ \Phi(c*)* & \Phi_2(d) \end{pmatrix}$$

is completely positive.

Now consider the elements, $X_i = \left(\begin{smallmatrix} 0 & U_i \\ 0 & 0 \end{smallmatrix}\right) \in M_2(\mathcal{A})$, $i = 1, 2, 3$. Since, $\Psi(X_i) = X_i$, $\Psi(X_i^* X_i) = X_i^* X_i$, $\Psi(X_i X_i^*) = X_i X_i^*$, the elements X_i, $i = 1, 2, 3$, belong to Choi's multiplicative domain [12, Theorem 3.18 and Corollary 3.19] of Ψ.

Consequently,

$$\begin{pmatrix} 0 & \Phi(U_1 U_2^* U_3) \\ 0 & 0 \end{pmatrix} = \Psi\bigl(X_1 X_2^* X_3\bigr) = X_1 \Psi\bigl(X_2^*\bigr) X_3 = \begin{pmatrix} 0 & U_1 U_2^* U_3 \\ 0 & 0 \end{pmatrix},$$

and the result follows. □

Theorem 2.9. *Let $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be a $\mathbb{Z}$-equivariant contractive, idempotent map. If $\lambda_1, \lambda_2, \lambda_3 \in \sigma(\Phi)$, then $\lambda_1 \overline{\lambda_2} \lambda_3 \in \sigma(\Phi)$. For any $\lambda \in \sigma(\Phi)$, the set $\bar{\lambda} \cdot \sigma(\Phi)$ is a subgroup of $\mathbb{T}$. If, in addition, Φ is unital, then $\sigma(\Phi)$ is a subgroup of $\mathbb{T}$.*

Proof. The first statement is obvious from the above theorem and the fact that x_λ is a unitary element of $\ell^\infty(\mathbb{Z})$. To see the second claim let $\lambda \in \sigma(\Phi)$, and set $G = \bar{\lambda} \cdot \sigma(\Phi)$. Then $1 \in G$

and, whenever $\alpha = \bar{\lambda}\lambda_1$, $\mu = \bar{\lambda}\lambda_2 \in G$, we have that $\alpha \cdot \mu = \bar{\lambda}(\lambda_1 \cdot \bar{\lambda} \cdot \lambda_2) \in G$ and $(\alpha)^{-1} = \bar{\alpha} = \bar{\lambda}(\lambda \cdot \bar{\lambda}_1 \cdot \lambda) \in G$. The final claim comes from choosing $\lambda = 1$. □

We will show later that $\sigma(\Phi)$ does not determine Φ. In fact, we will give an example of a $\mathbb{Z}$-equivariant, unital completely positive idempotent that is not the identity map for which $\sigma(\Phi) = \mathbb{T}$.

The following result is an analogue of Solel's ternary subalgebra result. We let $\mathbb{Z}_n = \{\lambda \in \mathbb{T}\colon \lambda^n = 1\}$ denote the cyclic subgroups of order n, and let $\mathcal{C}_n = \operatorname{span}\{x_\lambda\colon \lambda^n = 1\}$ denote the corresponding finite-dimensional C^*-subalgebras of $\ell^\infty(\mathbb{Z})$.

Corollary 2.10. *Let $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ be a $\mathbb{Z}$-equivariant contractive, idempotent map. If $\mathcal{R}(\Phi)$ is weak*-closed, then either $\sigma(\Phi) = \mathbb{T}$ and Φ is the identity map or there exist n and $\lambda \in \mathbb{T}$ such that $\sigma(\Phi) = \lambda \cdot \mathbb{Z}_n$ and $\mathcal{R}(\Phi) = x_\lambda \cdot \mathcal{C}_n$. In all of these cases, $\mathcal{R}(\Phi)$ is a ternary subalgebra of $\ell^\infty(\mathbb{Z})$.*

Proof. Let $M \subseteq \ell^\infty(\mathbb{Z})$ be a weak*-closed, $\mathbb{Z}$-invariant subspace and let $sp(M) = \{\lambda\colon x_\lambda \in M\}$, then M is the weak*-closed span of $\{x_\lambda\colon \lambda \in sp(M)\}$. To see this, recall that if we identify $\ell^1(\mathbb{Z})$ with the Wiener algebra, $A(\mathbb{T})$, then the predual, $M_\perp$, is a norm closed ideal in $A(\mathbb{T})$. If we let $k(\cdot)$ denote the kernel of an ideal and $h(\cdot)$ the hull of a set, then $h(E)^\perp = wk^*\text{-span}\{x_\lambda\colon \lambda \in E\}$. Now $sp(M) = k(M_\perp)$ and $M_\perp = h(sp(M))$ since $A(\mathbb{T})$ is regular and semisimple. Hence, $M = (M_\perp)^\perp = h(sp(M))^\perp = wk^*\text{-span}\{x_\lambda\colon \lambda \in sp(M)\}$.

Thus, if $\sigma(\Phi) = \mathbb{T}$, then $\mathcal{R}(\Phi) = \ell^\infty(\mathbb{Z})$ and so Φ is the identity map.

Note that since $\mathcal{R}(\Phi)$ is weak*-closed, $\sigma(\Phi)$ is a closed subset of the circle. Thus, if $\lambda \in \sigma(\Phi)$, then $\bar{\lambda} \cdot \sigma(\Phi) = G$ is a closed subgroup of $\mathbb{T}$ and hence, $G = \mathbb{Z}_n$ for some n and it follows that $\mathcal{R}(\Phi) = x_\lambda \cdot \mathcal{C}_n$. □

In spite of the above result we will later give an example of a $\mathbb{Z}$-equivariant, contractive, unital, idempotent map whose range is not a C^*-subalgebra.

We begin with one set of examples that is easy to describe, although as we will see their existence is a bit subtle. Let G denote a (discrete) group. By a *G-space*, we mean a compact, Hausdorff space, X, together with a homomorphism of G into the group of homeomorphisms of X. Given a G-space P and a closed G-invariant subset $Y \subseteq P$, a continuous, G-equivariant function $\gamma : P \to Y$ is called a *G-retraction* provided that $\gamma(P) = Y$ and $\gamma(y) = y$ for all $y \in Y$. In this case we also call Y a *G-retract* of P.

