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Rational Approximation with Locally Geometric Rates

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We investigate the rate of pointwise rational approximation of functions from two classes. The distinguishing feature of these classes is the essentially faster convergence of the best uniform rational approximants versus best uniform polynomial approximants. It is known that for piecewise analytic functions "near best" polynomials converging geometrically fast at every point of analyticity of the function exist. Here we construct rational approximants enjoying similar properties. We also show that our construction yields rates of convergence that are, in a certain sense, best possible. © 1998 Academic Press

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1. INTRODUCTION

Let $\mathbf{P}_n$ denote the set of all algebraic polynomials of degree at most n, $n \geqslant 0$, and let $\mathbf{R}_n$ be the class of all rational functions $r = p/q$, $p, q \in \mathbf{P}_n$, $q \not\equiv 0$. For any $f \in C[-1, 1]$, we denote by

$$E_n(f) := \inf_{p \in \mathbf{P}_n} \|f - p\|_{[-1,1]}, \qquad R_n(f) := \inf_{r \in \mathbf{R}_n} \|f - r\|_{[-1,1]},$$

the errors in best approximation of f on $[-1, 1]$ by elements of $\mathbf{P}_n$ and $\mathbf{R}_n$, respectively. Here and in what follows, $\|\cdot\|$ stands for the uniform norm on an indicated interval.

In the following, c_0, c, C, etc. denote positive constants, possibly different at each occurrence, which are either absolute or depend on certain parameters. When necessary, this dependence will be indicated.

Given sequences $a_n > 0$, $b_n > 0$, we write $a_n \asymp b_n$ if there exist c_1, c_2 such that $c_1 b_n \leqslant a_n \leqslant c_2 b_n$, for $n \geqslant 1$.

The famous theorem of D. J. Newman [7] states that

$$c_1 e^{-9\sqrt{n}} \leqslant R_n(|x|) \leqslant c_2 e^{-\sqrt{n}}, \tag{1.1}$$

while it is a well-known result of S. Bernstein that $E_n(|x|) \asymp n^{-1}$. Newman's surprising result (which was later refined by Vyacheslavov [17] and Stahl [11]) stimulated numerous investigations, and various classes of functions were found for which $R_n(f)$ tends to zero substantially faster than $E_n(f)$. In this paper we consider two of these classes.

The first is the class of *piecewise analytic functions*. Recall that f is piecewise analytic on $[-1, 1]$ if there exists a partition

$$-1 = x_0 < x_1 < \cdots < x_{s-1} < x_s = 1, \qquad s \geqslant 2, \tag{1.2}$$

such that the restriction of f to each $[x_j, x_{j+1}]$, $0 \leqslant j \leqslant s-1$, has an analytic continuation to a neighborhood of this *closed* interval, but f itself is not analytic at $x_1, ..., x_{s-1}$. For such f, it is known that $E_n \asymp n^{-k}$, for some $k \geqslant 1$ (cf. [14]). On the other hand, it was shown by Turán and Szűsz [16] that

$$R_n(f) \leqslant C e^{-c\sqrt{n}}, \qquad n \geqslant 0. \tag{1.3}$$

The second class that we shall investigate was originally considered by Gonchar [3] and, in the general case, by Szabados [12, 13]. Let $f \in C[-1, 1]$ and assume there exists a partition (1.2) such that the restriction of f to each *open* interval (x_j, x_{j+1}) has an analytic and bounded continuation to some open rhombus D_j with opposite vertices x_j, x_{j+1}, $0 \leqslant j \leqslant s-1$. Then we say that f belongs to the *Gonchar–Szabados class* ($f \in \mathrm{GS}$).

Further, let ω_f denote the modulus of continuity of $f \in \mathrm{GS}$ on $[-1, 1]$. Then (cf. [13, Theorem 3])

$$R_n(f) \leqslant C\omega_f(e^{-t_n}), \tag{1.4}$$

where t_n satisfies the relation

$$\omega_f(e^{-t_n}) = t_n e^{-cn/t_n}, \qquad c = c(f). \tag{1.5}$$

Note that the Gonchar–Szabados class contains all functions piecewise analytic on $[-1, 1]$. Moreover, if f is piecewise analytic, then $\omega_f(\delta) \asymp \delta$ and it can be seen that (1.4) and (1.5) yield (1.3). Gonchar also proved (cf. [2, 4]) that the bounds (1.3), (1.4) are, in general, sharp.

In view of the structures of the above functions, it is reasonable to expect that a sequence of polynomials (or rational functions) exists, such that it converges to f with a global rate close to the best one, and at the same time converges to f much faster (say, geometrically) at points of analyticity of f. This problem was investigated for the polynomial case in [1, 5, 10, 15]. For example, the following result was obtained by Saff and Totik.

THEOREM 1.1 [10]. *Let f be piecewise analytic on* $[-1, 1]$ *and belong to* C^{k-1}, *for some* $k \geqslant 1$. *Then given* $\beta > 1$, *there exist constants* C, c *and polynomials* $p_n \in \mathbf{P}_n$, $n = 1, 2, \ldots$, *such that*

$$|f(x) - p_n(x)| \leqslant Cn^{-k} \exp(-cn[d(x)]^{\beta}), \qquad x \in [-1, 1], \tag{1.6}$$

where $d(x)$ *is the distance from* x *to the nearest singularity* x_j *of* f *on* $(-1, 1)$.

Moreover, (1.6) *does not in general hold with* $\beta = 1$.

Since $E_n(f) \asymp n^{-k}$ for f in Theorem 1.1 (provided $f \notin C^k[-1, 1]$), we see that it is possible to construct "near best" polynomial approximants that converge to f geometrically fast at every regular point of f on $[-1, 1]$. Hence, to maintain the advantage of rational approximants with respect to polynomial ones, it is desirable to construct rational functions converging to f with the global rate (1.3) and geometrically at regular points of f.

In Section 2, we examine Newman's approximants to $|x|$ and show that they do *not* converge geometrically for $x \neq 0$. We modify Newman's construction in Section 3 and apply this to the approximation of the signum function. Having done this, we immediately get the desired approximation for $|x|$. We show, for example (this is a special case of Theorem 4.1 proved in Section 4), that given $\beta > 1$, there exist $r_n \in \mathbf{R}_n$, $n = 1, 2, \ldots$, and positive constants C, c depending only on β such that

$$||x| - r_n(x)| \leqslant Ce^{-c\sqrt{n}} \exp\left(-cn \Big/ \left(\log \frac{2}{|x|}\right)^{\beta}\right), \qquad x \in [-1, 1]. \tag{1.7}$$

Note that the second exponential factor in (1.7) decreases much faster than $\exp(-cn\,|x|^{\beta})$. Therefore (see (1.6)), the local geometric rate is also much better (for x close to the singularity $x = 0$ of $|x|$) than in the polynomial case. We also show that (1.7) is impossible with $\beta = 1$ (this is a special case of Theorem 4.2). Finally, in Section 5, we consider functions of the Gonchar–Szabados class.

