Abstract

We study the spectral shift function $s(\lambda, h)$ and the resonances of the operator $P(h) = -\Delta + V(x) + W(hx)$. Here $V$ is a periodic potential, $W$ a decreasing perturbation and $h$ a small positive constant. We give a representation of the derivative of $s(\lambda, h)$ related to the resonances of $P(h)$, and we obtain a Weyl-type asymptotics of $s(\lambda, h)$. We establish an upper bound $O(h^{-n+1})$ for the number of the resonances of $P(h)$ lying in a disk of radius $h$. © 2005 Elsevier Inc. All rights reserved.

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1. Introduction

The purpose of this paper is to study the spectral shift function $s(\lambda, h)$ and the resonances for the perturbed periodic Schrödinger operator

$$P(h) = P + W(hx), \quad P = -\Delta + V(x), \quad (h \searrow 0),$$

where $V$ is periodic with respect to a lattice $\Gamma$ in $\mathbb{R}^n$, and $W(x) = O(|x|^{-\tilde{n}})$, $(\tilde{n} > n)$. The operator $P(h)$ describes the quantum motion of an electron in a crystal placed in an external field.
There are many works devoted to the spectral properties of this model
\[2,10,13,18,19,20,22,24,46\].

The spectral shift function (SSF for short) is defined as a distribution on \( \mathbb{R} \) by the
relation
\[
\text{tr}\left( f(P(h)) - f(P) \right) = \langle s'(\cdot, h), f \rangle \quad \forall \ f \in C^\infty_0(\mathbb{R}).
\]

The SSF may be considered as a generalization of the eigenvalues counting function. Under suitable assumptions it can be identified with the scattering phase. For more
details, we refer to the survey paper [3].

The relation between the asymptotics of the SSF (or the scattering phase) and res-
onances was first investigated by Melrose [31] and, then with successive extensions,
by Guillopé, and Zworski [12] (see also [48]) and Petkov and Zworski [34]. All these
works use the scattering theory.

Recently, Bruneau and Petkov [6] have obtained a representation of the derivative
of the SSF of the Schrödinger operator with long-range perturbations going to 0 as
\( |x| \to +\infty \). The approach of Bruneau and Petkov does not require the scattering theory.
However, the method developed in [6] cannot be applied directly to operator like
\( P(h) \) since its symbol \( |\xi|^2 + V(x) + W(hx) \) does not converge to \( |\xi|^2 \) as \( |x| \to +\infty \), and
as it is not an \( h \)-pseudodifferential one.

In this paper we study the connexions between the resonances of \( P(h) \) and the SSF
associated to the pair \( (P(h), P) \). Our goal is to show that the derivative \( \frac{\partial s}{\partial \lambda}(\lambda, h) \) has
the same representation as the one obtained in [6]. We apply this representation to give
an upper bound for the number of resonances of \( P(h) \) in a disk of radius \( h \), and we
establish an \( h \)-local trace formula for the pair \( (P(h), P) \).

Moreover, we obtain a Weyl-type asymptotics for \( s(\lambda, h) \):
\[
s(\lambda, h) = \omega_0(\lambda) h^{-n} + \mathcal{O}(h^{-n+1}), \quad (h \searrow 0), \tag{1.2}
\]
where
\[
\omega_0(\lambda) = \int \left( \rho(\lambda - W(x)) - \rho(\lambda) \right) dx.
\]

Here \( \rho(\lambda) \) is the integrated density of states associated to the unperturbed Hamiltonian
\( P \), see Theorem 2.

Under the assumption that \( P(h) \) has no resonances in a small complex neighborhood
of an interval \( I \), we obtain a full asymptotic expansion in powers of \( h \) of \( s'(\lambda, h) \):
\[
s'(\lambda, h) \sim \sum_{j=0}^\infty b_j(\lambda) h^{j-n}, \quad (h \searrow 0), \tag{1.3}
\]
uniformly with respect to \( \lambda \in I \).
With the change of variable \( r = h x \), Eq. (1.1) becomes

\[
P_1(h) = -h^2 \Delta_r + V \left( \frac{r}{h} \right) + W(r), \quad P_0(h) = -h^2 \Delta_r + V \left( \frac{r}{h} \right).
\]  

(1.4)

In the case where \( V = 0 \), the semi-classical asymptotics of \( s(\lambda, h) \) and \( s'(\lambda, h) \) were studied by many authors [11,36,38,39,32]. The main ingredient in the proofs of these results is the construction of a long-time parametrix for the time-dependent Schrödinger equation [38].

For \( V \neq 0 \) our results are new. They are in harmony with the physical intuition which argues that, when \( h \) is sufficiently small, the main effect of the periodic potential \( V \) consists in changing the dispersion relation from the free kinetic energy \( |k|^2 \) to the modified kinetic energy \( \lambda_n(k) \) given by the \( n \)th band, see [46].

Let us describe the main problems encountered while trying to carry out the asymptotic analysis of Eq. (1.4). The first obstacle is related to the spectrum of the unperturbed Schrödinger operator \( P \). In fact, in the study of \( P \), one introduces usually the Fermi surface \( F(\lambda) \), which is the set of Bloch numbers \( k \) such that \( \lambda \) is an eigenvalue of the reduced operator \( P_k \), see the next section. In general, the geometry of \( F(\lambda) \) can be quite complicated [26].

The second obstacle is related to the perturbation \( W \). Notice that, the question of absence of eigenvalues is not completely solved for \( P(h) \). Up to our knowledge, in the multi-dimensional situation the only known result on the absence of embedded eigenvalues concerns the case of exponentially decaying perturbations (i.e. \( W(r) = O(e^{-|r|^l}), \ l > \frac{4}{3} \) see [28]. On the other hand, when \( h \to 0 \), there are two spatial scales in Eq. (1.4). The first one of the order of the linear dimensions \( \gamma \) of the periodicity cell and the second one of order \( \gamma/h \) on which the perturbation of the potential varies appreciably. Hence, it seems quite difficult to construct a parametrix for the propagators \( e^{itP(h)/h} \), see [24]. In particular, it is not clear how to apply Robert’s approach to prove (1.2) and (1.3).

This paper continues our previous works [12,13,17] on the resonances and the eigenvalues counting function for \( P(h) \). The starting point is the same as in [12,13,17]: we apply the two-scales expansion method to reduce the study of \( P(h) \) to the one of a system of \( h \)-pseudodifferential operators. More precisely, following [19] (see also [10,22] and Section 3.2), we can reduce the spectral study of \( P(h) \) near any fixed energy level \( z \) to the study of a finite system of \( h \)-pseudodifferential operators acting on \( L^2(\mathbb{T}^n; \mathbb{C}^N) \). Here, \( \mathbb{T}^n = \mathbb{R}^n/\Gamma^* \) is the flat torus and \( \Gamma^* \) is the dual lattice of \( \Gamma \). For the reduced problem, the dependence on the spectral parameter is non-linear, there does not seem to be natural well-posed evolution problems (see Section 5). For these reasons all our proofs are time independent.

Our representation of \( s'(\lambda, h) \) (see formula (2.2)) is similar to the one obtained in [6], although technical details of the proofs are rather different. The novelty here consisting in working with a system of \( h \)-pseudodifferential operators. It seems that the methods of the present work easily apply to more general problems in mathematical physics where the spectral problems can be reduced to similar problems in lower dimension but where the dependence on the spectral parameter becomes non-linear. For instance
we have in mind, Hamiltonians corresponding to strong magnetic fields, stark effect and Born–Oppenheimer approximation, for these problems the effective Hamiltonians are constructed in [47, 27, 29] respectively.

2. Preliminaries and main result

Let $\Gamma = e_1 \mathbb{Z} + \cdots + e_n \mathbb{Z}$ be a lattice in $\mathbb{R}^n$ with the unit cell $E = \{ \sum_{j=1}^n t_j e_j, \; t_j \in [0, 1] \}$, and let $\Gamma^* = e_1^* \mathbb{Z} + \cdots + e_n^* \mathbb{Z} \subset \mathbb{R}^n$ be the dual lattice with the unit cell $E^* = \{ \sum_{j=1}^n t_j e_j^*, \; t_j \in [0, 1] \}$, where $(e_j, e_k^*) = 2\pi \delta_{jk}, \; k, j = 1, \ldots, n$. We use the usual flat metrics on $\mathbb{T}^n := \mathbb{R}^n / \Gamma$ and $\mathbb{T}^n := \mathbb{R}^n / \Gamma^*$.

Let $V$ be a real-valued potential, $C^\infty$ and $-\Gamma$-periodic. For $k \in \mathbb{R}^n$, we define the operator $P_k$ on $L^2(\mathbb{T}^n)$ by

$$P_k = (D_x + k)^2 + V(x), \quad D_x = (D_{x_1}, \ldots, D_{x_n}), \quad D_{x_j} = \frac{1}{i} \partial_{x_j}.$$

The operator $P_k$ is a semi-bounded self-adjoint with $k$-independent domain $H^2(\mathbb{T}^n)$. As the resolvent of $P_k$ is compact, $P_k$ has a complete set of (normalized) eigenfunctions, called Bloch functions. The corresponding eigenvalues accumulate at infinity and we enumerate them according to their multiplicities, $\lambda_1(k) \leq \lambda_2(k) \leq \cdots$. The operator $P_k$ satisfies the identity $e^{-iy \cdot \gamma^*} P_k e^{iy \cdot \gamma^*} = P_{k+y^*}$, then for every $p \geq 1$, the function $k \mapsto \lambda_p(k)$ is $\Gamma^n$-periodic.

Standard perturbation theory shows that for every fixed $p$, $\lambda_p(k)$ are continuous functions on $k$ and $\lambda_p(k)$ is an even analytic function of $k$ near every point $k_0 \in \mathbb{T}^n$ such that $\lambda_p(k_0)$ is a simple eigenvalue of $P_{k_0}$. The function $\lambda_p(k)$ is called the band function, and the closed intervals $\Lambda_p = \lambda_p(\mathbb{T}^n)$ are called bands.

Consider the self-adjoint operator

$$P = -\Delta + V(x)$$

on $L^2(\mathbb{R}^n)$ with domain $H^2(\mathbb{R}^n)$. Under our assumptions on $V$, it is well known that the spectrum of $P$, $\sigma(P)$, consists of the bands $\Lambda_p$; these bands consist of purely absolutely continuous spectrum, see [35, Theorem XIII.90 and Theorem XIII.100].

Fix $\tau \in \sigma(P)$ and put $\mathcal{F}(\tau) = \{ k \in \mathbb{T}^n; \tau \in \sigma(P_k) \}$. We make the following hypothesis on the spectrum of the unperturbed operator:

(H1) for all $k_0 \in \mathcal{F}(\tau)$ and for all $n \in \mathbb{N}^*$ such that $\lambda_n(k_0) = \tau$, we assume that $\lambda_n(k_0)$ is a simple eigenvalue of $P_{k_0}$ and that $\nabla_k \lambda_n(k_0) \neq 0$.

We now consider the perturbed periodic Schrödinger operator:

$$P(h) = P + W(hx),$$
where \( W \in C^\infty(\mathbb{R}^n; \mathbb{R}) \). We assume that there exist positive constants \( a \) and \( C \) such that \( W \) extends analytically to \( \Gamma(a) := \{ z \in \mathbb{C}^n : |\Im(z)| \leq a(\Re(z)) \} \), and

\[
|W(z)| \leq C(z)^{-\tilde{n}}, \quad \text{uniformly on } z \in \Gamma(a), \tilde{n} > n, \tag{2.1}
\]

where \( \langle z \rangle = (1 + |z|^2)^{1/2} \). Here, \( \Re(z) \), \( \Im(z) \) denote, respectively, the real part and the imaginary part of \( z \).

Assumption (2.1) implies that for all \( f \in C^\infty_0(\mathbb{R}) \) the operator \( (f(P(h)) - f(P)) \) is of trace class, see Theorem B.1.

To formulate our first result, we need to introduce the resonances. Let us recall the definition. We follow essentially the presentation from [12] (see also [33]).