Also, recall that the *corona set* of a discrete group G, is the set $\mathcal{C}(G) = \beta(G)\backslash G$.

Proposition 2.11. *The following are equivalent*:

(i) *there exists a proper subset Y that is a $\mathbb{Z}$-retract of $\beta(\mathbb{Z})$,*
(ii) *there exists an idempotent, $\mathbb{Z}$-equivariant, *-homomorphism, $\pi : C(\beta(\mathbb{Z})) \to C(\beta(\mathbb{Z}))$, that is not the identity map,*
(iii) *there exists a point $\omega \in \mathcal{C}(\mathbb{Z})$, such that the closure of the orbit of ω is a $\mathbb{Z}$-retract of $\beta(\mathbb{Z})$ that is contained in $\mathcal{C}(\mathbb{Z})$.*

Moreover, in this case, $\sigma(\pi) = \mathbb{T}$.

Proof. Clearly, (iii) implies (i). Assuming (i), let $\gamma : \beta(\mathbb{Z}) \to Y$ be the retraction map and set $\pi = \gamma^*$, i.e., $\pi(f) = f \circ \gamma$, and note that the fact that γ is a $\mathbb{Z}$-retraction implies that π is a $\mathbb{Z}$-equivariant, idempotent homomorphism.

Conversely, assuming (ii), there exists a continuous function, $\gamma : \beta(\mathbb{Z}) \to \beta(\mathbb{Z})$ such that $\pi = \gamma^*$, and the fact that π is a $\mathbb{Z}$-equivariant, idempotent map, implies that γ is a $\mathbb{Z}$-retraction. Thus, (i) and (ii) are equivalent.

Finally, assuming (ii), we have that $\pi = \gamma^*$ with γ a $\mathbb{Z}$-retraction onto some set Y. Since π is $\mathbb{Z}$-equivariant, there exists, $\rho : C(\beta(\mathbb{Z})) \to \mathbb{C}$, such that $\pi = \Phi_\rho$. Since π is a homomorphism, ρ must be a homomorphism and hence there exists $\omega \in \beta(\mathbb{Z})$ such that $\rho(f) = f(\omega)$.

Since, for any $f \in C(\beta(\mathbb{Z}))$, we have $f(m \cdot \omega) = \pi(f)(m) = f(\gamma(m))$, we see that $\gamma(m) = m \cdot \omega$ and hence the range of γ must be the closure of the orbit of ω. Thus, the closure of the orbit of ω is the $\mathbb{Z}$-retract, Y, and γ is a $\mathbb{Z}$-retraction onto the orbit. Moreover, since π is idempotent and not the identity, it must be singular and so, $\pi(c_0(\mathbb{Z})) = (0)$, but this implies that $Y \cap \mathbb{Z}$ is empty and so the closure of the orbit of ω is contained in $\mathcal{C}(\mathbb{Z})$).

To see the final claim, note that since x_λ is a unitary element of $C(\beta(\mathbb{Z}))$, we have that $\pi(x_\lambda) \neq 0$, and hence, $\pi(x_\lambda) = x_\lambda$, for every $\lambda \in \mathbb{T}$. □

We will now prove that such points and homomorphisms exist and consequently provide an example of a homomorphism such that π is not uniquely determined by $\sigma(\pi)$.

We will show that the existence of such a point can be deduced, essentially, from the existence of *idempotent ultrafilters* [2,10]. We are grateful to Gideon Schechtman for introducing us to this theory. The usual proof of the existence of idempotent ultrafilters is done for the semigroup $\mathbb{N}$. Since we need to modify this to the case of $\mathbb{Z}$, we present a slightly different version of this theory that avoids any reference to ultrafilters. The following construction applies to any discrete group, but we shall stick to $\mathbb{Z}$ for simplicity.

Note that the homeomorphism $k \to k+1$ of $\mathbb{Z}$ extends to a unique homeomorphism of $\beta(\mathbb{Z})$ which we shall denote by φ. Note that $\varphi^{(n)}$ is the unique homeomorphic extension of $k \to k+n$. Also, note that $\varphi(\mathcal{C}(\mathbb{Z})) \subseteq \mathcal{C}(\mathbb{Z})$.

Given $\omega \in \beta(\mathbb{Z})$ the map $k \to \varphi^{(k)}(\omega)$ extends uniquely to a continuous function, $p_\omega : \beta(\mathbb{Z}) \to \beta(\mathbb{Z})$. Given $q \in \beta(\mathbb{Z})$ we set $p_\omega(q) = \omega * q$. Note that if $\omega \in \mathcal{C}(\mathbb{Z})$, then $\varphi^{(n)}(\omega) \in \mathcal{C}(\mathbb{Z})$ for all n. Hence when $\omega, q \in \mathcal{C}(\mathbb{Z})$, then $\omega * q \in \mathcal{C}(\mathbb{Z})$.

The following result, with $\mathbb{Z}$ replaced by $\mathbb{N}$, is contained in [2].

Proposition 2.12. *We have that* $(\mathcal{C}(\mathbb{Z}), *)$ *is a compact left-continuous, associative semigroup and there exists a point* $\omega \in \mathcal{C}(\mathbb{Z})$ *such that* $\omega * \omega = \omega$.

Proof. Left continuity means that if $q_\lambda \to q$ then $\omega * q_\lambda \to \omega * q$, which follows from the continuity of p_ω. Since $\mathcal{C}(\mathbb{Z})$ is compact all that remains is to show that the product is associative, i.e., that $\omega_1 * (\omega_2 * q) = (\omega_1 * \omega_2) * q$ for all $\omega_1, \omega_2, q \in \mathcal{C}(\mathbb{Z})$.

