A Controllability Technique for Nonlinear Systems*

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1. INTRODUCTION

In this paper we present sufficient conditions for complete controllability of the nonlinear control system

\[
\dot{x} = g(t, x) + k(t, u) \quad (\dot{x} = dx/dt)
\]

over a bounded interval \( I = [t_0, t_1] \). Our technique is to use the Fan fixed-point theorem to show the existence of an absolutely continuous function satisfying the generalized boundary value problem

\[
\dot{x}(t) \in R(t, x(t)) \quad \text{a.e. on } I
\]

\[
x(t_0) = x_0, \quad x(t_1) = x_1.
\]

\( R \) denotes a set valued mapping from \( I \times E^n \) into the set of nonempty closed subsets of Euclidean \( n \)-dimensional space \( E^n \) which is upper semicontinuous with respect to set inclusion.

Tarnove [1] used a fixed point theorem to obtain sufficient conditions for \( A \)-controllability of the nonlinear system \( \dot{x} = f(t, x, u) \), \( A \) a nonempty bounded closed convex set of continuous functions. (A system is said to be \( A \)-controllable if there exists a solution of the system belonging to \( A \).) Although Tarnove did not consider generalized differential equations in the notation of this paper, he proved that the system \( \dot{x} = f(t, x, u) \) is \( B_p \)-controllable if \( \Phi(y) \neq \emptyset \) for all \( y \in B_p \),

\[
\Phi(y) = \{ x \in B_p : \dot{x}(t) \in f(t, y(t), \Omega(t)) \text{ a.e. on } I \}.
\]

We shall be concerned with giving explicit conditions on \( g \) and \( k \) which imply controllability between two fixed points. We then demonstrate a technique which uses these conditions as a criterion for complete controllability for a given system.

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A CONTROLLABILITY TECHNIQUE

One of the hypotheses for complete controllability which results from a fixed point approach is that the set $k(t, \Omega(t))$ be convex for every point $(t, x) \in I \times E^n$. Although this assumption is common in results on the existence of optimal controls, it does not seem to be particularly desirable. By using generalized equations we are able to replace the convexity condition on $k$ by a Lipschitz condition on $g$ and obtain sufficient conditions for $\epsilon$-approximate controllability. This type of controllability is interesting in applications in, e.g., the theory of nonlinear oscillators. Also, we can combine this result with Markus' [2] result involving the linear approximation to a nonlinear system and obtain sufficient conditions for complete controllability for nonlinear systems.

2. THE GENERALIZED BOUNDARY VALUE PROBLEM

Fix $x_0, x_1 \in E^n$ and let $B$ be the set of all Lipschitz continuous functions $x$ such that $x(t_0) = x_0$ and $x(t_1) = x_1$. Consider the following two norms on $B$:

$$|x| = \max_{t \in I} |x(t)|,$$

$$\|x\| = \max_{t \in I} |x(t)| + \inf\{M_x\},$$

where $\{M_x\}$ is the set of all Lipschitz constants for the function $x$. For $p > 0$, let

$$B_p = \{x \in B : \|x - x_0\| \leq p\},$$

where $(x - x_0)(t) = x(t) - x_0$. Then $B_p$ is compact with the norm $\|\|$. Note that $R_p$ is convex and nonempty for sufficiently large $p$.

Throughout this section, we will assume that $R(t, x)$ is convex for $t \in I, x \in E^n$. Let the multifunction $\Phi$ be defined on $B_p$ by

$$\Phi(y) = \{z \in B_p : z(t) \in R(t, y(t)) \text{ a.e. on } I\}.$$ 

The following lemma is an extension of a result due to Filippov [3, p. 78]. This result has been proven by Lasota and Olech [4, p. 620] using orientator fields. We shall give a more standard proof. (Results similar to this can be found in [5-7].)

**Lemma 1.** $\Phi$ has closed graph, i.e., let $(y_n) \subseteq B_p$ and assume $z_n \in \Phi(y_n)$ for $n = 1, 2, \ldots$. If $x, y \in B_p$ are such that

$$|x_n - x| \to 0, \quad |y_n - y| \to 0,$$

then $z \in \Phi(y)$. 
Proof. Let \( t \in I \) be a point such that \( z(t) \) exists and let \( \varepsilon > 0 \) be given. Let \( M \) be a Lipschitz constant for \( y \). Since \( R \) is upper semicontinuous with respect to set inclusion, there exists \( \delta > 0 \) such that if \( |t - i| < \delta \) and \( |y_0 - y(i)| < 2M\delta \), then

\[
R(t, y_0) \subseteq U_{\varepsilon},
\]

the closed \( \varepsilon \)-neighborhood of \( R(i, y(i)) \). By possibly making \( \delta \) smaller we can assume that for \( |t - t_i| < \delta \) we have

\[
|z(t) - z(i)| < \varepsilon.
\]