2. NONGEOMETRIC CONVERGENCE OF NEWMAN'S APPROXIMANTS

We first recall Newman's construction [7]. Let

$$N_n(x) := \prod_{j=0}^{n-1} (\zeta^j + x), \qquad \zeta := \exp(-1/\sqrt{n}) \tag{2.1}$$

and set

$$r_n(x) := \frac{N_n(x) - N_n(-x)}{N_n(x) + N_n(-x)} \in \mathbf{R}_n. \tag{2.2}$$

Then, for $x \geqslant 0$, there holds

$$|x - xr_n(x)| = 2x \frac{|N_n(-x)/N_n(x)|}{1 + N_n(-x)/N_n(x)} \geqslant x \left| \frac{N_n(-x)}{N_n(x)} \right|, \tag{2.3}$$

since $|N_n(-x)/N_n(x)| < 1$, for $x > 0$.

Next, fix $\varepsilon \in (0, 1)$ and split the product in (2.1) as

$$N_n = \prod_{j \in I(\varepsilon)} (\zeta^j + x) \prod_{j \notin I(\varepsilon)} (\zeta^j + x) =: N_n^{(1)} N_n^{(2)},$$

where $I(\varepsilon) := \{ j : \zeta^j < \varepsilon/n \}$. Since

$$\left| \frac{\zeta^j - x}{\zeta^j + x} \right| \geqslant 1 - \frac{2}{n}, \qquad x \in [\varepsilon, 1], \quad j \in I(\varepsilon),$$

we have

$$|N_n^{(1)}(-x)/N_n^{(1)}(x)| \geqslant \left(1-\frac{2}{n}\right)^n, \qquad x \in [\varepsilon, 1]. \tag{2.4}$$

Furthermore, $\zeta^j \geqslant \varepsilon/n$ implies $j \leqslant \sqrt{n}\log(n/\varepsilon)$, and so $\deg N_n^{(2)} = O(\sqrt{n}\log n)$. Therefore, for any $[\varepsilon, \eta] \subset [\varepsilon, 1]$ we have, by Newman's inequality (cf. [7, Lemma 3]),

$$\begin{aligned}\max_{[\varepsilon,\eta]} \{\log |N_n^{(2)}(-x)/N_n^{(2)}(x)|\} &\geqslant \frac{1}{\log(\eta/\varepsilon)} \int_\varepsilon^\eta \log \left|\frac{N_n^{(2)}(-x)}{N_n^{(2)}(x)}\right| \frac{dx}{x} \\ &\geqslant -c(\varepsilon, \eta)\sqrt{n}\log n. \end{aligned}\tag{2.5}$$

Applying (2.4), (2.5), we deduce from (2.3) that

$$\lim_{n\to\infty} \| |x| - x r_n(x) \|_{[\varepsilon,\eta]}^{1/n} = 1.$$

Thus, the Newman sequence $\{x r_n(x)\}$ does not converge geometrically to $|x|$ on any fixed interval $[\varepsilon, \eta] \subset [0, 1]$. Since $x r_n(x)$ is even, the same is true for $[-\eta, -\varepsilon] \subset [-1, 0]$.

As the above argument reveals, the lack of geometric convergence is an inevitable consequence of the extreme crowding of Newman's nodes, ζ^j, near 0. To gain geometric convergence, the idea is to use, for a given n, only one-half of these nodes (to retain an $\exp(-c\sqrt{n})$ rate) and then choose the remaining $n/2$ nodes in order to get geometric rates for $x \neq 0$. This technique will be employed in subsequent sections. (In a subsequent paper [6] we shall give a finer analysis for the possible global rate of convergence when geometric rates hold for $x \neq 0$.)

One may naturally ask whether the *best* uniform rational approximants to $|x|$ on $[-1, 1]$ have the desired geometric convergence property. However, it was shown by Saff and Stahl [9] that the extreme points (alternation points) for this best approximation problem are *dense* in $[-1, 1]$, and so (1.1) implies that geometric convergence fails to hold on any subinterval.

3. RATIONAL APPROXIMATION OF sgn x

The importance of the signum function, sgn x, in both polynomial and rational approximation, is well known. Once a good approximation is obtained for sgn x, we easily get one for any step-function, and the extension to continuous functions is standard (see Section 4). A glance at the equality in (2.3) (divided by x) shows that given any polynomial P_n satisfying

$$|P_n(-x)/P_n(x)| \leqslant \begin{cases} \delta_n(x), & x \in [\varepsilon, 1] \\ 1, & x \in [0, \varepsilon) \end{cases} \tag{3.1}$$

and

$$1 + P_n(-x)/P_n(x) \geqslant \alpha > 0, \qquad x \in [0, 1], \tag{3.2}$$

the rational function

$$r_n(x) := \frac{P_n(x) - P_n(-x)}{P_n(x) + P_n(-x)} \tag{3.3}$$

is *odd* and satisfies

$$|\operatorname{sgn} x - r_n(x)| \leqslant \begin{cases} 2\alpha^{-1}\delta_n(|x|), & \varepsilon \leqslant |x| \leqslant 1 \\ 2\alpha^{-1}, & |x| \leqslant \varepsilon. \end{cases} \tag{3.4}$$

Therefore, given n, ε and a desired error bound function δ_n on $[\varepsilon, 1]$, it suffices to construct P_n with the above properties.

According to a result of Gonchar [2],

$$R_n(\operatorname{sgn} x, \{\varepsilon \leqslant |x| \leqslant 1\}) \geqslant \frac{1}{2} \exp\left(-\pi^2 n/2 \log \frac{1}{\varepsilon}\right).$$

Therefore, the best one can hope for is to construct, for given ε, n, a polynomial P_n that satisfies (3.1) with

$$\delta_n(x) \asymp \exp\left\{-cn\left(\frac{1}{\log 1/\varepsilon} + \varphi(x)\right)\right\}, \qquad x \in [\varepsilon, 1],$$

where φ is some positive increasing function on $(0, 1]$, such that

$$\varphi(\varepsilon) \asymp 1/\log \frac{1}{\varepsilon}, \qquad \text{as} \quad \varepsilon \to 0.$$

Unfortunately, this goal cannot be achieved (see Theorem 4.2 below), but we can come close.

LEMMA 3.1. *Let $\varphi(x)$ be a right continuous, nondecreasing function on $[0, 1]$, with $\varphi(0) = 0$, that satisfies*

$$\int_0^1 \frac{\varphi(x)}{x}\, dx < \infty. \tag{3.5}$$

Then, given any $\varepsilon \in (0, 1/2)$ *and any* $n \geqslant 1$, *there exists a polynomial* $M_n = M_{n,\varepsilon} \in \mathbf{P}_n$ *such that*

$$\left|\frac{M_n(-x)}{M_n(x)}\right| \leqslant c_1 \exp\left\{-cn\left(\frac{1}{\log 1/\varepsilon} + \varphi(x)\right)\right\}, \qquad x \in [\varepsilon, 1] \tag{3.6}$$

and

$$1 + M_n(-x)/M_n(x) \geqslant \alpha > 0, \qquad x \in [0, 1] \tag{3.7}$$

where c, c_1, *and* α *are independent of* ε, n.