Associativity is equivalent to proving that, $p_{\omega_1}(p_{\omega_2}(q)) = p_{\omega_1 * \omega_2}(q)$. Since $\mathbb{Z}$ is dense it will suffice to prove this equality for all $q = n \in \mathbb{Z}$. To this end choose a net of integers, $\{m_\alpha\}$ that converges to ω_2.

We have that, $p_{\omega_1}(p_{\omega_2}(n)) = p_{\omega_1}(\varphi^{(n)}(\omega_2)) = \lim_\alpha p_{\omega_1}(\varphi^{(n)}(m_\alpha)) = \lim_\alpha p_{\omega_1}(n + m_\alpha) = \lim_\alpha \varphi^{(n+m_\alpha)}(\omega_1)$. On the other hand, $p_{\omega_1 * \omega_2}(n) = \varphi^{(n)}(\omega_1 * \omega_2) = \varphi^{(n)}(p_{\omega_1}(\omega_2)) = \lim_\alpha \varphi^{(n)}(p_{\omega_1}(m_\alpha)) = \lim_\alpha \varphi^{(n)}(\varphi^{(m_\alpha)}(\omega_1)) = \lim_\alpha \varphi^{(n+m_\alpha)}(\omega_1)$ and so associativity follows.

The existence of the point ω now follows by [2, Theorem 3.3]. □

A point ω satisfying $\omega * \omega = \omega$ is called an *idempotent point* or an *idempotent ultrafilter*.

Theorem 2.13. *Let $\omega \in \mathcal{C}(\mathbb{Z})$ be an idempotent point, then the map, $p_\omega : \beta(\mathbb{Z}) \to \mathcal{R}(p_\omega)$ is a $\mathbb{Z}$-retraction onto a proper subset. Consequently, the map, $\pi_\omega : C(\beta(\mathbb{Z})) \to C(\beta(\mathbb{Z}))$ defined by, $\pi_\omega(f) = f \circ p_\omega$ is a $\mathbb{Z}$-equivariant idempotent *-homomorphism onto a proper subalgebra with $\sigma(\pi_\omega) = \mathbb{T}$.*

Proof. Since, $\mathcal{R}(p_\omega) \subseteq \mathcal{C}(\mathbb{Z})$ it is a proper subset. Note that $(p_\omega \circ p_\omega)(q) = \omega * (\omega * q) = (\omega * \omega) * q = \omega * q = p_\omega(q)$, and so the map p_ω is idempotent.

Finally, to see that it is $\mathbb{Z}$-equivariant it is enough to show that $p_\omega \circ \varphi = \varphi \circ p_\omega$. To this end, it is enough to consider a dense set. We have that $p_\omega(\varphi(n)) = p_\omega(n+1) = \varphi^{(n+1)}(\omega) = \varphi \circ \varphi^{(n)}(\omega) = \varphi(p_\omega(n))$. The rest of the proof follows from Proposition 2.11. $\square$

We now present an example of a completely positive, $\mathbb{Z}$-equivariant projection, Φ such that $\sigma(\Phi)$ is a dense subgroup of $\mathbb{T}$ that is *not* closed. A modification of this example will lead to the counterexample to Solel's conjecture.

The construction of this example uses Hamana's theory [6] of the G-injective envelope, $I_G(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ which we will outline below.

Recall that maps between two spaces equipped with a G-action are called *G-equivariant* if they satisfy, $\phi(g \cdot a) = g \cdot \phi(a)$. A C^*-algebra equipped with an action by a discrete group G is called *G-injective* provided that it has the property that G-equivariant completely positive maps into it have G-equivariant completely positive extensions. The *G-injective envelope* $I_G(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ (or operator system) equipped with a G-action is a "minimal" G-injective C^*-algebra $\mathcal{B}$ containing $\mathcal{A}$. To obtain $I_G(\mathcal{A})$, Hamana first shows that $\mathcal{A}$ can always be G-equivariantly embedded into a C^*-algebra that is G-injective and then constructs a minimal, G-equivariant idempotent map that fixes $\mathcal{A}$. It is easy to see, and is pointed out in Hamana [6], that $\ell^\infty(G)$ is always G-injective. When $\mathcal{A}$ is contained in a G-injective object, the key difference between $I(\mathcal{A})$ and $I_G(\mathcal{A})$, is that to obtain the latter object, one must restrict to maps that fix $\mathcal{A}$ and are G-equivariant. Thus, generally, $I(\mathcal{A})$ can be a smaller object than $I_G(\mathcal{A})$, since one has a larger family of idempotents to minimize over.

In [4] it is shown that if $\mathcal{A}$ is an Abelian C^*-algebra, then $I_G(\mathcal{A})$ is also an Abelian C^*-algebra. We give an ad hoc argument of this fact for the case that we are interested in.

Let $\mathbb{T}$ denote the unit circle in the complex plane, fix an irrational number, θ_0, with $0 < \theta_0 < 1$ and let $\lambda_0 = e^{2\pi i \theta_0}$ so that the set $\{\lambda_0^n\}_{n \in \mathbb{Z}}$ is dense in $\mathbb{T}$. We regard $\mathbb{T}$ as a $\mathbb{Z}$-space with the action given by $n \cdot z = \lambda_0^n z$. There exists a $\mathbb{Z}$-equivariant *-monomorphism $\Pi : C(\mathbb{T}) \to \ell^\infty(\mathbb{Z})$ given by $\Pi(f) = (f(\lambda_0^n))$. Dually, this *-monomorphism is induced by the continuous $\mathbb{Z}$-equivariant function $\gamma : \beta(\mathbb{Z}) \to \mathbb{T}$ that is given uniquely by $\gamma(n) = \lambda_0^n$.