Now

\[
\frac{z(t) - z(i)}{t - i} = \lim_{n \to \infty} \frac{z_n(t) - z_n(i)}{t - i} = \lim_{n \to \infty} \frac{1}{t - i} \int_{i}^{t} \dot{z}_n(s) \, ds.
\]

Using the definition of \( M \) and the convergence of \( \{y_n\} \), we have that for sufficiently large \( n \) and \( |\tau - t_i| < \delta \) the following are true

\[
|y_0 - y(t)| < M\delta,
\]

\[
|y_n(\tau) - y(\tau)| < M\delta.
\]

Hence

\[
|y_n(\tau) - y(t)| < 2M\delta,
\]

which implies

\[
R(\tau, y_n(\tau)) \subseteq U_{\varepsilon}
\]

for sufficiently large \( n \) and \( |\tau - t_i| < \delta \). Since \( z_n \in \Phi(y_n) \), we have

\[
z_n(i + (t - i)s) = \dot{z}_n(\tau) \in U_{\varepsilon} \quad \text{for a.e. } s \in [0, 1].
\]

Since \( R(i, y(i)) \) is convex, we have that \( U_{\varepsilon} \) is convex. Hence

\[
\int_{0}^{1} \dot{z}_n(i + (t - i)s) \, ds \in U_{\varepsilon}
\]

for sufficiently large \( n \). Thus, from (1) and (2) we have

\[
z(t) \in U_{2\varepsilon}.
\]
Since \( \epsilon > 0 \) is arbitrary and \( R(t, y(t)) \) is closed, we have
\[
\mathbf{z}(t) \in R(t, y(t)).
\]
We have \( \mathbf{z} \in \Phi(y) \) since \( \dot{\mathbf{z}}(t) \) exists a.e. on \( I \).

For \( x \in B_p \), let
\[
\int_I R(s, x(s)) \, ds - \left\{ \int_I r(s) \, ds : r \text{ is measurable}, r(s) \in R(s, x(s)) \text{ a.e. on } I \right\}.
\]

**THEOREM 1.** Assume \( p > 0 \) is such that \( B_p \neq \emptyset \) and such that
\[
\text{the closed ball in } E^n \text{ centered at } 0 \text{ with radius } M = \min\{p/2, p/2(t_1 - t_0)\}. \text{ If for every } y \in B_p \text{ we have}
\]
\[
x_1 - x_0 \in \int_I R(s, y(s)) \, ds,
\]
then there exists a function \( x \in B_p \) such that
\[
x(t_0) = x_0, \quad x(t_1) = x_1,
\]
and
\[
\dot{x}(t) \in R(t, x(t)) \quad \text{a.e. on } I.
\]

**Proof.** Let \( p > 0 \) satisfy the hypotheses of the theorem. We need only show that the multifunction \( \Phi \) has a fixed point \( x \in B_p \). To do this we are going to apply the Fan fixed point theorem [8, Theorem 1]. This theorem states that any upper semicontinuous multifunction from a compact, convex subset \( D \) of a locally convex linear topological space into the set of nonempty, closed, convex subsets of \( D \) has a fixed point in \( D \).

Let \( y \in B_p \), \( \Phi(y) \) is convex since \( B_p \) and \( R(t, x) \) are convex. We now show that \( \Phi(y) \neq \emptyset \). From condition (3) there exists a measurable function \( r \) such that \( r(t) \in R(t, y(t)) \) a.e. on \( I \) and such that
\[
x_1 = x_0 + \int_I r(s) \, ds.
\]
Setting
\[
x(t) = x_0 + \int_{t_0}^t r(s) \, ds,
\]
we have $z \in B$. Also,
\[ |z(t) - x_0| \leq \int_{t_0}^{t} \frac{p}{2(t_1 - t_0)} \, ds \leq \frac{p}{2} \]
for all $r \in I$ and
\[ |z(t) - z(i)| \leq \int_{i}^{t} \frac{p}{2} \, ds \leq \frac{p}{2} |t - i| \]
for all $t, i \in I$. Hence
\[ \|z - x_0\| = \max_{t \in I} |z(t) - x_0| + \inf \{M_z\} \leq \frac{p}{2} + \frac{p}{2} = p. \]
Thus $z \in \Phi(y)$ which implies $\Phi(y) \neq \emptyset$.

By Lemma 1, $\Phi$ has closed graph. Therefore, it has closed values and since the domain $B_p$ of $\Phi$ is compact, it follows that $\Phi$ is upper semicontinuous.

