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Quadrature methods for 2D and 3D problems ☆

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Abstract

In this paper we give an overview on well-known stability and convergence results for simple quadrature methods based on low-order composite quadrature rules and applied to the numerical solution of integral equations over smooth manifolds. First, we explain the methods for the case of second-kind equations. Then we discuss what is known for the analysis of pseudodifferential equations. We explain why these simple methods are not recommended for integral equations over domains with dimension higher than one. Finally, for the solution of a two-dimensional singular integral equation, we prove a new result on a quadrature method based on product rules.

1. Introduction

A major task in numerical analysis is to provide methods for the solution of integral equations. For instance, the popular boundary element method consists in transforming a boundary value problem for a partial differential equation into an equivalent boundary integral equation and in solving this boundary equation numerically. Usually, collocation or Galerkin schemes are applied for the discretization of integral equations. If no analytic formulas for the integrals appearing in the discretized matrix operators are known, then, in a further discretization step, the integrals are to be replaced by quadrature formulas. Therefore, methods like Galerkin's, collocation or qualocation are called semi-discretization schemes. To get efficient numerical methods, the question arises how to choose optimal quadrature rules. This essential question is discussed in a lot of papers in the engineering literature, and mathematicians have analyzed and systematized these quadrature algorithms (cf., e.g., [2,10,19–22,26,28,40,41,50,53,56,59]).

However, right from the start (cf., e.g., [39]) fully discrete schemes have been proposed. Applying these so-called quadrature methods, the integrals in the original integral equation are directly replaced by a quadrature rule. The entries of the resulting linear system can be expressed as linear combinations of kernel function values with quadrature weights as coefficients. The advantage of quadrature

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methods is that they require less time for writing codes and a little bit less time for computation. On the other hand, as a rule of thumb, the approximation errors of quadrature methods are a little bit larger than those of Galerkin or collocation schemes. Especially, the errors measured in negative Sobolev norms may be essentially larger than those for Galerkin methods. However, there are cases when quadrature methods can compete with the accuracy of other schemes. Quadrature methods can be recommended for univariate integral equations of the second kind with smooth kernels. For univariate equations of the first kind and nonsmooth kernels, quadrature methods often require modifications, and their analysis is much more involved. Note that first kind equations with smooth kernels are ill-posed, and the methods of their regularization will not be discussed in this paper. In case of higher-dimensional equations, the simple quadrature methods can be recommended only for second-kind equations with smooth kernels. For the general case, more complicated quadrature methods like methods based on product integration are needed. The latter, however, are very close to Galerkin or collocation methods with quadrature approximated entries in the stiffness matrix. Note that, in general, there is no big difference between a quadrature method and a collocation scheme combined with an efficient quadrature algorithm. Only the "singular" integrals in the main diagonal of the stiffness matrix and the "almost singular" integrals corresponding to the neighbor elements are treated differently. Unfortunately, this small difference is essential for the convergence analysis and the error estimates.

Similarly to the semi-discretized schemes, the quadrature methods can be divided into h-methods, p-methods, and h–p-methods according to the underlying quadrature rule. If the last is exact for high-order polynomials, i.e., a variant of a Gauß rule, then the quadrature algorithm is called a p-method. These p-versions of quadrature are known to be useful for second-kind equations, and they have been studied very extensively for Cauchy singular integral equations over the interval (cf. the results and references in [17,18,44]). Quite recently they have been applied to different one-dimensional operator equations as well (cf. [33,37] and see also [58] for a comparable approach). If the underlying quadrature rule is a low-order composite rule, i.e., if the domain of integration is subdivided into small domains of step size less or equal to h and if a low-order rule like the trapezoidal rule or Simpson's rule is applied to each subdomain, then we call the quadrature method an h-method. h-methods for second- and first-kind equations have been well analyzed (cf., e.g., [3,4,15,16,18,23,44] and the references in these publications). Clearly, due to the fixed polynomial accuracy, these h-versions of quadrature methods are designed for problems with finite degree of smoothness. Finally, a combination of the composite technique with quadrature rules over the subdomains of variable orders (cf. [55]) is called an h–p method. Note that p- and h–p-methods seem to be very promising even for equations with a finite degree of smoothness. The analysis of these methods for general equations, however, seems to be a challenging problem.

In this paper we give an overview on more or less well-known results for the h-version of quadrature methods. In Section 2 we shall introduce the notion of simple quadrature methods and that of quadrature methods with product integration. We shall formulate some convergence results for second-kind equations with smooth kernel functions and smooth solutions. In Section 3 we shall apply simple quadrature methods to pseudodifferential equations, i.e., to first-kind integral equations over smooth curves and over the torus. Note that, for these methods, a sort of "Fourier analysis" is required to derive stability and convergence. If the integral operator is defined over nonsmooth boundary curves, then the "Fourier analysis" of the approximation methods is much more involved, and we refer, e.g., to [16,44] for more details. In Section 4 we show how the concept of mesh gradings for higher-dimensional quadrature methods leads naturally to fully discretized collocation schemes. We explain why simple quadrature methods may not converge in case of second-kind boundary integral equations over curves and surfaces with corners and edges. Finally, we explain that, from the view point of complexity, simple quadrature methods over graded meshes are not optimal for the approximation of these higher-dimensional integral equations. In Section 5 we consider a quadrature method based on product integration for the numerical solution of two-dimensional strongly singular integral equations.

2. Quadrature methods and Fredholm integral equations of the second kind

A lot of boundary value problems over domains with smooth boundary can be converted into a Fredholm integral equation of the second kind (cf., e.g., [34]) the numerical theory for which is well known (cf., e.g., [3,4,23]). Let us begin with the simplest one-dimensional case. Suppose we have to solve the equation

$$x(t) + \int_0^1 k(t,\tau)x(\tau)\,\mathrm{d}\tau = y(t), \quad 0 \leqslant t \leqslant 1, \tag{2.1}$$

where y and k are given smooth functions and x is to be determined. Replacing the integral by the rectangle rule, we obtain the Nyström method for (2.1).

$$\tilde{x}(t) + \sum_{l=0}^{n-1} k\left(t, \frac{l+1/2}{n}\right)\tilde{x}\left(\frac{l+1/2}{n}\right)\frac{1}{n} = y(t), \quad 0 \leqslant t \leqslant 1. \tag{2.2}$$

The solution of this continuous equation over the interval $[0,1]$ consists of two steps. First, one has to solve the quadratic linear system for the values $\tilde{x}((j+\frac{1}{2})/n)$, $j = 0,\ldots,n-1$

$$\tilde{x}\left(\frac{j+1/2}{n}\right) + \sum_{l=0}^{n-1} k\left(\frac{j+1/2}{n}, \frac{l+1/2}{n}\right)\tilde{x}\left(\frac{l}{n}\right)\frac{1}{n} = y\left(\frac{j}{n}\right), \quad j = 0,1,\ldots,n-1. \tag{2.3}$$

Then, knowing the values $\tilde{x}(j/n)$, $\tilde{x}$ is to be computed via Nyström's interpolation

$$\tilde{x}(t) = y(t) - \sum_{l=0}^{n-1} k\left(t, \frac{l+1/2}{n}\right)\tilde{x}\left(\frac{l+1/2}{n}\right)\frac{1}{n}, \quad 0 \leqslant t \leqslant 1. \tag{2.4}$$

Using, e.g., the theory of collectively compact operators, one can prove that (2.2) is stable, i.e., that (2.3) has a unique solution for n large enough and that the spectral norm of the inverse matrix is uniformly bounded. The approximate solution $\tilde{x}$ converges to x with the same order as the quadrature rule in (2.2) approximates the integral in (2.1).

Next, we generalize this method. Suppose Γ is a compact manifold which is embedded in a Euclidean space and which is either closed or open. One should think of closed smooth curves or two-dimensional closed surfaces (i.e., boundary surfaces of open domains) or pieces of these two. Over the manifold we consider the integral equation

$$a(t)x(t) + \int_\Gamma k(t,\tau)x(\tau)\,\mathrm{d}_\Gamma\tau = y(t), \quad t \in \Gamma \tag{2.5}$$

including the kernel k and the coefficient function a. For first-kind equations, a is zero. If a is a bounded nonvanishing function, then we can divide the equation by a. Thus we may suppose that a is a constant. A good example for the kernel k is the two-dimensional double-layer kernel which corresponds to three-dimensional boundary value problems for Laplace's equation and which is defined by the formula $k(t,\tau):=(t-\tau)\cdot v_\tau/(4\pi||t-\tau||^3)$, where $||t-\tau||$ is the Euclidean distance from t to τ and v_τ stands for the unit normal to the manifold Γ taken at the point τ. Note that the integral operator corresponding to this double-layer kernel is a pseudodifferential operator of order minus one (cf., e.g., [9,25]). The double-layer equation including this integral operator is an equation of the second kind with $a=0.5$. In order to discretize (2.5) we introduce a partition $\Gamma=\bigcup_{k=1}^K \Gamma_k$ of Γ into small submanifolds Γ_k of diameter less than a prescribed small positive number h. Fixing a small integer L and choosing quadrature knots $t_{k,l}\in\Gamma_l$, $l=1,\dots,L$, and nonnegative quadrature weights $\omega_{k,l}$, $l=1,\dots,L$, for a quadrature over Γ_k, we arrive at the composite quadrature rule

$$\int_\Gamma f(\tau)\,\mathrm{d}_\Gamma\tau=\sum_{k=1}^K\int_{\Gamma_k} f(\tau)\,\mathrm{d}_{\Gamma_k}\tau\sim\sum_{k=1}^K\sum_{l=1}^L f(t_{k,l})\omega_{k,l}. \tag{2.6}$$