Proof. A slight modification of Newman's construction produces (cf. [3, Lemma 2]) a polynomial $P_n \in \mathbf{P}_n$ of the form

$$P_n(x) = P_{n,\varepsilon}(x) = \prod_{j=1}^{n} (\zeta^j + x), \qquad \zeta := \varepsilon^{1/n},$$

such that

$$|P_n(-x)/P_n(x)| \leqslant c_2 \exp\left(-c_3 n \Big/ \log\frac{1}{\varepsilon}\right), \qquad x \in [\varepsilon, 1] \tag{3.8}$$

and

$$0 < P_n(-x)/P_n(x) \leqslant 1, \qquad x \in [0, \varepsilon). \tag{3.9}$$

Here, the constants c_2, c_3 are independent of ε and n; moreover, one can take $c_2 = 1$ in (3.8), provided $n \geqslant \log 1/\varepsilon$.

Next, we note that

$$\int_{2^{-k-1}}^{2^{-k}} \frac{\varphi(x)}{x}\, dx \geqslant \varphi(2^{-k-1}) \log 2, \qquad k = 0, 1, ...,$$

since φ is increasing. Therefore

$$s := \sum_0^\infty \varphi(2^{-k}) \leqslant \varphi(1) + \frac{1}{\log 2}\int_0^1 \frac{\varphi(x)}{x}\, dx < \infty,$$

by our assumption (3.5).

Suppose first that n is large enough, namely

$$n \geqslant \max\left\{\log\frac{1}{\varepsilon}, \frac{s}{\varphi(1)}\right\}. \tag{3.10}$$

Define $N \geqslant 0$ by

$$\frac{n}{s}\,\varphi(2^{-N-1}) < 1 \leqslant \frac{n}{s}\,\varphi(2^{-N}) \tag{3.11}$$

and consider the polynomial

$$Q_n(x) := \prod_{k=0}^{N} (2^{-k}+x)^{m_k}, \qquad m_k := \left[\frac{n}{s}\,\varphi(2^{-k})\right], \tag{3.12}$$

where $[\cdot]$ denotes the greatest integer function. Then $Q_n \in \mathbf{P}_n$. Now, for $2^{-k-1} \leqslant x \leqslant 2^{-k}$, $k=0, 1, ..., N$, we have

$$\left|\frac{Q_n(-x)}{Q_n(x)}\right| \leqslant \left|\frac{2^{-k}-x}{2^{-k}+x}\right|^{m_k} \leqslant \left(\frac{1}{3}\right)^{m_k} < \exp\left\{-\left[\frac{n}{s}\,\varphi(2^{-k})\right]\right\}$$
$$< \exp\left\{1-\frac{n}{s}\,\varphi(x)\right\},$$

so that

$$|Q_n(-x)/Q_n(x)| < \exp\left\{1-\frac{n}{s}\,\varphi(x)\right\}, \qquad x \in [2^{-N-1}, 1]. \tag{3.13}$$

We also have

$$|Q_n(-x)/Q_n(x)| \leqslant \tfrac{1}{3} < e^{-1}, \qquad x \in [2^{-N-1}, 1]. \tag{3.14}$$

For $x \in [0, 2^{-N-1}]$ it follows from (3.11) that

$$\left|\frac{Q_n(-x)}{Q_n(x)}\right| \leqslant 1 < \exp\left\{1-\frac{n}{s}\,\varphi(2^{-N-1})\right\} \leqslant \exp\left\{1-\frac{n}{s}\,\varphi(x)\right\}. \tag{3.15}$$

Now, define $M_{2n} := P_n Q_n \in \mathbf{P}_{2n}$. Then (3.13), (3.15), and (3.8) give the required bound (3.6). Moreover, (3.14) and (3.8) (with $c_2 = 1$, since $n \geqslant \log(1/\varepsilon)$ by our restriction (3.10)) yield

$$\left|\frac{M_{2n}(-x)}{M_{2n}(x)}\right| \leqslant \max\{e^{-1}, e^{-c_3}\}, \qquad \text{for} \quad 1 \geqslant x \geqslant \min\{\varepsilon, 2^{-N}\}.$$

For $0 \leqslant x < \min\{\varepsilon, 2^{-N}\}$, the ratio $M_{2n}(-x)/M_{2n}(x)$ is positive (see (3.12), (3.9)), so that (3.7) holds on $[0, 1]$ with $\alpha := 1 - \max\{e^{-1}, e^{-c_3}\} > 0$. The passage from M_{2n} to $M_n \in \mathbf{P}_n$ is obvious, so that the lemma is proved, provided n satisfies (3.10).

The remaining case is simpler. If $n < \min\{\log(1/\varepsilon), s/\varphi(1)\}$, put $M_n \equiv 1$. If $\log(1/\varepsilon) \leqslant n < s/\varphi(1)$, put $M_n := P_n$. Then (3.8) (with $c_2 = 1$) and (3.9) give the desired result. Finally, if $s/\varphi(1) \leqslant n < \log(1/\varepsilon)$, put $M_n := Q_n$, and apply (3.13)–(3.15), and the positivity of $Q_n(\pm x)$ on $[0, 2^{-N}]$. ∎

We mention two simple facts concerning the behavior of $M_n(-z)/M_n(z)$ in the complex plane $\mathbf{C}$. By construction, this is a Blaschke product for the right half-plane, so that

$$|M_n(-z)/M_n(z)| = 1 \Leftrightarrow \operatorname{Re}(z) = 0. \tag{3.16}$$

Next, this Blaschke product includes the factor (see (3.12))

$$\left(\frac{1-z}{1+z}\right)^{m_0}, \qquad m_0 = \left[\frac{n}{s}\,\varphi(1)\right] \geqslant cn,$$

and its other factors are less than 1 (in absolute value) if $\operatorname{Re}(z) > 0$. Therefore, if we define, for $0 < \delta < 1$,

$$K_\delta := \{z \colon \delta \leqslant \operatorname{Re}(z) \leqslant \delta^{-1},\ |\operatorname{Im}(z)| \leqslant \delta^{-1}\},$$

we obtain the bound

$$|M_n(-z)/M_n(z)| \leqslant e^{-c_\delta n}, \qquad z \in K_\delta, \tag{3.17}$$

where $c_\delta > 0$ is independent of n, ε.

We are now ready to prove

THEOREM 3.2. *Let φ be as in Lemma* 3.1. *Then given any $\varepsilon \in (0, 1/2)$ and any $n \geqslant 1$, there exists a rational function $r_n = r_{n,\varepsilon} \in \mathbf{R}_n$ with poles on the imaginary axis such that*

$$|\operatorname{sgn} x - r_{n,\varepsilon}(x)| \leqslant C \exp\left\{-cn\left(\frac{1}{\log 1/\varepsilon} + \varphi(|x|)\right)\right\}, \qquad \varepsilon \leqslant |x| \leqslant 1, \tag{3.18}$$

and

$$|r_{n,\varepsilon}(x)| \leqslant C, \qquad |x| \leqslant \varepsilon. \tag{3.19}$$

The constants C, c are independent of n, ε.

Proof. Define

$$r_{n,\varepsilon}(x) := \frac{M_{n,\varepsilon}(x) - M_{n,\varepsilon}(-x)}{M_{n,\varepsilon}(x) + M_{n,\varepsilon}(-x)},$$

where $M_{n,\varepsilon}$ is the polynomial constructed in Lemma 3.1. The discussion at the beginning of this section then yields (3.18) and (3.19). The poles of $r_{n,\varepsilon}$ lie on the imaginary axis due to (3.16). ∎

The next result should be compared with Theorem 1 in [10].