Let \( v \in C^\infty(\mathbb{R}^n; \mathbb{R}^n) \) be \( \star \)-periodic. For \( t \in \mathbb{R} \), we introduce the spectral deformation family \( U_t \) defined by

\[
U_t u(r) = F_h^{-1}\{J_t^{1/2}(\cdot) F_h v_t(\cdot))}(r) \forall u \in S(\mathbb{R}^n),
\]

where \( v_t(k) = k - tv(k) \), and \( J_t(k) \) is its Jacobian. Here, \( S(\mathbb{R}^n) \) is the space of rapidly decreasing test functions, and \( F_h \) is the semi-classical Fourier transform:

\[
F_h u(\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} x \cdot \xi} u(x) dx \quad \forall u \in S(\mathbb{R}^n).
\]

For \( t \in \mathbb{R} \), consider the family of unitarily equivalent operators

\[
P_i(t, h) = U_t P_i(h) U_t^{-1}, \quad i = 0, 1.
\]

Here, \( P_1(h) \) is given by formula (1.4). In [12, Proposition 2.8], it was shown that \( P_1(t, h) \) extends to an analytic type-A family of operators on \( D(t_0) = \{ t \in \mathbb{C} : |t| < t_0 \} \) with domain \( H^2(\mathbb{R}^n) \). Moreover, according to [12] (see also Section 3.2), there exist a neighborhood \( \tilde{\Omega} \) of \( \tau \) and a small positive constant \( \eta \) such that, for \( t \in D(t_0) \) with \( \Im t > 0 \) the spectrum of \( P_0(t, h) \) in \( \Omega_t := \{ z \in \tilde{\Omega} : \Im z > -\eta \Im t \} \) is empty, and the one of \( P_1(t, h) \) consists of discrete eigenvalues of finite multiplicities that lie in the lower half plane. These eigenvalues are \( t \)-independent under small variations of \( \Im t > 0 \) and are called resonances. We will denote the set of resonances by \( \text{Res}(P(h)) \).

Fix \( t \in D(t_0) \) with \( \Im t > 0 \), and let \( \Omega \subset \Omega_t \) be an open, simply connected and relatively compact set. Let \( \Omega_t \subset \subset \Omega \) be a simply connected set. We assume that \( I = \Omega \cap \mathbb{R} \) is an interval. Our first result is the following.
Theorem 1. Assume (2.1) and (H1). There exists a small positive constant $h_0$ such that, for all $\lambda \in I$ and all $h \in [0, h_0]$ we have

$$s'(\lambda, h) = \frac{1}{\pi} \text{Im} g(\lambda, h) - \frac{1}{\pi} \sum_{\omega \in \text{Res}(P(h))^c \Omega} \frac{\text{Im} \omega}{|\lambda - \omega|^2} + \sum_{\omega \in I \cap \text{Res}(P(h))} \delta(\lambda - \omega),$$  \hspace{1cm} (2.2)

where $g(z, h)$ is a function holomorphic in $\Omega$, and $g(z, h)$ satisfies the estimate

$$|g(z, h)| \leq C h^{-n},$$  \hspace{1cm} (2.3)

with a constant $C$ independent on $(z, h) \in \Omega_1 \times [0, h_0]$.

Before stating the assumptions and the results concerning the spectral shift function we need some notations. Let $\mu \in \sigma(P)$. For $k_0 \in \mathcal{F}(\mu)$, we let $m = \dim \ker(P_{k_0} - \mu)$ such that

$$\lambda_{n-1}(k_0) < \lambda_n(k_0) = \mu = \cdots = \lambda_{n+m-1}(k_0) < \lambda_{n+m}(k_0).$$  \hspace{1cm} (2.4)

Let $u_j := u_{j, k_0}$, $1 \leq j \leq m$ be an orthonormal basis of $\ker(P_{k_0} - \lambda_n(k_0))$. For $T \in \mathbb{R}^n$, we denote by $M_T(k_0)$ the matrix

$$M_T(k_0) = \left( \int_E u_j(x) \langle T, D_x + k_0 \rangle u_i(x) \, dx \right)_{1 \leq i, j \leq m}.$$

Here $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{R}^n$.

As in the study of the eigenvalues counting function for $P(h)$ (see [13]), we need the following assumption.

Assumption (H2). Let $r_0 \in \mathbb{R}^n$ such that $\nabla W(r_0) = 0$, and set $\mu = \tau - W(r_0)$. We assume that, for all $k_0 \in \mathcal{F}(\mu)$ there exist $T_0 \in \mathbb{R}^n$ and $C > 0$ such that

$$(M_{T_0}(k_0) \omega, \omega) \geq \frac{1}{C} \|\omega\|^2, \forall \omega \in \mathbb{C}^m.$$  \hspace{1cm} (2.5)

Theorem 2. Assume (2.1). Let $E_0 < E_1$ and suppose that for all $\tau \in [E_0, E_1]$, (H1) and (H2) hold. We have

$$s(\lambda, h) = \omega_0(\lambda) h^{-n} + O(h^{-n+1}), \ (h \downarrow 0),$$  \hspace{1cm} (2.6)

uniformly with respect to $\lambda \in [E_0, E_1]$. Here

$$\omega_0(\lambda) = \int \left( \rho(\lambda - W(x)) - \rho(\lambda) \right) \, dx,$$
where \( \rho(\lambda) = (2\pi)^{-n} \sum_{j \geq 1} \int_{\{k \in E^*; \lambda_j(k) \leq \lambda\}} dk \) is the integrated density of states associated to \( P \).

Finding asymptotics like (2.6) is complicated by the fact that unlike the eigenvalue counting function, the spectral shift function needs not be monotone, so Tauberian theorems do not apply. To overcome this problem, as in [31,6], we will use Theorem 1. More precisely, we will use the fact that \( s'(\lambda, h) - \frac{1}{\pi} \text{Im} g(\lambda, h) \) is positive in the distribution sense (see Lemma 3).

**Remark 1.** Notice that, condition (2.5) is only needed for \( r_0 \) such that \( \nabla W(r_0) = 0 \).

In the case where \( \lambda_n(k) \) is a simple eigenvalue of \( P_{k_0} \), assumption (H2) is equivalent to

\[
(\nabla \lambda_n(k_0), \nabla W(r_0)) \neq (0, 0).
\]

To see this, assume that \( m = 1 \) in (2.5). Then the normalized eigenfunction \( u_{n,k} \) corresponding to \( \lambda_n(k) \)

\[
\left((D_x + k)^2 + V(x) - \lambda_n(k)\right)u_{n,k}(x) = 0, \quad \|u_{n,k}\|_{L^2(E)} = 1,
\]

can be chosen analytic near \( k_0 \). Thus, taking the gradient with respect to \( k \) in the above equation and taking the inner product with \( u_{n,k} \), we get

\[
2 \int_E \frac{u_{n,k}(x)(D_x + k)u_{n,k}(x)}{\lambda_n(k)} dx = \nabla_k \lambda_n(k).
\]

Thus, (2.5) is equivalent to \( \nabla \lambda_n(k_0) \neq 0 \).

**Remark 2.** Clearly, if \( \tau \) satisfies (H1), then for \( \eta > 0 \) small enough each \( \mu \in ]\tau - \eta, \tau + \eta[ \) satisfies (H1). This together with the above remark show that, if \( \|W\|_\infty \) is small enough then (H1) implies (H2).

In the next result, we give a full asymptotic expansion in powers of \( h \) of \( s'(\lambda, h) \) when there are no resonances near \( ]E_0, E_1[ \).

**Theorem 3.** Assume (2.1) and suppose that for \( \tau \in ]E_0, E_1[ \), (H1) and (H2) hold. Assume in addition that there exist \( \delta \in [0, 1[ \) and \( C > 0 \) such that for \( h \) small enough

\[
\text{Res}(P(h)) \cap \left\{ x + iy \in \mathbb{C}; \ E_0 < x < E_1, -\frac{h^\delta}{C} \leq y \leq 0 \right\} = \emptyset. \quad (2.7)
\]
We have

\[ s'(\lambda, h) \sim \sum_{j=0}^{\infty} b_j(\lambda) h^{j-n}, \quad (h \searrow 0), \quad (2.8) \]

uniformly for \( \lambda \in [a, b] \subset ]E_0, E_1[ \), where

\[ b_0(\lambda) = \omega_0'(\lambda) = \int \frac{d\rho}{d\lambda} (\lambda - W(x)) - \frac{d\rho}{d\lambda} (\lambda) \, dx. \]

According to [12, Theorem 1.3], there exist an \( h \)-independent neighborhood \( \Omega \) of \([E_0, E_1]\) and a small positive constant \( \beta \), such that if \( \sup_{z \in \Gamma(a)} |W(z)| \leq \beta \) then \( \Omega \cap \text{Res}(P(h)) = \emptyset \). Combining this with Remark 2 and Theorem 3, we get

**Corollary 1.** Assume that for all \( \tau \in [E_0, E_1] \) assumption (H1) holds. There exists \( \varepsilon > 0 \) such that, if \( \sup_{z \in \Gamma(a)} |W(z)| \leq \varepsilon \) then the asymptotic expansion (2.8) holds uniformly with respect to \( \lambda \in [E_0, E_1] \).

**Remark 3.** We do not insist here on the computation of the coefficient \( b_j(\lambda) \) in the asymptotics (2.8), which is a separate question. We only insist on the existence of these asymptotics. For a computation of the coefficients, one only needs to compute explicitly the coefficients in the weak asymptotics given by Theorem B.1.

The second application which we shall make of Theorem 1 is to resonances. In the next theorem, we give an upper bound on the number of resonances in a disk of radius \( h \).

We denote by \#A the number of elements of A, counted with their multiplicity.

**Theorem 4.** Assume (2.1). Let \( \tau \in \sigma(P) \) be satisfying (H1) and (H2). There exist \( h_0 > 0 \) and \( C > 0 \) such that for all \( h \in ]0, h_0[ \), we have

\[ \# \{ z \in \text{Res}(P(h)); |z - \tau| \leq h \} \leq C h^{-n+1}. \quad (2.9) \]

Throughout the paper we use the notation \( [a_j]_{j=0}^1 = a_1 - a_0 \). As a consequence of Theorem 4, we obtain the following \( h \)-local trace formula (see [4,5]):

**Theorem 5.** Let \( \Omega \subset \mathbb{C} \) be an open, simply connected neighborhood of 0, such that \( \Omega \cap \mathbb{R} \) is connected. Let \( \chi \in C_0^\infty(\mathbb{R}; [0, 1]) \) satisfy

\[ \chi(x) = \begin{cases} 0, & d(\Omega \cap \mathbb{R}, x) > 2\varepsilon, \\ 1, & d(\Omega \cap \mathbb{R}, x) < \varepsilon \end{cases} \]
and let $f$ be holomorphic in a neighborhood of $\Omega_h := \tau + h\Omega$. Under the assumptions of Theorem 4 there exist $C$ and $h_0 > 0$ such that for any $h \in ]0, h_0]$, we have

$$
\tr\left( \left[ \chi_h(P_j(h) - \tau) f(P_j(h)) \right]_{j=0}^{1} \right) = \sum_{z \in \text{Res}(P(h)) \cap \Omega_h} f(z) + E_{\Omega, f, \tau},
$$

$$
E_{\Omega, f, \tau} \leq C h^{-n+1} \sup \{|f(z)|; 0 < d(z, \Omega_h) < 2\varepsilon h; \text{Im} \, z \leq 0\}, \quad (2.10)
$$

where $\chi_h(x) = \chi(x/h)$.

Outline of the paper: The paper is organized as follows. In the next section, we recall the effective Hamiltonian method, and we give a representation of the derivative of the spectral shift function. Section 4 is devoted to the proof of Theorem 1. Theorems 2 and 3 are proved in Sections 5 and 6. Section 7 is devoted to the proofs of Theorems 4 and 5. In Section 8, we establish the semi-classical version of the Breit–Wigner approximation. Finally, in an Appendix, we recall some results on the $h$-pseudodifferential calculus and on trace formulas for $P(h)$.

3. Effective Hamiltonian approximation

3.1. Definitions and notations

Let $H$ be a Hilbert space. The scalar product in $H$ will be denoted by $(\cdot, \cdot)$. The set of linear bounded operators from $H_1$ to $H_2$ is denoted by $\mathcal{L}(H_1, H_2)$.

