Convergence for Degenerate Parabolic Equations

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We prove that any bounded global solution to a degenerate parabolic problem in one spatial dimension converges to a unique stationary state. © 1999 Academic Press

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1. INTRODUCTION

In this paper, we address the question of convergence as $t \to \infty$ of solutions to the problem

\[
\begin{align*}
&u_t(x, t) - \phi(u(x, t))_{xx} = f(u(x, t)), \quad x \in (-L, L), \quad t > 0, \\
&u(-L, t) = u(L, t) = 0, \\
&u(x, 0) = u_0(x) \geq 0 \quad \text{for} \quad x \in (-L, L).
\end{align*}
\]

(1.1) (1.2) (1.3)

For the quantities $\phi, f$, we shall assume the following hypotheses:

\[
\begin{align*}
&\phi \in C(R^+) \cap C^3(R^+ \setminus \{0\}), \\
&\phi(0) = 0, \quad \phi'(z) > 0 \quad \text{for all} \quad z > 0, \\
&f \in C^1(R^+), \\
&f(0) > 0.
\end{align*}
\]

(1.4) (1.5) (1.6) (1.7)

Thus the equation may degenerate and the solution $u$ may lose regularity in the region $\{u = 0\}$. 439
If the equation is strictly parabolic, Zelenyak [17] and Matano [15] proved independently that any bounded solution converges for \( t \to \infty \) to a single solution of the corresponding stationary problem. On the other hand, the problem of convergence for dimension \( N \geq 2 \) remains open except for some particular cases (see e.g. Haraux and Poláčik [10], Hale and Raugel [9]).

Unfortunately, however, all the methods used for nondegenerate problems depend essentially on the regularity of solutions as well as on the relatively simple structure of the set of stationary solutions. In fact, any \( \omega - \) limit set is homeomorphic to a subset of \( \mathbb{R}^2 \) (cf. Fiedler and Mallet-Parrot [7]).

Using the result of Langlais and Phillips [14] we already know that the \( \omega - \) limit set of any bounded solution of (1.11.3) is contained in the set of stationary states, i.e., the solutions of the problem:

\[
\begin{align*}
-\phi(w(x))_{xx} &= f(w(x)), \quad \text{for } x \in (-L, L), \\
w(x) &\geq 0 \quad \text{for } x \in (-L, L), \quad w(-L) = w(L) = 0.
\end{align*}
\]

(1.8)

(1.9)

It was shown by Aronson, Crandall, and Peletier [2] that the problem (1.8), (1.9) may admit a continuum of compactly supported solutions, in particular, they cannot be distinguished by the above methods, based on the fact that the stationary solutions of the regular problems are uniquely determined by the values of \( w(x), w_{xx}(x) \) at any point \( x \). Moreover, the \( \omega - \) limit set is no longer symmetric which excludes the use of the approach of Hess and Poláčik [11].

Nevertheless, the fact that the problem is posed in one spatial dimension forces the solutions to converge even if the equation is degenerate.

**Theorem 1.1.** Let \( \phi, f \) satisfy the hypotheses (1.4)-(1.7). Let \( u \) be a global in time (weak) solution of the problem (1.1)-(1.4) such that

\[
\limsup_{t \to \infty} \|u(t)\|_{L^\infty(-L,L)} \text{ is finite.}
\]

Then

\[
u(t) \to w \quad \text{in } C([-L, L]) \quad \text{as } t \to \infty,
\]

(1.10)

where \( w \) is a solution of the stationary problem (1.8), (1.9).

**Remark.** The weak solutions of the evolution problem are defined in the standard way (see Section 2).

The rest of the paper will be devoted to the proof of Theorem 1.1.
It turns out that one has to face two rather different possibilities for the structure of the $\omega$--limit set. In the first case, as in the classical situation, the $\omega$--limit set contains at least one solution $w$ such that
\[
\phi(w)_x(-L) = -\phi(w)_x(L) > 0
\]
(note that $w_x$ may not even exist at the boundary points). If we knew that $\phi(w)_x$ converges at the boundary points, we could use the method of Matano [15] to conclude that the $\omega$--limit set is necessarily a singleton. Unfortunately, however, this is known to be true only for strong solutions under some restrictive hypotheses concerning $\phi$ (see Simondon and Toure [16]). In the general case, the relevant information concerning the behaviour of the solution at the boundary points will be obtained by comparison with a suitable traveling wave (see Section 6).

The second possibility is that all the functions $w$ in the $\omega$--limit set satisfy
\[
\phi(w)_x(L) = \phi(w)_x(-L) = 0.
\]
Consequently, one has to eliminate the possibility of lateral oscillation of the solution. In the regular case, this may be done by determining the sign of the derivative of the solution $u$ at any interior point $s$ of the interval $(-L, L)$ (cf. Chen and Matano [5]). Specifically, one considers the zeros of the function
\[
v_s(x, t) = u(s + x, t) - u(s - x, t)
\]
satisfying a regular (linear) parabolic equation. It is known that the number of zeros of $v_s$ is finite and nonincreasing at any positive $t$ and that it drops strictly when there is a multiple zero (see Angenent [1]).

In the degenerate case, however, the equation satisfied by $v_s$ is no longer strictly parabolic in the regions where $u = 0$. To avoid this difficulty, we show that the solution may be decomposed into a singular part converging to zero and a finite number of components enjoying the positivity property in the sense of Bertsch, et al. [3] (see Section 5). Finally, each of these components may be treated separately using the theory of Angenent [1] restricted to a suitable set of intervals (see Section 7). In fact, this is the only part of the proof when we need the additional regularity property of $\phi \in C^4$. Moreover, one could relax slightly the hypotheses concerning the behaviour of $f$ in the neighbourhood of zero replacing (1.6) by
\[
f(z) \geq -Cz, \quad f(z_1) - f(z_2) \leq C(z_1 - z_2) \quad \text{for all } z, z_1 \geq z_2 \text{ small,}
\]
\[
f \in C(\mathbb{R}^1) \cap C^1(\mathbb{R}^1 \setminus \{0\}).
\]
2. EXISTENCE, UNIQUENESS, REGULARITY

If not specified otherwise, we shall assume throughout this Section that $\phi$ is defined on the whole of $\mathbb{R}^1$ and
$$\phi \in C(\mathbb{R}^1) \cap C^1(\mathbb{R}^1 \setminus \{0\}), \quad \phi(0) = 0, \quad \phi \text{ strictly increasing on } \mathbb{R}^1. \quad (2.1)$$

We start with the associated problem
$$v_t - \phi(v)_{xx} = g(x,t), \quad (x,t) \in Q = (a,b) \times (0,T)$$
$$v(a,t) = h_a(t), \quad v(b,t) = h_b(t), \quad t \in (0,T)$$
$$v(x,0) = v_0(x), \quad x \in [a,b]. \quad (2.2)$$

We shall say that $v$ is a solution of the problem (2.2) if the equality
$$\int_0^T \int_Q v(x,t) \psi_t(x,t) + \phi(v(x,t)) \psi_{xx}(x,t) + g(x,t) \psi(x,t) \, dx \, dt$$
$$= \int_0^T \int_a^b \psi(x,t) \psi_t(x,t) \, dx \, dt - \int_a^b v_0(x) \psi(0,x) \, dx$$
holds for any test function $\psi \in C^\infty(Q)$, $\psi(a,t) = \psi(b,t) = \psi(x,T) = 0$ for all $t \in [0,T], \, x \in [a,b]$.