THEOREM 3.3. *Let* φ *be as in Lemma* 3.1. *Then there exists* $R_n \in \mathbf{R}_n$, $n = 1, 2, \ldots$, *such that*

$$|\operatorname{sgn} x - R_n(x)| \leqslant C \exp\{-cn\varphi(|x|)\}, \qquad x \in [-1, 1]. \tag{3.20}$$

Moreover, as $n \to \infty$,

$$R_n(z) \to \begin{cases} 1, & \operatorname{Re}(z) > 0 \\ -1, & \operatorname{Re}(z) < 0 \end{cases}$$

uniformly (*and geometrically fast*) *on compact subsets of* $\mathbf{C} \backslash \{z \colon \operatorname{Re}(z) = 0\}$.

Proof. Given $n \geqslant 1$, set $R_n := r_{n, \varepsilon_n}$, where $\varepsilon_n = e^{-n}$ and r_{n, ε_n} is the rational function of Theorem 3.2. Then (3.18) yields (3.20) for $|x| \geqslant e^{-n}$. Next, since $\varphi(x)$ is increasing and satisfies (3.5), we obtain for any $0 < x < 1$,

$$\varphi(x) \log \frac{1}{x} \leqslant \int_x^1 \frac{\varphi(t)}{t}\, dt < \int_0^1 \frac{\varphi(t)}{t}\, dt,$$

that is

$$\varphi(x) \log \frac{1}{x} \leqslant C, \qquad x \in (0, 1). \tag{3.21}$$

Therefore,

$$n\varphi(x) \leqslant n\varphi(e^{-n}) \leqslant C, \qquad \text{if} \quad x \in [0, e^{-n}]$$

and we see that (3.19) of Theorem 3.2 yields (3.20) for $|x| \leqslant e^{-n}$.
The second assertion of Theorem 3.3 follows from (3.17). ▌

With the aid of Theorem 3.2, we can approximate the characteristic function $\chi_{[a, b]}$ of any interval.

COROLLARY 3.4. *Let* $[a, b] \subseteq [-1, 1]$ *and* φ *be as in Lemma* 3.1. *Then, given any* $\varepsilon \in (0, 1)$ *and any* $n \geqslant 1$, *there exists* $r_n = r_{n, a, b, \varepsilon} \in \mathbf{R}_n$ *such that for* $x \in [-1, 1]$ *there holds*:

$$|\chi_{[a, b]}(x) - r_n(x)| \leqslant c_1 \exp\left\{-c_0 n \left(\frac{1}{\log 1/\varepsilon} + \varphi(d(x))\right)\right\}, \qquad \textit{if} \quad d(x) \geqslant \varepsilon, \tag{3.22}$$

where

$$d(x) := \min\{|x - a|, |x - b|\}.$$

Also,

$$|r_n(x)| \leqslant c_2, \qquad \text{if} \quad d(x) \leqslant \varepsilon.$$

The constants c, c_1, *and* c_2 *are independent of* a, b, n, ε.

Proof. First, we observe that

$$\chi_{[a,b]}(x) = \tfrac{1}{2}(\operatorname{sgn}(x-a) - \operatorname{sgn}(x-b)), \qquad x \neq a, b.$$

Next, define

$$\tilde{\varphi}(x) := \begin{cases} \varphi(2x), & 0 \leqslant x < 1/2, \\ \varphi(1), & 1/2 \leqslant x \leqslant 1, \end{cases}$$

and apply Theorem 3.2 with $\tilde{\varphi}$ instead of φ, with $\varepsilon/2$ instead of ε, and with n replaced by $[n/2]$ to get a corresponding $\tilde{r}(x)$. Then the function

$$r_n(x) := \frac{1}{2}\left(\tilde{r}\left(\frac{x-a}{2}\right) - \tilde{r}\left(\frac{x-b}{2}\right)\right)$$

has the desired properties. ∎

Remark 1. If we apply Theorem 3.2 with the original φ, we obtain the required estimate with $\varphi(x/2)$ instead $\varphi(x)$. Since $\varphi(x/2)$ may not be $\asymp \varphi(x)$, the passage to $\tilde{\varphi}$ was needed.

Remark 2. Let $\{\varepsilon_n\}$, $0 < \varepsilon_n < 1$, be an arbitrary sequence, and let $r_n = r_{n,a,b,\varepsilon_n}$ be as above. Then (3.16), (3.17) show that the poles of r_n lie on the vertical lines $\operatorname{Re}(z) = a$, $\operatorname{Re}(z) = b$ and that $r_n(z) \to \chi_{[a,b]}(x)$, $x = \operatorname{Re}(z)$, uniformly (and geometrically fast) on compact subsets of $\mathbf{C} \setminus \{z\colon \operatorname{Re}(z) = a$ or $\operatorname{Re}(z) = b\}$.

4. RATIONAL APPROXIMATION OF PIECEWISE ANALYTIC FUNCTIONS

In this section, we construct a sequence of rational functions having the properties described in the Introduction. Namely, we prove the following.

THEOREM 4.1. *Let* f *be piecewise analytic on* $[-1, 1]$ *and belong to* $C^{k-1}[-1, 1]$, *and let* φ *be as in Lemma* 3.1. *Then there exist rational functions* $R_n \in \mathbf{R}_n$, $n = 1, 2, \ldots$, *such that for all* $x \in [-1, 1]$

$$|f(x) - R_n(x)| \leqslant C \exp\{-c\sqrt{kn} - cn\varphi(d(x))\}, \tag{4.1}$$

where $d(x)$ denotes the distance from x to the nearest singularity of f on $[-1, 1]$. The constants c, C are independent of x and n, and c is also independent of k.

Proof. A simple argument (cf. [10, Proof of Theorem 3]) shows that it suffices to verify the theorem for piecewise analytic functions of the form

$$f(x) = \begin{cases} g(x), & x \in [a, b] \\ 0, & \text{otherwise}, \end{cases} \tag{4.2}$$

where $g^{(i)}(a) = g^{(i)}(b) = 0$ for $i = 0, ..., k-1$. Note that for such f we have

$$d(x) = \min\{|x-a|, |x-b|\}. \tag{4.3}$$

We consider the case $-1 < a < b < 1$ (if either $a = -1$ or $b = 1$, the proof is similar). Let

$$f^*(x) := f(x)/[(x-a)(x-b)]^k := \begin{cases} g^*(x), & x \in [a, b] \\ 0, & \text{otherwise} \end{cases}. \tag{4.4}$$

By our assumptions, g^* is analytic on $[a - 2\tau, b + 2\tau]$, for some $\tau > 0$. Therefore, there exist polynomials $p_n \in \mathbf{P}_n$, $n = 1, 2, ...,$ such that

$$|g^*(x) - p_n(x)| \leqslant C_1 e^{-c_1 n}, \qquad x \in [a-\tau, b+\tau]. \tag{4.5}$$

In particular, the p_n are uniformly bounded on $[a-\tau, b+\tau]$ and so, by Bernstein's inequality (cf. [14]), we have

$$\| p_n \|_{[-1, 1]} \leqslant e^{c_2 n}. \tag{4.6}$$

We note for future reference that the constants c_1, c_2 are independent of k. The same will be true for all lower case constants that appear below.