Since $C(\mathbb{T})$ has been embedded into $\ell^\infty(\mathbb{Z})$ in a $\mathbb{Z}$-equivariant manner, we may obtain $I_{\mathbb{Z}}(C(\mathbb{T}))$ as the range of a minimal $\mathbb{Z}$-equivariant idempotent map, ϕ, that fixes the image of $C(\mathbb{T})$. A priori, we only know that this range is an operator subsystem of $\ell^\infty(\mathbb{Z})$, but we can give it a necessarily unique product via the Choi–Effros construction, i.e., for $\phi(a)$ and $\phi(b)$ in the range of ϕ, we set $\phi(a) \circ \phi(b) = \phi(ab)$. Note that since $\ell^\infty(\mathbb{Z})$ is Abelian, this product will be Abelian.

Thus, $I_{\mathbb{Z}}(C(\mathbb{T}))$ is an Abelian C^*-algebra and if we identify $I_{\mathbb{Z}}(C(\mathbb{T})) = C(Y)$ then there is a homeomorphism, $\eta : Y \to Y$, which gives the $\mathbb{Z}$-action on Y, $n \cdot y = \eta^{(n)}(y)$, and on $C(Y)$ by $(n \cdot f)(y) = f(n \cdot y)$. The inclusion of $C(\mathbb{T})$ into $C(Y)$ is given by a $\mathbb{Z}$-equivariant onto map $h : Y \to \mathbb{T}$.

Choose any point, $y_0 \in Y$ with $h(y_0) = 1$ and set $y_n = n \cdot y_0$. By the universal properties of $\beta(\mathbb{Z})$, there is a unique continuous function, $\Gamma : \beta(\mathbb{Z}) \to Y$, with $\Gamma(n) = y_n$. Since, $\Gamma(\varphi(n)) = \Gamma(n+1) = y_{n+1} = \eta(y_n) = \eta(\Gamma(n))$, we see that Γ is $\mathbb{Z}$-equivariant. Also, since $h(\Gamma(n)) = \gamma(n)$, we see that $h \circ \Gamma = \gamma$, i.e., Γ is a $\mathbb{Z}$-equivariant lifting of γ. Hence, $\Gamma^* : C(Y) \to \ell^\infty(\mathbb{Z})$ is a $\mathbb{Z}$-equivariant *-homomorphism, with $\Pi(f) = \Gamma^*(f \circ h)$, that is, the restriction of Γ^* to the image of $C(\mathbb{T})$ is Π.

Since Π is a $\mathbb{Z}$-equivariant *-monomorphism, Γ^* must also be a *-monomorphism by Hamana's [5] $\mathbb{Z}$-essential property of the $\mathbb{Z}$-injective envelope. Hence, even though $C(Y)$ was only assumed to be an operator subsystem of $\ell^\infty(\mathbb{Z})$, it can always be embedded $\mathbb{Z}$-equivariantly as a C^*-subalgebra.

The difficult problem, as we shall see shortly, is proving that $C(Y)$ can be embedded in such a way that it is *not* a C^*-subalgebra!

Since Γ^* is a *-monomorphism, Γ must be an onto map. Since $\Gamma(n) = \eta^{(n)}(y_0)$, we see that the range of Γ is just the closure of the set $\{y_n \colon n \in \mathbb{Z}\}$, and hence, the orbit of y_0 is dense in Y. Recall that y_0 was just an arbitrary point in Y satisfying $h(y_0) = 1$. Thus, every point in Y that is a pre-image of 1 has a dense orbit. Moreover, the *-monomorphism of $C(Y)$ into $\ell^\infty(\mathbb{Z}) = C(\beta(\mathbb{Z}))$ is given by $\Gamma^*(f) = (f(y_n)) \in \ell^\infty(\mathbb{Z})$.

Since $C(Y)$ is $\mathbb{Z}$-injective there will exist a completely positive $\mathbb{Z}$-equivariant idempotent map Φ on $\ell^\infty(\mathbb{Z})$ whose range is the image of $C(Y)$, $\Gamma^*(C(Y))$. By the above results, Φ is either the identity map or singular.

We now argue that Φ cannot be the identity map. To see this note that $\Pi(C(\mathbb{T})) \cap c_0(\mathbb{Z}) = (0)$. Hence, if we compose Π with the quotient map into $\ell^\infty(\mathbb{Z})/c_0(\mathbb{Z})$, then this composition will still be a *-monomorphism on $C(\mathbb{T})$ and hence will also be a *-monomorphism on $C(Y)$. Hence, the image of $C(Y)$ cannot be onto and hence Φ cannot be the identity map and thus is singular.

We claim that for this map, $-1 \notin \sigma(\Phi)$. First note that if we let $f_m(z) = z^m$, then $\Pi(f_m)(n) = f_m(\lambda_0^n) = \lambda_0^{mn}$. Hence, $\Pi(f_m) = x_{\lambda_0^m}$, and so $x_{\lambda_0^m}$ is in the range of Φ. Thus, by definition, $\lambda_0^m \in \sigma(\Phi)$ for all $n \in \mathbb{Z}$ and we have that $\sigma(\Phi)$ contains this dense subgroup of $\mathbb{T}$.

Now assume that $-1 \in \sigma(\Phi)$. Let $p_i \in \ell^\infty(\mathbb{Z})$, $i = 0, 1$, be the projections onto the even and odd integers, i.e., p_i, $i = 0, 1$, are the characteristic functions of these sets. Since $\pm 1 \in \sigma(\Phi)$, then $p_0 = (x_{-1} + x_{+1})/2$ is in the range of Φ, and hence, p_1 is also in the range of Φ. Consequently, there exists disjoint, cl-open sets Y_i, $i = 0, 1$, with $Y = Y_0 \cup Y_1$ such that $p_i = \Gamma^*(\chi_{Y_i})$, $i = 0, 1$.

Thus, $p_0(n) = \chi_{Y_0}(y_n)$, and we see that, $y_n \in Y_0$, for n even and $y_n \in Y_1$, for n odd.

Note that since Γ is a lifting of γ, we have that $h(y_0) = 1 \in \mathbb{T}$ and since Γ is equivariant, $h(y_n) = h(n \cdot y_0) = \lambda_0^n$.

Since θ_0 was irrational, there exists a sequence of odd integers n_k, such that $\lambda_0^{n_k}$ converges to 1. Since Y is compact, some subnet of $y_{n_k} \in Y_1$ will converge to a point $z_0 \in Y_1$, and $h(z_0) = 1$.