**Proposition 2.1.** [Brezis and Crandall [4]]. Assume that $\phi$ satisfies (2.1). Then given $v_0 \in L^\infty(a,b), \quad h_a, h_b \in L^1(0,T), \quad g \in L^1(Q)$

there is at most one solution $v$ of the problem (2.2) in the class
$$v \in C([0,T]; \quad L^1(a,b)) \cap L^\infty(Q). \quad (2.3)$$

**Remark.** In fact, the main concern of [4] is the problem on the spatial domain $\mathbb{R}^N$. However, as remarked in the last section of [4], the same method may be used for problems on bounded domains with suitable boundary conditions.

We shall say that the data are regular if
$$\|v_0\|_{L^\infty(a,b)}, \quad \|g\|_{L^1(Q)}, \quad \|h_a\|_{L^1(0,T)}, \quad \|h_b\|_{L^1(0,T)} \leq K_1, \quad (2.4)$$
$$\|\phi(h_a)\|_{W^{2,1}(0,T)}, \quad \|\phi(h_b)\|_{W^{2,1}(0,T)}, \quad \|G_s\|_{L^1(0,T)} \leq K_2, \quad (2.5)$$
$$\|\phi(v_0)\|_{L^2(a,b)} \leq K_3. \quad (2.6)$$
v_0 \in C([a, b]), h_r, h_l \in C([0, T]) and the compatibility conditions
\begin{align}
    v_0(a) &= h_l(0), \\
    v_0(b) &= h_r(0)
\end{align}
(2.7)
hold.

We set
\[\sigma(z) \equiv \int_0^z \eta(s) \, ds \quad \text{if} \quad z \neq 0, \quad \sigma(0) = 0,\]
where
\[\eta(z) \equiv \min\{z^2, \phi'(z), \sqrt{\phi''(z)}\}, \quad z \neq 0.\]

**Lemma 2.1. (Approximation Lemma).** Let \( \phi \) satisfy (2.1) and let the data be regular. Then there exists a unique solution \( v \) of the problem (2.2) such that
\begin{align}
    &\|v\|_{L^\infty(\bar{Q})} \leq C(K_1, T), \\
    &\|\phi(v)\|_{L^1(\bar{Q})} \leq C(K_1, K_2, T), \\
    &\|\sigma(v)\|_{L^2(\bar{Q})} \leq C(K_1, K_2, K_3, T),
\end{align}
(2.8) \quad (2.9) \quad (2.10)
and
\begin{align}
    &\|\phi(v)_x(t)\|_{L^2(a, b)} + \|\sigma(v)_x(t)\|_{L^2(a, b)} \\
    &\leq C(K_1, K_2, K_3, T) \quad \text{for all} \quad t \in [0, T].
\end{align}
(2.11)

Moreover, \( v \) may be obtained as a limit
\[v^* \to v \quad \text{in} \ C(\bar{Q})\]
where \( v^* \) are classical solutions of an approximate problem
\begin{align}
    v^*_t - \phi'(v^*_x)x &= g^* \\
    v^*(a, \cdot) &= h^*_r, \\
    v^*(b, \cdot) &= h^*_l, \\
    v^*(\cdot, 0) &= v_0^*,
\end{align}
(2.12)
with smooth data \( v_0^*, h^*_r, h^*_l, \) and
\[\phi^* \in C^\infty(R^1), \quad (\phi^*)' > c(x) > 0.\]
(2.13)

**Proof.** Let us find a sequence \( \phi^* \) satisfying (2.13) and such that
\[\phi^* \to \phi \quad \text{uniformly on compact sets.}\]
and

\[(\phi')'(z) \geq \sigma'(z) \quad \text{for a.e. } z \in \mathbb{R}^1 \text{ and all } \varepsilon > 0.\]

Next, we approximate the data by smooth functions so that

\[v_0 \rightarrow v_0, \quad h_l^\varepsilon \rightarrow h_l, \quad h_r^\varepsilon \rightarrow h_r \quad \text{pointwise} \quad g^\varepsilon \rightarrow g \quad \text{in } L^1(Q)\]

and the estimates (2.4–2.6) as well as the compatibility condition (2.7) may be satisfied by the \(\varepsilon\)-quantities with the constants \(K_i, i = 1, 2, 3\) independent of \(\varepsilon\). Note that it can be done as we may approximate any superposition \(\phi(h)\) by a sequence of smooth functions

\[\xi^\varepsilon \rightarrow \phi(h) \quad \text{pointwise}\]

and then take

\[h^\varepsilon = (\phi')^{-1} (\xi^\varepsilon) \rightarrow h \quad \text{pointwise as } \varepsilon \rightarrow 0\]

for \(h = v_0, h_l, h_r\) respectively.

Now, the standard theory of quasilinear parabolic equations yields the existence of a classical solution \(v^\varepsilon\) of the approximate problems (2.12). Moreover, by virtue of the maximum principle, we have

\[\|v^\varepsilon\|_{L^\infty(Q)} \leq C(K_1, T).\]

To deduce (2.9–2.11) consider an auxiliary function

\[\chi'(x, t) = \frac{b - x}{b - a} \phi'(h_l(t)) + \frac{x - a}{b - a} \phi'(h_r(t)),\]

which is nothing else as the extension of the boundary data to the whole \(Q\). Now, the estimate (2.9) may be obtained multiplying (2.12) by \(\phi(v^\varepsilon) - \chi^\varepsilon\) and integrating by parts while (2.10), (2.11) follow in a similar way using the multiplier \(\phi'(v^\varepsilon) - \chi'^\varepsilon\).

Finally, by virtue of (2.10), (2.11), the sequence \(\{\sigma(v^\varepsilon)\}_\varepsilon\) is precompact in \(C(Q)\) and it is a matter of routine to pass to the limit in (2.12) to complete the proof. Observe that \(\sigma\) is increasing together with \(\phi\).

Our next goal will be to show a comparison principle for the problem (2.2):


**Lemma 2.3.** Assume \( \phi \) satisfies (2.1). Let \( v, \bar{v} \) be two solutions of the problem (2.2) belonging to the class (2.3) and corresponding respectively to the data

\[
\begin{align*}
\bar{v}_0, v_0 \in L^\infty(a, b), \quad \check{h}_1, \check{h}_r, \check{h}_l, \check{h}_r \in C([0, T]), \quad g, \bar{g} \in L^\infty(Q)
\end{align*}
\]

such that

\[
\begin{align*}
\check{h}_1 \leq \check{h}_r, \quad \check{h}_r \leq \check{h}_s, \quad \text{on } [0, T].
\end{align*}
\]

Then

\[
\exp(At) \int_a^b \left[ \left( v(t) - \bar{v}(t) \right)^+ + \int_{t_0}^t \exp(As) \left[ A(v(s) - \bar{v}(s)) + g(s) - \bar{g}(s) \right]^+ \, ds \right] \, dx
\]

for all \( t \in (0, T) \) and \( A \geq 0 \).

**Remark.** Although we do not claim any originality of this result, we were not able to find a suitable reference in the literature. There are many comparison theorems for the problem in question, the closest results being probably that of Aronson, Crandal, Peletier [2] and Gilding [8]. However, one has to relax the hypothesis \( \phi \in C^1(R^1) \).