Note that, for finer and finer approximations, K tends to infinity, the maximum h of the diameters $\operatorname{diam}\Gamma_k$, $k=1,\dots,K$, tends to zero but L is supposed to be fixed. By m_Q we denote the order of convergence defined by

$$\left|\int_\Gamma f(\tau)\,\mathrm{d}_\Gamma\tau-\sum_{k=1}^K\sum_{l=1}^L f(t_{k,l})\omega_{k,l}\right|\leqslant Ch^{m_Q}. \tag{2.7}$$

For example, the partition could be a triangulation of a two-dimensional polyhedron and the quadrature rule the mid-point rule $\int_{\Gamma_k} f\sim f(t_{k,1})\omega_{K,1}$ with $t_{k,1}$ the centroid of triangle Γ_k and $\omega_{k,1}:=\int_{\Gamma_k}1$ or the three-point rule using the mid-points of the sides of the triangle as quadrature knots and the weights $\omega_{k,l}=\int_{\Gamma_k}1/3$. Note that the mid-point rule is exact for linear functions whereas the three-point rule is exact for quadratic functions over the subtriangles of the triangulation which leads to an order of convergence of $m_Q=2$ and 3, respectively. For polygons the subdomains are intervals, and one could take the trapezoidal rule and Simpson's rule, which are exact for linear and cubic polynomials, respectively. In other words $m_Q=2$ and 4, respectively. However, for periodic functions over the interval, the order m_Q of the trapezoidal rule is even ∞. In case of curved polygons or polyhedra, we can introduce parametrization mappings $\gamma:\Omega\to\Gamma$ to reduce the integral $\int_{\Gamma_k} f$ to the integral $\int_{\Omega_k} f\circ\gamma\cdot|\gamma'|$ over a subdomain $\Omega_k=\gamma^{-1}(\Gamma_k)$ of Ω which is a subtriangle or subinterval. Applying the just mentioned rules to the transformed integral, we end up with a rule of the form (2.6).

Now, we replace the integration in (2.5) by quadrature (2.6) and arrive at the corresponding simple quadrature method (cf. (2.3))

$$a\tilde{x}_h(t_{k',l'})+\sum_{k=1}^K\sum_{l=1}^L k(t_{k',l'},t_{k,l})\tilde{x}_h(t_{k,l})\omega_{k,l}=y(t_{k',l'}),\quad k'=1,\dots,K,\ l'=1,\dots,L. \tag{2.8}$$

If the constant coefficient a is not zero and if the linear system (2.8) is solved, then we even can define the Nyström interpolant (cf. (2.4))

$$\tilde{x}_h(t):=\frac{1}{a}\left\{y(t)-\sum_{k=1}^K\sum_{l=1}^L k\left(t,t_{k,l}\right)\tilde{x}_h\left(t_{k,l}\right)\omega_{k,l}\right\},\quad t\in\Gamma. \tag{2.9}$$

Theorem 1. *Suppose that the compact manifold Γ is m_Q+1 times continuously differentiable and the right-hand side y is m_Q times continuously differentiable. Furthermore, suppose that the kernel*

k is m_Q times continuously differentiable with respect to each of its variable such that even the mixed derivatives $\partial_t^\alpha \partial_\tau^\beta k(t,\tau)$ with order α and β less than or equal to m_Q are bounded. Finally, assume that the constant a is not zero and that, for $y \equiv 0$, the integral equation (2.5) *has only the trivial solution $x \equiv 0$. Then the linear system of the quadrature method* (2.8) *is uniquely solvable for any right-hand side y at least if the step size of discretization h is sufficiently small. The approximate solution $\tilde{x}_h$ converges uniformly to the exact solution x and*

$$\sup_{t \in \Gamma} |\tilde{x}_h(t) - x(t)| \leqslant C h^{m_Q} \tag{2.10}$$

with a constant C independent of the discretization parameters h and K.

Note that in case of quasi-uniform partitions, i.e., in case that there exists a constant $c > 1$ with $c^{-1}h \leqslant \operatorname{irad} \Gamma_k \leqslant \operatorname{diam} \Gamma_k \leqslant ch$,

$$\operatorname{diam} \Gamma_k := \sup\{|t-\tau|:\ t,\tau \in \Gamma_k\},$$

$$\operatorname{irad} \Gamma_k := \sup\{\epsilon:\ \exists \tau \in \Gamma_k \text{ s.t. } |t-\tau| \leqslant \epsilon \Rightarrow t \in \Gamma_k\},$$

then the number of degrees of freedom is of order $O(h^{-d})$ with $d = 1$ and 2 for boundary curves and two-dimensional surfaces, respectively.

Theorem 2. *Suppose that k is the kernel of a classical pseudodifferential operator of negative order $-m$. Furthermore, suppose that $m_Q \geqslant m > 0$, that the compact manifold Γ is $m+1$ times continuously differentiable, and the right-hand side y is m times continuously differentiable. Finally, assume that a is a nonzero constant and that, for $y \equiv 0$, the integral equation* (2.5) *has only the trivial solution $x \equiv 0$. Then the linear system of the quadrature method* (2.8) *is uniquely solvable for any right-hand side y at least if the step size of discretization h is sufficiently small. The approximate solution $\tilde{x}_h$ converges uniformly to the exact solution x and*

$$\sup_{t \in \Gamma} |\tilde{x}_h(t) - x(t)| \leqslant C \log h^{-1} h^m \tag{2.11}$$

with a constant C independent of the discretization parameters h and K.

In particular, the quadrature method applied to the double-layer equation over a two-dimensional boundary manifold converges with order $O(h \log h^{-1})$. To prove the results of the last two theorems, one first shows stability of the discretized operators on the right-hand side of (2.8). This can be done, for instance, by the principle of collective compactness. Once stability is shown, the convergence order is derived from the order of convergence of the quadrature. For details we refer, e.g., to [3,4,18,23]. The reason for the restrictive order of convergence in Theorem 2 is the singular behavior of the kernel which can be characterized by the so called Calderón–Zygmund estimate

$$|\partial_t^\alpha \partial_\tau^\beta k(t,\tau)| < C\ |t-\tau|^{-d+m-|\alpha|-|\beta|}, \tag{2.12}$$

valid for all derivatives of order α and β such that $-d+m-|\alpha|-|\beta| < 0$. Here $-m$ is the order of the pseudodifferential operator and d the dimension of the underlying manifold Γ. The order in Theorem 2 can be improved if a slightly modified quadrature method is considered. This modification is called singularity subtraction or regularization (cf., e.g., [18]). To introduce this method we write (2.5) as

$$b(t)x(t) + \int_\Gamma k(t,\tau)[x(\tau)-x(t)]\,\mathrm{d}_\Gamma\tau = y(t), \quad t\in\Gamma,$$

$$b(t) := a + \int_\Gamma k(t,\tau)\,\mathrm{d}_\Gamma\tau. \tag{2.13}$$

Thus, we assume that we are able to compute the function b explicitly. For example, for the double-layer equation over smooth surfaces, constant functions are known to be eigenfunctions of the integral operator corresponding to the eigenvalue one-half, and (2.13) takes the form

$$x(t) + \frac{1}{4\pi}\int_\Gamma \frac{n_\tau\cdot(t-\tau)}{||\tau-t||^3}[x(\tau)-x(t)]\,\mathrm{d}_\Gamma\tau = y(t), \quad t\in\Gamma. \tag{2.14}$$

If we replace the integration in (2.13) by quadrature (2.6), we obtain the following quadrature method and the following Nyström interpolation step:

$$b(t_{k',l'})\tilde{x}_h(t_{k',l'}) + \sum_{k=1}^{K}\sum_{l=1}^{L} k(t_{k',l'},t_{k,l})[\tilde{x}_h(t_{k,l}) - \tilde{x}_h(t_{k',l'})]\omega_{k,l} = y(t_{k',l'}),$$

$$k'=1,\dots,K,\ l'=1,\dots,L. \tag{2.15}$$

$$\tilde{x}_h(t) := \frac{y(t) - \sum_{k=1}^{K}\sum_{l=1}^{L} k(t,t_{k,l})\tilde{x}_h(t_{k,l})\omega_{k,l}}{b(t) - \sum_{k=1}^{K}\sum_{l=1}^{L} k(t,t_{k,l})\omega_{k,l}}, \quad t\in\Gamma. \tag{2.16}$$

Theorem 3. *Suppose that k is the kernel of a classical pseudodifferential operator of negative order $-m$. Furthermore, suppose that $m_Q \geqslant m+1>0$, that the compact manifold Γ is $m+2$ times continuously differentiable, and that the right-hand side y is $m+1$ times continuously differentiable. Finally, assume that a is a nonzero constant and that, for $y\equiv 0$, the integral equation* (2.5) *has only the trivial solution $x\equiv 0$. Then the linear system of the quadrature method* (2.15) *is uniquely solvable for any right-hand side y and the denominator in* (2.16) *does not vanish at least if the step size of discretization h is sufficiently small. The approximate solution $\tilde{x}_h$ converges uniformly to the exact solution x and*

$$\sup_{t\in\Gamma}|\tilde{x}_h(t)-x(t)| \leqslant C\log h^{-1}h^{m+1} \tag{2.17}$$

with a constant C independent of the discretization parameters h and K.

Another way to improve quadrature methods for nonsmooth kernels is to apply quadrature rules of product type (cf., e.g., [3,18,30]). Indeed, in many applications the kernel function $k(t,\tau)$ is singular but it admits a factorization

$$k(t,\tau) = k_{\mathrm{sm}}(t,\tau)k_{\mathrm{si}}(t,\tau), \tag{2.18}$$

where the first factor k_{sm} has at least a finite degree of smoothness and where the singularity of k is contained in k_{si}. Moreover, we suppose that the singular kernel k_{si} is simpler such that the integral of k_{si} can be computed by analytic formulae. Or we suppose that $k_{\mathrm{si}}(t,\tau)$ is analytic with respect to τ for $\tau\neq t$ such that the integral of k_{si} can be computed by higher-order Gauß rules and other techniques (cf., e.g., [28,53,55,56]). Note that an additional additive perturbation by a smooth kernel function can be treated easily. For the sake of simplicity, however, we drop this additional term.