For $m \in \mathbb{R}$ and $N \in \mathbb{N}$ we denote by $S^m(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(C))$ the space of $P \in C^\infty(\mathbb{R}_k^n ; \mathcal{M}_N(C))$, $\Gamma^*$-periodic with respect to $k$, such that for all $\alpha$ and $\beta$ in $\mathbb{N}^n$ there exists $C_{\alpha, \beta}$ such that

$$
\| \partial_{\alpha}^r \partial_{k}^\beta P(k, r) \|_{\mathcal{M}_N(C)} \leq C_{\alpha, \beta} (r)^{-m}, \quad (r) = (1 + |r|^2)^{1/2}, \quad (3.1)
$$

where $\mathcal{M}_N(C)$ is the set of $N \times N$-matrices.

If $P$ depends on a semi-classical parameter $h \in ]0, h_0]$ and possibly on other parameters as well, we require (3.1) to hold uniformly with respect to these parameters. For $h$ dependent symbols, we say that $P(k, r; h)$ has an asymptotic expansion in powers of $h$, and we write

$$
P(k, r; h) \sim \sum_{j=0}^{\infty} P_j(k, r) h^j,
$$

if for every $N \in \mathbb{N}, h^{-(N+1)}(P - \sum_{j=0}^{N} P_j h^j) \in S^m(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(C))$. 
For $P \in S^m(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))$, the $h$-Weyl operator $P = P^w(k, hD_k; h) = \text{Op}^w_h(P)$ is defined by

$$P^w(k, hD_k; h)u(k) = (2\pi h)^{-n} \int \int e^{i(k-y)r/h} P \left( \frac{k+y}{2}, r; h \right) u(y) dy dr.$$

### 3.2. Effective Hamiltonian

In this subsection, we describe the effective Hamiltonian method. The reader can find more details and the proofs of the results of this subsection in [12] (see also [23,19]).

Denote by $T^\gamma$ the distribution in $S'_{\mathbb{R}^{2n}}$ defined by

$$T^\gamma (r, x) = \frac{1}{\text{vol}(E) h^n} \sum_{\beta \in \Gamma^*} e^{i(r-hx)\beta/h}.$$

We introduce the following Hilbert spaces with their natural norms:

$$H_{m,k} = \{ u \in L^2(\mathbb{T}^n); (D_x + k)^2 u \in L^2(\mathbb{T}^n) \forall |x| \leq m \}$$

and

$$L^m = \{ u(r)T^\gamma (r, x); \hat{c}^2 u \in L^2(\mathbb{R}^n) \forall |x| \leq m \}.$$

Let us first recall the following result:

**Proposition 1** (*Helffer and Sjöstrand* [23, Theorem 3.1]). Fix $[a, b] \subset \mathbb{R}$. There exist $N \in \mathbb{N}$, a complex neighborhood $\Omega_0$ of $[a, b]$ and analytic functions $\Phi_j : \mathbb{T}^n \rightarrow H_{2,k}$, $1 \leq j \leq N$, such that for each $k \in \mathbb{T}^n$ and each $z \in \Omega_0$ the operator

$$\mathcal{P}(k, z) = \begin{pmatrix} (k+D_x)^2 + V(x) - z & R-(k) \\ R+(k) & 0 \end{pmatrix}$$

is bijective from $H_{2,k}(\mathbb{T}^n) \times \mathbb{C}^N$ into $L^2(\mathbb{T}^n) \times \mathbb{C}^N$ with an inverse

$$\mathcal{E}(k, z) = \begin{pmatrix} E_0(k, z) & E_{0,+}(k, z) \\ E_{0,-}(k, z) & E_{0,-+}(k, z) \end{pmatrix}$$

uniformly bounded with respect to $(k, z) \in \mathbb{T}^n \times \Omega_0$. Here $R_-(k) = \sum_{j=1}^N \Phi_j(\cdot, k)$.

Using that $\mathcal{P}(k, z)$ is a left and right inverse of $\mathcal{E}(k, z)$ as well as the fact that $R_-(k)$ is independent of $z$ we get the well-known results [23] (see also [13, Section 4])

$$z \in \sigma((k+D_x)^2 + V(x)) \iff 0 \in \sigma(E_{0,-+}(k, z)).$$
Combining this with the fact that \( \sigma(P_k) = \bigcup_{j=1}^{\infty} \{ \lambda_j(k) \} \), we get

\[
0 \in \sigma(E_{0,-+}(k, z)) \iff \exists j \in \mathbb{N}^*; \lambda_j(k) = z.
\] (3.4)

Let us now consider the operator \( P_1(t, h) \). We recall that \( P_1(t, h) = U_t P_t(h) U^{-1}_t \). A simple computation shows that

\[
P_1(t, h) = (v_t(h D_r))^2 + V \left( \frac{r}{h} \right) + W_t(r, h D_r; h),
\]

where \( W_t \in S^\alpha(\mathbb{T}^*n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})) \) (see [12, (2.25)]).

As pointed out in the introduction, when \( h \to 0 \) there are two spatial scales in the operator \( P_1(t, h) \): \( x = r/h \) and \( r \). This makes it possible to use the method of two-scale expansions (\( x = r/h \) and \( r \) are regarded as independent variables) (cf. [10,19,14,22]). To be more precise, consider the following operator on \( \mathcal{D}'(\mathbb{R}^n_r x) \):

\[
P_1(t, h) = (v_t(h D_r) + D_x)^2 + V(x) + W_t(r, h D_r; h).
\]

According to Lemma 3.2 of [12], \( P_1(t, h) \) acting on \( L^2(\mathbb{R}^n) \) with domain \( H^2(\mathbb{R}^n) \) is unitarily equivalent to \( P_1(t, h) \) acting on \( \mathbb{L}^0 \) with domain \( \mathbb{L}^2 \).

Thus, by considering \( P_1(t, h) \) as an \( h \)-pseudodifferential operator in the \( r \) variables with an operator-valued symbol:

\[
P_1(t, h) = \text{Op}_h^w \left( (v_t(k) + D_x)^2 + V(x) + W_t(r, k; h) \right),
\]

we can use Proposition 1 to construct the following Grushin problem for \( P_1(t, h) \):

**Proposition 2 (Dimassi [12, Theorem 3.7]).** Assume (2.1). Fix \( \tau \) in \( \sigma(P) \) and let \( t \in D(t_0), t_0 \) small. there exist \( N \in \mathbb{N} \) and a complex neighborhood \( \tilde{\Omega} \) of \( \tau \) such that for all \( z \in \tilde{\Omega} \) and \( 0 < h < h_0 \) small enough, the operator

\[
P_1(t, h) = \left( \begin{array}{cc}
P_1(t, h) - z & R_- (v_t(h D_r)) \circ \mathcal{F}_h^{-1} \\
\mathcal{F}_h \circ R^*_- (v_t(h D_r)) & 0
\end{array} \right) : \mathbb{L}^2 \times L^2(\mathbb{T}^*n; \mathbb{C}^N)
\rightarrow \mathbb{L}^0 \times L^2(\mathbb{T}^*n; \mathbb{C}^N)
\] (3.5)

is bijective with bounded two-side inverse

\[
E_1'(z, h) = \left( \begin{array}{cc}
E_1^l(z, h) & E_1^l(z, h) \\
E_1^l(z, h) & E_1^l(z, h)
\end{array} \right).
\]
Here, \( E^i_{1,-+}(z, h) = E^w_{1,-+}(k, hD_k, z, t, h) \) is an \( h \)-pseudodifferential operator with symbol
\[
E^j_{1,-+}(k, r, z, t, h) \sim \sum_{j=0}^{\infty} E^j_{1,-+}(k, r, z, t) h^j, \quad \text{in } S^0(\mathbb{T}^*n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})). \quad (3.6)
\]

We recall that \( \mathcal{F}_h \) is the semi-classical Fourier transform.

We denote by \( P^0_j(z, h) \) and \( E^0_j(z, h) \) the operators given in Proposition 2 for \( W = 0 \). In the following, we drop the exponent 0 in the \( P^j_0(z, h) \) and \( E^0_j(z, h), i = 0, 1 \). In particular we denote \( E^i_{1,-+}(z, h) \) instead of \( E^0_{1,-+}(z, h) \).

**Proposition 3** (Dimassi [12]). Let \( E^i_{j,-+}(z, h) := E^w_{j,-+}(k, hD_k, z, t, h) : L^2(\mathbb{T}^*n; \mathbb{C}^N) \to L^2(\mathbb{T}^*n; \mathbb{C}^N), \tilde{\Omega} \) and \( D(t_0) \) be as above. We have

(i) \[ z \in \sigma(P_j(t, h)) \iff z \in \sigma(P_j(t, h)) \iff 0 \in \sigma(E^i_{j,-+}(z, h)). \]

(ii) \[
\left[ E^i_{1,-+}(z, h) - E^i_{0,-+}(z, h) \right] \in \text{Op}_h^w(S^{\tilde{n}}(\mathbb{T}^*n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))), \quad (3.7)
\]

where \( \tilde{n} \) is given in (2.1).

(iii) For \( t \in D(t_0) \cap \mathbb{R} \), \( E^i_{j,-+}(z, h) \) is unitarily equivalent to \( E_{j,-+}(z, h) \) and satisfies
\[
(E^i_{j,-+}(z, h))^* = E^i_{j,-+}(\bar{z}, h). \quad (3.8)
\]

Moreover, for \( z \) non-real, \( E^i_{j,-+}(z, h)^{-1} \) exists and satisfies
\[
\| E^i_{j,-+}(z, h)^{-1} \| \leq C|\text{Im } z|^{-1}. \quad (3.9)
\]

(iv) \((z, t) \to E^i_{j,-+}(z, h)\), is holomorphic on \( \tilde{\Omega} \times D(t_0) \).

(v) Assume (H1), and fix \( t \in D(t_0) \cap \{ \text{Im } t > 0 \} \). Then there exists \( \eta > 0 \) such that:

(a) \( z \to E^{i-1}_{0,-+}(z, h) \) is analytic on \( \Omega_t := \{ z \in \tilde{\Omega}; \text{Im } z \geq -\eta \text{ Im } t \} \) and \( \| (E^i_{0,-+}(z, h))^{-1} \| = O(1) \) uniformly for \( z \in \Omega_t \).

(b) \( z \to E^{i-1}_{1,-+}(z, h) \) is analytic on \( z \in \tilde{\Omega} \cap \{ \text{Im } z > 0 \} \) and extends meromorphically to \( \Omega_t \).

The first assertion is proved in [12, Theorem 3.8]. For the proof of (3.7) we refer to [12, Lemma 4.2]. Assertion (iv) follows from Propositions 3.4 and 3.5 in [12].
To see (3.8), we observe that $\mathcal{P}_1^t(z, h)^* = \mathcal{P}_1^t(\zeta, h)$ when $t$ is real, which implies that $E_1^t(z, h)^* = E_1^t(\zeta, h)$. From this we deduce (3.8).

Inequality (3.9) follows from the well-known formula [23] (see also [13,19])

$$E_{i,-+}^t(z, h)^{-1} = R_+^t(v_t(k))(z - \mathcal{P}_1(t, h))^{-1} R_-(v_t(k))$$

and the fact that $\mathcal{P}_1(t, h)$ is unitarily equivalent to $P_1(h)$ for real $t$ as well as the fact that $R_-$ is bounded. For the proof of (v) we refer to [12, Section 4].

**Remark 4.** Notice that, Proposition 1 is useful for the computation of the symbols of the effective Hamiltonian. In particular, in the case where $t = 0$, the principal symbol of $E_{1,-+}(z, h)$ (resp. the symbol of $E_{0,-+}(z, h)$) is equal to $E_{0,-+}(k, z - W(r))$ (resp. $E_{0,-+}(k, z)$), see [12, Theorem 4.2].

### 3.3. Representation of the derivative of the spectral shift function

Let $E_{j,-+}^t(z, h)$ be the effective Hamiltonian given in Propositions 2 and 3. We recall that $E_{j,-+}(z, h)$ denotes $E_{j,-+}^0(z, h)$. In this subsection, we give a representation of $s'(\lambda, h)$ related to the effective Hamiltonian $E_{j,-+}(z, h)$.