**Proof.** Step 1. To begin with, observe that the conclusion of the lemma holds provided the data are regular. Indeed, the formula (2.16) may be verified directly for the approximate problems (2.1) and the result follows passing to the limit for \( \epsilon \to 0 \) and using Lemma 2.2.

Step 2: Now, keeping \( g \) fixed we may approximate any continuous data \( v_0, h_1, h_r \) satisfying the compatibility conditions (2.7) by regular data so that

\[
v_0 \approx v_0, \quad h_1 \approx h_1, \quad h_r \approx h_r
\]

pointwise everywhere on the parabolic boundary of \( Q \). Applying the result of Step 1 for \( A = 0 \) we conclude

\[
v^{*} \approx v \quad \text{pointwise in } \bar{Q}.
\]

Now, it is easy to show that \( v \) solves (2.2) in the sense of distributions. Moreover, the relation (2.16) holds for any couple \( v, \bar{v} \) resulting from the above procedure provided the approximation has been taken such that (2.15) holds for all \( \check{h}_1, \check{h}_r, \check{h}_r, \check{h}_r \).
Thus the conclusion of the Lemma will be verified for any continuous boundary data provided we show that the above procedure yield the unique solution of the problem, i.e., in view of Proposition 2.1, we have to show

\[ v \in C([0, T]; L^1(a, b)). \]  

(2.18)

To this end, observe first that

\[ v \in C([0, T]; L^2_{\text{weak}}(a, b)) \]  

(2.19)

which follows from the fact that \( v \) is bounded everywhere on \( \bar{Q} \) and solves (2.2) in the sense of distributions which means, in particular, that any quantity of the form

\[ \int_a^b v(t) \psi \, dx, \quad \psi \in \mathcal{D}(a, b) \]

is a continuous function of \( t \).

Consequently, the function

\[ \chi(t) = \|v(t)\|_{L^2(a, b)}^2 \]

is lower semicontinuous on \([0, T]\).

On the other hand, by virtue of (2.17) and the monotone convergence theorem, we have

\[ \int_a^b (v'(t) - C)^2 \, dx \leq \int_a^b (v(t) - C)^2 \, dx \]

for all \( t \in [0, T] \) and a suitable constant \( C \) therefore the function

\[ t \mapsto \int_a^b (v(t) - C)^2 \, dx \]

is upper semicontinous on \([0, T]\).

Consequently, the function \( \chi \) is continuous on \([0, T]\) which along with (2.19) completes the proof of (2.18).

Thus we have proved that for any continuous data \( v_0, h_l, h_r \) and any regular \( g \), the problem (2.2) possesses a unique solution \( v \in C([0, T]; L^1(a, b)) \) and the conclusion of the lemma holds.

Step 3. Finally, we have to verify (2.16) for any data \( v_0 \in L^\infty(a, b) \), \( g \in L^\infty(\bar{Q}) \). To this end, keeping \( h_l, h_r \) fixed we approximate

\[ v_0^e \to v_0 \quad \text{in} \ L^1(a, b), \quad g^e \to g \quad \text{in} \ L^1(\bar{Q}) \]
where \( g^r \) are regular and \( v^r \) continuous, uniformly bounded and satisfying the compatibility conditions (2.7). Using (2.16) with \( A = 0 \) we deduce
\[
\|v^r(t) - v^{r_0}\|_{L^1(a,b)} \leq \|v^r_0 - v^{r_0}\|_{L^1(a,b)} + \|g^r - g^{r_0}\|_{L^1(Q)}
\]
for all \( t \in [0, T] \); hence
\[
v^r \to v \in C([0, T]; L^1(a, b)), \quad v \in L^\infty(Q), \tag{2.20}
\]
where \( v \) is the unique solution of (2.2). Now, the validity of (2.16) for the limiting solutions follows from (2.20) which completes the proof of Lemma 2.2.

We shall say that a function \( u \) is a solution of the problem (1.1)-(1.3) on the time interval \((0, T)\) if
\[
u \in C([0, T); L^1(-L, L) \cap L^\infty((-L, L) \times (0, T))
\]
and \( u \) solves the problem (2.2) with
\[
a = -L, \quad b = L, \quad v_0 = u_0, \quad h = h = 0, \quad \text{and} \quad g = f(u).
\]

Note that in the course of the proof of Lemma 2.2, we have also proved existence of the solution \( v \) for any data satisfying (2.14) and its approximation by more regular functions. Using the same arguments, we can prove an analogous result for the problem (1.1)-(1.3).

**Proposition 2.4. (Existence, Uniqueness, Regularity).** Let the nonlinearity \( \phi \) satisfy (2.1) and \( f \in C(R^1) \) be globally Lipschitz continuous on \( R^3 \).

Then for any initial data \( u_0 \in L^\infty(\mathbb{R}^3) \) and any \( T > 0 \) there exists a unique solution \( u \) of the problem (1.1)-(1.3).

Moreover, for any \( t_0 > 0 \) we have
\[
u(t) \in C([-L, L])
\]
and
\[
\int_{t_0}^t \|\sigma(u)(s)\|_{L^2(-L, L)}^2 ds \leq C(t_0, \|u\|_{L^\infty(Q)}), \tag{2.21}
\]
\[
\|\sigma(u)(t)\|_{L^2(-L, L)}, \|\phi(u)(t)\|_{L^2(-L, L)} \leq C(t_0, \|u\|_{L^\infty(Q)}) \tag{2.22}
\]
for all \( t \geq t_0 > 0 \).
Remark. Under the hypotheses of Theorem 1.1 and given \( u_0 \) as in (1.3), there always exists a unique local solution \( u \geq 0 \) defined on some maximal interval \((0, T_{\text{max}})\). Moreover, if

\[
\limsup_{t \to T_{\text{max}}} \| u(t) \|_{L^\infty(-L, L)} \quad \text{is finite,}
\]

then, necessarily, \( T_{\text{max}} = \infty \). This is automatically true for any data \( u_0 \) if \( f \) satisfies certain growth conditions, e.g., if

\[
\text{sgn} \ z f(z) \leq C |\phi(z)|^q \quad \text{with} \quad 0 < q < 1 \quad \text{for large} \quad z.
\]

Proof. The existence may be proved as in the proof of Lemma 2.2, Lemma 2.3 approximating first \( \phi \) and then the data. Observe that the condition

\[
g_t \in L^1(Q)
\]

in (2.5) is irrelevant as in our case \( g = f(u) \) and we obtain

\[
\int_0^t \int_{-L}^L f(u) \phi(u)_x \, dx \, dt = \int_{-L}^L H(u(t)) - H(u_0) \, dx,
\]

where

\[
H(z) = \int_0^z \phi'(s) \, ds
\]

with the latter quantity bounded. Moreover, it follows from (2.9) that any interval \((0, t_0)\) contains a time \( t \) such that (2.6) holds. Consequently, the regularity estimates (2.21), (2.22) follow from (2.10)-(2.11) applied on the interval \((t_0, T)\).