Applying Corollary 3.4 with the above a, b and with $\varepsilon := e^{-\sqrt{n/k}}$, we get $r_n \in \mathbf{R}_n$, $n = 1, 2, ...,$ such that for all $x \in [-1, 1]$

$$|\chi_{[a, b]}(x) - r_n(x)| \leqslant C \exp\{-c_3 \sqrt{kn} - c_3 n\varphi(d(x))\}, \qquad \text{if} \quad d(x) \geqslant e^{-\sqrt{n/k}}, \tag{4.7}$$

and

$$|r_n(x)| \leqslant C, \qquad \text{if} \quad d(x) \leqslant e^{-\sqrt{n/k}}. \tag{4.8}$$

Set

$$r^*(x) := p_n(x)\, r_{cn}(x), \tag{4.9}$$

where the constant c will be chosen later.

Case 1: $d(x) \geqslant e^{-\sqrt{n/k}}$. First, let $x \notin [a-\tau, b+\tau]$. Then $d(x) \geqslant \tau \geqslant e^{-\sqrt{n/k}}$ provided $n \geqslant k \log^2(1/\tau)$. Applying (4.6), (4.7) we obtain:

$$|r^*(x)| \leqslant C \exp\{-c_3 \sqrt{kcn} - c_3 cn\varphi(d(x)) + c_2 n\}. \tag{4.10}$$

Choose c large enough to ensure that

$$c_3 c\varphi(\tau) \geqslant 2c_2.$$

With such a choice for c we get from (4.10) that

$$|r^*(x)| \leqslant C \exp\{-c_4 \sqrt{kn} - c_4 n\varphi(d(x))\}.$$

Note that this relation persists on $[a-\tau, b+\tau] \backslash [a, b]$, since $|p_n(x)| \leqslant C$ there, by (4.5). Also, $f^*(x) = 0$ for $x \notin [a.b]$, and we conclude that

$$|f^*(x) - r^*(x)| \leqslant C \exp\{-c_4 \sqrt{kn} - c_4 n\varphi(d(x))\}. \tag{4.11}$$

For $x \in [a, b]$ we have (recall (4.9))

$$|f^*(x) - r^*(x)| \leqslant |f^*(x)(1 - r_{cn}(x))| + |(f^*(x) - p_n(x))\, r_{cn}(x)|.$$

Applying the estimates (4.5), (4.7) we see that the relation (4.11) persists for $x \in [a, b]$ (possibly, with a different c_4).

Case 2: $d(x) \leqslant e^{-\sqrt{n/k}}$. Then $x \in [a-\tau, b+\tau]$ (see the restriction on n, made at the beginning of Case 1), so that $|p_n(x)| \leqslant C$. Then (4.8), (4.9) yield a trivial estimate:

$$|f^*(x) - r^*(x)| \leqslant \|f^*\|_{[-1, 1]} + |r^*(x)| \leqslant C. \tag{4.12}$$

Next, set

$$R(x) := r^*(x)(x-a)^k\,(x-b)^k \in \mathbf{R}_{n(1+c)+2k} \tag{4.13}$$

and note that $|(x-a)(x-b)|^k \leqslant 4^k$, $x \in [-1, 1]$, while $|(x-a)(x-b)|^k \leqslant 2^k \exp(-\sqrt{kn})$ if $d(x) \leqslant \exp(-\sqrt{n/k})$. Therefore, by multiplying (4.11) and (4.12) by $|(x-a)(x-b)|^k$ and recalling (4.4) we obtain

$$|f(x) - R(x)| \leqslant C \exp(-c_4 \sqrt{kn} - c_4 n\varphi(d(x))), \qquad d(x) \geqslant e^{-\sqrt{n/k}} \tag{4.14}$$

and

$$|f(x) - R(x)| \leqslant C \exp(-\sqrt{kn}), \qquad d(x) \leqslant e^{-\sqrt{n/k}}. \tag{4.15}$$

Finally, (3.21) implies that

$$n\varphi(d(x)) \leqslant n\varphi(e^{-\sqrt{n/k}}) \leqslant C\sqrt{kn}, \qquad \text{if} \quad d(x) \leqslant e^{-\sqrt{n/k}}.$$

This inequality, together with (4.15), (4.14), shows that R satisfies the required estimate (4.1), except that R is a rational function of order $\asymp n$ (see (4.13)) and is not of precise order n. This difficulty can be circumvented by using a standard argument. ▮

Remark 3. If g^* of (4.4) is entire, one can use Maclaurin polynomials to replace (4.5), (4.6) by

$$|g^*(z) - p_n(z)| \leqslant c_\rho \rho^{-n}, \qquad |z| \leqslant \rho$$

$$|p_n(z)| \leqslant \tilde{c}_\rho, \qquad |z| \leqslant \rho,$$

where $\rho > 1$ is arbitrary. Applying Remark 2 at the end of Section 3, we see that the rational functions constructed above (with above choice of p_n) converge to

$$\tilde{f}(z) = \begin{cases} g(z), & a < \mathrm{Re}(z) < b \\ 0, & \mathrm{Re}(z) > b \text{ or } \mathrm{Re}(z) < a. \end{cases}$$

The convergence is uniform (and geometrically fast) on compact subsets of $\mathbf{C} \backslash \{z\colon \mathrm{Re}(z) = a \text{ or } \mathrm{Re}(z) = b\}$. Similar remarks apply to any piecewise entire function f.

Our next result shows that the condition (3.5) imposed on φ is *necessary* in order to get geometric convergence of R_n to a given f.

THEOREM 4.2. *Let f be continuous and piecewise analytic on $[-1, 1]$, but not analytic on $[-1, 1]$. Given a right-continuous, nondecreasing function φ on $[0, 1]$ with $\varphi(0) = 0$, assume there exist $R_n \in \mathbf{R}_n$, $n = 1, 2, ...$, such that*

$$|f(x) - R_n(x)| \leqslant C \exp\{-n\varphi(d(x))\}, \qquad x \in [-1, 1]. \tag{4.16}$$

Then (3.5) *holds.*

Proof. Assume first that (4.16) holds with $f(x) = |x|^{2k+1}$, for some integer $k \geqslant 0$. Then (4.16) becomes

$$||x|^{2k+1} - R_n(x)| \leqslant C \exp(-n\varphi(|x|)), \qquad x \in [-1, 1]. \tag{4.17}$$

Replacing R_n by $R_{2n}(x) := (R_n(x) + R_n(-x))/2$, we see that

$$|x^{2k+1} - R_{2n}(x)| \leqslant Ce^{-n\varphi(x)}, \qquad x \in [0, 1], \tag{4.18}$$

and R_{2n} is even. Applying the classical Newman's method we deduce from (4.18) that a polynomial $p \in \mathbf{P}_{2n+2k+1}$ exists such that

$$x^{2k+1} \left| \frac{p(-x)}{p(x)} \right| \leqslant Ce^{-n\varphi(x)}, \qquad x \in [0, 1]. \tag{4.19}$$

(See [8, pp. 75–76] for details. The proof is given there for $k=0$, but it remains the same for $k \geqslant 1$.) Now, take the log of both sides of (4.19), divide by x, and integrate from $e^{-\sqrt{n}}$ to 1 to obtain

$$-(2k+1)\frac{n}{2} + \int_{e^{-\sqrt{n}}}^{1} \log \left| \frac{p(-x)}{p(x)} \right| \frac{dx}{x} \leqslant (\log C)\sqrt{n} - n \int_{e^{-\sqrt{n}}}^{1} \frac{\varphi(x)}{x}\, dx.$$

Since $p \in \mathbf{P}_{2n+2k+1}$, we obtain via Newman's inequality (cf. Section 2) that

$$\int_{e^{-\sqrt{n}}}^{1} \frac{\varphi(x)}{x}\, dx \leqslant c_1 + O(n^{-1/2}), \qquad \text{as} \quad n \to \infty.$$

Therefore $\int_0^1 (\varphi(x)/x)\, dx$ converges.