Let $\Omega \subset \tilde{\Omega}$ be an open, simply connected and relatively compact set. Put $\Omega_{\pm} = \Omega \cap \{z \in \mathbb{C}; \pm \text{Im} z > 0\}$. We may assume that $I = \Omega \cap \mathbb{R}$ is an interval.

According to Proposition 3, $z \to \left[E_{j,-+}(z, h)\right]_{j=0}^1$ is holomorphic, it follows then from (3.7) and Lemma A.4 that

$$\left[\partial_z E_{j,-+}(z, h)\right]_{j=0}^1 \in \mathcal{O}_{ph}^w\left(S^q(\mathbb{T}^*n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))\right).$$

(3.10)

Obviously,

$$\left[E_{j,-+}^{-1}(z, h)\partial_z E_{j,-+}(z, h)\right]_{j=0}^1 = E_{0,-+}^{-1}(z, h)\left[\partial_z E_{j,-+}(z, h)\right]_{j=0}^1$$

$$+ E_{1,-+}^{-1}(z, h)\left[E_{0,-+}(z, h) - E_{1,-+}(z, h)\right]$$

$$\circ E_{0,-+}^{-1}(z, h)\partial_z E_{1,-+}(z, h).$$

(3.11)

It follows from (3.7), (3.10) and Theorem A.3 that each term of the r.h.s. of (3.11) is of trace class. Consequently, the cyclicity of the trace yields

$$\text{tr}\left(\left[E_{j,-+}^{-1}(z, h)\partial_z E_{j,-+}(z, h)\right]_{j=0}^1\right) = \text{tr}\left(\left[\partial_z E_{j,-+}(z, h)\right]_{j=0}^1 E_{0,-+}^{-1}(z, h)\right)$$

$$+ \text{tr}\left(\partial_z E_{1,-+}(z, h) E_{1,-+}^{-1}(z, h)\right)$$
\[ \times \left[ E_{0, \to} (z, h) - E_{1, \to} (z, h) \right] E_{0, \to}^{-1} (z, h) \]

\[ = \text{tr} \left( [\partial_z E_{j, \to} (z, h) E_{j, \to}^{-1} (z, h)]^1_{j=0} \right). \]  \hspace{1cm} (3.12)

On the other hand, from Theorem A.3 and (3.9) we deduce that

\[ \| [E_{j, \to}^{-1} (z, h) \partial_z E_{j, \to} (z, h)]^1_{j=0} \|_{\text{tr}} = \mathcal{O} (h^{-n} |\text{Im} z|^{-2}). \]  \hspace{1cm} (3.13)

Introduce the functions

\[ \sigma_\pm (z) = \text{tr} \left( [E_{j, \to}^{-1} (z, h) \partial_z E_{j, \to} (z, h)]^1_{j=0} \right), \quad z \in \Omega_\pm. \]

Using again that \( z \to E_{j, \to} (z, h) \) is holomorphic, we deduce from (3.8)

\[ \partial_z E_{j, \to} (z, h)^* = \partial_z E_{j, \to} (\bar{z}, h). \]

Exploiting the fact that \( \text{tr}(A) = \text{tr}(A^*) \) and using (3.8), (3.12) and the above equality we obtain

\[ \overline{\sigma_+ (z)} = \sigma_- (\bar{z}), \quad \text{for} \quad z \in \Omega_+. \]  \hspace{1cm} (3.14)

The following representation plays a crucial role in the proof of Theorem 1.

**Lemma 1.** In \( \mathcal{D}'(I) \), we have

\[ s' (\lambda, h) = \lim_{\varepsilon \searrow 0} \frac{i}{2\pi} \left( \sigma_+ (\lambda + i\varepsilon) - \overline{\sigma_+ (\lambda + i\varepsilon)} \right). \]

**Proof.** Let \( f \in C_{0}^\infty (I) \), and \( \tilde{f} \in C_{0}^\infty (\Omega) \) be an almost analytic extension of \( f \) such that \( \tilde{f}_R = f \) and

\[ \overline{\partial} \tilde{f} (z) = \mathcal{O} (|\text{Im} z|^N) \text{ for all } N \in \mathbb{N}, \quad \overline{\partial} = \frac{\partial}{\partial \bar{z}}. \]  \hspace{1cm} (3.15)

We refer to [15] for the existence of \( \tilde{f} \).

In Proposition 2.6 of [17], it was shown that

\[ \langle s', f \rangle := \text{tr} \left( [f (P_j (h))]^1_{j=0} \right) \]

\[ = -\frac{1}{\pi} \int_{\mathbb{C}} \overline{\partial} \tilde{f} (z) \text{tr} \left( [E_{j, \to}^{-1} (z, h) \partial_z E_{j, \to} (z, h)]^1_{j=0} \right) L(dz). \]  \hspace{1cm} (3.16)

Here \( L(dz) = dx \, dy \) denotes the Lebesgue measure on \( \mathbb{C} \).
Since \( \sigma_{\pm}(z) = \mathcal{O}(h^{-n}|\text{Im } z|^{-2}) \) and \( \bar{\partial} f(z) = \mathcal{O}(|\text{Im } z|^2) \), the integral in (3.16) converges. Thus the r.h.s. of (3.16) can be written as

\[
\langle s', f \rangle = \lim_{\varepsilon \searrow 0} \left( -\frac{1}{\pi} \int_{\text{Im } z > 0} \bar{\partial} f(z) \sigma_+(z + i\varepsilon) L(dz) + \int_{\text{Im } z < 0} \bar{\partial} f(z) \sigma_-(z - i\varepsilon) L(dz) \right).
\]

(3.17)

According to Proposition 3, \( \sigma_+(z + i\varepsilon) \) (resp. \( \sigma_-(z - i\varepsilon) \)) is holomorphic in \( \Omega_+ \) (resp. \( \Omega_- \)), Green’s formula then gives

\[
\langle s', f \rangle = \lim_{\varepsilon \searrow 0} \frac{i}{2\pi} \int f(\lambda) \left[ \sigma_+(\lambda + i\varepsilon) - \sigma_-(\lambda - i\varepsilon) \right] d\lambda.
\]

(3.18)

Thus Lemma 1 follows from (3.14) and (3.18). □

Lemma 2. For all \( t \in D(t_0) \cap \{ \text{Im } t \geq 0 \} \) and all \( z \in \Omega_+ \), we have

\[
\sigma_+(z) = \text{tr} \left( \left[ E^t_{j,-+}(z,h) - 1 \partial_z E^t_{j,-+}(z,h) \right]_{j=0} \right).
\]

(3.19)

Proof. According to Proposition 3, \( E^t_{j,-+}(z,h) - 1 \partial_z E^t_{j,-+}(z,h) \) is unitarily equivalent to

\[
E_{j,-+}(z,h) - 1 \partial_z E_{j,-+}(z,h)
\]

when \( t \in D(t_0) \cap \mathbb{R} \). Consequently, the cyclicity of the trace yields

\[
\sigma_+(z) = \text{tr} \left( \left[ E^t_{j,-+}(z,h) - 1 \partial_z E^t_{j,-+}(z,h) \right]_{j=0} \right)
\]

(3.20)

for all \( z \in \Omega_+ \), \( t \in D(t_0) \cap \mathbb{R} \).

Fix \( \delta > 0 \) and let \( z \in \Omega_\delta = \{ z \in \Omega ; \text{Im } z \geq \delta \} \). Since \( t \to E^t_{j,-+}(z,h) \) is analytic and \( E_{j,-+}(z,h) \) is uniformly invertible in \( z \in \Omega_\delta \) (see (3.9) and Proposition 3 (iv)), then \( t \to E^t_{j,-+}(z,h) - 1 \partial_z E^t_{j,-+}(z,h) \) is analytic in \( D(t_0) \), for \( t_0 \) small enough. This shows that the r.h.s. of (3.20) extends by analytic continuation in \( t \) to the disc \( D(t_0) \). By uniqueness of analytic continuation, (3.20) remains true for all \( z \in \Omega_\delta \) and all \( t \in D(t_0) \).

Now, fix \( t \in D(t_0) \) with \( \text{Im } t \geq 0 \). According to Proposition 3(v), both of the terms of (3.20) are analytic in \( \Omega_+ \) and coincides on \( \Omega_\delta \). Thus (3.20) remains true for all \( (z,t) \) in \( \Omega_+ \times D_+(t_0) \). □
4. Proof of Theorem 1

From now on, we fix \( t \in D(t_0) \) with \( 0 < \text{Im} \, t \), and we let \( \Omega_1 \subset \Omega \subset \Omega_t = \{ z \in \tilde{\Omega} : \text{Im} \, z \geq -\eta \text{Im} \, t \} \) be as in Theorem 1.

Now, as in the proof of Theorem 1.5 of [17], we will reduce the study of the r.h.s. of (3.19) to the study of a finite rank operator.

**Proposition 4.** There exists a finite rank operator \( K(z,h) \) from \( L^2(\mathbb{T}^n; \mathbb{C}^{N}) \) into its self such that: \( \text{rank}(K(z,h)) = O(h^{-n}) \), \( \| K(z,h) \| = O(1) \) and

\[
\sigma_+(z) = \text{tr}((I + K(z,h))^{-1} \delta_z K(z,h)) + k(z,h) \quad \forall z \in \Omega_+,
\]

where \( k(z,h) \) is holomorphic in \( \Omega \) and satisfies the estimate

\[
|k(z,h)| \leq C(\Omega) h^{-n},
\]

with \( C(\Omega) > 0 \) independent on \( h \in ]0, h_0[ \).

**Proof.** Let \( \chi \in C_0^\infty(\mathbb{R}; [0, 1]) \) be equal to 1 near \([-C, C] \), where \( C > 0 \) will be chosen later. Let us introduce the operator

\[
A = \chi^w(-h^2 \Delta_y) I_{\mathbb{C}^N} : L^2(\mathbb{T}^n; \mathbb{C}^{N}) \to L^2(\mathbb{T}^n; \mathbb{C}^{N}).
\]

For \( u = (u_j)_{1 \leq j \leq N} \in L^2(\mathbb{T}^n, \mathbb{C}^{N}) \) where \( u_j(y) = \sum_{\gamma \in \Gamma} c_{j,\gamma} e^{i\gamma \cdot y} \in L^2(\mathbb{T}^n) \), we have

\[
\chi^w(-h^2 \Delta_y) u(y) = \left( \sum_{\gamma \in \Gamma} c_{j,\gamma} (h^2 |\gamma|^2) e^{i\gamma \cdot y} \right)_{1 \leq j \leq N},
\]

which implies that \( \text{rank} \, A = \# \{ \gamma ; h^2 |\gamma|^2 \in \text{supp} \, \chi \} = O(h^{-n}) \).

Set

\[
K_1(z,h) = (\text{Id} - A) \left[ E_{1,++}^t(z,h) - E_{0,++}^t(z,h) \right] E_{0,++}^t(z,h)^{-1}
\]

and

\[
K_2(z,h) = A \left[ E_{1,++}^t(z,h) - E_{0,++}^t(z,h) \right] E_{0,++}^t(z,h)^{-1}.
\]

Since the symbol of \( \text{Id} - A \) is equal to 0 near \( \{ r ; |r|^2 \leq C \} \) and

\[
\left[ E_{1,++}^t(z,h) - E_{0,++}^t(z,h) \right] \in \text{Op}_h^w \left( \mathcal{S}_h^{\tilde{n}}(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})) \right),
\]

(4.2)
it follows from Theorem A.2 that
\[ \| (\text{Id} - A) \left[ E^t_{1,-+}(z, h) - E^t_{0,-+}(z, h) \right] \| = O(C^{-n/2}). \]

On the other hand, Proposition 3 shows that \( \| E^t_{0,-+}(z, h)^{-1} \| = O(1) \) uniformly for \( z \in \Omega \). Thus, for \( C \) large enough, \( z \to (I + K_1(z, h))^{-1} \) is well defined and analytic in \( \Omega \).

Now consider the operator
\[ K(z, h) = K_2(z, h)(\text{Id} + K_1(z, h))^{-1}. \]

Clearly, \( \text{rank}(K(z, h)) = O(h^{-n}) \) and \( \| K(z, h) \| = O(1) \).