Now let, $z_n = \eta^{(n)}(z_0)$, so that $h(z_n) = \lambda_0^n$ and define another $\mathbb{Z}$-equivariant lifting of γ, $\Gamma_1 : \beta(\mathbb{Z}) \to Y$ by $\Gamma_1(n) = z_n$. The homomorphism Γ_1^* also extends Π and so it too must be a *-monomorphism and the orbit of z_0 must also be dense.

Note that, since z_0 is a limit of odd y_m's, we have that $z_n \in Y_0$ for n odd and $z_n \in Y_1$ for n even. Thus, $\Gamma_1^*(\chi_{Y_0}) = p_1$.

Finally, let $\Psi : C(Y) \to \ell^\infty(\mathbb{Z})$ be defined by $\Psi = (\Gamma^* + \Gamma_1^*)/2$. Then Ψ is completely positive, $\mathbb{Z}$-equivariant, and its restriction to $C(\mathbb{T})$ is Π, so again by the properties of the $\mathbb{Z}$-injective envelope, Ψ must be a complete order injection onto its range.

But, $2\Psi(\chi_{Y_0} - \chi_{Y_1}) = p_0 - p_1 + p_1 - p_0 = 0$, a contradiction. Thus, $-1 \notin \sigma(\Phi)$.

A similar argument can be used to show that no root of unity can be in $\sigma(\Phi)$.

We summarize some of these results below.

Theorem 2.14. *There exists a $\mathbb{Z}$-equivariant, unital completely positive, idempotent, $\Phi:\ell^\infty(\mathbb{Z})\to\ell^\infty(\mathbb{Z})$ with $\mathcal{R}(\Phi)$ a C^*-subalgebra, such that Φ is not a homomorphism, $\Phi(c_0(\mathbb{Z}))=0$ and $\sigma(\Phi)$ is a dense, proper subgroup of $\mathbb{T}$.*

Proof. Let Φ be the projection onto $\Gamma(C(Y))$, as above. Then we have shown that Φ has the last two properties. But we have also seen that the spectrum of a homomorphism must be the entire circle. Thus, Φ cannot be a homomorphism. □

We now present an example to show that the analogue of Solel's theorem is not true in this setting.

Theorem 2.15. *There exists a $\mathbb{Z}$-equivariant, unital completely positive, idempotent $\Phi:\ell^\infty(\mathbb{Z})\to\ell^\infty(\mathbb{Z})$ whose range is not a C^*-subalgebra.*

Proof. We retain the notation of the above discussion. Let $C(Y)=I_{\mathbb{Z}}(C(\mathbb{T}))$, $h:Y\to\mathbb{T}$, and $\eta:Y\to Y$ be as above. It is easy to see that if $h^{-1}(\{1\})$ was a singleton, then necessarily, h is one-to-one. But this is impossible since $C(\mathbb{T})$ is not $\mathbb{Z}$-injective. So let $y_0\neq w_0$ be points in Y with $h(y_0)=h(w_0)=1$ and let $y_n=\eta^{(n)}(y_0)$, $w_n=\eta^{(n)}(w_0)$.

These points yield two continuous $\mathbb{Z}$-equivariant maps, $\Gamma_1,\Gamma_2:\beta(\mathbb{Z})\to Y$ by setting, $\Gamma_1(n)=y_n$, $\Gamma_2(n)=w_n$ and corresponding $\mathbb{Z}$-equivariant *-homomorphisms $\Gamma_i^*:C(Y)\to\ell^\infty(\mathbb{Z})$, $i=1,2$. Since both of these *-homomorphisms extend, $\Pi=\gamma^*:C(\mathbb{T})\to\ell^\infty(\mathbb{Z})$ which is a *-monomorphism, then by the properties of the injective envelope Γ_i, $i=1,2$, will both be *-monomorphisms.

Consider the unital, $\mathbb{Z}$-equivariant, completely positive map, $\Psi=\frac{\Gamma_1+\Gamma_2}{2}$. Since Γ_i, $i=1,2$, are both extensions of Π, we have that the restriction of Ψ to $C(\mathbb{T})$ is a *-monomorphism. Again using Hamana's property [5] that $C(Y)$ is a $\mathbb{Z}$-essential extension of $C(\mathbb{T})$, we have that Ψ will be a complete order isomorphism of $C(Y)$ into $\ell^\infty(\mathbb{Z})$.

If $\mathcal{R}(\Psi)$ was a C^*-subalgebra of $\ell^\infty(\mathbb{Z})$, then by the Banach–Stone theorem [8] Ψ would be a *-isomorphism onto its range. We now argue that Ψ cannot be a *-homomorphism.

Since, $C(Y)$ is injective, it is generated by its projections. Now let, $p\in C(Y)$ be any projection, then $\Gamma_i^*(p)=\chi_{E_i}$, $i=1,2$, and $\Psi(p)=\chi_{E_3}$ must be the characteristic functions of three sets. But since $\chi_{E_3}=\frac{\chi_{E_1}+\chi_{E_2}}{2}$, examining points where both sides are 0 or 1, it follows that $E_3=E_1\cap E_2$ and $E_1^{\mathrm{c}}=(E_1\cup E_2)^{\mathrm{c}}$ and hence, $E_1=E_2=E_3$. Thus, $\Psi=\Gamma_1^*=\Gamma_2^*$, which contradicts the choice of Γ_1 and Γ_2.

Thus, $\mathcal{R}(\Psi)$ is not a C^*-subalgebra, but since it is completely order isomorphic to $C(Y)$ it is a $\mathbb{Z}$-injective operator subsystem of $\ell^\infty(\mathbb{Z})$ and so we may construct a $\mathbb{Z}$-equivariant, completely positive, idempotent projection Φ of $\ell^\infty(\mathbb{Z})$ onto it. □

We are grateful to W.B. Arveson for the following argument.