As a straightforward consequence of Lemma 2.2, we obtain the following:

**Proposition 2.5. (Comparison Theorem).** Let the nonlinearity \( \phi \) satisfy (2.1) and \( f \) be locally Lipschitz continuous on \( R^1 \). Let \( u, \bar{u} \in C(Q) \) satisfy

\[
\frac{\partial u}{\partial t} - \phi(u)_{xx} - f(u) = g \leq \bar{g} = \frac{\partial \bar{u}}{\partial t} - \phi(\bar{u})_{xx} - f(\bar{u})
\]

in \( \Omega(Q) \), where \( Q \equiv [a, b] \times [t_1, t_2] \), \( g, \bar{g} \in L^\infty(Q) \) and

\[
u \leq \bar{u} \quad \text{on the parabolic boundary of} \quad Q.
\]
Then

Then

\[ u \leq u \text{ in } Q. \]

We conclude this section by presenting interior regularity estimates which follow directly from Proposition 2.2, the additional regularity hypotheses (1.4), (1.6) and the Schauder interior estimates (see for example Ladyzhenskaya et al. [13, Theorems III.10.1, III.12.2]).

**Proposition 2.4.** Let the nonlinearities \( \phi, f \) satisfy the hypotheses (1.4)-(1.6). Let \( u \) be a solution of (1.1)-(1.3) such that

\[ 0 < u(x, t) \leq \rho_2 > 0 \text{ for all } (x, t) \in Q \]

for a certain \( Q \equiv [a, b] \times [t_1, t_2] \).

Then

\[ \|u_{xx}\|_{L^p(Q)}, \|u_{xt}\|_{L^p(Q)}, \|u_t\|_{L^p(Q)}, \|u_x\|_{L^p(Q)} \leq C(\rho_1, \rho_2, K) \]

for any compact \( K \subset (a, b) \times (t_1, t_2) \).

3. ASYMPTOTIC BEHAVIOR—FIRST APPROXIMATION

We introduce the \( \omega \)-limit set of the solution \( u \):

\[ \omega(u) \equiv \{ w \mid \text{there is } t_n \to \infty \text{ such that } u(t_n) \to w \text{ in } C([-L, L]) \}. \]

Observe that by virtue of the regularity estimates (2.21), (2.22), the set \( \omega(u) \) is nonvoid, compact and connected for any bounded solution \( u \). Moreover we can use Theorem 1.1 of Langlais and Phillips [14] to obtain:

**Proposition 3.1.** Under the hypotheses of Theorem 1.1, the \( \omega \)-limit set \( \omega(u) \) is contained in the set of stationary solutions, i.e., any \( w \in \omega(u) \) solves the problem (1.8), (1.9).

Moreover, the convergence is stronger at the points where the limit function is positive, i.e., in the parabolic regime, as shown in the following lemma:

**Lemma 3.2.** Let \( w \in \omega(u) \) and \( x_0 \in (-L, L) \) such that

\[ w(x_0) > 0. \]
Then there is a sequence \( t_n \to \infty \) such that

\[
\lim_{n \to \infty} u_n(x_0, t_n) = w(x_0).
\]

**Proof.** Let \( t_n \to \infty \) be such that

\[
u(t_n) \to w \quad \text{in } C([-L, L]).
\]

Since \( w(x_0) > 0 \), there is a closed interval \( J \) such that

\[
x_0 \in \text{int } J, \quad w > 0 \quad \text{on } J
\]
and

\[
\sup_{(x, t) \in J \times [-1, 1]} |u(x, t_n + t) - w(x)| \to 0 \quad \text{as } t_n \to \infty.
\]

Now the conclusion of the lemma follows from the interior regularity estimates stated in Proposition 2.4.

4. THE STATIONARY SOLUTIONS

We concentrate on the stationary problem (1.8), (1.9). For the purpose of this section, we shall always assume

\[
f \circ \phi^{-1} \quad \text{locally Lipschitz continuous on } (0, \phi(\infty)), \quad (4.1)
\]
\[
f \circ \phi^{-1} \quad \text{continuous on } [0, \phi(\infty)) \quad (4.2)
\]

where

\[
\phi(0) = 0, \quad 0 < \phi(\infty) = \lim_{z \to \infty} \phi(z) \leq \infty,
\]

which clearly follows from (1.4)-(1.6).

We shall look for solutions \( w \) satisfying

\[
f(w) \in L^1(-L, L),
\]

hence, by the standard bootstrap arguments, \( \phi(w) \in C^2([-L, L]) \) and the Eq. (1.8) as well as the boundary conditions (1.9) hold in the classical sense.
Proposition 4.1. Let \( f, \phi \) satisfy (4.1), (4.2). Then the set of solutions of (1.8), (1.9) may be characterized as follows:

(i) Given \( \mu > 0 \) there is at most one solution \( w \) of (1.8), (1.9) such that

\[
\phi(w)_x (-L) = \mu. \tag{4.3}
\]

Moreover, \( w \) is positive on \((-L, L)\),

\[ w(-x) = w(x), \quad w \text{ decreasing on } (0, L). \]

If \( w_1, w_2 \) are two solutions of (1.8), (1.9) satisfying (4.3) with \( \mu_1, \mu_2, 0 \leq \mu_1 < \mu_2 \), then

\[ w_2(x) > w_1(x) \quad \text{for all } x \in (-L, L). \tag{4.4} \]

(ii) If \( w \) is a solution of (1.8), (1.9) such that

\[ \phi(w)_x (-L) = 0, \tag{4.5} \]

then either

\[ w \equiv 0 \]

or

\[ w(x) = \sum_{i=1}^{k} w_g(x - y_i) \quad \text{for all } x \in [-L, L], \]

where \( k \) is a finite integer, \( y_i \in (-L, L) \), \( i = 1, ..., k \), and \( w_g \) is the so-called ground state solution of the Eq. (1.8), (1.9) defined on the whole \( \mathbb{R}^1 \) and enjoying the properties

\[
\supp(w_g) = (-m, m), \quad m \text{ finite}, \quad w_g(x) > 0 \text{ for all } x \in (-m, m), \\
w_g(-x) = w_g(x), \quad w_g(x) \text{ strictly decreasing for all } x \in (0, m)
\]

and

\[ \phi(w_g)_x (-m) = w_g(-m) = 0. \]

Moreover, one has \( \text{dist}(y_i, y_j) \geq 2m, i \neq j \).

The rest of this section will be devoted to the proof of Proposition 4.1. To begin, observe that \( w \) is a solution of (1.8), (1.9) if and only if \( v = \phi(w) \) solves

\[
-v_{xx} = f \cdot \phi^{-1}(v), \quad v(x) \in [0, \phi_{\infty}) \quad \text{for } x \in (-L, L) \quad v(-L) = v(L) = 0. \tag{4.6}
\]
Lemma 4.2. Let \( y \in (-L, L) \) be such that
\[
v_y(y) = 0, \quad v(y) > 0.
\]
Then \( f \circ \phi^{-1}(v(y)) > 0 \), i.e., \( y \) is a point of a strong local maximum of \( v \).

Proof. (i) If \( f \circ \phi^{-1}(v(y)) = 0 \), then, by virtue of the uniqueness theorem for second order equations, we have \( v \equiv v(y) > 0 \) on \((-L, L)\).

(ii) If \( f \circ \phi^{-1}(v(y)) < 0 \), then \( v \) attains its strong local minimum at \( y \). Using the same argument as above, we conclude the solution \( v \) is symmetric around \( y \) on a certain interval \([y-a, y+a]\) and it attains its local maxima at the points \( y-a, y+a \). Thus, necessarily, \( v \) is periodic with the period \( 2a \), \( v \geq v(y) \) on \([-L, L]\).