Notice that, \( K_1 + K_2 + I = (I + K)(I + K_1) = E^t_{1,-+}(z, h)E^t_{0,-+}(z, h)^{-1} \). Thus,
\[ E^t_{1,-+}(z, h) = (I + K(z, h))(I + K_1(z, h))E^t_{0,-+}(z, h). \quad (4.3) \]

Differentiating the above equality with respect to \( z \) and multiplying it from the left by \( E^t_{1,-+}(z, h)^{-1} \), we get
\[
\left[ E^t_{j,-+}(z, h)^{-1} \partial_z E^t_{j,-+}(z, h) \right]_{j=0}^1
= E^t_{0,-+}(z, h)^{-1}(\text{Id} + K_1(z, h))^{-1} \partial_z K_1(z, h)E^t_{0,-+}(z, h)
+ E^t_{0,-+}(z, h)^{-1}(\text{Id} + K_1(z, h))^{-1}(\text{Id} + K(z, h))^{-1} \circ \partial_z K(z, h)(\text{Id} + K_1(z, h))E^t_{0,-+}(z, h).
\]

Using the cyclicity of the trace we obtain
\[ \sigma_{\pm}(z) = \text{tr}( \left( (I + K(z, h))^{-1}\partial_z K(z, h) \right) + \text{tr}( (I + K_1(z, h))^{-1}\partial_z K_1(z, h) )) \quad (4.4) \]

Since \( z \to K_1(z, h) \) is holomorphic, it follows from (4.2) and Lemma A.4 that \( \partial_z K_1(z, h) \in \text{Op}_h^{\mathfrak{w}}(S^\mathfrak{a}(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))) \). Combining this with Theorem A.3 we get \( \| \partial_z K_1(z, h) \|_{\mathfrak{w}} = O(h^{-n}) \). Hence, the last term in (4.4) is holomorphic in \( \Omega \) and is equal to \( O(h^{-n}) \), and the proof of Proposition 4 is complete. \( \square \)

Set
\[ D(z, h) := \det(1 + K(z, h)). \]

It follows from Proposition 3 (see also [12]) that the resonances of \( P(h) \) lie in the lower half plane and that \( z \) is a resonance of \( P(h) \) if and only if \( 0 \in \sigma(E^t_{1,-+}(z, h)) \).
with the same multiplicity. Combining this with the fact that \((I + K_1(z, h))E_{0, +}(z, h)\) is invertible, we deduce from (4.3) that the zeros of \(D(z, h)\) are the resonances of \(P(h)\) and that the multiplicities agree.

Let

\[
D(z, h) = G(z, h) \prod_{\omega \in \text{Res}(P(h)) \cap \Omega} (z - \omega),
\]

(4.5)

where \(G(z, h)\) and \(\frac{1}{G(z, h)}\) are holomorphic in \(\Omega\). According to [17, (4.31)]

\[
\left| \frac{\partial}{\partial z} \log(G(z, h)) \right| \leq C(\Omega_1)h^{-n},
\]

(4.6)

where \(\Omega_1 \subset \subset \Omega\) is an arbitrary open simply connected set.

Since

\[
\hat{\partial}_z \log D(z, h) = \text{tr}((I + K(z, h))^{-1} \hat{\partial}_z K(z, h)),
\]

it follows from (4.5), (4.6) and Proposition 4 that

\[
\sigma_+(z) = g_1(z, h) + \sum_{\omega \in \text{Res}(P(h)) \cap \Omega} \frac{1}{z - \omega}, \quad z \in \Omega_+,
\]

(4.7)

where \(g_1(z, h)\) is holomorphic in \(\Omega\) and satisfies

\[
|g_1(z, h)| \leq C(\Omega_1)h^{-n}, \quad z \in \Omega_1,
\]

(4.8)

with \(C(\Omega_1) > 0\) independent on \(h \in [0, h_0]\).

From (4.7) and Lemma 1, we have

\[
s'(\lambda, h) = \frac{i}{2\pi} \left( g_1(\lambda, h) - \overline{g_1(\lambda, h)} \right)
\]

\[
+ \frac{i}{2\pi} \sum_{\omega \in \text{Res}(P(h)) \cap \Omega} \lim_{\varepsilon \to 0^+} \left[ \frac{1}{\lambda + i\varepsilon - \omega} - \frac{1}{\lambda - i\varepsilon - \overline{\omega}} \right].
\]

(4.9)

Notice that, for \(\omega \in \Omega \cap \mathbb{R}\),

\[
\frac{i}{2\pi} \lim_{\varepsilon \to 0^+} \left( \frac{1}{\lambda + i\varepsilon - \omega} - \frac{1}{\lambda - i\varepsilon - \omega} \right) = \delta(\lambda - \omega),
\]
while for \( \omega \in \Omega \cap \{ \text{Im } w < 0 \}, \)

\[
\frac{i}{2\pi} \lim_{\varepsilon \to 0} \left( \frac{1}{\lambda + ie - \omega} - \frac{1}{\lambda - ie - \omega} \right) = -\frac{1}{\pi |\lambda - \omega|^2} \text{Im } \omega
\]

where the limits are taken in the sense of distributions. Thus, Theorem 1 follows from (4.8) and (4.9).

For future use we recall here an upper bound on the number of resonances obtained in [17] (see also [43–45]). For convenience we give the proof.

Since \( \text{rank} (K(z,h)) = O(h^{-n}) \) (see Proposition 4), it follows from [35, Chapter XIII.17, Lemma 4] that

\[
|D(z,h)| \leq e^{C(\Omega)h^{-n}}.
\]

Next, the proof of Lemma 2 shows that \( \|E^1_{1,-+}(z,h)^{-1}\| = O(1) \), uniformly for \( z \in \Omega_\delta = \Omega \cap \{ \text{Im } z \geq \delta \} \), which together with (4.3) yields \( (I + K(z,h))^{-1} = O(1) \) uniformly with respect to \( z \in \Omega_\delta \). Thus, \( \text{rank} \left( K(z,h)(I + K(z,h))^{-1} \right) = \mathcal{O}(h^{-n}) \). We write \( (I + K(z,h))^{-1} \) in the form

\[
(I + K(z,h))^{-1} = \left( I - K(z,h)(I + K(z,h))^{-1} \right)
\]

and we obtain

\[
|\det \left( (I + K(z,h))^{-1} \right)| \leq e^{Ch^{-n}}, \; z \in \Omega_\delta,
\]

which implies

\[
|D(z,h)| \geq C e^{Ch^{-n}},
\]

uniformly with respect to \( z \in \Omega_\delta \).

Now, applying the Jensen inequality in a slightly larger domain [42], we obtain

\[
\# \{z \in \text{Res}(P(h)) \cap \Omega \} \leq C(\Omega)h^{-n}.
\]  

(4.10)

5. Proof of Theorem 2

As stated in the introduction, in the case where \( V = 0 \), Theorem 2 was obtained in [39] under the assumption that \( J \) is a non-critical compact interval for \( p_1(x, \xi) = |\xi|^2 + W(x) \), (i.e., \( \nabla_{x,\xi} p_1(x, \xi) \neq 0, \forall (x, \xi) \in \{(y, \eta): p_1(y, \eta) \in J\} \)).
Let us explain briefly why one needs assumption (H2) and why it is difficult to apply the Robert’s approach [38] to the $h$-pseudodifferential operator $E_{1,-+}(z, h)$.

First of all, $z \rightarrow E_{1,-+}(z, h)$ is not linear and it is defined only in a bounded set $\Omega$. Hence, an evolution equation of the form

$$E_{1,-+}(hD_t, h)U(t) = 0, \ U(0) = I,$$

is not well-posed.

The second, more serious problem is due to the fact that, the principal symbol $E_{0,-+}^0(k, r, z)$ of $E_{1,-+}(z, h)$ has eigenvalues with non-constant multiplicity. Let us explain this.

According to (3.4) and Remark 4, the characteristic set $\Sigma_\tau$ of the principal symbol $E_{1,-+}(k, r, z) = E_{0,-+}(k, z - W(r))$ of $E_{1,-+}(z, h)$ is given by

$$\Sigma_\tau = \{(k, r) \in E^* \times \mathbb{R}^n; \ \det(E_{1,-+}^0(k, \tau - W(r))) = 0\}
= \bigcup_{j \geq 1} \{(k, r) \in E^* \times \mathbb{R}^n; \ \lambda_j(k) + W(r) = \tau\}.$$

In general, the set $\mathcal{G}$ of $(k_0, r_0) \in \Sigma_\tau$ such that $\dim \ker(P_{k_0} - \lambda_j(k_0)) > 1$ is nonempty. Near $(k_0, r_0) \in \mathcal{G}$, the eigenvalues of $E_{0,-+}^0(k, \tau - W(r))$ are degenerate. In this situation, it is quite difficult or impossible to construct explicitly the solutions of (5.1), (see [24]).

Thus, we are studying a system of $h$-pseudodifferential operators whose principal symbol has degenerate eigenvalues. In particular, near $(k_0, r_0) \in \mathcal{G}$, as in the works of [15, 25], one needs a microhyperbolicity condition for $E_{1,-+}^0(k, \tau - W(r))$. This will be assured by assumption (H2) (see [12]).

**Remark 5.** As $W$ is bounded, there exists $M > 0$ such that

$$P(h) \geq M. \quad (5.2)$$

The SSF is determined modulo a constant and from (5.2) we deduce that $s(\lambda, h)$ is constant on $]-\infty, M]$. In the following, without loss of generality, we may choose $s(\lambda, h)$ so that $s(\lambda, h) = 0$ on $]-\infty, M]$.

From now on, we assume the assumptions of Theorem 2. Fix $\tau \in [E_0, E_1]$. Applying Theorem 1 to $\tau$, we deduce that there exists two simply connected neighborhoods $\Omega_1 \subset \subset \Omega$ of $\tau$ such that for $\lambda \in I = \Omega \cap \mathbb{R}$, we have

$$s'(\lambda, h) = \frac{1}{\pi} \operatorname{Im} g(\lambda, h) - \frac{1}{\pi} \sum_{\omega \in \text{Res}(P(h)) \cap \Omega} \frac{\operatorname{Im} \omega}{|\lambda - \omega|^2} + \sum_{\omega \in I \cap \text{Res}(P(h))} \delta(\lambda - \omega), \quad (5.3)$$
where $g(z, h)$ is holomorphic in $\Omega$ and satisfies the estimate

$$|g(z, h)| \leq C h^{-n}, \quad (5.4)$$

with a constant $C$ independent on $(z, h) \in \Omega_1 \times [0, h_0]$. The proof of Theorem 2 will be a consequence of the two following lemmas:

**Lemma 3.** Let $\varepsilon$ be a small constant such that $]\tau - 2\varepsilon, \tau + 2\varepsilon[ \subset I$. Let $\varphi_1, \varphi_2 \in \mathcal{C}_0^\infty(\mathbb{R}; \mathbb{R}^+)$ be such that $\text{supp}(\varphi_1) \subset ]-\infty, \tau - \varepsilon[,$ $\text{supp}(\varphi_2) \subset ]\tau - 2\varepsilon, \tau + 2\varepsilon[,$ $\varphi_2 = 1$ on $]\tau - \varepsilon, \tau + \varepsilon[,$ and $\varphi_1 + \varphi_2 = 1$ near $[M, \tau + \varepsilon[.$ The constant $M$ is given by (5.2). Then for all $\lambda \in ]\tau - \varepsilon, \tau + \varepsilon[,$ we have

$$s(\lambda, h) = \text{tr} \left( \left[ \varphi_1(P_j(h)) \right]_{j=0}^1 \right) + M_{\varphi_2}(\lambda, h) + G_{\varphi_2}(\lambda, h), \quad (5.5)$$

where

$$G_{\varphi_2}(\lambda, h) = \int_{-\infty}^{\lambda} \varphi_2(\mu) \text{Im} g(\mu, h) \frac{d\mu}{\pi},$$

$$M_{\varphi_2}(\lambda, h) = \sum_{\omega \in \text{Res}(P(h)) \cap \Omega, \text{Im} \omega < 0} \int_{-\infty}^{\lambda} \varphi_2(\mu) \frac{\text{Im} \omega}{|\mu - \omega|^2} \frac{d\mu}{\pi} + \sum_{\omega \in \text{Res}(P(h)) \cap ]\tau - 2\varepsilon, \lambda]} \varphi_2(\omega). \quad (5.6)$$