Recall that $\mathbb{T}$ is the set of characters of $\mathbb{Z}$ and that for each $\mu\in\mathbb{T}$ the corresponding character is the function, $f_\mu(n)=\mu^n$, i.e., $f_\mu=x_\mu$ under the identification of functions with vectors. Thus, the C^*-subalgebra generated by the set $\{x_\mu\colon\ \mu\in\mathbb{T}\}$ is nothing more than the C^*-subalgebra generated by the characters, which is the C^*-algebra, $AP(\mathbb{Z})$ of almost periodic functions on $\mathbb{Z}$. By [1] $AP(\mathbb{Z})=C(b\mathbb{Z})$, where $b\mathbb{Z}$ is the Bohr compactification of $\mathbb{Z}$. Recall, $b\mathbb{Z}=\widetilde{\mathbb{T}}_d$, the group of characters of the Abelian group, $\mathbb{T}_d$ where $\mathbb{T}_d$ denotes $\mathbb{T}$ with the discrete topology.

Recall that a topological space, X, is called 0-*dimensional* if it is Hausdorff and the cl-open sets are a basis for the topology of X. It is fairly easy to show that every extremally disconnected, compact Hausdorff space is 0-dimensional. For a proof, see the text [10, Proposition 10.69].

If $C(b\mathbb{Z})$ was injective, then $b\mathbb{Z}$ would be extremally disconnected and, consequently, 0-dimensional. Hence, by [7, Theorem 24.26], the character group of $b\mathbb{Z}$ would be a torsion group. But [7, Theorem 26.12] $\mathbb{T}_d$ is the character group of $b\mathbb{Z}$ which is not a torsion group, contradiction.

Therefore, $AP(\mathbb{Z})$ is not an injective C^*-subalgebra of $\ell^\infty(\mathbb{Z})$. This leads to the following observation.

Theorem 2.16. *The $\mathbb{Z}$-injective envelope of $AP(\mathbb{Z})$ is strictly contained in $\ell^\infty(\mathbb{Z})$. There is a $\mathbb{Z}$-equivariant, unital completely positive projection, $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ that fixes $AP(\mathbb{Z})$, annihilates $c_0(\mathbb{Z})$, whose range is a C^*-subalgebra that is *-isomorphic to $I_{\mathbb{Z}}(AP(\mathbb{Z}))$ and $\sigma(\Phi) = \mathbb{T}$.*

Proof. Since $AP(\mathbb{Z}) \cap c_0(\mathbb{Z}) = 0$, the quotient map into $\ell^\infty(\mathbb{Z})/c_0(\mathbb{Z})$ is $\mathbb{Z}$-equivariant and a complete isometry on $AP(\mathbb{Z})$. Hence, again using the $\mathbb{Z}$-essential property, the quotient map must be a complete isometry on the $\mathbb{Z}$-injective envelope.

Thus, $I_{\mathbb{Z}}(AP(\mathbb{Z}))$ must be properly contained in $\ell^\infty(\mathbb{Z})$ and we may choose a unital completely positive, $\mathbb{Z}$-equivariant projection onto it that annihilates $c_0(\mathbb{Z})$. Because $x_\lambda \in AP(\mathbb{Z})$, we will have that $\Phi(x_\lambda) = x_\lambda$, for every $\lambda \in \mathbb{T}$. □

We remark that the same proof yields the following result.

Theorem 2.17. *The injective envelope of $AP(\mathbb{Z})$ is strictly contained in $\ell^\infty(\mathbb{Z})$. There is a unital completely positive projection $\Phi : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ that fixes $AP(\mathbb{Z})$ and whose range is a C^*-subalgebra, that is, *-isomorphic to $I(AP(\mathbb{Z}))$.*

Problem 2.18. Is $I(AP(\mathbb{Z})) = I_{\mathbb{Z}}(AP(\mathbb{Z}))$?

Note that if $\omega \in \mathcal{C}(\mathbb{Z})$ is an idempotent point, then the induced idempotent, $\mathbb{Z}$-equivariant *-homomorphism, π_ω given by Theorem 2.13 has $\sigma(\pi_\omega) = \mathbb{T}$. Thus, π_ω is a projection that fixes $AP(\mathbb{Z})$.

Problem 2.19. If ω is an idempotent point, then is the range of π_ω a copy of $I_{\mathbb{Z}}(AP(\mathbb{Z}))$, that is, is the range completely isometrically isomorphic to $I_{\mathbb{Z}}(AP(\mathbb{Z}))$ via a $\mathbb{Z}$-equivariant map that fixes $AP(\mathbb{Z})$?

This problem is equivalent to asking if π_ω is a minimal element in the set of all $\mathbb{Z}$-equivariant idempotent maps that fix $AP(\mathbb{Z})$.

3. MASA bimodule projections

A subspace $\mathcal{T} \subseteq B(\mathcal{K})$ is called a *ternary subalgebra* provided that, $A, B, C \in \mathcal{T}$ implies that $AB^*C \in \mathcal{T}$. It is known that if $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ is a completely contractive, idempotent map, then the range $\mathcal{R}$ of Φ is completely isometrically isomorphic to a ternary subalgebra of operators on some Hilbert space.

Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a maximal Abelian subalgebra (MASA). It is also known that if $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ is a MASA bimodule map, then $\|\Phi\| = \|\Phi\|_{cb}$.

Solel [14] proves that if $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ is a weak*-continuous, contractive, idempotent $\mathcal{M}$-bimodule map, then the range of Φ is a ternary subalgebra of $B(\mathcal{H})$. Thus, under these stronger hypotheses, the completely isometric isomorphism can be taken to be the identity.

In particular, Solel's result implies that the range of any weak*-continuous, unital, completely positive MASA bimodule idempotent, must be a C^*-subalgebra of $B(\mathcal{H})$.

We will prove that the analogue of Solel's result is false in the non-weak*-continuous case. When the MASA is discrete, then MASA bimodule maps are known to be automatically weak*-continuous, so the main case of interest is when the MASA is, for example, $L^\infty(\mathbb{T})$ represented as multiplication operators on $B(L^2(\mathbb{T}))$. This subalgebra is maximal.