Lemma 4.2. Let a solution \( v \) of (4.6) attains its strong local maximum at a point \( y \in (-L, L) \).
Then there exists \( a > 0 \) such that
\[
-L < y - a < y + a < L,
\]
\( v(y - x) = v(y + x) \) for all \( x \in [0, a] \), and there is a constant \( \lambda > 0 \) such that
\[
\frac{1}{2}v_x^2(x) + F(v(x)) = \lambda \quad \text{for all} \quad x \in [y, y + a], \tag{4.7}
\]
where
\[
F(z) = \int_0^z f \circ \phi^{-1}(s) \, ds.
\]

Proof. We take \( a = \sup \{ x \mid v(y + x) > 0 \} \). Consequently, \( v \) is even with respect to \( y \) on \((y-a, y+a)\) by virtue of the uniqueness of solutions to (4.6). By virtue of Lemma 4.1, we have \( v_x < 0 \) on \( (y, y + a) \) so we can multiply the equation by \( v_x \) to deduce (4.7).

Corollary 4.4. If \( v_1, v_2 \) are two solutions such that
\[
(v_2)_x (-L) \geq (v_1)_x (-L) > 0,
\]
then
\[
v_2 \geq v_1 \quad \text{on} \quad (-L, L).
\]
Moreover, if the former inequality is sharp, then so is the latter.
Proof. By virtue of Lemmas 4.1, 4.2, any solution \( v \) such that \( v_x(-L) > 0 \) must attain its unique local maximum \( p \) at \( x = 0 \). Furthermore, \( v \) satisfies the Eq. (4.7) with \( v_x(-L) = \sqrt{2\lambda} \) on \([0, L]\). Thus, necessarily,
\[
\lambda - F(z) > 0 \quad \text{for all } \quad z \in (0, p), \quad F(p) = \lambda.
\]
Thus \( p = p(\lambda) \) is uniquely characterized as
\[
p = \inf\{ z \mid z > 0, F(z) = \lambda \}
\]
(4.8) and \( p \) is an increasing function of \( \lambda \) which proves that \( v_1, v_2 \) must coincide provided they have the same positive derivative at \(-L\).

Moreover, two solutions with different maxima at \( x = 0 \) cannot intersect, as they are both decreasing and satisfy (4.7) on \((0, L)\) with different \( \lambda \) so
\[
v_1(x) = v_2(x) \quad \text{would always yield} \quad (v_2)_x(x) > (v_1)_x(x).
\]
It remains to prove the second part of Proposition 4.1 concerning the solutions with zero derivative at the boundary points as well as the relation (4.4) for \( \mu_1 = 0 \).

Let \( v \) be a solution of (4.6) such that \( v_x(0) = 0 \) and let \( y \in (-L, L) \) be a point of its strong local maximum \( m_0 > 0 \). By virtue of of Lemma 4.2, \( v \) satisfies (4.7) with \( \lambda = 0 \) on the maximal neighborhood of \( y \) on which it is strictly positive. Reasoning as above, we conclude that \( m_0 > 0 \) is characterized by the property
\[
m_0 = \inf\{ z \mid z > 0, F(z) = 0 \}, \quad F < 0 \quad \text{on } (0, m_0).
\]
(4.9)
It follows from (4.8), (4.9) that \( p > m_0 \) if the latter is finite which completes the proof of (4.4) for \( \mu_1 = 0 \).

Moreover, \( v \) has at most a finite number \( k \) of strong local maxima \(-L < y_1 < \cdots < y_k < L \) and there is \( m > 0 \) and a solution \( v_g \) such that
\[
v(x) = \sum_{i=1}^{k} v_g(x - y_i), \quad \text{dist}(y_i, y_j) \geq 2m, \quad i \neq j,
\]
where \( v_g \) is even and strictly decreasing with respect to the origin, \( \text{supp} v_g = (-m, m) \) and
\[
(v_g)_x(-m) = v_g(-m) = 0.
\]
We have proved Proposition 4.1.

Now observe, that if the stationary problem admits the ground state solution, then \( f(0) = 0 \) by virtue of (4.9). To conclude, we present a result concerning the stability of the zero solution:
4.3. Assume that $\phi, f$ are such that the stationary problem (1.8), (1.9) admits the ground state solution $w_g$.

Then there exists $\alpha > 0$ such that

$$0 \leq u(x, t_0) \leq \alpha \quad \text{for all} \quad x \in [-L, L] \quad \text{and a certain} \quad t_0$$

implies

$$\sup_x u(x, t) \to 0 \quad \text{as} \quad t \to \infty$$

for any solution $u$ of the problem (1.1)-(1.3) with $L > 0$ arbitrary.

Proof. By virtue of (4.7), the ground state solution $w_g$ satisfies

$$\frac{1}{2}(\phi(w_g)_x)^2 + H(w_g) = 0 \quad \text{for all} \quad x \in [0, \max_x w_g].$$

As $\phi' > 0$ on $R^+ \setminus \{0\}$, there exists $\alpha > 0$ such that

$$f(\alpha) = 0, \quad \alpha = \max_x w_g.$$

Consequently, by comparison theorem,

$$0 \leq u(x, t) \leq \alpha \quad \text{for all} \quad t \geq t_0. \quad (4.10)$$

Now, by virtue of Proposition 3.1, the set $\omega(u)$ is formed by stationary solutions and the only stationary solution with maximum less than $\alpha$ is necessarily zero.

5. THE DECOMPOSITION LEMMA

We shall say that a function $v$ has the weak positivity property if

$$v(x_0, t_0) > 0 \quad \text{implies} \quad v(x, t) > 0 \quad \text{for all} \quad t > t_0. \quad (5.1)$$

The weak positivity of solutions to the problem (1.1)-(1.3) was studied e.g., in Kersner [12]. Here we present a very simple condition assuring the weak positivity property for solutions of (1.1)-(1.3):

**Lemma 5.1.** Under the hypotheses of Theorem 1.1, let the stationary problem (1.8), (1.9) admit the ground state solution $w_g$ defined on $R^1$ and such that $\text{supp}(w_g) \subset [-L, L]$.

Then any solution $u$ of (1.1)-(1.3) has the weak positivity property.
\textbf{Proof.} Let \( u(x_0, t_0) > 0 \) for certain \( x \in (-L, L) \), \( t_0 \geq 0 \). Then we can find a function \( v_0(x) \) such that
\[
0 \leq v_0(x) \leq u(x, t_0) \quad \text{for all} \quad x \in [-L, L], \tag{5.2}
\]
\[
0 \leq v_0(x) \leq w_g(x - y) \quad \text{for all} \quad x \in \mathbb{R}^1, \tag{5.3}
\]
where \( y \) may be found such that
\[
\sup \{ w_g(\cdot - y) \} \subset [-L, L],
\]
\[
0 < \max_x v_0(x) = v_0(x_0), \quad v_0(x_0 - x) = v_0(x_0 + x) \quad \text{for all} \quad x,
\]
\[
v_0 \text{ nonincreasing on } [x_0, \infty).
\]

Now, let \( v \) be a solution of the problem
\[
v_t - \Delta v = -Mv, \quad v = v(x, t), \quad x \in \mathbb{R}^1, \quad t > t_0, \quad v(t_0) = v_0 \tag{5.4}
\]
where \( M \) has the property
\[
f(z) \geq -Mz \quad \text{for all} \quad z \in [0, v_0(x_0)]. \tag{5.5}
\]
Such a solution is unique in the class of distributional solutions satisfying
\[
v \in C(R^+; L^1(R^1)),
\]
and may be constructed using the nonlinear semigroup theory (see Brezis and Crandall [4]).