One example for a factorization of the form (2.18) is the representation of one-dimensional potential kernels for the Helmholtz equation. In particular, the single-layer kernel $k_{\boldsymbol{k}}$ corresponding to the equation with wave number $\boldsymbol{k}$ and transformed to the 2π periodic interval (cf. [30]) takes the form

$$\begin{aligned} k_{\boldsymbol{k}}(t,\tau) &= M_1(t,\tau)\log\left|4\sin^2\frac{t-\tau}{2}\right| + M_2(t,\tau),\\ M_1(t,\tau) &:= -\frac{1}{2\pi}J_0(\boldsymbol{k}|\gamma(t)-\gamma(\tau)|),\\ M_2(t,\tau) &:= \frac{\mathrm{i}}{2}H_0^{(1)}(\boldsymbol{k}|\gamma(t)-\gamma(\tau)|) - H_1(t,\tau)\log\left|4\sin^2\frac{t-\tau}{2}\right|, \end{aligned} \tag{2.19}$$

where $\gamma:[0,2\pi]\to\Gamma$ is the parametrization of the boundary curve, J_0 is the Bessel function of order zero, and $H_0^{(1)}$ is the Hankel function of order one. The factors M_1 and M_2 in (2.19) are analytic (resp. smooth) functions if the parametrization γ is analytic (resp. smooth). Another example for a factorization is the representation $k(t,\tau)=k_0(t,\tau)|t-\tau|^{-\alpha}$ for a typical boundary integral kernel over a smooth boundary curve $\tilde{\Gamma}$, where k_0 is an analytic function and where $\alpha>0$ is a certain degree of singularity. If $\gamma:\Gamma=[0,2\pi]\to\tilde{\Gamma}$ denotes the parametrization of the boundary manifold and if γ_0 is the parametrization of the unit circle, then we get a factorization of the form (2.18) for the kernel transformed to the 2π-periodic interval setting

$$\begin{aligned} k_{\mathrm{sm}}(\gamma(t),\gamma(\tau)) &:= k_0(\gamma(t),\gamma(\tau))\frac{|\gamma(t)-\gamma(\tau)|^{\alpha}}{|\gamma_0(t)-\gamma_0(\tau)|^{\alpha}}|\gamma'(\tau)|,\\ k_{\mathrm{si}}(\gamma(t),\gamma(\tau)) &:= |\gamma_0(t)-\gamma_0(\tau)|^{-\alpha}. \end{aligned} \tag{2.20}$$

Unfortunately, such a factorization does not work for the higher-dimensional case. In the higher-dimensional case, the structure of singularity is more involved and depends strongly on the geometry. Thus factorization (2.18) is to be defined by $k_{\mathrm{sm}}(t,\tau)=k_0(t,\tau)$ and $k_{\mathrm{si}}(t,\tau)=|t-\tau|^{-\alpha}$ (cf., Section 5 for more details). Then, in the case of curved boundaries, there are no analytic formulas available for the integration of k_{si}. However, if the boundary manifold is piecewise analytic, then the integral of k_{si} can be computed by tensor products of Gaussian quadratures. For general boundaries of finite degree of smoothness, the parametrization γ can be replaced by a piecewise polynomial interpolant $\tilde{\gamma}$ which is polynomial at least over each subdomain Γ_k of the corresponding partition of the quadrature method. After this substitution the integral over the kernel $k_{\mathrm{si}}(\tilde{\gamma}(t),\tilde{\gamma}(\tau))=|\tilde{\gamma}(t)-\tilde{\gamma}(\tau)|^{-\alpha}$ can again be computed by tensor products of Gaussian quadratures (for more details in some special case cf., e.g., [14]).

Now, we choose points $\tau_{k,l}\in\Gamma_k$ and interpolating polynomials $\varphi_{k,l}$ over Γ_k such that $\varphi_{k,l}(\tau_{k,l'})=\delta_{l,l'}$. Polynomial means here polynomial with respect to a given parametrization of the boundary manifold. We consider the quadrature rule

$$\int_\Gamma k(t,\tau)x(\tau)\,\mathrm{d}\tau = \int_\Gamma k_{\mathrm{si}}(t,\tau)[k_{\mathrm{sm}}(t,\tau)x(\tau)]\,\mathrm{d}\tau$$

$$\sim \sum_{k=1}^{K} \int_{\Gamma_k} k_{\mathrm{si}}(t,\tau) \sum_{l=1}^{L} [k_{\mathrm{sm}}(t,\tau_{k,l})x(\tau_{k,l})]\varphi_{k,l}(\tau)\,\mathrm{d}\tau$$

$$= \sum_{k=1}^{K} \sum_{l=1}^{L} k_{\mathrm{sm}}(t,\tau_{k,l})x(\tau_{k,l})\omega^p_{k,l}, \quad \omega^p_{k,l} := \int_{\Gamma_k} k_{\mathrm{si}}(t,\tau)\varphi_{k,l}(\tau)\,\mathrm{d}\tau. \tag{2.21}$$

In order to simplify the assumptions, we assume that the manifold Γ is a curve or a surface given by a single parametrization $\gamma : \Omega \to \Gamma$ and that the preimages $\Omega_k := \gamma^{-1}(\Gamma_k)$ of the subdomains Γ_k are intervals or triangles. Moreover, we suppose that the L parameter points $\sigma_{k,l}$ corresponding to the quadrature knots $\tau_{k,l} = \gamma(\sigma_{k,l}) \in \Gamma_k$ are defined as the affine images of fixed points σ_l, $l = 1,\dots,L$ in the standard interval $[0,1]$ (resp., in the standard triangle $\{(s_1,s_2)\colon 0 \leqslant s_2 \leqslant s_1 \leqslant 1\}$). Likewise, the polynomials $\varphi_{k,l}$ are supposed to be the pull backs of interpolatory polynomials φ_l defined over the standard interval or triangle. If this basis spans a space containing all polynomials of degree less than m_p, than the convergence order of the quadrature rule is $m_Q = m_p$. Applying the corresponding product rule to (2.5), we arrive at the quadrature method

$$a\tilde{x}_h(\tau_{k',l'}) + \sum_{k=1}^{K} \sum_{l=1}^{L} k_{\mathrm{sm}}(\tau_{k',l'},\tau_{k,l})\tilde{x}_h(\tau_{k,l})\omega^p_{k,l} = y(\tau_{k',l'}), \quad k' = 1,\dots,K, \ l' = 1,\dots,L. \tag{2.22}$$

Let us note that, for the special choice $k_{\mathrm{sm}} \equiv 1$, method (2.16) coincides with the piecewise polynomial collocation method, where the trial space is the span of the $\{\varphi_{k,l},\ k = 1,\dots,K,\ l = 1,\dots,L\}$. In other words, the quadrature method with product rule is already a compromise between quadrature and collocation method.

Theorem 4. *Suppose that the kernel k admits a factorization* (2.18), *where k_{sm} is m_p times continuously differentiable with respect to both variables such that even the mixed derivatives $\partial_t^\alpha \partial_\tau^\beta k(t,\tau)$ with order α and β less than or equal to m_p are bounded. For $k_{\mathrm{si}}(t,\tau)$, we suppose the same degree of differentiability for $t \neq \tau$ and, for $t \to \tau$ and the same orders of differentiation, estimates* (2.12) *where $m > 0$. Furthermore, suppose that the compact manifold Γ is $m_p + 1$ times continuously differentiable, and that the exact solution x and the right-hand side y are m_p times continuously differentiable. Finally, assume that a is a nonzero constant and that, for $y \equiv 0$, the integral equation* (2.5) *has only the trivial solution $x \equiv 0$. Then the linear system of the quadrature method* (2.22) *is uniquely solvable for any right-hand side y at least if the step size of discretization h is sufficiently small. The approximate solution $\tilde{x}_h$ converges uniformly to the exact solution x and*

$$\sup_{t \in \{\tau_{k,l}: k=1,\dots,K,\, l=1,\dots,L\}} |\tilde{x}_h(t) - x(t)| \leqslant C h^{m_p} \tag{2.23}$$

with a constant C independent of the discretization parameters h and K.