In this section, we show how $\mathbb{Z}$-equivariant idempotent maps on $\ell^\infty(\mathbb{Z})$ can be used to construct L^∞-bimodule idempotent maps on $B(L^2(\mathbb{T}))$. The idea of the construction can be traced back to Arveson's construction of a concrete projection of $B(L^2(\mathbb{T}))$ onto $L^\infty(\mathbb{T})$.

Let $z^n = e^{in\theta}$, $n \in \mathbb{Z}$, denote the standard orthonormal basis for $L^2(\mathbb{T})$. This basis defines a Hilbert space isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$. We identify bounded operators on $\ell^2(\mathbb{Z})$ with the infinite matrices $(a_{i,j})$, $i, j \in \mathbb{Z}$, and using this isomorphism, the multiplication operator for a function f is identified with the bounded, *Laurent matrix*, $(a_{i,j})$, where $a_{i,j} = \hat{f}(i-j)$, the Fourier coefficient. In particular, the operator of multiplication by z corresponds to the bilateral shift operator B.

We further identify $\ell^\infty(\mathbb{Z})$ with the bounded, diagonal operators $\mathcal{D} \subseteq B(\ell^2(\mathbb{Z}))$. Note that under this identification, the action of $\mathbb{Z}$ on $\ell^\infty(\mathbb{Z})$ induced by translation is implemented by conjugation by B. We define $\alpha(n) : \mathcal{D} \to \mathcal{D}$ by $\alpha(n)(D) = B^n D B^{-n}$. Thus, a map $\Phi : \mathcal{D} \to \mathcal{D}$ is $\mathbb{Z}$-equivariant if and only if $\Phi(B^n D B^{-n}) = B^n \Phi(D) B^{-n}$, for all D and all n, which is if and only if $\Phi(BDB^{-1}) = B\Phi(D)B^{-1}$, for all D.

We define, $E : B(\ell^2(\mathbb{Z})) \to \mathcal{D}$, by letting $E((a_{i,j}))$ be the diagonal operator, with diagonal entries, $a_{i,i}$. It is well known that, E is a completely positive projection from $B(\ell^2(\mathbb{Z}))$ onto $\mathcal{D}$. Given $A \in B(\ell^2(\mathbb{Z}))$ we set $\hat{A}(n) = E(AB^{-n}) \in \mathcal{D}$ and we call $\sum_{n\in\mathbb{Z}} \hat{A}(n)B^n$ the *formal Fourier series* for A.

We should remark that just as with L^∞-functions, the formal Fourier series uniquely determines A, but does not need to converge to A in any reasonable sense. In fact, it need not converge to A in even the weak operator topology.

The key fact, whose proof we defer until later, is the following theorem.

Theorem 3.1. *Let $\Phi : \mathcal{D} \to \mathcal{D}$ be a $\mathbb{Z}$-equivariant, unital completely positive map. Then there is a well-defined unital completely positive, L^∞-bimodule map, $\Gamma : B(\ell^2(\mathbb{Z})) \to B(\ell^2(\mathbb{Z}))$ satisfying $\Gamma(\sum_{n\in\mathbb{Z}} D_n B^n) = \sum_{n\in\mathbb{Z}} \Phi(D_n) B^n$. Moreover, if Φ is idempotent, then Γ is idempotent.*

Corollary 3.2. *There exists a unital, completely positive, idempotent $L^\infty(\mathbb{T})$-bimodule map $\Gamma : B(L^2(\mathbb{T})) \to B(L^2(\mathbb{T}))$ whose range is not a C^*-subalgebra.*

Proof. Let $\Phi : \mathcal{D} \to \mathcal{D}$ be the $\mathbb{Z}$-equivariant, unital completely positive, idempotent map given by Theorem 2.15 whose range is not a C^*-subalgebra and let $\Gamma : B(\ell^2(\mathbb{Z})) \to B(\ell^2(\mathbb{Z}))$ be the map given by the above theorem. If $\mathcal{R}(\Gamma)$ was a C^*-subalgebra, then $\mathcal{R}(\Gamma) \cap \mathcal{D} = \mathcal{R}(\Phi)$ would also be a C^*-subalgebra. Hence, $\mathcal{R}(\Gamma)$ is not a C^*-subalgebra. The proof is completed by making the identification of $\ell^2(\mathbb{Z})$ with $L^2(\mathbb{T})$ which carries the Laurent matrices to the multiplication operators. $\square$

Note that when Φ is singular, Γ will also not be weak*-continuous. Thus, using the example from the previous section, we see that there exists a unital, completely positive, idempotent L^∞-bimodule map Γ, such that not only is $\Gamma(K) = 0$ for every compact operator K, but $\Gamma(A) = 0$, whenever $\hat{A}(n) \in c_0(\mathbb{Z})$ for all n.

Problem 3.3. Does there exist a unital, completely positive, idempotent, $\mathbb{Z}$-equivariant map $\Phi : \mathcal{D} \to \mathcal{D}$, whose range is a C^*-subalgebra, but such that the range of $\Gamma : B(\ell^2(\mathbb{Z})) \to B(\ell^2(\mathbb{Z}))$ is not a C^*-subalgebra?

Before proving the above theorem, we will need a few results about cross-products. Recall that in general we can form two crossed-products, a full and reduced crossed-product, but when the group is amenable, then these crossed-products agree [13, Theorem 7.7.7]. Since $\mathbb{Z}$ is amenable, we let $\mathbb{Z} \times_\alpha \mathcal{D}$ be this C^*-algebra, where α is the action defined earlier. A dense set of elements of the crossed product is given by finitely supported functions, $f : \mathbb{Z} \to \mathcal{D}$. Given any pair consisting of a *-homomorphism $\pi : \mathcal{D} \to B(\mathcal{H})$ and a unitary $U \in B(\mathcal{H})$ such that $U^n \pi(D) U^{-n} = \pi(\alpha(n)(D))$ (such a pair is called a *covariant pair*), there exists a *-homomorphism $\Pi : \mathbb{Z} \times_\alpha \mathcal{D} \to B(\mathcal{H})$, satisfying $\Pi(f) = \sum_n f(n) U^n$.