Hence, by the Fan fixed point theorem, $\Phi$ has a fixed point $x \in B_p$. The function $x$ is the desired function.

Remark. The boundedness assumption on $R$ in Theorem 1 is more relaxed than that assumed by Hermes in his existence results [9, 10]. For example, consider $R(t, x) = \{g(t, x) + k(t, \Omega(t))\}$ where $g$ is bounded on $I \times E^n$ but $k$ is not a bounded function on $I \times E^n$. In Section 3, we will use this type of system with $\Omega(t) = E^2$ in an example.

3. The Control Problem

We now want to apply Theorem 1 to the problem of controllability of nonlinear control systems. Let $\Omega$ be a multifunction defined from $I$ into the set of nonempty, compact subsets of $E^m$. Assume $\Omega$ is upper semicontinuous with respect to set inclusion. Let $\mathcal{M}(\Omega)$ be the set of all measurable functions $u : I \to E^m$ such that $u(t) \in \Omega(t)$ for all $t \in I$. From Kuratowski and Ryll–Nardzewski [11, p. 398] we have that $\mathcal{M}(\Omega) \neq \emptyset$.

For the control system
\[ \dot{x} = g(t, x) + k(t, u(t)) \quad \text{a.e. on } I, \]
\[ x(t_0) = x_0, \]
with $u \in \mathcal{M}(\Omega)$ (solutions being absolutely continuous), let the multifunction $R$ be defined by
\[ R(t, x) = \{g(t, x) + k(t, \bar{u}) : \bar{u} \in \Omega(t)\}. \]
Theorem 2. Consider the system

\[ \dot{x} = g(t, x) + k(t, u), \]

where \( g, k \) are continuous and \( k(t, \Omega(t)) \) is convex. Assume \( p > 0 \) is such that \( B_p \neq \emptyset \) and

\[
\sup_{t \in I, u \in \Omega(t)} |k(t, u)| \leq M
\]
\[
\sup_{t \in I} |g(t, x)| \leq N
\]

where

\[
M + N \leq \min \left\{ \frac{p}{2}, \frac{p}{2(t_1 - t_0)} \right\}.
\]

If

\[
\int_I k(s, \Omega(s)) \, ds \supseteq \left\{ x_1 - x_0 - \int_I g(s, y(s)) \, ds : y \in B_p \right\},
\]

then there exists \( u \in M(\Omega) \) such that the solution of

\[
\dot{x} = g(t, x) + k(t, u(t)),
\]

\[
x(t_0) = x_0
\]

satisfies

\[
x(t_1) = x_1.
\]

Proof. Since \( g + k \) is continuous and \( \Omega \) is upper semicontinuous with respect to set inclusion and has compact values, it follows that \( R \) is upper semicontinuous with respect to set inclusion and has closed values. Hence we can apply Theorem 1 to get the existence of an absolutely continuous function \( x \) such that \( \dot{x}(t) \in R(t, x(t)) \) a.e. on \( I \), \( x(t_0) = x_0 \) and \( x(t_1) = x_1 \). By Filippov's lemma [3, p. 78] there exists \( u \in M(\Omega) \) such that \( \dot{x}(t) = g(t, x(t)) + k(t, u(t)) \) a.e. on \( I \).

Remarks. Richter [12] (see Aumann [13, Theorem 1]), has shown that the set

\[
\left\{ \int_I k(s, \Omega(s)) \, ds \right\} = \left\{ \int_I k(s, u(s)) \, ds : u \in M(\Omega) \right\}
\]

is convex. This fact is helpful in applying Theorem 2. Also, note that if \( k \) is linear in \( u \) and \( \Omega(t) \) is closed and convex for \( t \in I \), then \( R(t, x) \) is closed and convex.
Theorem 2 could be stated for systems of the form $\dot{x} = f(t, x, u)$. The hypotheses would be of the form of those in Theorem 1.

**Example.** The system

\[
\begin{align*}
\dot{x}(t) &= a e^{t-y^2(t)} \sin x(t) + cu^2(t) \\
\dot{y}(t) &= \frac{bt}{1 + x^2(t)} + dv^2(t) \cos v(t)
\end{align*}
\]

is completely controllable on $[0, 1]$ where $\Omega(t) = E^2$ for $0 \leq t \leq 1$ and the constants $c$ and $d$ are nonzero.