Moreover, using (5.3), (5.5) and the comparison principle we deduce
\[
\text{supp}(v(t)) \subset [-L, L] \quad \text{for all} \quad t \geq t_0;
\]
hence, by virtue of (5.2) and by comparison,
\[
u(x, t) \geq v(x, t) \quad \text{for all} \quad x \in [-L, L], \quad t \geq t_0.
\]
Thus it suffices to prove
\[
v(x_0, t) > 0 \quad \text{for all} \quad t \geq t_0,
\]
which is equivalent, in view of the symmetry properties of \( v \), to proving
\[
v(t) \neq 0 \quad \text{for all} \quad t \geq t_0. \tag{5.6}
\]
To show (5.6), we integrate (5.4) over $R^1$ to obtain
\[\int_{R^1} v(t) \, dx = \int_{R^1} v(t_0) \, dx - M \int_{t_0}^t v(x, s) \, dx \, ds\]  
which is nothing else as the principle of conservation of “mass” for (5.4) which may be verified for any weak solution approximating
\[\phi(v)_{xx} + Mv \approx \phi'(v)_{xx} + Mv\]
and making use of the approximation theorem of Crandal and Pierre [6]. Thus (5.6) follows from (5.7).

Remark. Note that if the stationary problem admits the solution $v_{eq}$, then necessarily $\phi'$ must be bounded at least for a certain sequence of arguments going to zero. So the above result does not contradict to the well-known extinction property for problems with $\phi(z) = z', r < 1$ (see, e.g., Kersner [12]).

The main result of this section is the following lemma:

**Lemma 5.2. (Decomposition Principle).** Let
\[Q \equiv (-L, L) \times (0, \infty)\].

Let $v \in C(\overline{Q})$, $v \geq 0$ be a function having the weak positivity property.

Then given $x > 0$ there exist a finite number of disjoint open intervals
\[(a_i, b_i) \subset (-L, L), i = 1, \ldots, k\] such that the following holds:

For any compact $C \subset \bigcup_{i=1}^k (a_i, b_i)$, there exists a finite $t(C) \geq 0$ such that
\[v(x, t) > 0 \quad \text{for all } x \in C, \ t \in (t(C), \infty), \quad (5.8)\]
\[0 \leq v(x, 0) \leq x \quad \text{for all } x \in \partial \equiv [-L, L] \bigcup_{i=1}^k (a_i, b_i), \quad (5.9)\]

and
\[v(x, t) = 0 \quad \text{for all } x \in \partial \equiv [-L, L], \ t \in [0, \infty). \quad (5.10)\]

**Proof.** Consider a function $\tau: [-L, L] \to [0, \infty]$
\[\tau(x) = \inf \left\{ t \geq 0 \mid v(x, t) > 0 \right\} \quad \text{where } \inf(\partial) \equiv \infty. \quad (5.11)\]

Since $v$ is continuous on $\overline{Q}$, $\tau$ is upper semicontinuous. Moreover, we consider
\[\mathcal{M}_\infty \equiv \{ x \mid \tau(x) = \infty \} = \{ x \mid v(x, t) = 0 \text{ for all } t \in [0, \infty) \}.\]
As $\mathcal{M}_\alpha$ is compact and $v(\cdot, 0)$ continuous on $[-L, L]$, there is a finite number of closed disjoint intervals $J_i$, $i = 1, \ldots, m$ such that

$$\mathcal{M}_\alpha \subset \bigcup_{i=1}^m J_i,$$

$$0 \leq v(x, 0) \leq x \quad \text{for all} \quad x \in \bigcup_{i=1}^m J_i.$$

Now, take the points

$$c_i = \inf(\mathcal{M}_\alpha \cap J_i), \quad d_i = \sup(\mathcal{M}_\alpha \cap J_i), \quad i = 1, \ldots, m.$$

Define the set

$$\mathcal{F} = \bigcup_{i=1}^m [c_i, d_i]$$

and observe that $v$ restricted to $\mathcal{F}$ enjoys the properties (5.9), (5.10) claimed above.

Now, the complement of $\mathcal{F}$ in $[-L, L]$ may be written as a union of a finite number of open intervals $(a_i, b_i)$ and $\tau |(a_i, b_i)$ is finite for all $i = 1, \ldots, k$. Consequently, as $\tau$ is upper semicontinuous, it attains its finite maximum $\tau(\mathcal{C})$ for any compact

$$\mathcal{C} \subset \bigcup_{i=1}^k (a_i, b_i).$$

Thus (5.8) follows from the positivity property (5.1).

6. ASYMPTOTIC BEHAVIOR AT BOUNDARY POINTS

The main result of this section reads as follows:

**Lemma 6.1.** Under the hypotheses of Theorem 1.1, let $w \in \omega(u)$ and denote

$$\phi(w)_+(L) = -\phi(w)_+(L) = \lambda \geq 0. \quad (6.1)$$

Then for any $\mu < \lambda$, there exists $d = d(\mu) > 0$ having the following property: Given $\delta > 0$, there is $t_\delta$ such that

$$|u(x, t_\delta) - w(x)| \leq \delta \quad \text{for all} \quad x \in [-L, L], \quad (6.2)$$

$$\phi(u)(x, t_\delta) \geq \mu(x + L) \quad \text{for all} \quad x \in [-L, -L + d] \quad (6.3)$$

$$\phi(u)(x, t_\delta) \geq -\mu(x - L) \quad \text{for all} \quad x \in [L - d, L]. \quad (6.3)$$

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Proof. As \( u > 0 \) the assertion is trivial for \( \mu \leq 0 \). In what follows, we shall assume \( 0 < \mu < \lambda \). We shall construct a suitable “travelling wave” solution and use comparison theorem. Since \( u \) is bounded, we can always assume that \( \phi \) and \( \phi^{-1} \) are asymptotically linear functions for large arguments taking a suitable truncation as the case may be. Moreover, we prolong \( \phi \) on \( R^1 \) as

\[
\phi(-z) = -\phi(z) \quad \text{for all} \quad z < 0.
\]

Consider the equation

\[
\phi(V)_x = Mx + V + \mu, \quad V(V(0)) = V(0) = 0,
\]

where \( M \) is taken such that

\[
|f(u)| \leq M \quad \text{for all} \quad x, t.
\]

As \( \phi^{-1} \) is continuous, the problem (6.4) admits a (not necessarily unique) solution \( V \) enjoying the following properties:

\[
V, \phi(V)_x \in C(R^1), \quad \phi(V)_x(0) = \mu, \quad \phi(V) \text{ strictly increasing and convex on } [0, \infty), \quad V < 0 \quad \text{on } (-a, 0) \quad \text{for a certain } a > 0.
\]

Consider a function \( v(x, t) = V(c + x + t) \) where \( c \) is a constant to be chosen later. We have

\[
v_x(x, t) = V_x(c + x + t), \quad \phi(v)_x(x, t) = \phi(V)_x(c + x + t)
\]

and, since \( V \) solves (6.4), we conclude

\[
v_x - \phi(v)_xx = -M,
\]

where all the equalities hold in the sense of distributions.