3. Quadrature methods for pseudodifferential equations over smooth boundaries

Boundary integral operators over smooth boundaries belong to the class of classical pseudodifferential operators (cf., e.g., [9,25]). If the order of such an operator is nonnegative, then the kernels of the integral operators are strongly singular or even hypersingular. The convergence of simple quadrature methods applied to such functions is not guaranteed. In fact, in many situations the straightforward quadrature methods do not converge. We present here convergent variants of quadrature methods, only. All these methods rely on a singularity subtraction step (cf., e.g., (3.7), (3.8), and [6,49] for operators of order minus one) though, at first glance, this may not be visible. Let us start with the simplest case, i.e., with a Cauchy singular integral equation over the unit circle $\mathbb{T}$

$$Ax(t):=a(t)x(t)+b(t)\frac{1}{\pi \mathrm{i}}\int_{\mathbb{T}}\frac{x(\tau)}{\tau-t}\,\mathrm{d}\tau+\int_{\mathbb{T}}k(t,\tau)x(\tau)\,\mathrm{d}\tau=y(t),\quad t\in\mathbb{T}. \tag{3.1}$$

Here a,b,k, and y are given functions and x is to be determined. Using the ideas developed for second-kind integral equations, it is not hard to reduce the problem for arbitrary kernel functions k to the case $k\equiv 0$. Moreover, for simplicity, we suppose a and b to be continuous. We choose an even positive integer n, set $t_k:=\mathrm{e}^{\mathrm{i}2\pi k/n}$, and consider the following quadrature rules:

$$\int_{\mathbb{T}} f(\tau)\,\mathrm{d}\tau=\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}s})\mathrm{i}\mathrm{e}^{\mathrm{i}s}\,\mathrm{d}s\sim\sum_{l=0}^{n-1} f(t_l)t_l\frac{2\pi \mathrm{i}}{n}, \tag{3.2}$$

$$\int_{\mathbb{T}} f(\tau)\,\mathrm{d}\tau\sim\sum_{\substack{l=0,\ldots,n-1\\ l\equiv k+1 \bmod 2}} f(t_l)t_l\frac{4\pi \mathrm{i}}{n}. \tag{3.3}$$

Note that rule (3.3) has doubled step size in comparison with (3.2). However, it is appropriate to functions f having a singularity at t_k and will lead to optimal quadrature methods. Thus, we consider (3.1) for $t=t_k$, $k=0,\ldots,n-1$, replace the integral by rule (3.3) to obtain the quadrature method

$$a(t_k)\tilde{x}(t_k)+b(t_k)\frac{1}{\pi \mathrm{i}}\sum_{\substack{l=0,\ldots,n-1\\ l\equiv k+1 \bmod 2}}\frac{\tilde{x}(t_l)}{t_l-t_k}t_l\frac{4\pi \mathrm{i}}{n}=y(t_k),\quad k=0,\ldots,n-1. \tag{3.4}$$

We call this quadrature method stable if, at least for sufficiently large n, Eq. (3.4) are uniquely solvable for any right-hand side and if the Euclidean matrix norms of the matrices of the linear systems in (3.4) and the norms of their inverses are uniformly bounded with respect to n. The method is called convergent if the trigonometric interpolation

$$L_n\tilde{x}(t):=\sum_{k=1}^{n-1}\tilde{x}(t_k)\frac{1}{n}\sum_{l=-n/2}^{n/2-1}\frac{t^l}{t_k^l} \tag{3.5}$$

tends in the L^2 norm to the exact solution x of (3.1) for all continuous right-hand sides y. Note that stability is an important condition for solving the linear system of equations. Moreover, it is necessary for the method to be convergent. We get (cf. [44] and compare the analogous results in [5,32]).

Theorem 5. *If the singular integral operator A is invertible, then the quadrature method* (3.4) *is stable and convergent. For a right-hand side which is m times differentiable such that the mth derivative is square integrable, the L^2 error $\|L_n\tilde{x}-x\|$ is less than a constant times n^{-m}.*

Proof. We assume a and b to be constant. The general case can be treated by well-known localization techniques (cf., e.g., [44]). Set $e_k(t):=t^k$, denote the span of the e_k, $k=-n/2,\ldots,n/2-1$ by T_n, and recall that L_n stands for the interpolation projection of (3.5). Now it is well known that e_k is an eigenfunction of A corresponding to the eigenvalue $a+b\,\mathrm{sign}(k+\frac{1}{2})$. Hence, T_n is an invariant subspace for A. The collocation solution $x_n \in T_n$ is defined by $Ax_n(t_k) = y(t_k),\ k = 0,\dots,n-1$, i.e., by $L_n A x_n = L_n y$. Consequently, we get $Ax_n = L_n y$ and the collocation solution $x_n = A^{-1} L_n y$ converges to the exact solution $x = A^{-1} y$. Thus in order to prove our theorem, it is sufficient to show the equivalence of method (3.4) and the collocation method.

The solution $\tilde{x}$ of (3.4) is a discrete function over $\{t_k,\ k = 0,\dots,n-1\}$. We identify $\tilde{x}$ with the linear interpolation $L_n \tilde{x}$. Then our proof is finished if we show

$$Ax_n(t_k) = a x_n(t_k) + b \frac{1}{\pi \mathrm{i}} \sum_{\substack{l = 0,\dots,n-1 \\ l \equiv k+1 \bmod 2}} \frac{x_n(t_l)}{t_l - t_k} t_l \frac{4\pi \mathrm{i}}{n}.$$

We have to prove that, for $x_n \in T_n$,

$$\frac{1}{\pi \mathrm{i}} \int_{\mathbb{T}} \frac{x_n(\tau)}{\tau - t_k} \mathrm{d}\tau = \frac{1}{\pi \mathrm{i}} \sum_{\substack{l = 0,\dots,n-1 \\ l \equiv k+1 \bmod 2}} \frac{x_n(t_l)}{t_l - t_k} t_l \frac{4\pi \mathrm{i}}{n}. \tag{3.6}$$

We arrive at

$$\begin{aligned}\frac{1}{\pi \mathrm{i}} \int_{\mathbb{T}} \frac{x_n(\tau)}{\tau - t_k} \mathrm{d}\tau &= \frac{1}{\pi \mathrm{i}} \int_{\mathbb{T}} \frac{x_n(\tau) - x_n(t_k)}{\tau - t_k} \mathrm{d}\tau + x_n(t_k) \frac{1}{\pi \mathrm{i}} \int_{\mathbb{T}} \frac{1}{\tau - t_k} \mathrm{d}\tau \\ &= \frac{1}{\pi \mathrm{i}} \sum_{\substack{l = 0,\dots,n-1 \\ l \equiv k+1 \bmod 2}} \frac{x_n(t_l) - x_n(t_k)}{t_l - t_k} t_l \frac{4\pi \mathrm{i}}{n} + x_n(t_k), \end{aligned} \tag{3.7}$$

where we have used that $e_0 \equiv 1$ is an eigenfunction corresponding to the eigenvalue 1, that $\{x_n(t) - x_n(t_k)\}/\{t - t_k\}$ is in the span$\{e_k,\ k = -n/2,\dots,n/2-2\}$ and that (3.3) is exact on span$\{e_k,\ k = -n/2,\dots,n/2-2\}$. Note that the exactness of (3.3) is a simple consequence of the formula for the geometric series. Now (3.6) follows from (3.7) by a straightforward computation. The convergence order can be derived by standard methods (cf., e.g., [44]). □

Theorem 5 can be generalized to nonuniform partitions (cf. [8,38,54]). An analogous result holds for the one-dimensional hypersingular equation ([27], cf. also [7,11,29]). However, the singularity subtraction step (3.7) is to be replaced by the following regularization of the finite part integral:

$$\begin{aligned}\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\tilde{x}(\tau)}{|\tau - t_k|^2} |\mathrm{d}\tau| &= \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\tilde{x}(\tau) - \tilde{x}(t_k) - \tilde{x}'(t_k)(\tau - t_k)}{|\tau - t_k|^2} |\mathrm{d}\tau| \\ &\quad + \tilde{x}(t_k) \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1}{|\tau - t_k|^2} |\mathrm{d}\tau| + \tilde{x}'(t_k) \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\tau - t_k}{|\tau - t_k|^2} |\mathrm{d}\tau|. \end{aligned} \tag{3.8}$$

Applying (3.3) to the first integral on the right-hand side, computing the others and performing some easy calculations, we arrive at the quadrature approximation

$$\frac{1}{2\pi} \int_{\mathbb{T}} \frac{\tilde{x}(\tau)}{|\tau - t_k|^2} |\mathrm{d}\tau| \sim \frac{n}{8} \tilde{x}(t_k) + \sum_{\substack{l = 0,\dots,n-1 \\ l \equiv k+1 \bmod 2}} \frac{\tilde{x}(t_l)}{|t_l - t_k|^2} t_l \frac{2}{n}, \tag{3.9}$$

which, again, is exact for $\tilde{x} = x_n \in T_n$. Note that a regularization like in (3.8) is necessary in order to obtain a convergent quadrature method.

Now, let us consider the generalized single-layer equation

$$Ax(t) := \int_{\mathbb{T}} k(t,\tau)x(\tau)|\mathrm{d}\tau| = y(t), \quad t \in \mathbb{T}, \tag{3.10}$$

$$k(t,\tau) := a(t)k_0(t,\tau) + b(t)k_1(t,\tau) + k_2(t,\tau),$$

$$k_0(t,\tau) := -\frac{1}{\pi}\log|t-\tau|, \quad k_1(t,\tau) := k_1(t/\tau),$$

$$k_1(\mathrm{e}^{\mathrm{i}2\pi u}) := \begin{cases} \mathrm{i}[u-0.5] & \text{if } 0<u<1, \\ 0 & \text{if } u=0, \end{cases} \tag{3.11}$$

where a, b and k_2 are smooth functions. Operator A is a pseudodifferential operator with principal symbol $\sigma_A(t,\xi) = [a(t) + b(t)\,\mathrm{sign}(\xi)]|\xi|^{-1}$. Replacing integration by quadrature (3.2), we arrive at the quadrature method

$$\sum_{l=0}^{n-1} k(t_k,t_l)\tilde{x}(t_l)\frac{2\pi}{n} = y(t_k), \quad k=0,\ldots,n-1. \tag{3.12}$$