To see this, let $(x_0, y_0), (x_1, y_1) \in E^2$ be given initial and terminal points. Set

\[
k = 24 \max\{|x_0|, |y_0|, |x_1|, |y_1|, 1, |b|, |c|, |d|, |e|, |f|, |g|, |h|
\]

and let

\[
\tilde{\Omega}(t) = \left\{(u, v) \in S_{g, \varphi}(0) : \|(cu^2, dv^2 \cos v)\| \leq \frac{k}{4}\right\}.
\]

Then, $k(t, \tilde{\Omega}(t)) = S_{g, \varphi}(0)$ is convex and $g, k$ satisfy the boundedness conditions of Theorem 2. Also,

\[
\int_0^1 k(t, \tilde{\Omega}(t)) \, dt = S_{g, \varphi}(0).
\]

Let

\[
\begin{align*}
\bar{\Omega}(t) &= \left\{(u, v) \in S_{g, \varphi}(0) : \|(cu^2, dv^2 \cos v)\| \leq \frac{k}{4}\right\}.
\end{align*}
\]

Then $|\bar{\Omega}(t)| \leq k/4$. Hence the theorem applies for the points $(x_0, y_0), (x_1, y_1)$ and the set of controllers $M(\bar{\Omega})$.

4. **The $\varepsilon$-Approximate Boundary Value Problem**

We now wish to consider systems where $R(t, x)$ is not convex. The following result is due to Filippov [14, Theorem 3]. Let $h$ denote the Hausdorff metric.

**Lemma 2.** Assume that $R$ is continuous, closed valued, and satisfies the Lipschitz condition

\[
h(R(t, x), R(t, \bar{x})) \leq w(t) |x - \bar{x}|,
\]
with \( w \in L^1(I) \). Suppose \( y \) is an absolutely continuous function satisfying
\[
y'(t) \in \text{convex hull of } R(t, y(t)) \quad \text{a.e. on } I,
\]
y(\( t_0 \)) = x_0.

Assume that there exists \( p > 0 \) such that \( y(t) \in S_p(x_0) \) for all \( t \in I \) and that the set \( R(t, x) \) is bounded for all \( t \in I, x \in S_p(x_0) \). Then given any \( \epsilon > 0 \) there exists an absolutely continuous function \( x \) satisfying
\[
x'(t) \in R(t, x(t)) \quad \text{a.e. on } I,
\]
x(\( t_0 \)) = x_0.

such that
\[
\max_{t \in I} |y(t) - x(t)| < \epsilon.
\]

Theorem 3. Assume that \( R \) is continuous, closed valued, and satisfies the Lipschitz condition
\[
h(R(t, x), R(t, \tilde{x})) \leq w(t) |x - \tilde{x}|
\]
with \( w \in L^1(I) \). Suppose \( p > 0 \) is such that \( B_p \neq \emptyset \) and such that
\[
\bigcup_{x \in S_p(x_0)} R(t, x) \subseteq S_{M}(0),
\]
where \( M = \min\{p/2, p/(t_1 - t_0)\} \). If for every \( y \in B_p \) we have
\[
x_1 - x_0 \in \int_I R(s, y(s)) \, ds,
\]
then given any \( \epsilon > 0 \) there exists an absolutely continuous function \( x \) such that
\[
x'(t) \in R(t, x(t)) \quad \text{a.e. on } I,
x(\( t_0 \)) = x_0, \quad \text{and} \quad |x(t_1) - x_1| < \epsilon.
\]

Proof. Let \( H(t, x) \) be the convex hull of the closed set \( R(t, x) \). By the hypotheses on \( R \) we have that \( H \) satisfies the hypotheses of Theorem 1. Hence there exists an absolutely continuous function \( y \in B_p \) such that
\[
y(\( t_0 \)) = x_0, \quad \text{and} \quad y'(t) \in H(t, y(t)) \quad \text{a.e. on } I.
\]

The result follows from Lemma 2.
COROLLARY. Consider the system

$$\dot{x} = g(t, x) + k(t, u),$$

where $g$, $k$ and $\Omega$ are continuous. Suppose there exists a function $w \in L^1(I)$ such that

$$|g(t, x) - g(t, \bar{x})| \leq w(t) |x - \bar{x}|$$

for all $t \in I$, $x$, $\bar{x} \in \mathbb{R}^n$. Assume $p > 0$ is such that $B_p \neq \emptyset$ and

$$\sup_{t \in I} |k(t, u)| \leq M,$$

$$\sup_{t \in I} |g(t, x)| \leq N,$$

where

$$M + N = \min \left\{ \frac{p}{2}, \frac{p}{2(t_1 - t_0)} \right\}.$$

If

$$\int_I k(s, \Omega(s)) \, ds \supseteq \left\{ x_1 - x_0 - \int_I g(t, y(s)) \, ds : y \in B_p \right\},$$

then given any $\epsilon > 0$ there exists $u \in \mathcal{M}(\Omega)$ such that the solution of

$$\dot{x} = g(t, x) + k(t, u(t))$$

$$x(t_0) = x_0$$

satisfies

$$|x(t_1) - x_1| < \epsilon.$$

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