On the other hand, we have

\[
0 < \phi(V)_x(0) = \mu < \lambda = \phi(w)_x(-L)
\]

and

\[
\phi(V), \phi(w) \in C^1 \quad \text{with} \quad \phi(V)(0) = \phi(w)(-L) = 0.
\]

Consequently, using (6.7) we conclude there are \( d(\mu), \beta(\mu) \) such that

\[
0 < d(\mu) < a, \quad \beta(\mu) > 0
\]

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and
\[ w(-L+d) > V(x) + \beta, \quad \text{for all } x \in [-a, d]. \quad (6.11) \]

Since \( w \in \omega(u) \), for given \( \delta \in (0, \beta/2) \) there exists \( i > 0 \) such that
\[ |u(x, t) - w(x)| \leq \delta \quad \text{for all } x \in [-L, L], \quad t \in [i, i+d]. \quad (6.12) \]

Now, take the function \( v \) defined above with \( c = L - d - i \). We want to compare the functions \( u, v \) on the rectangle \( \Omega = [-L, -L+d] \times [i, i+d] \).

For \( t = i \) we obtain
\[ v(x, i) = V(x + L - d) \leq 0 \leq u(x, i) \quad \text{for all } x \in [-L, -L+d] \]
by virtue of (6.8). Similarly, one deduces
\[ v(-L, t) = V(-d + t - i) \leq 0 \leq u(-L, t) \quad \text{for all } t \in [i, i+d] \]
and, by virtue of (6.11), (6.12),
\[ v(-L+d, t) = V(t - i) < w(-L+d) - \beta \leq u(-L+d, t) \]
for all \( t \in [i, i+d] \).

Thus \( v \leq u \) on the parabolic boundary of \( \Omega \) and \( v, u \) solves the Eq. (6.9), (1.1) respectively inside \( \Omega \) with \( M \) satisfying (6.5). Using Proposition 2.3 we conclude \( v \leq u \) on \( \Omega \), in particular,
\[ \phi(u)(x, i + d) \geq \phi(v)(x, i + d) = \phi(V)(x + L) \geq \mu(x + L) \]
for all \( x \in [-L, -L+d] \) \quad (6.13)
where the last inequality follows from (6.6) and convexity of \( \phi(V) \) on \( R^+ \).

Now, using similar arguments with \( v(x, t) \) replaced by \( v(-x, t) \) we can obtain a similar estimate at the boundary point \( x = L \). Taking \( t_a = i + d \) we observe that (6.12) yields (6.2) while (6.3) follows from (6.13).

7. MONOTONICITY PROPERTIES OF SOLUTIONS

We shall say that \( u \) has the strong positivity property on an interval \((-L, L)\) if it enjoys the weak positivity property and the following holds:

For any \( x \in (-L, L) \) there is a time \( \tau(x) \) such that \( u(x, t) > 0 \) for all \( t > \tau(x) \).

The strong positivity property of solutions to the problem (1.1)-(1.3) was studied by Bertsch et al. [3].
Proposition 7.1. Let \( \phi, f \) satisfy \( (1.4)-(1.7) \) and let \( u \) be a bounded weak solution of the problem \( (1.1)-(1.3) \) which has the strong positivity property on \( (-L, L) \).

Then

\[
\lim_{t \to \infty} \text{sgn}(u_x(s, t))
\]

exists and it is either \( 1 \) or \(-1\) for all \( s \neq 0, -L, L \).

The rest of this section is devoted to the proof of Proposition 7.1. The main tool will be the so called zero number theory developed for nondegenerate parabolic equations.

To begin with, observe that \( u_x(x, t) \) exists for all \( x \) and \( t > \tau(x) \) in view of Proposition 2.4 where \( \tau \) is defined by (5.11). Moreover, it is enough to give the proof for \( s > 0 \) as the problem is invariant with respect to the reflection.

Since \( u \) is continuous and has the strong positivity property, there exists a finite time

\[
t_0 = \sup \{ \tau(x) \mid x \in [-L + \delta, L - \delta] \}
\]

such that

\[
u(x, t) > \rho(t) > 0 \quad \text{for all} \quad x \in [-L + \delta, L - \delta], \quad t \geq t_0,
\]

where

\[
0 < \delta < \min \{ s, L - s \}.
\]

We can assume \( t_0 = 0 \) taking a time shift if necessary.

Now, consider an auxiliary function

\[
v_j(x, t) \equiv u(s + x, t) - u(s - x, t)
\]

defined for all

\[
x \in [-L + s, L - s].
\]

Consequently by virtue of (7.1), we have

\[
v_j(-L + s, t) = u(2s - L, t) - u(L, t) > \rho(t) > 0 \quad \text{for all} \quad t > 0,
\]

and

\[
v_j(L - s, t) = u(L, t) - u(2s - L, t) < -\rho(t) < 0 \quad \text{for all} \quad t > 0.
\]

Next, we shall need the following auxiliary result.
**Lemma 7.2.** Let \( v_s(x, t) = 0 \) for certain \( x \in (-L+s, L-s) \) and \( t > 0 \). Then \( u(x+s, t) = u(s-x, t) > 0 \).

**Proof.** Arguing by contradiction, we shall assume

\[
u(x+s, t) = u(s-x, t) = 0 \quad \text{for certain} \quad x \in (-L+s, L-s) \quad \text{and} \quad t > 0.
\]

Then, by virtue of (7.1) we have

\[
x + s \in (-L, -L + \delta) \cup (L - \delta, L)
\]
\[
s - x \in (-L, -L + \delta) \cup (L - \delta, L)
\]

which implies

\[
x \in (L - s - \delta, L - s) \cap (s - L, s - L + \delta) = \emptyset
\]

by virtue of (7.2).

For \( t > 0 \) let us denote

\[
\mathcal{Z}(v_s)(t) = \{ x \in (-L+s, L-s) | v_s(x, t) = 0 \}
\]

the set of zeros of \( v_s \) at a fixed time \( t > 0 \). According to (7.3), (7.4), the set \( \mathcal{Z} \) is compact in \((-L+s, L-s)\). Consequently, by virtue of Lemma 7.1 there is a finite system of disjoint open intervals \( J_i \subset (-L+s, L-s) \) such that

\[
\mathcal{Z}(v_s)(t) \subset \bigcup_{i=1}^{k} J_i,
\]

\[
u(s+x, t), u(s-x, t) \geq \rho > 0 \quad \text{for all} \quad x \in \overline{J_i}, \quad j = 1, ..., k.
\]