Note that $k_0(t,t) := \lim_{\tau\to t} k_0(t,\tau) = \infty$. Thus, in the last formula, we need to fix an artificial finite value for $k_0(t,t) = k_0(1,1)$. Due to the factor $2\pi/n$ this value is of no importance for the consistency of the quadrature. However, the choice of this value is essential for the stability and the order of convergence. We take $k_0(1,1) = -\log n/\pi$ which corresponds to the quadrature method modified by singularity subtraction [6,49]. For quadrature methods applied to the general pseudodifferential equation (3.10) of order -1 and analogous methods applied to other pseudodifferential equations of negative order, the method of the proof to Theorem 5 fails. The theory of collectively compact operators is helpful to treat the compact perturbations $\int k_2(t,\tau)x(\tau)|\mathrm{d}\tau|$. The stability of the main part of the equation, however, requires new techniques. Of course, stability is to be understood not in terms of the Euclidean matrix norm but in terms of a more general operator norm induced by the norms of the Sobolev spaces in which A and its inverse are bounded. The first method of proof is the so-called localization principle. The second is the Fourier analysis or circulant technique. For example, the stability of the discretized weakly singular operator defined by the left-hand side of (3.12) can be reduced by localization to the stability of the corresponding matrices with frozen functions a, b, and k_2. The matrix with constant a, b, and k_2 is a circulant and takes the form

$$V_n := \left(-a\log|1-t_{k-l}|\frac{2}{n} + bk_1(t_{k-l})\frac{2\pi}{n} + k_2\frac{2\pi}{n}\right)_{k,l=0}^{n-1}. \tag{3.13}$$

In general, a matrix $(a_{k,l})_{k,l=0}^{n-1}$ is called a circulant if $a_{k,l} = a_{k-l}$ and $a_{k-l} = a_{k-l\pm n}$. The eigenvalues $\{\lambda_l,\ l=-n/2,-n/2+1,\ldots,n/2-1\}$ of the circulant $(a_{k-l})_{k,l=0}^{n-1}$ are connected with the entries by

$$\lambda_k = \sum_{l=0}^{n-1} \mathrm{e}^{\mathrm{i}2\pi lk/n}a_l. \tag{3.14}$$

Using (3.14), writing $|1-t_l| = 2|\sin(\pi l/n)|$, and substituting $\sin(\pi x)$ by $\pi x\prod_{j=1}^{\infty}(1-x^2/j^2)$, it is not hard to verify that the matrix V_n has the eigenvalues

$$\lambda_l^n = \begin{cases} [s(t_l)]/n & \text{if } l=-\frac{n}{2},\ldots,-1,\ 1,\ldots,\frac{n}{2}-1, \\ 2\pi k_2 & \text{if } l=0, \end{cases} \tag{3.15}$$

where the numerical symbol function $\boldsymbol{s}$ is defined by $\boldsymbol{s}(t){:=}a\boldsymbol{f}(t)+b\,\boldsymbol{g}(t)$,

$$\boldsymbol{f}(t){:=}2\log(2\pi)-2\sum_{l\in\mathbb{Z},\ l\neq 0}\log|l|\,t^l,\qquad \boldsymbol{g}(\mathrm{e}^{\mathrm{i}2\pi u}){:=}\pi\cot(\pi u).$$

Note that function $\boldsymbol{f}$ is smooth except at $t=1$. Multiplying $\boldsymbol{f}(t)$ by $(t-1)^2$ we get an absolutely convergent series. By the way, $\boldsymbol{f}$ is the symbol function of the Toeplitz matrix $(-2\log|k-l|)_{k,l=-\infty}^{\infty}$ which is the quadrature discretization of step size one for the logarithmic equation over the real axis. Comparing eigenvalues (3.15) with the eigenvalues $\mu_0=2\pi k_2$ and $\mu_l=(a+\mathrm{sign}(l+\frac{1}{2}))|l|^{-1}$, $l=\pm 1,\pm 2,\dots$ corresponding to operator A, we observe the consistency property $\lambda_l^n/\mu_l\to 1$ for any fixed l and for $n\to\infty$. The stability is equivalent to the existence of a constant $c>1$ such that $c^{-1}|\mu_l|\leqslant|\lambda_l^n|\leqslant c|\mu_l|$ holds for sufficiently large n and $l=-n/2,\dots,n/2-1$. We arrive at the following typical theorem.

Theorem 6. *If the pseudodifferential operator A of order -1 is invertible, if $k_0(1,1)$ is chosen to be $-\log n/\pi$, and if $a(t)+\lambda b(t)\neq 0$ for all $t\in\mathbb{T}$ and $-1\leqslant\lambda\leqslant 1$, then the quadrature method* (3.4) *is stable and convergent. For a right-hand side which is four times continuously differentiable or at least contained in the Sobolev space $H^4(\mathbb{T})$, we get the estimate* (*cf.* (3.5))

$$\sup_{t\in\mathbb{T}}|L_n\tilde{x}-x|\leqslant C\,||L_n\tilde{x}-x||_{H^1(\mathbb{T})}\leqslant Cn^{-2}\,||y||_{H^4(\mathbb{T})}. \tag{3.16}$$

The assumption $a(t)+\lambda b(t)\neq 0$, $-1\leqslant\lambda\leqslant 1$ means that operator A is strongly elliptic. Note that the interval $[-1,1]$ in this condition originates from $[-1,1]=\{\boldsymbol{g}(t)/\boldsymbol{f}(t)\colon t\in\mathbb{T}\}$. The convergence order two in estimate (3.16) can be derived from the symbol function $\boldsymbol{s}$, too. Namely, if there exists a constant $\alpha>0$ such that $|x|\boldsymbol{f}(x)=1+\mathrm{O}(|x|^{\alpha})$ and $x\boldsymbol{g}(x)=1+\mathrm{O}(|x|^{\alpha})$ for $x\to\pm 0$, then $|\lambda_l^n-\mu_l|/|\mu_l|\leqslant\mathrm{O}(n^{-\alpha})$ and the order of convergence is α. In fact this constant exists, and is equal to two. As it is well known (cf. [6,49]), the convergence order is even three in the case that the coefficient b vanishes identically. To improve the order of convergence, one can use, for instance, an end-point correction for the rectangle rule (cf. [1]). More details and different modifications to improve convergence can be found in [6,18,31,35,36,44,49,52,57].

In general, for the stability of the quadrature method applied to a first-kind integral operator, the invertibility of the operator is not sufficient. Often strong ellipticity turns out to be the necessary and sufficient stability condition. For one-dimensional pseudodifferential operators of order less than -1, the quadrature method can also be considered as a Galerkin method with Dirac-δ ansatz functions (cf. [51]). In this case standard techniques for the Galerkin approximation of strongly elliptic operators can be applied.

Finally, let us remark that there is not much known for quadrature methods applied to pseudodifferential equations over the boundaries of higher-dimensional domains. The only paper in this direction we know about is due to Saad Abdel-Fattah [48]. To report this result, we consider the two-dimensional pseudodifferential operator of order zero over the torus $\mathbb{T}^2:=\{(t_1,t_2)\in\mathbb{R}^2\colon 0\leqslant t_i<1,\ i=1,2\}$

$$Ax(t):=a(t)x(t)+\int_{\mathbb{T}^2}k(t,\tau)x(\tau)\mathrm{d}\tau=y(t),\quad t\in\mathbb{T}^2,$$

$$k(t,\tau):=k_0(t,t-\tau)+k_2(t,\tau), \tag{3.17}$$

where the coefficient a and the kernel function k_2 are supposed to be smooth and one-periodic functions and where the singular kernel $k_0(t,\sigma)$ satisfies

$$k_0(t,\varrho\sigma) = \operatorname{sign}(\varrho) k_0(t,\sigma)\varrho^{-2}, \quad \varrho \neq 0 \tag{3.18}$$

and is smooth and one-periodic with respect to $t \in \mathbb{T}^2$ and smooth with respect to σ for $|\sigma| = 1$. Applying the tensor product trapezoidal rule to (3.17), we arrive at the quadrature method

$$a\left(\frac{k_1}{n},\frac{k_2}{n}\right) x\left(\frac{k_1}{n},\frac{k_2}{n}\right) + \sum_{l_1,l_2=0}^{n-1} k\left(\left(\frac{k_1}{n},\frac{k_2}{n}\right),\left(\frac{l_1}{n},\frac{l_2}{n}\right)\right) x\left(\frac{l_1}{n},\frac{l_2}{n}\right)\frac{1}{n^2}$$

$$= y\left(\frac{k_1}{n},\frac{k_2}{n}\right), \quad k_1,k_2 = 0,\dots,n-1. \tag{3.19}$$

Here the singular value $k(t,t)$ is set to zero. Using localization techniques and two-dimensional Fourier analysis, Saad proved the following theorem.

Theorem 7. *If the singular integral operator A is invertible and satisfies condition* (3.18), *then method* (3.19) *is stable and convergent in the same sense as method* (3.4) *in Theorem* 5.

The convergence order for smooth right-hand sides is one. Note that this result is not important as a result for the artificial torus but it is important as a local analysis of the quadrature method over regular tensor product grids. Of course, for a full understanding of quadrature methods a lot of further local cases have to be studied, and these cases seem to be much more involved.