**Proof.** The proof is similar to the one in [6, Lemma 2]. Let $f \in \mathcal{C}_0^\infty(]\tau - \varepsilon, \tau + \varepsilon[)$ and set

$$F(\lambda) = \varphi_1(\lambda) \int_{\lambda}^{+\infty} f(\mu) \frac{d\mu}{\lambda} + \varphi_2(\lambda) \int_{-\infty}^{\lambda} f(\mu) \frac{d\mu}{\lambda} =: F_1(\lambda) + F_2(\lambda).$$

Since $\text{supp} f \subset ]\tau - \varepsilon, \tau + \varepsilon[,$ and $(\varphi_1 + \varphi_2)(\lambda) = 1$ on $]\tau - \varepsilon, \tau + \varepsilon[,$ we have

$$F'(\lambda) = -f(\lambda) + (\varphi_1' + \varphi_2')(\lambda) \int_{-\infty}^{+\infty} f(\mu) \frac{d\mu}{\lambda},$$

where the last term vanishes on $[M, +\infty[.$ Our choice of $s(\cdot, h)$ (see Remark 5) allows us to write

$$\langle s(\cdot, h), f \rangle = -\langle s(\cdot, h), F' \rangle = \langle s'(\cdot, h), F \rangle. \quad (5.7)$$
By construction, $\varphi_1(\lambda) = 0$ on $]\tau - \varepsilon, +\infty[$. Consequently, $F_1(\lambda) = \varphi_1(\lambda) \int_{\lambda}^{+\infty} f(\mu) \, d\mu = \varphi_1(\lambda) \int_{\mathbb{R}} f(\mu) \, d\mu$, which implies

$$
\langle s'(\cdot, h), F_1 \rangle = \langle \int_{\mathbb{R}} f(\mu) \, d\mu \rangle \langle s'(\cdot, h), \varphi_1(\cdot) \rangle
= \left( \int_{\mathbb{R}} f(\mu) \, d\mu \right) \text{tr} \left( \varphi_1(P_1(h)) - \varphi_1(P_0(h)) \right). \tag{5.8}
$$

For the term involving $\varphi_2$, we apply (5.3) and we integrate by parts, we get

$$
\langle s'(\cdot, h), F_2 \rangle = \langle M_{\varphi_2}(\cdot, h), f \rangle + \langle G_{\varphi_2}(\cdot, h), f \rangle
$$

which together with (5.7) and (5.8) give (5.5).

Lemma 4. There exists a small positive constant $c$ such that for $\varphi \in C_c^\infty(]-c, c[; \mathbb{R})$ satisfying $F_{-1}^{-1}\theta(\tau) \geq 0$, $F_{-1}^{-1}\theta(0) > 0$ and $\theta(0) = 1$, we have

$$
F_{-1}^{-1}\theta * G_{\varphi_2}(\lambda, h) = G_{\varphi_2}(\lambda, h) + O(h^{-n+1}), \tag{5.9}
$$

$$
F_{-1}^{-1}\theta * M_{\varphi_2}(\lambda, h) = M_{\varphi_2}(\lambda, h) + O(h^{-n+1}), \tag{5.10}
$$

as $h \searrow 0$ uniformly for $\lambda \in \mathbb{R}$. Moreover, there exists $k(\lambda) \in C_c^\infty(\mathbb{R})$ such that

$$
M_{\varphi_2}(\lambda, h) + G_{\varphi_2}(\lambda, h) = k(\lambda)h^{-n} + O(h^{-n+1}), \quad h \searrow 0 \tag{5.11}
$$

uniformly for $\lambda \in \mathbb{R}$. Here

$$
F^{-1}\theta(\tau) = (2\pi)^{-1} \int_{\mathbb{R}} e^{it\tau} \theta(t) \, dt, \quad \text{and} \quad F_{-1}^{-1}\theta(\tau) = \frac{1}{h} F^{-1}\theta \left( \frac{\tau}{h} \right).
$$

First we are going to prove Theorem 2 assuming Lemma 4.

Proof of Theorem 2. Let $]\tau - \varepsilon, \tau + \varepsilon[$ be as in Lemma 3. By Theorem B.1, the first term of the r.h.s. of (5.5) has an asymptotic expansion in powers of $h$. Combining this with (5.11) we get

$$
 s(\lambda, h) = \omega_0(\lambda)h^{-n} + O(h^{-n+1}), \quad \text{as} \quad h \searrow 0 \tag{5.12}
$$

uniformly with respect to $\lambda \in ]\tau - \varepsilon, \tau + \varepsilon[$. Then (2.6) follows by applying the local result (5.12) and covering the compact interval $[E_0, E_1]$ by a finite number of small intervals.
The expression of $\omega_0(\lambda)$ follows from the weak convergence result given by Theorem B.1. □

To prove Lemma 4 we need the following result:

**Lemma 5.** Let $\theta \in C_0^\infty(\mathbb{R})$ and let $f$ be a $C^\infty$ function in $\mathbb{R}$, depending on a parameter $h \in ]0, h_0]$. We suppose that, there exist $\delta \in [0,1[$ and $m \in \mathbb{R}$ such that for all $k \in \mathbb{N},$

$$
\left( \frac{\partial}{\partial \lambda} \right)^k f(\lambda, h) = \mathcal{O}(h^{-\delta k+m}), \text{ as } h \downarrow 0 \text{ uniformly for } \lambda \in \mathbb{R}.
$$

(5.13)

We have

$$
\mathcal{F}^{-1}_{\theta} f(\lambda, h) = \sum_{k=0}^{N} \frac{(-ih)^k}{k!} \theta^{(k)}(0) \left( \frac{\partial}{\partial \lambda} \right)^k f(\lambda, h) + \mathcal{O}(h^{N(1-\delta)+m}), \text{ as } h \downarrow 0
$$

(5.14)

uniformly with respect to $\lambda \in \mathbb{R}$. In particular, if $\theta = 1$ near zero, then

$$
\mathcal{F}^{-1}_{1} f(\lambda, h) = f(\lambda, h) + \mathcal{O}(h^\infty).
$$

(5.15)

**Proof.** By a change of variable, we have

$$
\mathcal{F}^{-1}_{\theta} f(\lambda, h) = \int_{\mathbb{R}} \mathcal{F}^{-1}_{\theta(t)} f(\lambda - ht, h) \, dt.
$$

(5.16)

Applying Taylor’s formula to the function $t \rightarrow f(\lambda - ht, h)$ at $t = 0$, and using (5.13), we get

$$
f(\lambda - ht, h) = \sum_{k=0}^{N} f^{(k)}(\lambda, h) \frac{(-ht)^k}{k!} + \mathcal{O}(h^{N(1-\delta)+m} t^N).
$$

(5.17)

Inserting (5.17) in (5.16) and using that $\int_{\mathbb{R}} (-it)^k \mathcal{F}^{-1}_{\theta(t)} \, dt = \theta^{(k)}(0)$ we obtain (5.14). □

**Proof of Lemma 4.** From Gauchy’s inequalities and estimate (5.4) it follows that

$$
\frac{\partial}{\partial z} g(z, h) = \mathcal{O}(h^{-n}), \text{ uniformly for } z \in \Omega_1 \subset \subset \Omega
$$
which implies
\[ \partial_\lambda^k G_{\varphi_2}(\lambda, h) = \mathcal{O}(h^{-n}), \quad (5.18) \]
uniformly with respect to \( \lambda \in \mathbb{R} \). Thus, Lemma 5 applies (with \( f(\lambda, h) = G_{\varphi_2}(\lambda, h) \), \( m = n, \delta = 0 \)) and yields \( (5.9) \).

Similarly, using \( (5.18) \) and applying Lemma 5 to \( \partial_\lambda G_{\varphi_2}(\lambda, h) \), we obtain
\[ \partial_\lambda (\mathcal{F}_h^{-1} \theta \ast G_{\varphi_2})(\lambda, h) = \partial_\lambda G_{\varphi_2}(\lambda, h) + \mathcal{O}(h^{-n+1}). \quad (5.19) \]

In order to prove \( (5.10) \), we introduce
\[ S_{\varphi_2}(\lambda, h) = G_{\varphi_2}(\lambda, h) + M_{\varphi_2}(\lambda, h). \]

By the expressions of \( G_{\varphi_2}(\lambda, h) \) and \( M_{\varphi_2}(\lambda, h) \), we have
\[ S_{\varphi_2}'(\lambda, h) = \varphi_2(\lambda)s'(\lambda, h). \]

On the other hand, the definition of \( s'(\lambda, h) \) yields
\[ \mathcal{F}_h^{-1} \theta \ast (\varphi_2 s')(\lambda, h) = (\mathcal{F}_h^{-1} \theta(\lambda - \cdot), \varphi_2(\cdot)s'(\cdot, h)) \]
\[ = \text{tr} \left( \left[ \mathcal{F}_h^{-1} \theta(\lambda - P_j(h))\varphi_2(P_j(h)) \right]_{j=0}^1 \right). \quad (5.20) \]

Putting together the two above equalities and using Theorem B.2, we obtain
\[ (\mathcal{F}_h^{-1} \theta h \ast S_{\varphi_2})'(\lambda, h) = \varphi_2(\lambda)c(\lambda)h^{-n} + \mathcal{O}(h^{-n+1}/(\lambda)^2), \text{ as } (h \searrow 0), \quad (5.21) \]
uniformly for \( \lambda \in \mathbb{R} \), where \( c(\lambda) \) is \( C^\infty \) in \( \lambda \in ]-2\varepsilon, \tau + 2\varepsilon[ \).

From \( (5.18) \), \( (5.19) \) and \( (5.21) \), we deduce that
\[ (\mathcal{F}_h^{-1} \theta M_{\varphi_2})'(\lambda, h) = \mathcal{O}(h^{-n}), \quad (5.22) \]
uniformly with respect to \( \lambda \in \mathbb{R} \). On the other hand, \( (5.6) \) implies
\[ M_{\varphi_2}(\lambda, h) \leq \#\{\omega; \omega \in \text{Res}P(h) \cap \Omega\} \sup |\varphi_2(\mu)| \left[ 1 + \int_{-\infty}^{\lambda} |\text{Im} w| \frac{du}{|u - \omega|^2} \right], \]
which together with \( (4.10) \) and the obvious estimate
\[ \left| \int_{-\infty}^{\lambda} \frac{|\text{Im} w|}{|u - \omega|^2} \, du \right| \leq \int_{-\infty}^{\infty} \frac{1}{u^2 + 1} = \frac{1}{\pi}, \]
yield
\[
M_{\phi_2}(\lambda, h) = \mathcal{O}(h^{-n}).
\] (5.23)

Now, (5.10) follows from (5.22) and (5.23) by applying a Tauberian theorem [37, Theorem V-13] to the monotone function \(M_{\phi_2}(\lambda, h)\).

Let us prove (5.11). It follows from (5.18) and (5.23) that
\[
S_{\phi_2}(\lambda, h) = \mathcal{O}(h^{-n}) \quad \text{uniformly for } \lambda \in \mathbb{R}.
\]
By Lebesgue dominated convergence, we get
\[
F^{-1}_h \ast S_{\phi_2}(-\infty, h) = 0.
\]
Combining this with (5.21), we obtain
\[
F^{-1}_h \ast S_{\phi_2}(\lambda, h) = \int_{-\infty}^{\lambda} \varphi_2(\mu) c(\mu) d\mu h^{-n} + \mathcal{O}(h^{-n+1}).
\] (5.24)

which together with (5.9) and (5.10) give (5.11). We recall that
\[
S_{\phi_2}(\lambda, h) = G_{\phi_2}(\lambda, h) + M_{\phi_2}(\lambda, h).
\]

6. Proof of Theorem 3

Fix \(\tau \in \{E_0, E_1\}\). Applying Theorem 1 to \(\tau\) and taking into account (2.7), we deduce that there exists two simply connected neighborhoods \(\Omega_1 \subset \subset \Omega\) of \(\tau\) such that for \(\lambda \in I = \Omega \cap \mathbb{R}\), we have

\[
s'(\lambda, h) = \frac{1}{\pi} \text{Im} \ g(\lambda, h) - \frac{1}{\pi} \sum_{\omega \in \text{Res}(P(h)) \cap \Omega, \text{Im} \omega < -\frac{h^0}{c}} \frac{\text{Im} \ \omega}{|\lambda - \omega|^2},
\] (6.1)

where \(g(z, h)\) is holomorphic in \(\Omega\) and satisfies the estimate

\[
|g(z, h)| \leq C h^{-n},
\] (6.2)
with a constant \(C\) independent on \((z, h) \in \Omega_1 \times [0, h_0]\).