Now, the set

\[
\mathcal{M} = [ -L+s, L-s ] \setminus \bigcup_{i=1}^{k} J_i
\]

is compact and \( v_s|_{\mathcal{M}} \neq 0 \). Since \( v_s \) is continuous, there are \( t_1 < t < t_2 \) such that

\[
v_s(x, t) \neq 0 \quad \text{for all} \quad x \in \mathcal{M}, \quad t \in (t_1, t_2)
\]

hence

\[
\mathcal{Z}(v_s)(t) \subset \bigcup_{i=1}^{k} J_i \quad \text{for all} \quad t \in (t_1, t_2)
\]
and
\[ u(s + x, t), u(s - x, t) \geq \rho > 0 \quad \text{for all} \quad x \in J_i, \quad t \in (t_1, t_2), \quad i = 1, \ldots, k. \]
\[ (7.7) \]
Moreover, \( v_s \) solves the linear equation
\[ (v_s)_t - a(x, t)(v_s)_x + b(x, t)(v_s)_x + c(x, t) v_s = 0, \quad x \in J_i, \quad t \in (t_1, t_2) \]
\[ (7.8) \]
with
\[
 a(x, t) = \varphi'(u(s + x, t)), \quad b(x, t) = \varphi''(u(s - x, t))(u_x(x + s) + u_x(s - x)), \n\]
\[
 c(x, t) = -u_{xx}(x - s)[\varphi'] - u_x^2(s - x)[\varphi'' - \frac{f}{\nu}] \]
where we have denoted
\[
 [h](x, t) = \begin{cases} 
 0 & \text{if } u(s + x, t) = u(s - x, t), \\
 \frac{h(u(s + x, t)) - h(u(s - x, t))}{u(s + x, t) - u(s - x, t)} & \text{otherwise}. 
\end{cases} 
\]
By virtue of (7.7) and Proposition 2.4, \( v_s \) is a classical solution of (7.8) on each \( J_i \times (t_1, t_2) \) and the coefficients satisfy
\[
 a, a^{-1}, a_t, a_x, a_{xx}, b, b_t, b_x, c \in L^\infty(J_i \times (t_1, t_2)), \quad a > 0. 
\]
By virtue of (7.6), all zeros of \( v_s \) lie in the interior of the intervals \( J_i \) and, consequently, we may apply Theorem D of Angenent [1] on each interval \( J_i \) separately to conclude
\[
 \mathcal{Z}(v_s)(t) \text{ is finite for any } t > 0 
\]
and, letting \( z(v_s)(t) = \text{card} \mathcal{Z}(v_s)(t) \) we have \( z(v_s) \) a nonincreasing function of \( t \). Moreover, if \( v_s \) has a double zero at \( t \), i.e.,
\[
 v_s(x, t) = (v_s)_x(x, t) = 0, 
\]
then
\[
 z(v_s)(t - \varepsilon) > z(v_s)(t + \varepsilon) \quad \text{for all} \quad \varepsilon > 0 \quad \text{small.} 
\]
Now, it suffices to observe that
\[ v_s(0, t) = 0, \quad (v_s)_x(0, t) = 2u_x(s, t) \quad \text{for all} \quad t > 0. \]

Consequently, \( u_x(s, t) \) may vanish for only finitely many \( t \).

Proposition 7.1 has been proved.

8. CONVERGENCE

In this section we prove Theorem 1.1 stated in the introduction. It will be done in two steps.

**Lemma 8.1.** Under the hypotheses of Theorem 1.1, assume the set \( \omega(u) \) contains a stationary solution \( w \) such that
\[ \phi(w)_x(-L) > 0. \]

Then \( \omega(u) \) is a singleton.

**Proof.** Arguing by contradiction we assume there exist, since \( \omega(u) \) is connected, three solutions \( w_1, w_2, w_3 \in \omega(u) \) such that
\[ \mu_i < \mu_j \quad \text{for} \quad i < j, \]
where
\[ \mu_i = \phi(w_i)_x(-L), \quad i = 1, 2, 3. \]

Now, take \( w = w_3, \mu = (\mu_2, \mu_3) \) in Lemma 6.1 and let \( d = d(\mu) \) be the quantity for which (6.2), (6.3) hold.

Now, we can find \( a, \eta > 0, 0 < a < d(\mu) \) such that
\begin{align*}
\phi(w_3)(x) &< \mu(x + L) \quad \text{for all} \quad x \in [-L, -L + a], \\
\phi(w_3)(x) &< -\mu(x - L) \quad \text{for all} \quad x \in [L - a, L], \quad (8.2) \\
w_2(x) &< w_3(x) - \eta \quad \text{for all} \quad x \in [-L + a, L - a].
\end{align*}

where the last inequality follows from Proposition 4.1.

Thus taking \( \delta < \eta \) in Lemma 6.1 and comparing (6.2), (6.3) with (8.1) we conclude there is \( t_\delta > 0 \) such that
\[ u(x, t_\delta) \geq w_2(x) \quad \text{for all} \quad x \in [-L, L]; \]
hence, by comparison, the same holds for any $t > t_\delta$. But this is impossible as

$$w_2(x) > w_1(x) \quad \text{for all } x \in (-L, L)$$

and $w_1 \in \omega(u)$. 

**Lemma 8.2.** Under the hypotheses of Theorem 1.1, let all solutions $w \in \omega(u)$ satisfy

$$\phi(w)_x (-L) = 0.$$

Then $\omega(u)$ is a singleton.

**Proof.** Assume the contrary, i.e., the set $\omega(u)$ contains at least two different solutions. In this situation, necessarily, the stationary problem admits the ground state solution $w_\gamma$ and, by virtue of Lemma 5.1, $u$ has the weak positivity property.

Thus $u$ satisfies the hypotheses of the decomposition Lemma 5.2 and we obtain

$$u = \sum_{i=1}^k u_i + u'$$

where $u_i$ solve the problem (1.1), (1.2) on intervals $(a_i, b_i) \subset (-L, L)$,

$$u_i(a_i, t) = u_i(b_i, t) = 0 \quad \text{for all } t \geq 0$$

and

$$u' (x, 0) \leq x \quad \text{on } \mathcal{O} \equiv (-L, L) \bigcup_{i=1}^k (a_i, b_i),$$

$$u'(t)_{|_{S^k}} \equiv 0,$$

where $\alpha > 0$ is taken as in Lemma 4.3.

By virtue of Lemma 4.3, we have

$$\sup_x u' \to 0 \quad \text{as } t \to \infty$$

and the problem reduces to showing convergence for each component $u_i$, $i = 1, \ldots, k$. Thus we may assume $u = u_i$, $(a_i, b_i) = (-L, L)$ where, by virtue of (5.8), $u$ has the strong positivity property on $(-L, L)$. 

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Consequently, we can apply Proposition 7.1 to obtain
\[
\lim_{t \to \infty} \text{sgn}(u_x(x, t)) \text{ exists}
\]
(8.2)
for any \(x \neq 0, -L, L\).

Since we assume that \(\omega(u)\) is not a singleton, it must contain a solution \(w_1\) of the form
\[
w_1(x) = \sum_{i=1}^{k} w_i(x - y_i^1)
\]
(see Proposition 4.1). Moreover, as \(\omega(u)\) is connected, there must exists a solution \(w_2\) such that
\[
w_2(x) = \sum_{i=1}^{k} w_i(x - y_i^2),
\]
\[|y_i^1 - y_i^2| < \frac{m}{2} \quad \text{for all} \quad i = 1, \ldots, k,
\]
and
\[y_i^1 \neq y_j^2 \quad \text{at least for one } j
\]
where \(m\) is the length of the support of \(w_x\).

Using the properties of \(w_x\) we deduce there is an open (nonvoid) interval \(J \subseteq (-L, L)\) such that
\[
w_1(x), w_2(x) > \rho > 0, \quad (w_1)_x(x) > 0 > (w_2)_x(x) \quad \text{for all } x \in J.
\]
(8.3)
Combining (8.3) with Lemma 3.1 we obtain a contradiction with (8.2).

Theorem 1.1 has been proved.

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