4. Negative results for quadrature methods applied to higher-dimensional equations

Now, let us have a look at quadrature methods for the solution of general integral equations over two-dimensional manifolds. If the kernel function and the manifold are smooth, then we have nice results for second-kind equations like in Theorem 1 and a lot of problems with first-kind equations which are severely ill-posed and not discussed here. In many important applications, however, the kernel $k(t,\tau)$ is singular in the sense of (2.12). In this case, even if the quadrature method is stable, the convergence of the quadratures and, consequently, that of the approximate solutions is very poor (cf., e.g., Theorem 7) if the method is not, in fact, diverging. We only mention here the lack of convergence for the simplest quadrature method applied to the double-layer equation over polyhedra. Without loss of generalization, we choose the simplest example and consider the equation

$$2\,[1 - d_c(t)]\,x(t) + \frac{1}{2\pi}\int_\Gamma \frac{v_\tau\cdot(t-\tau)}{|t-\tau|^3} x(\tau)\,\mathrm{d}_\tau\Gamma = y(t), \quad t \in \Gamma \tag{4.1}$$

over the boundary Γ of $C = \{(x_1,x_2,x_3) \in \mathbb{R}^3\colon 0 \leqslant x_i \leqslant 1,\ i = 1,2,3\}$. Here v_τ is the unit normal to Γ at τ and $d_C(t)$ is the normalized solid angle of C at the boundary points, i.e., $d_C(t) = \frac{1}{8}$ for vertex points, $d_C(t) = \frac{1}{4}$ for edge points, and $d_C(t) = \frac{1}{2}$ else. Note that the double layer kernel is strongly singular at the edge and vertex points and (2.12) holds with $m = 0$. For simplicity, we choose $\Gamma = \bigcup_{k=1}^K \Gamma_k$ to be the partition of Γ into $K = 6n^2$ uniform squares of side length $h = 1/n$, and we suppose that rule (2.6) in method (2.8) is the mid-point rule. Then the error $\sup_{k,l} |\tilde{x}_h(t_{k,l}) - x(t_{k,l})|$ need not tend to zero even if the right-hand side y is smooth. This follows from the fact that the quadrature error does not turn to zero uniformly. Indeed, choose $t_{k',l'}=(0.5h,0,0.5h)$ and x to be one over $\{(x_1,x_2,0)\in\mathbb{R}^3\colon 0\leqslant x_i\leqslant 1,\ i=1,2\}$ and zero over the rest of Γ and consider the quadrature error

$$\int_\Gamma k(t_{k',l'},\tau)x(\tau)\mathrm{d}_\Gamma\tau-\sum_{k=1}^{K}\sum_{l=1}^{1}k(t_{k',l'},t_{k,l})x(t_{k,l})\omega_{k,l}$$

$$=\frac{1}{2\pi}\int_0^1\int_0^1\frac{(0,0,1)\cdot((0.5h,0,0.5h)-(x_1,x_2,0))}{|(0.5h,0,0.5h)-(x_1,x_2,0)|^3}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$$-\frac{1}{2\pi}\sum_{k_1,k_2=1}^{n}\frac{(0,0,1)\cdot((0.5h,0,0.5h)-([k_1-0.5]h,[k_2-0.5]h,0))}{|(0.5h,0,0.5h)-([k_1-0.5]h,[k_2-0.5]h,0)|^3}h^2$$

$$=\frac{1}{2\pi}\int_0^n\int_0^n\frac{0.5}{|(0.5,0,0.5)-(x_1,x_2,0)|^3}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$$-\frac{1}{2\pi}\sum_{k_1,k_2=1}^{n}\frac{0.5}{|(0.5,0,0.5)-([k_1-0.5],[k_2-0.5],0)|^3}.$$

Obviously, this tends to

$$\frac{1}{2\pi}\int_0^\infty\int_0^\infty\frac{0.5}{|(0.5,0,0.5)-(x_1,x_2,0)|^3}\,\mathrm{d}x_1\,\mathrm{d}x_2$$

$$-\frac{1}{2\pi}\sum_{k_1,k_2=1}^{\infty}\frac{0.5}{|(0.5,0,0.5)-([k_1-0.5],[k_2-0.5],0)|^3},$$

i.e., to the quadrature error over an unbounded conical boundary manifold. This quadrature error with step size $h=1$ is different from zero. Now, the convergence properties of the quadrature method correspond to those of the quadrature rule, and method (2.8) does not converge with respect to the supremum norm. Similar homogeneity arguments apply to quadrature methods including special graded meshes and double-layer equations over general piecewise smooth boundary manifold. To get a converging quadrature method, it is sufficient to choose the version with singularity subtraction (2.15) (cf. [47]). Analogous arguments can be used also for disproving the convergence in the case of strongly singular integral equations.

Now turn again to general quadrature methods over two-dimensional manifolds. To improve a low order of convergence, one has to adapt the quadrature to the singular behavior of the kernel function $\tau\mapsto k(t,\tau)$. We shall discuss two methods, mesh gradings in this section and product rules in Section 5. The first and simplest way of adaption is to use a mesh grading towards the singularity point t of the kernel. In other words, the quadrature rule employed for the numerical method should not be a fixed rule but it should depend on the source point t. Such an improved method seeks approximate values $\tilde{x}(t)$ for the unknown solution x over the points t of a fixed grid G. For each point $t\in G$, we have to approximate the integral $\int k(t,\tau)x(\tau)\,\mathrm{d}\tau$ in (2.5) by a quadrature rule over a refined grid G_t the points of which accumulate around t. Hence function values $x(\tau)$ at the quadrature knots $\tau\in G_t$ of this refined grid are required, and these can be obtained by interpolating the fixed set of approximate values $\{\tilde{x}(t),\ t\in G\}$. If the interpolant is $\tilde{x}_I$, i.e., if the values $\tilde{x}(\tau)$, $\tau\in G_t$ are approximated by $\tilde{x}_I(\tau)$, $\tau\in G_t$, then the quadrature approximation to $\int k(t,\tau)x(\tau)\,\mathrm{d}\tau$ is a discretization of $\int k(t,\tau)\tilde{x}_I(\tau)\,d\tau$. In other words, the resulting scheme in its simplest form is rather not a quadrature method but rather a fully discretized collocation method.

An exception, where a refined mesh can lead to an improved quadrature method in the sense of (2.15), is the case of second-kind integral equations over nonsmooth but piecewise smooth surfaces. Let the piecewise smooth surface $\Gamma \in \mathbb{R}^3$ take the form $\bigcup_{m=1}^{m_\Gamma} \Gamma^m$, where all the patches Γ^m are smooth. Then, for instance, the kernel $k(t,\tau)$ of the double-layer equation satisfies (2.12) with $m=0$ for points t,τ from different patches Γ^m and $\Gamma^{m'}$. For points of the same smooth boundary patch Γ^m, estimate (2.12) holds with $m=1$. Moreover, if the patches Γ^m are planar, then the double-layer kernel vanishes. Hence, one can choose a fixed mesh G graded towards the boundaries of the patches Γ^m, and, for each $t \in G$, the grids G_t can be chosen to be G. The mesh grading means that the diameter of the partition domain has to be small when the domain is close to the edge, i.e., to the boundary of the smooth patches Γ^m. Unfortunately, a partition with subdomains small only in the direction toward the edge and larger in the direction parallel to the edge is not sufficient. The resulting quadrature methods take the form (2.15). The number of subdomains and the corresponding number of degrees of freedom corresponding to such gradings is usually in the order $[h^{-1}]^\alpha$ where h is the maximal mesh size and where $\alpha>2$ depends on the smoothness of the solution or, equivalently, on the geometry of Γ. Thus, substantially more degrees of freedom are necessary than the $[h^{-1}]^2$ for methods over uniform grids. The corresponding quadrature methods are analyzed in [45,47].

To evaluate this quadrature method over graded meshes we turn to the complexity. Let us suppose that β is the order of complexity for Nyström's methods over regular grids, i.e., suppose that the number of necessary arithmetic operations to compute an approximate solution with a supremum norm error less than a prescribed $\varepsilon>0$ is less than $O([\varepsilon^{-1}]^\beta)$. Here the β depends on the singularities of the exact solution to the double-layer equation. It turns out that, using an appropriately graded mesh, the order of complexity of Nyström's method can be reduced to $\beta/2$. In contrast to this higher-dimensional result for quadrature methods, the complexity order of the univariate quadrature method and that of higher-dimensional discretized collocation or Galerkin methods can be reduced to an arbitrarily small number if only a quadrature rule (resp. a trial space) of sufficiently high order is used and if the mesh is appropriately graded. In particular, in case of the two-dimensional collocation method, the graded meshes can be chosen to include subdomains which are of small size in direction to the closest edge and which have a larger size in the perpendicular direction. Hence, the number of degrees of freedom can be estimated by $[h^{-1}]^2$ and, at least asymtotically, the order of complexity can be reduced to an arbitrarily small number. Consequently, even in the case of second-kind equations, the fully discretized collocation or Galerkin methods are more efficient than the simple quadrature methods (2.8) and (2.15).

5. Product quadrature for two-dimensional singular equations

Suppose Γ is a smooth two-dimensional manifold. Over Γ we consider the integral equation $Ax=y$ from (2.5) with A an operator invertible in the space $L^2(\Gamma)$. We suppose that the kernel k admits a factorization $k(t,\tau)=k_{\mathrm{sm}}(t,\tau)k_{\mathrm{si}}(t,\tau)$ with the factor k_{sm} of finite degree of smoothness and with the singularity factor k_{si}, which satisfies (2.12) with $m=0$ and $d=2$. We assume that, in contrast to the integration of k, the integration of k_{si} is easy to perform. Using the factorization, we can consider the product quadrature rule (2.21) of order m_p and the corresponding quadrature method (2.22) from Section 2.

Let us discuss one important example. Operator A could be a classical pseudodifferential operator of order zero. Clearly, the corresponding equation is of the form (3.17) with $\mathbb{T}^2$ replaced by Γ. To enable an explicit factorization, we consider singular kernels k_0 (cf. (3.17)) of the form

$$k_0(t,\tau)=k_{00}(t,\tau)\frac{p(t-\tau)}{|t-\tau|^{\alpha}}, \tag{5.1}$$

where α is an integer greater or equal to two and where p is a homogeneous polynomial of degree $\alpha-2$. Using (5.1), we define the factorization $k_0(t,\tau)=k_{\mathrm{sm}}(t,\tau)k_{\mathrm{si}}(t,\tau)$ by

$$\begin{aligned} k_{\mathrm{sm}}(\gamma(s),\gamma(s')) &:= k_{00}(\gamma(s),\gamma(s'))\cdot|\gamma'(s')|,\\ k_{\mathrm{si}}(\gamma(s),\gamma(s')) &:= \frac{p(\gamma(s)-\gamma(s'))}{|\gamma'(s')|\cdot|\gamma(s)-\gamma(s')|^{\alpha}}, \end{aligned} \tag{5.2}$$

where $\gamma:\Omega\to\Gamma$ is the parametrization of Γ and where $|\gamma'(s)|$ with $s=(s_1,s_2)$ stands for the Jacobian determinant $|\partial_{s_1}\gamma(s)\times\partial_{s_2}\gamma(s)|$ of the parametrization. To simplify the formulas, for the case that there is no global parametrization, we suppose that Ω is the disjoint union of the parameter domains corresponding to local parametrization patches.