Choose \(\varepsilon > 0\) small enough such that \([\tau - \varepsilon, \tau + \varepsilon] \subset I\). Let \(\varphi \in C^\infty_0(I)\) be equal to 1 near \([\tau - \varepsilon, \tau + \varepsilon]\), and let \(\theta \in C^\infty_0([-c, c[, \mathbb{R})\) with \(\theta = 1\) on \([-\xi, \xi]\).

For \(\omega \in \Omega\) with \(\text{Im} \ \omega < -\frac{h^0}{c}\), consider the function

\[
f_\omega(\lambda) = -\varphi(\lambda) \frac{\text{Im} \ \omega}{|\lambda - \omega|^2}.
\]

Obviously, for \(p \in \mathbb{N}\) we have

\[
\hat{f}_\lambda^p f_\omega(\lambda) = \mathcal{O}(h^{-p \delta - \delta}).
\]
Since \( \theta = 1 \) near 0, it follows from (5.15) that

\[
\mathcal{F}_h^{-1} \theta \ast f_{a}(\lambda) = f_{a}(\lambda) + O(h^\infty), \quad \text{as } h \downarrow 0 \text{ uniformly for } \lambda \in \mathbb{R}.
\]  

(6.3)

Applying again Gauchy's inequalities to \( g(z,h) \), and using (6.2) we deduce that

\[
\mathcal{F}_h^{-1} \theta \ast (\varphi \text{ Im } g)(\lambda, h) = \varphi(\lambda) \text{ Im } g(\lambda, h) + O(h^\infty), \quad \text{as } h \downarrow 0 \text{ uniformly for } \lambda \in \mathbb{R}.
\]  

(6.4)

It follows from (6.1), (6.3), (6.4), (5.20) and (4.10) that

\[
\text{tr}
\left[
\left[ \mathcal{F}_h^{-1} \theta(\lambda - P_j(h))\varphi(P_j(h)) \right]_{j=0}^1
\right] = \mathcal{F}_h^{-1} \theta \ast (\varphi s')(\lambda, h) = \varphi(\lambda)s'(\lambda, h) + O(h^\infty),
\]  

(6.5)

uniformly with respect to \( \lambda \in \mathbb{R} \).

Now, Theorem B.2 and (6.5) give (2.8) uniformly for \( \lambda \in [\tau - \varepsilon, \tau + \varepsilon] \). Then, Theorem 3 follows by applying the above local result and covering the compact interval \([a,b]\) by small intervals.

**Remark 6.** When the perturbation \( W \) is non globally analytic it is important to replace the condition (2.7) by the following one, see [9,30,40]: \( \forall N \in \mathbb{N} \exists h(N) \) such that

\[
\text{Res}(P(h)) \cap \left\{ z = x + iy \in \mathbb{C}; \; E_0 < x < E_1, \; -Nh \log \left( \frac{1}{h} \right) \leq y \leq 0 \right\} = \emptyset \text{ for all } h \in \mathbb{N}, h(N) \}.
\]

Let us show why Theorem 3 remains true under the above condition. We use the same notations as in the proof of Theorem 3. As \( \mathcal{F}_h^{-1} \theta(\frac{2a}{(\gamma^2 + a^2)})(\mu) = e^{-a|\mu|/h}, (a > 0) \), we have

\[
\left( \mathcal{F}_h^{-1} \theta \ast \frac{-\text{Im } w}{|\cdot - w|^2} \right)(\mu) = \int e^{i(\mu - \text{Re } \omega)t/h} e^{-|\text{Im } \omega t|/h} \frac{dt}{2h}.
\]

Using that \( \theta' = 0 \) near \([-c/2, c/2]\), we get after an integration by parts

\[
\left( \mathcal{F}_h^{-1} \theta \ast \frac{-\text{Im } w}{|\cdot - w|^2} \right)(\mu) = \frac{-\text{Im } \omega}{|\mu - \omega|^2} + O\left( \frac{1}{|\text{Im } \omega|} e^{-\frac{c}{2h}|\text{Im } \omega|} \right).
\]
Since for all resonances $w \in \Omega$ we have the lower bound $|\text{Im } w| \geq Nh \log(1/h)$, we may estimate the exponent involving $|\text{Im } w|$. Thus we conclude that, $\forall N \in \mathbb{N}$ $\exists h(N) > 0$ such that

$$\left(\mathcal{F}_h^{-1} \theta * -\frac{\text{Im } w}{|\lambda - w|^2}\right)(\lambda) = -\frac{\text{Im } w}{|\lambda - w|^2} + O\left(\frac{h^{\frac{n}{2}}}{h \log(1/h)}\right) \text{ for all } h \in [0, h(N)].$$

Now we proceed as in the proof of Theorem 3 by exploiting (6.6) instead of (6.3). □

7. Proofs of Theorems 4 and 5

7.1. Proof of Theorem 4

Let $\theta$ and $\varphi_2 = \varphi$ be as in Lemma 4. Since $\mathcal{F}^{-1} \theta(0) > 0$, there exists $\varepsilon, \eta > 0$ such that $\mathcal{F}^{-1} \theta(t) > \varepsilon$ for all $|t| \leq \eta$. Consequently,

$$M_{\varphi}(\lambda + \eta h, h) - M_{\varphi}(\lambda - \eta h, h)$$

$$= \int_{\lambda - \eta h}^{\lambda + \eta h} dM_{\varphi}(t, h) \leq \frac{1}{\varepsilon} \int_{\lambda - \eta h}^{\lambda + \eta h} \mathcal{F}^{-1} \theta \left(\frac{\lambda - t}{h}\right) dM_{\varphi}(t, h)$$

$$\leq \frac{h}{\varepsilon} \int_{\mathbb{R}} \mathcal{F}_h^{-1} \theta(t) dM_{\varphi}(t, h)$$

$$= \frac{h}{\varepsilon} \frac{d}{d\lambda} \mathcal{F}_h^{-1} \theta * M_{\varphi}(\lambda, h),$$

where we used that $\mathcal{F}_h^{-1} \theta(t) \geq 0$ for all $t$. Combining this with estimate (5.22), we get

$$M_{\varphi}(\lambda + \eta h, h) - M_{\varphi}(\lambda - \eta h, h) = O(h^{-n+1}),$$

uniformly with respect to $\lambda \in \mathbb{R}$. Next, divide $[\tau - 2h, \tau + 2h]$ into $N(\eta)$ subintervals of length $2\eta$ and apply the above equality to each of them we obtain

$$M_{\varphi}(\tau + 2h, h) - M_{\varphi}(\tau - 2h, h) = O(h^{-n+1}).$$

(7.1)

We now claim that

$$M_{\varphi}(\tau + 2h, h) - M_{\varphi}(\tau - 2h, h) \geq \frac{1}{2} \#\{z \in \text{Res}(P(h)); \ |z - \tau| \leq h\},$$

which together with (7.1) give Theorem 4.
Let us prove (7.2). From (5.6), we have

\[ M_\varphi(\tau + 2h, h) - M_\varphi(\tau - 2h, h) \]

\[ = \sum_{\omega \in \Omega \cap \text{Res}(P(h)) \cap \text{Im} \omega < 0} \int_{\tau - 2h}^{\tau + 2h} \frac{-\text{Im} \omega}{|\mu - \omega|^2} \frac{d\mu}{\pi} + \sum_{\omega \in \text{Res}(P(h)); \tau - 2h \leq \omega < \tau + 2h} \varphi(\omega) \]

\[ = \sum_{\omega \in \Omega \cap \text{Res}(P(h)) \cap \text{Im} \omega < 0} \int_{\tau - 2h}^{\tau + 2h} \frac{-\text{Im} \omega}{|\mu - \omega|^2} \frac{d\mu}{\pi} + \#\{\omega; \omega \in \text{Res}(P(h)) \cap [\tau - 2h, \tau + 2h]\}. \]

(7.3)

We recall that \( \varphi = 1 \) near \( \tau \).

Using that \(-\frac{\text{Im} \omega}{|\mu - \omega|^2} > 0\), we obtain

\[ \geq \sum_{\omega \in \Omega \cap \text{Res}(P(h)) \cap \text{Im} \omega < 0, |\omega - \tau| < h} \int_{\tau - 2h}^{\tau + 2h} \frac{-\text{Im} \omega}{|\mu - \omega|^2} \frac{d\mu}{\pi} = \sum_{\omega \in \Omega \cap \text{Res}(P(h)) \cap \text{Im} \omega < 0, |\omega - \tau| < h} \int_{\tau - \text{Re} \omega - 2h}^{\tau + \text{Re} \omega + 2h} \frac{1}{|\mu - \omega|^2} \frac{d\mu}{\pi} \]

\[ \geq \sum_{\omega \in \Omega \cap \text{Res}(P(h)) \cap \text{Im} \omega < 0, |\omega - \tau| < h} \int_{-1}^{1} \frac{1}{u^2 + 1} \frac{du}{\pi} \geq \frac{1}{2} \sum_{\text{Res}(P(h)); \text{Im} z < 0, |z - \tau| < h} \#\{z \in \text{Res}(P(h)); \text{Im} z < 0, |z - \tau| < h\}, \]

(7.4)

where we used the fact that \( \tau - \text{Re} \omega + 2h > |\text{Im} \omega| \) and \( \tau - \text{Re} \omega - 2h < -|\text{Im} \omega| \) if \( |\tau - \omega| < h \) and that \( \int_{-1}^{1} \frac{1}{u^2 + 1} \frac{du}{\pi} = \frac{1}{2} \).

Combining (7.3) and (7.4) we obtain (7.2). \( \square \)

7.2. Proof of Theorem 5

For \( h \) small enough we can use representation (5.3) and get

\[ \text{tr}\left( \left[ \chi_h(P_j(h) - \tau) f(P_j(h)) \right]_{j=0}^{1} \right) = \langle \chi_h(\cdot - \tau) f, s'(\cdot, h) \rangle \]

\[ = \int \text{Im} g(\lambda, h) \chi_h(\lambda - \tau) f(\lambda) \frac{d\lambda}{\pi} \]

\[ + \sum_{\omega \in \Omega \cap \text{Res}(P(h))} \chi_h(\omega - \tau) f(\omega) \]

\[ + \sum_{\omega \in \text{Res}(P(h)) \cap \Omega \cap \text{Im} \omega < 0} \int \frac{-\text{Im} \omega}{|\lambda - \omega|^2} \chi_h(\lambda - \tau) f(\lambda) \frac{d\lambda}{\pi}. \]

(7.5)
Let \( \tilde{\chi} \) be an almost analytic extension of \( \chi \) satisfying \( \tilde{\chi} = 1 \) on \( \Omega \) and

\[
\supp(\partial_z \tilde{\chi}) \subset \{ z; \ v \leq d(z, \Omega) \leq 2v \}. \tag{7.6}
\]

Fix \( \omega \in \Omega \) with \( \Im \omega < 0 \). An application of Green’s formula yields

\[
-\frac{1}{\pi} \int_{C^-} \partial_z \tilde{\chi}(z) \frac{1}{z - \omega} L(dz) + \tilde{\chi}(\omega) = -\frac{1}{2\pi i} \int_{\mathbb{R}} \tilde{\chi}(\lambda) \frac{1}{\lambda - \omega} d\lambda
\]

and

\[
-\frac{1}{\pi} \int_{C^-} \partial_z \tilde{\chi}(z) \frac{1}{z - \omega} L(dz) = -\frac{1}{2\pi i} \int_{\mathbb{R}} \tilde{\chi}(\lambda) \frac{1}{\lambda - \omega} d\lambda.
\]