Lemma 3.4. *Let $\pi : \mathcal{D} \to B(\ell^2(\mathbb{Z}))$ be the identity inclusion and let $B \in B(\ell^2(\mathbb{Z}))$ denote the bilateral shift. Then these are a covariant pair and the map, $\Pi : \mathbb{Z} \times_\alpha \mathcal{D} \to B(\ell^2(\mathbb{Z}))$ is a *-monomorphism.*

Proof. By [13, Theorem 7.7.5], if $\lambda : \mathbb{Z} \to B(\ell^2(\mathbb{Z}))$ denotes the left regular representation, then $\tilde{\pi} = \pi \otimes id : \mathcal{D} \to B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}))$ and $\tilde{\lambda} = id \otimes \lambda : \mathbb{Z} \to B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}))$ are a covariant pairs and the induced representation $\pi \times \lambda : \mathbb{Z} \times_\alpha \mathcal{D} \to B(\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}))$ is faithful, i.e., a *-monomorphism.

Let $\mathcal{H}_n = e_n \otimes \ell^2(\mathbb{Z})$ so that $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) = \sum_{n\in\mathbb{Z}} \bigoplus \mathcal{H}_n$. Each of these subspaces is a reducing subspace for the image of $\mathbb{Z} \times_\alpha \mathcal{D}$ and the restrictions to any pair of them are unitarily equivalent. Moreover, the restriction to $\mathcal{H}_0$ is Π. Hence, Π must be a *-monomorphism. □

Thus, we may identify $\mathbb{Z} \times_\alpha \mathcal{D}$ with the norm closure in $B(\ell^2(\mathbb{Z}))$ of the operators that are finite sums of the form $\sum D_n B^n$ with $D_n \in \mathcal{D}$.

Theorem 3.5. *Let G be a discrete group, let A be a C^*-algebra and let $\alpha : G \to Aut(A)$ be a group action. If $\Phi : A \to B(\mathcal{H})$ is a completely positive map, $\rho : G \to B(\mathcal{H})$ is a unitary representation, such that $\rho(g)\Phi(a)\rho(g^{-1}) = \Phi(\alpha(g)(a))$, i.e., a covariant pair, then there is a completely positive map $\rho \times \Phi : G \times_\alpha A \to B(\mathcal{H})$, satisfying $\rho \times \Phi(f) = \sum_{g\in G} \Phi(f(g))\rho(g)$, for any finitely supported function $f : G \to A$.*

Proof. This is a restatement of the covariant version of Stinespring's theorem [11, Theorem 2.1]. Let $\pi : A \to B(\mathcal{K})$, $\tilde{\rho} : G \to B(\mathcal{K})$ and $V : \mathcal{H} \to \mathcal{K}$ be the covariant pair that dilates Φ, ρ, then $(\rho \times \Phi)(f) = V^*((\tilde{\rho} \times \pi)(f))V$ and the result follows. □

We now turn our attention to the proof of Theorem 3.1. By the above results, given any $\Phi : \mathcal{D} \to \mathcal{D}$, that is, completely positive and $\mathbb{Z}$-equivariant, we have a well-defined completely positive map Γ, satisfying for any finite sum $\Gamma(\sum D_n B^n) = \sum \Phi(D_n) B^n$ whose domain is the norm closure of such finite sums and we wish to extend it to all of $B(\ell^2(\mathbb{Z}))$.

Consider for any $0 \leqslant r < 1$, the matrix, $P_r = (r^{|i-j|}) = (I - rB)^{-1} + (I - rB^*)^{-1} - I \geqslant 0$, since it is the "operator Poisson kernel" [12, Exercise 2.15]. Thus, the corresponding Schur product map $S_r : B(\ell^2(\mathbb{Z})) \to B(\ell^2(\mathbb{Z}))$ given by $S_r(a_{i,j}) = (r^{|i-j|}a_{i,j})$ is completely positive and unital. Writing $A = (a_{i,j}) \sim \sum D_n B^n$ in its formal Fourier series, we see that $A_r \equiv S_r(A) = \sum r^{|n|} D_n B^n$, where in the latter case we have absolute norm convergence of the partial sums. Hence, for any A, $S_r(A) \in \mathbb{Z} \times_\alpha \mathcal{D}$.

This shows that $\Gamma(S_r(A)) = \sum r^{|n|} \Phi(D_n) B^n$.

Note that, for any $A \in B(\ell^2(\mathbb{Z}))$ we have that $A \geqslant 0$ if and only if $A_r \geqslant 0$ for all $0 \leqslant r < 1$, and $\|A\| = \sup_{0 \leqslant r < 1} \|A_r\|$. Finally, given any formal matrix $A = (a_{i,j})$ it is easily checked that A defines a bounded operator if and only if $\sup_{0 \leqslant r < 1} \|A_r\|$ is finite and that this supremum equals the norm.

Hence, if $A \sim \sum D_n B^n$ is bounded, then $\| \sum r^{|n|} \Phi(D_n) B^n \| \leqslant \|\Phi\| \|A\|$ for all $0 \leqslant r < 1$ and hence, $\sum \Phi(D_n) B^n$ is bounded. This shows that we may extend Γ to all of $B(\ell^2(\mathbb{Z}))$ and the norm of the extended map will be at most $\|\Phi\|$. Using the positivity properties of A_r we see that the extended map Γ will be positive. Complete positivity of the extended map follows similarly.

Finally, since in Theorem 3.1, we are assuming that Φ is unital, Γ will fix every Laurent matrix. Since the Laurent matrices form a C^*-subalgebra, by Choi's theory of multiplicative domains [3] (see also [12]), we have that Γ is a bimodule map over the Laurent matrices.

This completes the proof of Theorem 3.1.

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