The case of general f can be reduced to the case just considered. By our assumption, f has a singularity at some $x_0 \in (-1, 1)$. Then, by restricting (4.16) to $x \in [x_0 - \delta, x_0 + \delta]$, with $\delta > 0$ small enough, and applying the transformation $x - x_0 = \delta t$ we obtain

$$|\tilde{f}(t) - \tilde{R}_n(t)| \leqslant Ce^{-n\tilde{\varphi}(|t|)}, \qquad t \in [-1, 1], \tag{4.16$'$}$$

where $\tilde{\varphi}(t) := \varphi(\delta t)$,

$$\tilde{f}(t) := \begin{cases} \tilde{f}_+(t), & 0 \leqslant t \leqslant 1 \\ \tilde{f}_-(t), & -1 \leqslant t \leqslant 0, \end{cases}$$

and $\tilde{f}_+, \tilde{f}_-$ both are analytic on $[-1, 1]$ and agree at 0. Next, replacing $\tilde{f}$ by $(\tilde{f}(t) + \tilde{f}(-t))/2$ and $\tilde{R}_n$ by $(\tilde{R}(t) + \tilde{R}(-t))/2$, we may assume that $\tilde{f}$ in (4.16$'$) is even. Therefore, the above analyticity properties imply

$$\tilde{f}(t) = |t|^{2k+1} g(t^2) + h(t^2), \qquad t \in [-1, 1], \tag{4.20}$$

where $k \geqslant 0$, $g(0) \neq 0$, and g, h are analytic on $[-1, 1]$. We may assume $g \neq 0$ on $[-1, 1]$ (otherwise, we restrict (4.20) to some $[-\delta, \delta]$ and stretch it back to $[-1, 1]$). Thus, we may divide (4.16$'$) by g and approximate h/g by an even polynomial p_n of order n and $1/g$ by an even polynomial q_n of order n to obtain

$$||t|^{2k+1} + p_n(t) - q_{(n)}(t)\, \tilde{R}_n(t)| \leqslant c_1 e^{-c_2 n} + ce^{-n\tilde{\varphi}(|t|)} \leqslant c_3 e^{-n\tilde{\varphi}(|t|)}.$$

Therefore, we get (4.17) for some R_n of order $\asymp n$ and φ replaced by $\tilde{\varphi}$. Thus condition (3.5) holds for $\tilde{\varphi}$ and hence for φ. ∎

5. FUNCTIONS OF THE GONCHAR–SZABADOS CLASS

In this section we extend Theorem 4.1 to a collection of functions in the GS class described in the Introduction. Let $f \in C[-1, 1]$ and assume $f \in \mathrm{GS}$, that is, there exists a partition $-1 = x_0 < x_1 < \cdots < x_s = 1$ such that the restriction of f on each open interval (x_j, x_{j+1}) has an analytic and bounded continuation in an open rhombus D_j with opposite vertices x_j, x_{j+1}. If $s \geqslant 2$, we assume that every interior point x_j is a singularity of f, but f may be regular at the endpoints ± 1. If $s = 1$, one of the endpoints may be regular, but not both. In other words, the exact set of singularities of f on $[-1, 1]$ is $\{x_0, ..., x_s\}$, with the possible exception of x_0, x_s. We denote this set by S_f.

Further, we assume that the extended function f is not only bounded in $D := \bigcup_0^{s-1} D_j$, but is continuous on the closure $\bar{D}$. We define the local modulus of continuity of f on S_f, with respect to $\bar{D}$, by

$$\omega_f^*(t) := \max_{x_j \in S_f} \max_{|z - x_j| \leqslant t,\, z \in \bar{D}} |f(z) - f(x_j)|, \tag{5.1}$$

and impose the following restriction on ω_f^*:

$$\int_0^1 \omega_f^*(t) \frac{dt}{t} < \infty. \tag{5.2}$$

If f satisfies all the above assumptions, we write $f \in \mathrm{GS}^*$.

Finally, the local modulus of continuity of f on S_f, with respect to $[-1, 1]$, is defined by

$$\tilde{\omega}_f(t) := \max_{x_j \in S_f} \max_{|x - x_j| \leqslant t,\, x \in [-1, 1]} |f(x) - f(x_j)|. \tag{5.3}$$

THEOREM 5.1. *Let φ be as in Lemma* 3.1, *and assume additionally that, for x small enough,*

$$\varphi(2x) \leqslant (2 - \alpha)\, \varphi(x) \tag{5.4}$$

for some $0 < \alpha < 1$. Then, given $f \in \mathrm{GS}^*$, there exist $R_n \in \mathbf{R}_n$, $n = 1, 2, ...,$ such that

$$|f(x) - R_n(x)| \leqslant C \exp\{-cn(t_n^{-1} + \varphi(d(x)))\}, \qquad x \in [-1, 1], \tag{5.5}$$

where t_n is defined for n large enough by

$$\tilde{\omega}_f(e^{-t_n}) = \exp(-c_1 n / t_n) \tag{5.6}$$

and $d(x)$ is the distance from x to S_f.

Remark 4. $\tilde{\omega}_f(t)$ may be much smaller (as $t \to 0$) than the ordinary modulus of continuity, $\omega_f(t)$. In such a case, even the uniform part of (5.5) improves the estimate given in (1.4). For example, it can be shown that, for $\beta > 0$,

$$R_n(\exp(-|x|^{-\beta})) \leqslant C \exp(-c_\beta n/\log n).$$

Proof of Theorem 5.1. As in Section 4, we may assume that f has two singularities, say at a, b, $-1 < a < b < 1$, while $f = 0$ on $[-1, a] \cup [b, 1]$. Since $f \in \mathrm{GS}^*$, there exists a $\mu > 0$ such that f is analytic in the open rhombus D, bounded by the lines

$$z = x + iy, \qquad y = \begin{cases} \mu(-1)^k\,(x-a), & a \leqslant x \leqslant (a+b)/2 \\ \mu(-1)^k\,(x-b), & (a+b)/2 \leqslant x \leqslant b \end{cases}, \qquad k = 0, 1,$$

and f is continuous in $\bar{D}$. By Cauchy's formula,

$$f(x) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\tau)}{\tau - x}\, d\tau, \qquad x \in [-1, 1]. \tag{5.7}$$

(Note that (5.7) holds at $x = a$ and $x = b$ because of (5.2).) Thus our problem is reduced to the approximation of Cauchy-type integrals

$$F(x) := \int_\gamma \frac{f(\tau)}{\tau - x}\, d\tau, \qquad x \in [-1, 1], \tag{5.8}$$

where γ is one of the sides of D, say the side

$$\gamma := \left\{ t + \mu i (t - a) : a \leqslant t \leqslant \frac{a+b}{2} \right\}.$$

Since the linear transformation $x \to (x-a)/2$ transforms $[-1, 1]$ into $[-(1+a)/2, (1-a)/2] \subset [-1, 1]$, it is enough to approximate the function

$$\tilde{F}(x) := \int_{\tilde{\gamma}} \frac{\tilde{f}(\tau)}{\tau - x}\, d\tau, \qquad x \in [-1, 1], \tag{5.9}$$

where $\tilde{F}(x) := F(2x + a)$, $\tilde{f}(\tau) := f(2\tau + a)$, and $\tilde{\gamma} := \{(1 + \mu i)\, t : 0 \leqslant t \leqslant (b-a)/4\}$. To this end, we need the following generalization of Lemma 3.1 which will be proved later.