Applying the above formulas to the last term in equality (7.5) and using that \( \tilde{\chi}(\lambda) = \chi(\lambda) \) for \( \lambda \in \mathbb{R} \), we obtain

\[
\sum_{\omega \in \Res(P(h)) \cap \Omega, \ \Im \omega < 0} \int \frac{-\Im \omega}{|\lambda - \omega|^2} \chi_h(\lambda - \tau) f(\lambda) \frac{d\lambda}{\pi}
\]

\[
= \sum_{\omega \in \Res(P(h)) \cap \Omega, \ \Im \omega < 0} \tilde{\chi} \left( \frac{\omega - \tau}{h} \right) f(\omega)
\]

\[
+ \sum_{\omega \in \Res(P(h)) \cap \Omega, \ \Im \omega < 0} \frac{1}{\pi h} \int_{C^-} (\partial_z \tilde{\chi})(z - \tau / h) f(z)
\]

\[
\times \left( \frac{1}{z - \omega} - \frac{1}{z - \omega} \right) L(dz). \tag{7.7}
\]

It follows from Theorem 4 and (7.6) that

\[
\sum_{\omega \in \Res(P(h)) \cap \Omega, \ \Im \omega < 0} \tilde{\chi} \left( \frac{\omega - \tau}{h} \right) f(\omega)
\]

\[
= \sum_{\omega \in \Res(P(h)) \cap \Omega, \ \Im \omega < 0} f(\omega)
\]

\[
+ \mathcal{O}(h^{-n+1}) \sup \{|f(z)|; 0 < d(z, \Omega_h) \leq 2\varepsilon h, \ \Im z \leq 0 \}.
\]
On the other hand, since $\chi(\lambda) = \bar{\chi}(\lambda)$ for $\lambda \in \mathbb{R}$ then

$$\sum_{\omega \in \text{Res}(P(h)) \cap \Omega \atop \text{Im} \omega < 0} \tilde{\chi} \left( \frac{\omega - \tau}{h} \right) f(\omega) = \sum_{\omega \in \text{Res}(P(h)) \cap \Omega \atop \text{Im} \omega < 0} \tilde{\chi} \left( \frac{\omega - \tau}{h} \right) f(\omega)$$

$$+ \sum_{\omega \in \text{Res}(P(h)) \cap I} \chi \left( \frac{\omega - \tau}{h} \right) f(\omega).$$

In View of (7.5), (7.6) and the two above equalities, the proof of Theorem 5 is reduced to show that

$$\sum_{\omega \in \text{Res}(P(h)) \cap \Omega \atop \text{Im} \omega < 0} \tilde{\chi} \left( \frac{\omega - \tau}{h} \right) f(\omega) \leq \sum_{\omega \in \text{Res}(P(h)) \cap \Omega \atop \text{Im} \omega < 0} \tilde{\chi} \left( \frac{\omega - \tau}{h} \right) f(\omega).$$

Due to (7.6) and (5.4), the integral of the r.h.s. is estimated by

$$h^{-n-1} \sup \{|f(z)|; 0 < d(z, \Omega_h) \leq 2\varepsilon h, \ \text{Im} \ z \leq 0\} \int_{\mathbb{C}_-} |(\bar{\partial} z \bar{\chi})(z - \tau/h)| \frac{L(dz)}{\pi}.$$
We write (7.8) as follows

\[
\sum_{\omega \in \text{Res}(P(h)) \cap \Omega} \frac{1}{\pi h} \int_{C_{-}} (\tilde{\phi} \tilde{z}) (z - \tau/h) f(z) \left( \frac{1}{z - \omega} - \frac{1}{z - \bar{\omega}} \right) L(dz)
\]

+ \sum_{\omega \in \text{Res}(P(h)) \cap \Omega, \Im \omega < 0, |\omega - \tau| < Ch} \frac{1}{\pi h} \int_{C_{-}} (\tilde{\phi} \tilde{z}) (z - \tau/h) f(z) \left( \frac{1}{z - \omega} - \frac{1}{z - \bar{\omega}} \right) L(dz) = (1) + (2).

Due to (7.10), the second term is estimate by

\[
(2) \leq C_{2} \frac{1}{\pi h^{n+1}} \int_{C_{-}} |(\tilde{\phi} \tilde{z}) (z - \tau/h) f(z)| L(dz) \leq \tilde{C}_{f, \tau} h^{-n+1},
\]

where \( \tilde{C}_{f, \tau} \) satisfies (2.10).

By using Theorem 4 and the elementary inequality [34, (5.3)]

\[
\int_{\Omega} \frac{1}{|z - \omega|} L(dz) = \leq 2(2\pi \text{vol}(\Omega))^{\frac{1}{2}},
\]

we get the same estimate for the first term.

This completes the proof of (7.8) and also the proof of Theorem 5. \( \square \)

8. Breit–Wigner approximation

We now establish the semi-classical version of the Breit–Wigner approximation. Fix \( \tau \) in \( \sigma(P) \), and assume (H1) and (H2). Let \( \varepsilon > 0 \) be as in Lemma 3. We introduce the real function

\[
\eta(\lambda, h) = s(\lambda, h) - \#\{\mu \in [\tau - \varepsilon, \lambda]; \mu \in \text{Res}(P(h))\}.
\]

Theorem 6. Under the assumptions of Theorem 4, there exists \( 0 < D_{0}, D \) such that for all \( \delta \in ]0, h/D[ \), we have

\[
\eta(\tau + \delta, h) - \eta(\tau - \delta, h) = \sum_{\omega \in \text{Res}(P(h)), \Im \omega < 0, |\omega - \tau| < Ch/D_{0}} \int_{\tau - \delta}^{\tau + \delta} \frac{-\Im \omega}{|\mu - \omega|^{2}} \frac{d\mu}{\pi} + O(\delta)h^{-n}. \tag{8.1}
\]
Proof. Let \( \varphi_j, j = 1, 2, \) be as in Lemma 3. It follows from (5.6) that

\[
\eta(\tau + \delta, h) - \eta(\tau - \delta, h) = \int_{\tau - \delta}^{\tau + \delta} \text{Im} \ g(\mu, h) \frac{d\mu}{\pi} + \sum_{\omega \in \text{Res}(P(h)) \cap \Im \omega < 0} \int_{\tau - \delta}^{\tau + \delta} \frac{-\text{Im} \ \omega}{|\mu - \omega|^2} \frac{d\mu}{\pi},
\]

(8.2)

We recall that \( \varphi_2 = 1 \) near \( \tau \).

The first term of the r.h.s. of (8.2) is \( O(\delta)h^{-n} \), due to (5.4). Combining this with (7.10) and (8.2) we obtain (8.1). □

Appendix A.

In this appendix, we recall some well known results on the \( h \)-pseudodifferential calculus. For the proof we refer to [37, 15, 20]. We use the same notations as in Section 3.1.

Theorem A.1. Let \( P \in S^\delta(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})) \) and \( Q \in S^{\delta'}(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})) \), then

\[
P^w(k, hD_k, h) \circ Q^w(k, hD_k, h) = R^w(k, hD_k, h),
\]

with \( R \in S^{\delta + \delta'}(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})) \) given through

\[
R(k, r, h) = e^{ih(\nabla_r \cdot \nabla_x - \nabla_\xi \cdot \nabla_k)} P(k, r, h) Q(x, \xi, h)|_{x=r, \xi=k}.
\]

Theorem A.2. Let \( P \in S^0(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})) \), then \( P^w(k, hD_k, h) \in \mathcal{L}(L^2(\mathbb{T}^n; \mathbb{C}^N)) \) and

\[
\| P^w(k, hD_k, h) \| \leq C_n \sup_{|\alpha|+|\beta| \leq c_n} \sup_{(k,r) \in E^* \times \mathbb{R}^n} \| \hat{\partial}_k^\alpha \hat{\partial}_r^\beta P(r, k, h) \|_{\mathcal{M}(\mathbb{C}^N)}.
\]

The constants \( C_n \) and \( c_n \) depend only on the dimension \( n \).

Theorem A.3. Let \( P \in S^\delta(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C})) \), with \( \delta > n \). Then \( P^w(k, hD_k, h) \) is trace class operator. Moreover,

\[
\text{tr} \left( P^w(k, hD_k, h) \right) = (2\pi h)^{-n} \int_{E^* \times \mathbb{R}^n} \hat{\text{tr}}(P(k, r, h)) \, dk \, dr
\]

and

\[
\| P^w(k, hD_k, h) \|_{\text{tr}} \leq C_n h^{-n} \sum_{|\beta|+|\alpha| \leq 2n+1} \int_{E^* \times \mathbb{R}^n} \| \hat{\partial}_k^\alpha \hat{\partial}_r^\beta P(r, k, h) \|_{\mathcal{M}(\mathbb{C}^N)} \, dk \, dr.
\]
The constant $C_n$ depends only on the dimension $n$. Here, $\hat{\text{tr}}$ denotes the trace in the set of $N \times N$-matrices.

**Lemma A.4.** Let $P \in S^\delta(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))$ depending holomorphically in $z \in \Omega$, where $\Omega$ is an open set in $\mathbb{C}$. Then, $\partial_z P \in S^\delta(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))$ uniformly for $z \in \Omega_1$, $\Omega_1 \subset \Omega$.

**Proof.** By Gaucho formula, we have

$$\partial_z P(k, r, z, h) = \frac{1}{2\pi i} \int_{\gamma_z(z)} \frac{P(k, r, w, h)}{(z - \omega)^2} dw,$$

where

$$\gamma_z(z) = \{z + e^{it}; 0 \leq t \leq 2\pi\}, 0 < \varepsilon < 1,$

which implies that $\partial_z P \in S^\delta(\mathbb{T}^n \times \mathbb{R}^n; \mathcal{M}_N(\mathbb{C}))$ uniformly on $\{z + \eta e^{it}; 0 \leq \eta \leq \varepsilon/2, 0 \leq t \leq 2\pi\}$. Then the lemma follows by applying the local result and covering the compact set $\Omega_1$ by a finite number of disks $\{z_j + \eta e^{it}; 0 \leq \eta \leq \varepsilon_j/2, 0 \leq t \leq 2\pi\}$. $\square$

**Appendix B.**

In this appendix, we recall some results from [17,13,15,16]. For the details we refer to [17,13].

**Theorem B.1** (Dimassi and Zerzeri [17, Theorem 1.3]). Assume (2.1). For $f \in C^\infty_0(\mathbb{R})$, the operator $[f(P_1(h)) - f(P_0(h))]$ is of trace class. Moreover,

$$\text{tr}[\left[f(P_j(h))\right]_{j=0}^{\infty}] \sim \sum_{j=0}^{\infty} a_j h^{-n+j}, \text{ as } (h \searrow 0),$$

where

$$a_0 = \iint f'(\lambda)\left[\rho(\lambda) - \rho(\lambda - W(x))\right] dx d\lambda.$$

Here

$$\rho(\lambda) := (2\pi)^{-n} \sum_{j \geq 1} \int_{\{k \in E^*: \lambda_j(k) \leq \lambda\}} dk$$

is the integrated density of states associated to $P = -\Delta + V(x)$.

**Theorem B.2** (Dimassi [13, Section 4]). Assume (2.1). Let $\tau$ be satisfying (H1) and (H2). There exist $C >> 1$ and $\sigma > 0$ such that for $0 \in C^\infty(\mathbb{R}) - [\frac{1}{\epsilon}, \frac{1}{\epsilon}]; \mathbb{R}$ and
for all \( m, N \in \mathbb{N} \) we have:

\[
\text{tr} \left( \left[ F_h^{-1}(\lambda - P_j(h)) f(P_j(h)) \right] \right)_{j=0}^1 = \sum_{j=0}^N b_j(\lambda) h^{-n-j} + O(h^N(\lambda)^{-m}), \quad \text{as} \quad (h \searrow 0),
\]

(10.1)

uniformly with respect to \( \lambda \in \mathbb{R} \). The coefficients \( b_j(\lambda) \) are of the form

\[
b_j(\lambda) = \sum_{k=0}^j \frac{1}{j! k!} (f(\gamma_k)^k(\lambda) \theta^k(0),
\]

where \( \gamma_k \) is \( C^\infty \) near \( \text{supp} \ f \).

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References