For our example, we now consider a quadrature partition $\Gamma=\bigcup_{k=1}^{K}\Gamma_k$ which corresponds to a triangulation of the parameter domain Ω. We may suppose that the parametrization γ is analytic over each panel Γ_k since otherwise we can replace γ by a piecewise polynomial parametrization which is polynomial over the parametrization domain $\gamma^{-1}(\Gamma_k)$ (for an estimate of such an replacement cf., e.g., [14]). Note that the integrand $s'\mapsto k_{\mathrm{si}}(\gamma(s),\gamma(s'))|\gamma'(s')|$ is analytic over all triangular subdomains $\gamma^{-1}(\Gamma_k)$ with a possible singularity at $s'=s$. The degree of smoothness of k_{sm} is determined by the degree of smoothness of k_{00} and of γ.

Let us turn back to the general case. To simplify the notation, we suppose from now on, that the knots $\tau_{k,l}$ are located in the interior of the triangular panels Γ_k. Moreover, we shall call the triangulation $\Gamma=\bigcup_{k=1}^{K}\Gamma_k$ locally quasi-uniform if

(i) There is an $\varepsilon>0$ such that the interior angles of the triangles $\gamma^{-1}(\Gamma_k)$ are all bounded between ε and $\pi-\varepsilon$.

(ii) There exists constants $c>0$ and $\beta\geqslant 1$ such that the quadrature step size $h:=\max\{\operatorname{diam}\Gamma_k\colon k=1,\dots,K\}$ satisfies the estimate $ch^{\beta}\leqslant\min\{\operatorname{diam}\Gamma_k\colon k=1,\dots,K\}$.

(iii) There is a constant $C>0$ such that, for any two nonneighbor subdomains Γ_k and $\Gamma_{k'}$, we have $\operatorname{diam}\Gamma_k\leqslant C\operatorname{dist}(\Gamma_k,\Gamma_{k'})$.

As before, we call method (2.22) stable, if (2.22) is uniquely solvable for any right-hand side at least for sufficiently small h and if the norm of the matrix

$$[(a\delta_{(k,l),(k',l')}+k_{\mathrm{sm}}(\tau_{k,l},\tau_{k',l'})\omega^{p}_{k',l'})_{(k,l),(k',l')}]^{-1}$$

inverse to the matrix of system (2.22) is uniformly bounded for all locally quasi-uniform partitions with sufficiently small step size h. The norm of the matrix is the one induced by the L^2 space. Since

$$\left\|\sum_{k=1}^{K}\sum_{l=1}^{L}\xi_{k,l}\varphi_{k,l}\right\|_{L^2(\Gamma)}\sim\sqrt{\sum_{k=1}^{K}\sum_{l=1}^{L}\varrho_k^2|\xi_{k,l}|^2},\quad \varrho_k:=\sqrt{\int_{\Gamma_k}1\,\mathrm{d}_{\Gamma}t} \tag{5.3}$$

holds for any sequence of numbers $\xi_{k,l}$, the norm of the matrix is the Euclidean matrix norm of

$$[aI+(\varrho_k k_{\mathrm{sm}}(\tau_{k,l},\tau_{k',l'})\omega^{p}_{k',l'}\varrho_{k'}^{-1})_{(k,l),(k',l')}]^{-1}.$$

As mentioned in Section 2, the quadrature method based on product integration is a perturbation of the collocation method where the trial functions are functions spanned by $\varphi_{k,l}$. More precisely, the collocation method seeks an approximate solution $\tilde{x}$ for the exact solution x of (2.5) in the span of the functions $\varphi_{k,l}$ such that $A\tilde{x}(\tau_{k,l}) = y(\tau_{k,l})$ holds for any point $\tau_{k,l}$. The coefficients of $\tilde{x}$ with respect to the basis functions $\varphi_{k,l}$ are to be determined from a system of linear equations including the so-called stiffness matrix $(A\varphi_{k',l'}(\tau_{k,l}))_{(k,l),(k',l')}$. Analogously to the quadrature method, the collocation is called stable if the stiffness matrix is invertible at least for small step size h and if the Euclidean matrix norm of the inverse matrices $(\varrho_k A\varphi_{k',l'}(\tau_{k,l})\varrho_{k'}^{-1})^{-1}_{(k,l),(k',l')}$ are uniformly bounded. The stability analysis of these collocation methods for two-dimensional manifolds is a difficult task. It seems, there exist only very few results for special cases (cf. [24,42] and, for similar operator equations, cf. [3,12,13,43,46,60]). On the other hand, many engineers use collocation methods successfully without observing any stability problem. If stability is true, then the derivation of the usual convergence results for the collocation is not difficult.

Theorem 8. *We suppose that the partition* $\Gamma = \bigcup_{k=1}^{K} \Gamma_k$ is *locally quasi-uniform. Furthermore, we suppose that the parametrization* γ *is analytic over each subdomain* Γ_k *and* $m_p + 1$ *times continuously differentiable. Recall that* $m_p \geqslant 2$ *is the order of approximation of the interpolation* $f \mapsto \sum_{k,l} f(\tau_{k,l})\varphi_{k,l}$ *and the order of the product rule* (2.21). *We assume that the kernel of* (2.5) *admits a factorization* $k(t,\tau) = k_{\mathrm{sm}}(t,\tau)k_{\mathrm{si}}(t,\tau)$ *such that the factor* k_{sm} *is* m_p *times continuously differentiable and that* k_{si} *satisfies* (2.12) *with* $m = 0$ *and* $d = 2$. *For the exact solution* x *of* (2.5), *we suppose the existence of square integrable derivatives up to order* m_p. *Finally, we suppose that the integral operator on the right-hand side of* (2.5) *is invertible and that the collocation method based on the trial basis functions* $\varphi_{k,l}$ *and the collocation points* $\tau_{k,l}$ *is stable. Then the quadrature method* (2.22) *based on product quadrature is stable, too. Moreover, we get the error estimate*

$$||\tilde{x}_h - x||_{L^2(\Gamma)} \leqslant Ch^{m_p} \log h^{-1}, \quad \tilde{x}_h(t) := \sum_{l=1}^{L} \tilde{x}_h(\tau_{k,l})\varphi_{k,l}(t) \quad \text{if } t \in \Gamma_k. \tag{5.4}$$

Proof. We have to show two things. First, to obtain stability, we have to prove that the matrix of the quadrature method is a small perturbation of the collocation matrix with respect to the norm. Second, to show the error estimate, we have to derive consistency, i.e., we have to consider the difference of the quadrature discretized operator applied to the exact solution minus the operator applied to the exact solution and to prove that the result can be estimated by the right-hand side of the estimate in (5.4).

For the difference of the matrix entries corresponding to the quadrature and collocation matrices, we get

$$\begin{aligned} d_{(k,l),(k',l')} &:= k_{\mathrm{sm}}(\tau_{k,l}, \tau_{k',l'})\omega^{p}_{k',l'} - \int_{\Gamma_{k'}} k(\tau_{k,l}, t)\varphi_{k',l'}(t)\,\mathrm{d}_\Gamma t \\ &= \int_{\Gamma_{k'}} [k_{\mathrm{sm}}(\tau_{k,l}, t_{k',l'}) - k_{\mathrm{sm}}(\tau_{k,l}, t)]k_{\mathrm{si}}(\tau_{k,l}, t)\varphi_{k',l'}(t)\,\mathrm{d}_\Gamma t. \end{aligned} \tag{5.5}$$

In view of the local uniformness of the mesh, we conclude, for $t \in \Gamma_k$ and $t' \in \Gamma_{k'}$ with disjoint Γ_k and $\Gamma_{k'}$, that (cf. condition (iii) of the local uniformness)

$$\mathrm{dist}(\Gamma_k, \Gamma_{k'}) \leqslant |t - t'| \leqslant \mathrm{dist}(\Gamma_k, \Gamma_{k'}) + \mathrm{diam}\,\Gamma_k + \mathrm{diam}\,\Gamma_{k'}$$

$$\leqslant (1+2C)\mathrm{dist}(\Gamma_k,\Gamma_{k'}),$$

$$|t-t'|^{-2} \sim |\tau_{k,l}-\tau_{k',l'}|^{-2}. \tag{5.6}$$

Similarly, for neighbors Γ_k and $\Gamma_{k'}$, the definition of the points $\tau_{k,l}=\gamma(\sigma_{k,l})$ as affine images of interior points σ_k in the standard triangle, implies

$$|t-\tau_{k',l'}|^{-2} \sim |\tau_{k,l}-\tau_{k',l'}|^{-2} \tag{5.7}$$

for any $t\in\Gamma_k$. Using this, the estimate $|\varphi_{k,l'}(t)|\leqslant C\,\mathrm{diam}(\Gamma_k)^{-1}|t-\tau_{k,l}|$ valid for $l\neq l'$, condition (2.12), representation (5.5), and the differentiability of kernel k_{sm}, we arrive at

$$|d_{(k,l),(k',l')}|\leqslant C\begin{cases} h\,|\,\tau_{k,l}-\tau_{k',l'}|^{-2}\varrho_{k'}^2 & \text{if } k\neq k',\\ h & \text{if } k=k'.\end{cases} \tag{5.8}$$

We estimate the norm of the corresponding matrix by Schur's lemma to get

$$n:=||(\varrho_k d_{(k,l),(k',l')}\varrho_{k'}^{-1})_{(k,l),(k',l')}||$$

$$\leqslant \sup_{k,l}\left\{\sum_{k',l'}|d_{(k,l),(k',l')}|\right\}\sup_{k',l'}\left\{\sum_{k,l}\varrho_k^2|d_{(k,l),(k',l')}|\varrho_{k'}^{-2}\right\}. \tag{5.9}$$

Now, inequality (5.8) together with (5.6),(5.7) and property (iii) of local uniformness of the quadrature partition lead to

$$\sum_{k',l'}|d_{(k,l),(k',l')}| \leqslant Ch+Ch\sum_{k',l'}|\tau_{k',l'}-\tau_{k,l}|^{-2}\varrho_{k'}^2$$

$$\leqslant Ch+Ch\int_{\Gamma\setminus\Gamma_k}|t-\tau_{k,l}|^{-2}\,\mathrm{d}_\Gamma t\leqslant Ch\log h^{-1},$$

$$\sum_{k,l}\varrho_k^2|d_{(k,l),(k',l')}|\varrho_{k'}^{-2}\leqslant Ch+Ch\sum_{k,l}|\tau_{k',l'}-\tau_{k,l}|^{-2}\varrho_k^2\leqslant Ch\log h^{-1}.$$

Hence, the difference of the quadrature discretized operator minus the collocation discretized operator has a norm n less than $Ch\log h^{-1}$.