CLAIM. *Denote by* z^* *the reflection of* $z \in \mathbf{C}$ *about* $\tilde{\gamma}$. *Given* $\varepsilon \in (0, 1/2)$ *and* $n \geqslant 1$, *set* $\zeta = e^{1/n}$ *and*

$$M(z) := \prod_{j=1}^{n} \frac{z - \zeta^j}{z - (\zeta^j)^*} \prod_{j=0}^{N} \left(\frac{z - 2^{-j}}{z - (2^{-j})^*} \right)^{m_j} \in \mathbf{R}_{2n}. \tag{5.10}$$

Then $M(z)$ *satisfies*

$$|M(x)| \leqslant \begin{cases} C \exp\left\{ -c_1 n \left(\dfrac{1}{\log 1/\varepsilon} + \varphi(x) \right) \right\}, & x \in [\varepsilon, 1] \\ C, & x \in [0, \varepsilon] \\ C \exp\left\{ +c_2 n \left(\dfrac{1}{\log 1/\varepsilon} + \varphi(x) \right) \right\}, & x \in [-1, 0], \end{cases} \tag{5.11}$$

where C, c_1, c_2 *are independent of* n, ε.

Obviously, we also have

$$|M(z)| = 1, \qquad z \in \tilde{\gamma}. \tag{5.12}$$

Next let

$$S(\tau, z) := \frac{1}{\tau - z} \frac{M(\tau) - M(z)}{M(\tau)} = \frac{1}{\tau - z} - \frac{1}{\tau - z} \frac{M(z)}{M(\tau)}.$$

This is a rational function of z, of degree $2n$, whose poles coincide with those of $M(z)$, which interpolates the Cauchy kernel at the zeros of $M(z)$. It can be easily verified that

$$\tilde{\pi}(z) := \int_{\tilde{\gamma}} S(\tau, z)\, \tilde{f}(\tau)\, d\tau$$

is a rational function (of degree $\leqslant 2n$). Since

$$|\tau - x| \geqslant \operatorname{Im}(\tau) = \frac{\mu}{\sqrt{1+\mu^2}} |\tau|, \qquad \tau \in \tilde{\gamma}, \quad x \text{ real},$$

we obtain from (5.9), (5.12), and (5.2) that for all $x \in [-1, 1]$,

$$\begin{aligned} |\tilde{F}(x) - \tilde{\pi}(x)| &\leqslant |M(x)| \int_{\tilde{\gamma}} \frac{|f(\tau)|}{|\tau - x|} |d\tau| \leqslant C_1 |M(x)| \int_{\tilde{\gamma}} \frac{\omega_f^*(|2\tau|)}{|\tau|} |d\tau| \\ &\leqslant C_2 |M(x)|. \end{aligned} \tag{5.13}$$

Applying a similar procedure to the integrals (5.8) along the other three sides of D, and making the corresponding inverse substitutions in (5.13), (5.11), we obtain a rational function $\pi(x) \in \mathbf{R}_{8n}$ that satisfies for all $x \in [-1, 1]$

$$|f(x)-\pi(x)| \leqslant \begin{cases} C \exp\left\{-c_1 n \left(\dfrac{1}{\log 1/\varepsilon} + \varphi(d(x))\right)\right\}, \\ \qquad x \in [a, b], \quad d(x) > 2\varepsilon \\ C, \qquad x \in [a, b] \quad d(x) \leqslant 2\varepsilon \\ C \exp\left\{+c_2 n \left(\dfrac{1}{\log 1/\varepsilon} + \varphi(d(x))\right)\right\}, \\ \qquad x \notin [a, b], \end{cases} \tag{5.14}$$

where $d(x) := \min\{|x-a|, |x-b|\}$.

Next, let $a' := a + 3\varepsilon$, $b' := b - 3\varepsilon$, $d'(x) := \min\{|x-a'|, |x-b'|\}$, and let $r_n = r_{n, a', b', \varepsilon}$ be the rational function of Corollary 3.4. Consider the function

$$R(x) := r_{cn}(x)\, \pi(x) \in \mathbf{R}_{(c+8)n},$$

where $c \geqslant 1$ will be chosen later. To estimate the difference $f-R$ we proceed as in proof of Theorem 4.1, but now we use (5.14) instead of (4.5), (4.6). First, let $x \in [-1, a] \cup [b, 1]$. Then, $f(x) = 0$, while $d'(x) \geqslant 3\varepsilon$ and also $d'(x) \geqslant d(x)$. Applying (3.22), (5.14), and setting

$$c := (c_2 + 1)/c_0,$$

we obtain

$$|f(x) - R(x)| = |R(x)| \leqslant C \exp\left\{-n\left(\frac{1}{\log 1/\varepsilon} + \varphi(d(x))\right)\right\}. \tag{5.15}$$

For $x \in [a, b]$, write

$$|f(x) - R(x)| \leqslant |f(x)| \cdot |1 - r_{cn}(x)| + |r_{cn}(x)| \cdot |f(x) - \pi(x)| =: L.$$

If $x \in [a, a+2\varepsilon] \cup [b - 2\varepsilon, b]$, then $d(x) \leqslant 2\varepsilon$, $d'(x) \geqslant \varepsilon$ and we get the estimate

$$\begin{aligned} L &\leqslant C_1 \left[\tilde{\omega}_f(2\varepsilon) + \exp\left\{-c_0 n \left(\frac{1}{\log 1/\varepsilon} + \varphi(d'(x))\right)\right\}\right] \\ &\leqslant C_2 \left[\tilde{\omega}_f(2\varepsilon) + \exp\left\{-c_0 n \Big/ \log \frac{1}{\varepsilon}\right\}\right]. \end{aligned} \tag{5.16}$$

If $x \in [a+2\varepsilon, a+4\varepsilon] \cup [b-4\varepsilon, b-2\varepsilon]$, then $d(x) > 2\varepsilon$, $d'(x) < \varepsilon$, so that

$$L \leqslant C\left[\tilde{\omega}_f(4\varepsilon) + \exp\left\{-c_1 n \Big/ \log\frac{1}{\varepsilon}\right\}\right]. \tag{5.17}$$

Finally, if $x \in [a+4\varepsilon, b-4\varepsilon]$, then $d'(x) \geqslant \frac{1}{4}d(x)$ and we obtain

$$L \leqslant C \exp\left\{-c_3 n\left(\frac{1}{\log 1/\varepsilon} + \varphi\left(\frac{1}{4}d(x)\right)\right)\right\}. \tag{5.18}$$

In view of (5.4) we have, for t small enough, $\varphi(t) \asymp \varphi(t/4)$, $t \in (0, 1]$, and we may replace $d(x)/4$ in (5.18) by $d(x)$. Applying (5.15)–(5.18) with ε replaced by $\varepsilon/4$ and n replaced by $n/(c+8)$, we obtain $R_n \in \mathbf{R}_n$, which satisfies, for all $x \in [-1, 1]$,

$$|f(x) - R_n(x)| \leqslant \begin{cases} \exp\left\{-c_4 n\left(\dfrac{1}{\log 1/\varepsilon} + \varphi(d(x))\right)\right\}, & d(x) \geqslant \varepsilon \\ \tilde{\omega}_f(\varepsilon) + \exp\{-c_5 n/\log 1/\varepsilon\}, & d(x) < \varepsilon. \end{cases}$$

Therefore, on choosing $\varepsilon := e^{-t_n}$, with t_n as defined by (5.6), we get the required estimate (5.5). (Recall from (3.21) that $\varphi(t) \leqslant C/\log(1/t)$.)

