Preface

The fields of special functions, orthogonal polynomials, random matrices and integrable systems have much in common and their interactions in recent years have been mutually beneficial. It is the purpose of this volume to give an overview of some although selective areas in these fields in order to have a compact and up to date edition of issues and current problems in these interesting areas of research. We hope that our volume will serve as a valuable resource of information for all researchers working in these areas, in particular young researchers and graduate students.

The 10 articles in this book are written by leading experts who gave talks at the two special sessions “Integrable Systems and Special Functions” and “Integrable Systems, Orthogonal Polynomials, and Random Matrices: Interdisciplinary Aspects”, Joint Meetings, Atlanta, January 2005. The authors were asked to provide an appropriate, although not comprehensive, list of references. We thank all the contributors for their participation at our special sessions in Atlanta, for their support in this ambitious project and for their great efforts to make this collection of articles a valuable piece of work. We also thank our co-organizers in Atlanta Jinho Baik, Andras Balogh, Mourad Ismail, Peter Miller and Zhijun Qiao for their support.

A short description of each chapter follows.

Lie algebraic aspects of the finite nonperiodic Toda flows constitutes Chapter 1 and reviews extensively various Lie algebraic aspects of finite nonperiodic full and tridiagonal Toda flow theory. For example, gradient structures in the dynamics and geometry of these systems are studied and comparisons are given for numerous examples of nonperiodic Toda equations.

The second chapter The Widom–Dyson constant for the gap probability in random matrix theory gives a new proof to the large $s$ asymptotics of the probability $P(s)$ that there are no eigenvalues in the interval $(0, 2s)$, for the bulk scaling limit of the Gaussian Unitary Ensemble. It is well-known that the probability $P(s)$ is expressed as the Fredholm determinant, $P(s) = \det(I - Ks)$, where $Ks$ is the trace-class operator with the kernel, $K(x, y) = \frac{\sin(x - y)}{x - y}$, acting on $L^2(0, 2s)$, and the problem reduces to the large $s$ asymptotics of the Fredholm determinant.

The chapter Special functions arising from discrete Painlevé equations: a survey is a survey dealing with recent studies on special solutions of the discrete Painlevé equations, especially on hypergeometric solutions of the $q$-Painlevé equations. In addition, an overview is given of the present status of studies on difference Painlevé equations as a source of special functions.

In the chapter Singular structure of Toda lattices and cohomology of certain compact Lie groups the authors present a thorough study of the singularities of the Toda lattice associated with a real split semisimple Lie algebra $g$. The blow-ups of the Toda lattice are given by the zero set of certain $\tau$ functions.

Chapter 5 Tau functions, Grassmannians and rank one conditions gives a beautiful account of $\tau$ functions which are closely related to Schur polynomials and how they relate to Grassmannians and rank one conditions.

The next chapter Structure of certain Chebyshev-type polynomials in Onsager’s algebra representation presents a systematic account of mathematical structures of certain special polynomials arising from the energy study of the superintegrable $N$-state chiral Potts model with a finite number of sizes. The polynomials of low-lying sectors are represented in two different forms, one of which is directly related to the energy description of superintegrable chiral Potts $\mathbb{Z}_N$-spin chains via the representation theory of Onsager’s algebra.

Special functions arising in the study of semi-linear equations in circular domains is a chapter devoted to the study of Rayleigh functions. These functions appear in classical problems such as vibrating circular membranes, heat conduction in cylinders and diffraction through circular apertures. In the present paper, it is shown that a new family of special
functions built out of convolutions of Rayleigh functions arise in constructing solutions of certain semi-linear evolution equations over a class of circular domains.

The title of Chapter 8 is *The generalized Weierstrass system inducing surfaces of constant and nonconstant mean curvature in Euclidean three space*. One of the classical problems of differential geometry is the study of the connection between the geometry of submanifolds and nonlinear partial differential equations. In this paper, a generalized Weierstrass system is introduced and its correspondence with the associated two-dimensional nonlinear sigma model is established. The method of symmetry reduction is discussed and used to derive classes of invariant solutions for this system.

The chapter *2D Toda lattice equation with self-consistent sources: Casoratian type solutions, bilinear Bäcklund transformation and Lax pair* deals with the interesting subject of soliton equations with self-consistent sources. More precisely, the current paper studies in detail the two dimensional Toda lattice equation with self-consistent sources.

The final chapter, Chapter 10, is entitled *Coalescent random walks on graphs* and is motivated by the interesting theory of coalescence in biology. The paper introduces a stochastic model called “multi-person simple random walks” on a graph $G$. In this model, there are a finite number of persons distributed randomly at the vertices of the graph $G$ and in each step of this discrete time Markov chain, a person is picked up and moved to a random adjacent vertex. To study this model, the authors introduce tensor powers of graphs and tensor products of Markov processes.

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