Next, we turn to the estimation of the difference of the quadrature discretized operator minus the full operator applied to the exact solution. Thus, we have to estimate the norm of $\sum_{k,l} d_{k,l}\varphi_{k,l}$ with

$$d_{k,l}:=\sum_{k',l'}k_{\mathrm{sm}}(\tau_{k,l},\tau_{k',l'})x(\tau_{k',l'})\omega^p_{k',l'}-\int_\Gamma k(\tau_{k,l},t)x(t)\,\mathrm{d}_\Gamma t$$

$$=\int_\Gamma k_{\mathrm{si}}(\tau_{k,l},t)\{L[k_{\mathrm{sm}}(\tau_{k,l},t)x(t)]-[k_{\mathrm{sm}}(\tau_{k,l},t)x(t)]\}\,\mathrm{d}_\Gamma t,$$

where L stands for the interpolatory projection, i.e., $L[k_{\mathrm{sm}}(\tau_{k,l},t)x(t)]:=\sum_{k',l'}k_{\mathrm{sm}}(\tau_{k,l},\tau_{k',l'})x(\tau_{k',l'})\varphi_{k',l'}(t)$. We split $d_{k,l}=d^1_{k,l}+d^2_{k,l}$ with

$$d^1_{k,l}:=\int_{\Gamma_k}k_{\mathrm{si}}(\tau_{k,l},t)\{L[k_{\mathrm{sm}}(\tau_{k,l},t)x(t)]-[k_{\mathrm{sm}}(\tau_{k,l},t)x(t)]\}\,\mathrm{d}_\Gamma t,$$

$$d^2_{k,l}:=d_{k,l}-d^1_{k,l}$$

and estimate the norms of $\sum_{k,l} d_{k,l}^1\varphi_{k,l}$ and $\sum_{k,l} d_{k,l}^2\varphi_{k,l}$ separately. Using the approximation property of the interpolation as well as the smoothness assumptions for $k_{\rm sm}$ and x, we arrive at

$$|d_{k,l}^2| \leqslant C \sum_{k'\neq k,l'} |k_{\rm si}(\tau_{k,l},\tau_{k',l'})|\varrho_{k'}\sqrt{\int_{\Gamma_{k'}} |L[k_{\rm sm}(\tau_{k,l},t)x(t)] - [k_{\rm sm}(\tau_{k,l},t)x(t)]|^2\,{\rm d}_\Gamma t}$$

$$\leqslant C \sum_{k'\neq k,l'} |k_{\rm si}(\tau_{k,l},\tau_{k',l'})|\varrho_{k'}h^{m_p}\sqrt{\sum_{n=0}^{m_p}\int_{\Gamma_{k'}} |\nabla^n x(t)|^2\,{\rm d}_\Gamma t}.$$

This expression can be looked at as the result of multiplying the vector $(\sqrt{\sum_n\int_{\Gamma_{k'}}|\nabla^n x(t)|^2\,{\rm d}t})_{(k',l')}$ by a matrix. Hence, in view of (2.12) and (5.3), the norm of $\sum_{k,l} d_{k,l}^2\varphi_{k,l}$ is less than

$$Ch^{m_p}||(\varrho_k|\tau_{k,l}-\tau_{k',l'}|^{-2}\varrho_{k'})_{(k,l),(k',l')}||\sqrt{\sum_{n=0}^{m_p}\int_{\Gamma} |\nabla^n x(t)|^2\,{\rm d}_\Gamma t}. \qquad (5.10)$$

Analogously to the estimate $h\log h^{-1}$ for (5.9), we get the estimate $C\log h^{-1}$ for the matrix norm in (5.10). Finally, the norm of $\sum_{k,l} d_{k,l}^2\varphi_{k,l}$ is less than the expression $Ch^{m_p}\log h^{-1}$ on the right-hand side of the estimate in (5.4).

Let us turn to $\sum_{k,l} d_{k,l}^1\varphi_{k,l}$. Over an arbitrary smooth and bounded two-dimensional manifold $\tilde{\Gamma}$, the functions of the Sobolev space H^2 are known to be Lipschitz, and we get that, for a fixed constant $C>0$, for any $\tilde{\tau}\in\tilde{\Gamma}$, and for any function $\tilde{f}$ on $\tilde{\Gamma}$,

$$\int_{\tilde{\Gamma}} |\tilde{\tau}-\tilde{t}|^{-2}|\tilde{f}(\tilde{\tau})-\tilde{f}(\tilde{t})|\,{\rm d}_{\tilde{\Gamma}}\tilde{t}\leqslant C\sqrt{\int_{\tilde{\Gamma}} |\nabla\tilde{f}(\tilde{t})|^2\,{\rm d}_{\tilde{\Gamma}}\tilde{t}}+C\sqrt{\int_{\tilde{\Gamma}} |\nabla^2\tilde{f}(\tilde{t})|^2\,{\rm d}_{\tilde{\Gamma}}\tilde{t}}.$$

Choosing $\tilde{\Gamma}:=\{\tilde{t}=t/{\rm diam}\,\Gamma_k\colon\ t\in\Gamma_k\}$, substituting the variable of integration $\tilde{t}$ by $t/{\rm diam}\,\Gamma_k$, and setting $f(t)=\tilde{f}(\tilde{t})$ and $f(\tau)=\tilde{f}(\tilde{\tau})$, we arrive at

$$\int_{\Gamma_k} |\tau-t|^{-2}|f(\tau)-f(t)|\,{\rm d}_{\Gamma_k}t \leqslant C\sqrt{\int_{\Gamma_k} |\nabla f(t)|^2\,{\rm d}_{\Gamma_k}t}$$

$$+\,C\,{\rm diam}\,\Gamma_k\sqrt{\int_{\Gamma_k} |\nabla^2 f(t)|^2\,{\rm d}_{\Gamma_k}t}, \quad \tau\in\Gamma_k.$$

Using this and the approximation property for projection L, we obtain that

$$|d_{k,l}^1| \leqslant C\left|\int_{\Gamma_k} |\tau_{k,l}-t|^{-2}|L[k_{\rm sm}(\tau_{k,l},t)x(t)]-[k_{\rm sm}(\tau_{k,l},t)x(t)]|\,{\rm d}_\Gamma t\right|$$

$$\leqslant C\sqrt{\int_{\Gamma_k} |\nabla\{L[k_{\rm sm}(\tau_{k,l},t)x(t)]-[k_{\rm sm}(\tau_{k,l},t)x(t)]\}|^2\,{\rm d}_\Gamma t}$$

$$+\,C\,h\sqrt{\int_{\Gamma_k} |\nabla^2\{L[k_{\rm sm}(\tau_{k,l},t)x(t)]-[k_{\rm sm}(\tau_{k,l},t)x(t)]\}|^2\,{\rm d}_\Gamma t}$$

$$\leqslant C\,h^{m_p-1}\sum_{n=0}^{m_p}\sqrt{\int_{\Gamma_k} |\nabla^n x(t)|^2\,{\rm d}_\Gamma t}.$$

Hence, in view of $\varrho_k \leqslant Ch$ and (5.3), we get that the norm of $\sum_{k,l} d^1_{k,l}\varphi_{k,l}$ is less than Ch^{m_p}, and the consistency order of (5.4) is shown. □

Note that, in view of the last proof we can relax the assumptions of Theorem 8. The global differentiability of γ and k_{sm} can be replaced by differentiability over each subdomain Γ_k together with the global boundedness of these local derivatives. This weaker assumption holds true when a parametrization is replaced by its piecewise polynomial interpolation. Furthermore, Theorem 8 remains true if the solution x has a weak singularity at a finite number of points. In this case, the mesh should be graded toward these points such that the larger values for the $\sqrt{\int_{\Gamma_k} |\nabla^{m_p} x|^2}$ in the estimates for the interpolation error $x - \sum x(\tau_{k,l})\varphi_{k,l}$ are compensated by the factors $[\operatorname{diam}\Gamma_k]^{m_p}$ which are smaller than h^{m_p}.

Further, we remark that the logarithm in the error estimate (5.4) can be dropped if the integral operator with the kernel function $|k(t,\tau)|$ is bounded in L^2. This last assumption holds true, e.g., for operators of double-layer type defined over non-smooth domains. Finally, a generalization of Theorem 8 to operators of order minus one and to piecewise linear collocation over regular grids has been treated in [14]. In that paper even a fast quadrature algorithm for a wavelet approach has been derived.

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