It remains to prove the estimate (5.11). It is easy to see that for x real and $t > 0$, there holds

$$\left|\frac{t-x}{t-x^*}\right| = \left|\frac{x-t}{x-t^*}\right| \leqslant \begin{cases} e^{-c_1 u}, & \text{if} \quad x > 0 \\ e^{c_2 u}, & \text{if} \quad x < 0, \end{cases} \tag{5.19}$$

where $u := \min\{t^{-1}|x|, t\,|x|^{-1}\}$ and c_1, c_2 depend only on μ. Let

$$r_n^{(1)}(x) := \prod_{j=1}^{n} [(\zeta^j - x)/((\zeta^j)^* - x)], \qquad \zeta := \varepsilon^{1/n}.$$

For $x \in (\varepsilon, 1]$ define $1 \leqslant k \leqslant n$ from the condition $x \in (\zeta^k, \zeta^{k-1}]$ and let $k := n+1$, if $x \in [0, \varepsilon]$. Then (5.19) yields

$$|r_n^{(1)}(x)| \leqslant \exp\left\{-c_1\left(x \sum_{j=1}^{k-1} \zeta^{-j} + x^{-1} \sum_{j=k}^{n} \zeta^j\right)\right\}$$

(one of the sums may be empty)

$$= \exp\left\{-c_1\left(\frac{x}{\zeta^{k-1}}\,\frac{1-\zeta^{k-1}}{1-\zeta} + \frac{\zeta^k}{x}\,\frac{1-\zeta^{n-k+1}}{1-\zeta}\right)\right\}.$$

Applying the estimates (cf. [2])

$$1-\zeta \asymp \log\frac{1}{\zeta}=\frac{1}{n}\log\frac{1}{\varepsilon}, \qquad \text{if} \quad n \geqslant \log\frac{1}{\varepsilon}$$

(that is, if $\zeta \geqslant e^{-1}$), and

$$1-\zeta \asymp 1, \qquad \text{if} \quad n<\log\frac{1}{\varepsilon},$$

we obtain

$$|r_n^{(1)}(x)| \leqslant \begin{cases} C\exp\left(-c_3 n \Big/ \log\frac{1}{\varepsilon}\right), & x\in[\varepsilon, 1] \\ C\exp\left(-c_3 nx \Big/ \varepsilon\log\frac{1}{\varepsilon}\right), & x\in[0,\varepsilon]. \end{cases} \tag{5.20}$$

For $x\in[-1,0]$ we proceed similarly and get estimates

$$\begin{aligned} |r_n^{(1)}(x)| &\leqslant \begin{cases} C\exp\left(c_4 n \Big/ \log\frac{1}{\varepsilon}\right), & x\in[-1,-\varepsilon] \\ C\exp\left(c_4 n\,|x| \Big/ \varepsilon\log\frac{1}{\varepsilon}\right), & x\in[-\varepsilon, 0], \end{cases} \\ &\leqslant C\exp\left(c_4 n \Big/ \log\frac{1}{\varepsilon}\right), \qquad x\in[-1,0]. \end{aligned} \tag{5.21}$$

Next, set

$$r_n^{(2)}(x) := \prod_{j=0}^{N} [(x-2^{-j})/(x-(2^{-j})^*)]^{m_j}.$$

The estimate

$$|r_n^{(2)}(x)| \leqslant C\exp(-c_5 n\varphi(x)), \qquad x\in[0,1], \tag{5.22}$$

follows exactly as in the proof of Lemma 3.1 (apply (5.19) and we note that $|(x-2^{-j})/(x-(2^{-j})^*)|<1$ for $x>0$). Given $x\in[-1,0]$, define k $(0\leqslant k\leqslant N-1)$ from the condition $|x|\in(2^{-k-1}, 2^{-k}]$, if $|x|\in(2^{-N}, 1]$; and set $k=N$ if $|x|\in[0, 2^{-N}]$. Then, from (5.19),

$$\begin{aligned} |r_n^{(2)}(x)| &\leqslant \exp\left\{c_2\,|x| \sum_{j=0}^{k}\left[\frac{n}{s}\varphi(2^{-j})\right]2^{j} + c_2\,|x|^{-1}\sum_{j=k+1}^{N}\left[\frac{n}{s}\varphi(2^{-j})\right]2^{-j}\right\} \\ &=: \exp\{L_1+L_2\}. \end{aligned}$$

In L_2, $\varphi(2^{-j}) \leqslant \varphi(|x|)$, so that

$$L_2 \leqslant c_2 \, |x|^{-1} \, \varphi(|x|) \frac{n}{s} \, 2^{-k} \leqslant 2c_2 \frac{n}{s} \, \varphi(|x|).$$

Next,

$$\varphi(2^{-j}) \, 2^j \leqslant \int_{2^{-j-1}}^{2^{-j}} \frac{\varphi(t)}{t^2} \, dt.$$

Therefore,

$$L_1 \leqslant c_2 \frac{n}{s} \, |x| \int_{2^{-k-1}}^{1} \frac{\varphi(t)}{t^2} \, dt. \tag{5.23}$$

Since we may alter φ on any fixed interval $[\delta, 1]$, $\delta > 0$, getting $\varphi^* \asymp \varphi$, we may assume that our assumption (5.4) holds everywhere on $[0, 1]$. Then a straightforward estimation yields

$$\int_{\tau}^{1} \frac{\varphi(t)}{t^2} \, dt \leqslant C_\alpha \frac{\varphi(\tau)}{\tau}.$$

Putting here $\tau = 2^{-k-1}$, we get from (5.23)

$$L_1 \leqslant c_3 n \, |x| \, \varphi(2^{-k-1}) \, 2^{k+1} \leqslant 2c_3 n\varphi(2^{-k-1}) \leqslant 2c_3 n\varphi(|x|),$$

provided $|x| \geqslant 2^{-N}$; that is, $0 \leqslant k \leqslant N-1$. If $|x| < 2^{-N}$, we may replace the lower limit in (5.23) by $|x|$ and then proceed as above. We have thus proved that

$$|r_n^{(2)}(x)| \leqslant C \exp(c_6 n\varphi(|x|)), \qquad x \in [-1, 0]. \tag{5.24}$$

Collecting the estimates (5.20)–(5.22), and (5.24), we get (5.11). ▌

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